**The Mutual Dressing of Bare Relativistic Heavy Ions
During Transit in Collider Orbits**

I. General Considerations

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I. Introduction

In the interaction regions of a relativistic heavy-ion collider, bare, highly charged nuclei pass at great velocities, oppositely moving similar nuclei. Each nucleus, then, sees strong, time varying electromagnetic fields, and it is not surprising that there is a large probability for electron-positron pair creation (a process evaluated many years ago).⁽¹⁾ There is also a smaller probability that the electron of the pair is created, not in a continuum state, but in one of the bound orbits, the K or L or higher orbits, of the Coulomb field of the previously bare nuclei. This dressing is most unfortunate for a collider device, since the charge change leaves the nucleus in an orbit that is no longer appropriate for continuing circulation. This note evaluates the probability for such dressing, and also examines whether a subsequent sweep-off process can undo the first step with any sizeable likelihood.

II. Review of the Semi-Classical Method

The very high energies of relative motion guarantee the validity of a classical approach, in which the heavy-ions are taken as moving on straight line trajectories with constant velocity. The probability of the process under consideration can then be evaluated as a function of impact parameter; this is particularly useful in estimating multi-step processes, such as the sweep-off possibility in which the effects of the first step are undone. We have felt the need to examine these secondary effects because the basic (bound- e^- ,

continuum- e^+) cross section is small compared to the photoelectric cross-section that can remove the bound- e^- .

Although the semi-classical Weizsacker-Williams method, together with closely related improvements, are all very well described(2), we put forward a brief rederivation of the key-points to insure coverage of the special assumptions needed here and to insure a uniformity between first and higher orders. The calculation is carried through in the rest-frame of one of the heavy ions, which will be called the target ion. The other, projectile, ion is moving at very high relative energy ($E_p/M_p = 20,000$) on a trajectory with impact parameter b . The fundamental interactions are: (1) the Moller interaction between the projectile and the electron-positron system (2) the Coulomb interaction between the stationary, target ion and the electron-positron system. The cross-section for the given process, here the creation of a bound-electron and a positron, is just

$$\sigma = \int d^2b P(b), \quad (1a)$$

$$P(b) = |\langle B-e^-, e^+ | S(b) - 1 | 0 \rangle|^2, \quad B = K, L, \dots, \quad (1b)$$

$$S(b) = 1 + \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt' V(t') + \left(\frac{1}{i\hbar}\right)^2 \int_{-\infty}^{\infty} dt' \int_{-\infty}^{t'} dt'' V(t') V(t'') + \dots; \quad (1c)$$

In this calculation, $V(t)$ will be restricted in all orders to the Moller interaction, ignoring crossed photon diagrams and radiative corrections of many kinds. The lowest order is illustrated in Figure 1a; the only higher orders, to be discussed later, are shown in Figure 1b, 1c. Then,

$$V(t) = 4\pi \int d\tau_p d\mathbf{r}_p \frac{d^3q}{(2\pi)^3} \frac{cdq_0}{2\pi} \frac{e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}_p) - i\epsilon_0 c(t - t_p)}}{q^2 - q_0^2} \left(\rho\rho_p - \frac{1}{c^2} \mathbf{J} \cdot \mathbf{J}_p \right), \quad (2)$$

$$\rho_p = e\delta(\mathbf{r}_p - \mathbf{v}_p t - \mathbf{b}), \quad \mathbf{J}_p = e\mathbf{v}_p \delta(\mathbf{r}_p - \mathbf{v}_p t - \mathbf{b}).$$

Standard Approximation

As a first step, the lowest order, Figure 1a, will be explored. The evaluation of the S-matrix element involves spatial and time integrals over the electron-positron variables; it is useful to hold-off the spatial integrals but to carry out the time-integral at the beginning. With

$$w = E_-(B) + E_+, \quad B = K, L, \dots$$

$$\int dt' V(t') = 4\pi \int d\tau_p d\mathbf{r}_p \frac{d^3q}{(2\pi)^3} \frac{e^{i\mathbf{q} \cdot \mathbf{r} - \mathbf{r}_p} e^{iwt_p}}{q^2 - w^2/c^2} \left(\rho\rho_p - \frac{1}{c^2} \mathbf{J} \cdot \mathbf{J}_p \right) dt_p d\tau_p. \quad (3)$$

Separation of longitudinal and transverse contributions is made explicit by the introduction of the familiar unit vectors, $\mathbf{e}$, defined as orthogonal to $\mathbf{q}$; then

$$\frac{\rho\rho_p - \frac{1}{c^2} \mathbf{J} \cdot \mathbf{J}_p}{q^2 - w^2/c^2} = \frac{\rho\rho_p}{q^2} - 1/c^2 \frac{\mathbf{J} \cdot \hat{\mathbf{e}}_1}{q^2} \frac{\mathbf{J}_p \cdot \hat{\mathbf{e}}_1}{q^2}. \quad (4)$$

It will be seen that the leading contribution, proportional to $\ln(E_p/M_p c^2)$, comes about from the near singularity in the photon-propagator denominator, at $q^2 = w^2/c^2$. Since the first term, the static Coulomb, lacks this important capability it cannot contribute in this leading order, and is henceforth dropped, along

with other lower orders; also, the Coulomb matrix element between positive and negative energy states is small. Then, carrying out the integrals over the projectile coordinates leads to $\delta(w - v_p q_z)$, and with $q = q_z \hat{z} + q_T$,

$$\int_{-\infty}^{\infty} dt' V(t') \Big|_T = -4\pi e Z_p \frac{1}{c} \int \frac{d^2 q_T}{(2\pi)^2} \frac{e^{i w/v_p z} e^{i \vec{q}_T \cdot \vec{\rho}} e^{-i \vec{q}_T \cdot \vec{b}} \frac{1}{c} \vec{J} \cdot \hat{e}_1 \hat{z} \cdot \hat{e}_1}{q_T^2 + \frac{w^2}{v_p^2} \frac{1}{\gamma^2}}, \quad (5a)$$

it has been recognized that,

$$\frac{w^2}{v_p^2} - \frac{w^2}{c^2} = \frac{w^2}{v_p^2} \left(1 - \frac{v_p^2}{c^2}\right) = \frac{w^2}{v_p^2} \frac{1}{\gamma^2}; \quad (5b)$$

also a variable $\vec{\rho}$ is used that is just the projection of $\vec{r}$ on the plane perpendicular to v_p , which has been taken as the Z-axis. A definite choice for the $\hat{e}_1$ -vectors completes the formal calculation; a simplifying choice consists of:

$$\begin{aligned} \hat{e}_1 &= \frac{\vec{q} \times \hat{z}}{|\vec{q} \times \hat{z}|} \\ \hat{e}_2 &= \frac{\hat{e}_1 \times \vec{q}}{|\hat{e}_1 \times \vec{q}|} = \frac{-q_z \frac{\vec{q}_T}{|q_T|} + q_T \hat{z}}{[q_T^2 + q_z^2]^{1/2}} = \frac{-\frac{w}{v_p} \frac{\vec{q}_T}{|q_T|} + q_T \hat{z}}{[q_T^2 + (w/v_p)^2]^{1/2}}; \end{aligned} \quad (6)$$

further since $\hat{e}_1$ is orthogonal to $\hat{z}$, only $\hat{e}_2 \equiv \hat{e}$ contributes; then,

$$\begin{aligned} \vec{J} \cdot \hat{e} &= \frac{1}{\sqrt{2}} \{J_+ e^{-i\phi} - J_- e^{i\phi}\} \frac{w/v_p}{[q_T^2 + (w/v_p)^2]^{1/2}} \\ &+ J_z \frac{q_T}{[q_T^2 + (w/v_p)^2]^{1/2}}, \end{aligned} \quad (7)$$

where ϕ is the polar angle of $\vec{q}_T$. Of these two terms, only the first involving J_+ , J_- will be seen to contribute in leading order; it will, therefore, be examined first. We have, therefore, to consider:

$$\int_{-\infty}^{\infty} dt' V(t') \Big|_{T, \pm} = -4\pi \frac{e}{c} z_p e^{i w/v_p z} \int \frac{d\phi dq_T}{(2\pi)^2} q_T \left[\frac{1}{\sqrt{2}} (\pm) \frac{1}{c} J_{\pm} e^{\mp i \phi} \right] \frac{w/v_p}{[q_T^2 + (w/v_p)^2]^{1/2}} \frac{q_T}{[q_T^2 + (w/v_p)^2]^{1/2}} \frac{1}{q_T^2 + (w/v_p)^2} e^{i \vec{q}_T \cdot \vec{\rho}} e^{-i \vec{q}_T \cdot \vec{b}} \frac{1}{\gamma^2} \quad (8)$$

The important point to note is that, for $\gamma \gg 1$, the factor

$[q_T^2 + (w/v_p)^2 (1/\gamma^2)]^{-1}$ weights small q_T , $q_T \ll w/v_p$, very importantly, then, $(\vec{q}_T \cdot \vec{\rho}) \ll (w/v_p) z$ and the factor $e^{i \vec{q}_T \cdot \vec{\rho}} \approx$

1. A more quantitative proof will be presented below, but first, in order to connect with familiar results, let us assume the validity of these arguments and evaluate the consequences of using the approximate form:

$$\begin{aligned} \langle B^- e^-, e^+ \Big|_{w} \Big| \int_{-\infty}^{\infty} dt' V(t') \Big| 0 \rangle &= (\mp) \frac{4\pi e}{\sqrt{2}} z_p \langle B^- e^-, e^+ \Big|_{w} \Big| \frac{1}{c} J_{\pm} e^{i w/v_p z} \Big| 0 \rangle \times \\ &\quad \left(\frac{w}{v_p} \right) \int \frac{d\phi dq_T}{(2\pi)^2} e^{-i \vec{q}_T \cdot \vec{b}} e^{\mp i \phi} \frac{q_T^2}{q_T^2 + (w/v_p)^2} \frac{1}{q_T^2 + (w/v_p)^2} \frac{1}{\gamma^2} \\ &= (\mp) \frac{2\pi e}{c} z_p \langle B^- e^-, e^+ \Big|_{w} \Big| \frac{1}{c} J_{\pm} e^{i w/v_p z} \Big| 0 \rangle \left(\frac{w}{v_p} \right) \int dq_T J_1(b q_T) \\ &\quad \frac{q_T^2}{[q_T^2 + (w/v_p)^2] [q_T^2 + (w/v_p)^2 \frac{1}{\gamma^2}]} \\ &= (\mp) \sqrt{2} \frac{e}{c} z_p \langle B^- e^-, e^+ \Big|_{w} \Big| \frac{1}{c} J_{\pm} e^{i w/v_p z} \Big| 0 \rangle \left\{ \frac{\gamma}{\gamma^2 - 1} K_1 \left(\frac{b w/v_p}{\gamma} \right) - \frac{\gamma^2}{\gamma^2 - 1} K_1(b w/v_p) \right\}, \quad (9) \end{aligned}$$

where K_1 is the standard form of the modified Bessel function. Then, in lowest order and to the approximations outlined,

$$P_1(b) = 2\alpha Z_p^2 \frac{1}{c} \int d\omega \rho(\omega) \sum_{\pm} \left| \langle B^- e^-, e^+ | \frac{1}{c} J_{\pm} e^{\frac{i\omega}{V} p^z} | 0 \rangle \right|^2 \times \left| \frac{\gamma}{\gamma^2 - 1} K_1\left(\frac{b\omega}{\gamma V}\right) - \frac{\gamma^2}{\gamma^2 - 1} K_1\left(\frac{b\omega}{V}\right) \right|^2. \quad (10)$$

Since only very high projectile energies are being considered, $V_p = c$; then, the matrix element that appears above is just that corresponding to a real photon process, and, in fact, the photon-induced cross-section is just

$$\phi_{\gamma}(\omega) = \frac{(2\pi)^2}{\omega/c} \rho(\omega) \frac{1}{2} \sum_{\pm} \left| \langle B^- e^-, e^+ | \frac{1}{c} J_{\pm} e^{\frac{i\omega z}{c}} | 0 \rangle \right|^2. \quad (11)$$

Then,

$$P_1(b) = \frac{\alpha Z_p^2}{\pi^2} \int \frac{d\omega}{c} \phi_{\gamma}(\omega) \frac{\omega}{c} \left| \frac{\gamma}{\gamma^2 - 1} K_1\left(\frac{b\omega}{\gamma c}\right) - \frac{\gamma^2}{\gamma^2 - 1} K_1\left(\frac{b\omega}{c}\right) \right|^2. \quad (12)$$

It is worth noting that the $b \rightarrow 0$ limit of $P_1(b)$ goes as $b^2 \ln^2 b$ and not $1/b^2$; even more to the point, on carrying through the integration over impact parameters, the cross-section takes the form:

$$\sigma = \int_0^{\infty} d^2b P_1(b) = \frac{2}{\pi} \alpha Z_p^2 \left(\frac{\gamma^2}{\gamma^2 - 1}\right)^2 \left\{ \left(1 + \frac{2}{\gamma^2 - 1}\right) \ln \gamma - 1 \right\} \int \frac{d\omega}{\omega} \phi_{\gamma}(\omega), \quad (13a)$$

$$\approx \frac{2}{\pi} \alpha Z_p^2 \ln \gamma / e \int \frac{d\omega}{\omega} \phi_{\gamma}(\omega), \quad (13b)$$

which differs from the usual Weizsacker-Williams result,

$$\ln \gamma / \omega b_{\min}, \quad (14)$$

in the replacement of the logarithm by one that differs in significant detail. Since this is a well known result, no general comment is

needed; an important consequence of the Weizsacker-Williams form has to do with a related concept, a cut-off in the w -integration: the mechanism that produces the particular logarithmic dependence also requires that $\gamma/wb_{\min} > 1$ or

$$w < \gamma/b_{\min}. \quad (15)$$

Such a w -cut-off makes no practical difference for the present problem of bound- e^- , e^+ production, since the cross-sections peak at finite w and decrease as $w/m_e c^2$ becomes very large. However, the continuum- e^- , e^+ (pair) production cross-section does not have such an inherent cut-off, and in fact, increases as $\ln w/m_e c^2$; the existence of the cut-off in the excitation mechanism and its nature is then a more pressing problem. The cut-off will be seen below to come about from a q_T -dependence of the matrix element for very large $w/m_e c^2$, a regime in which it is not proper to discard such dependences without careful consideration; for the present problem the approximation used above will be seen to be adequate.

Before going on we can readily dispose of the J_z terms, that appeared in eq. (7). The equivalent of eq.(9) will involve the factor,

$$\left\{ \frac{\gamma^2}{\gamma^2-1} K_0(bw/v_p) - \frac{1}{\gamma^2-1} K_0\left(\frac{b(w/v_p)}{\gamma}\right) \right\}, \quad (16)$$

to replace the curly bracket. Because of the differing rotational (about z) properties, the J_z and $J_{\pm}$ terms contribute incoherently. On carrying out the integration over impact parameter

it is immediately seen that the J_z contribution is of order $1/\gamma^2$ smaller than that of the $J_{\pm}$ terms; the order of magnitude can be read out of eq (7) on noting that q_T is effectively limited within the region $((w/v_p)/\gamma)$ and therefore, the relative weighting of J_z and $J_{\pm}$ is $\sim 1/\gamma$. These terms J_z may be safely dropped; in fact, they must be dropped along with the q_T -dependence, $e^{iq_T \cdot \rho}$, since they are linked by the requirement of transversality.

Form Factor and q_T -Dependences

In order to explore the effects of the omitted q_T -dependence (omitted in dropping the factor $e^{iq_T \cdot \rho}$), hypothesize that the full matrix element, including the q_T -dependence, depends on the momentum transfer as $(1/\mu^2 + q^2)$; μ is a spatial scale parameter, which for the present problem is to be assumed as of the order of the bound- e^- orbit radius; $q^2 = [q_z^2 + q_T^2] = [(w/c)^2 + q_T^2]$.

Normalizing at $q_T^2 = 0$ results in the form factor

$$\frac{\mu^2 + (w/c)^2}{\mu^2 + (w/c)^2 + q_T^2} \quad (17)$$

The specific task at hand is, then, calculation of the effects of inclusion of this form factor into eq. (9); that is, to compare

$$\frac{w}{c} \int dq_T J_1(bq_T) \frac{q_T^2}{[q_T^2 + (w/c)^2] \{ [q_T^2 + (w/c)^2] \frac{1}{\gamma^2} \}} \left\{ \frac{\mu^2 + (w/c)^2}{\mu^2 + (w/c)^2 + q_T^2} \right\} \quad (18)$$

with the calculations already carried out, that is, with the last factor absent. The integral (18) is easily found to be:

$$\begin{aligned}
 & \left\{ -\frac{\gamma}{\gamma^2-1} K_1\left(\frac{bw/c}{\gamma}\right) - K_1(bw/c) \right\} \\
 & + \frac{(w/c)^2 \frac{1}{\gamma^2-1}}{\mu^2 + (w/c)^2 (1-1/\gamma^2)} \frac{1}{\gamma} K_1\left(\frac{bw/c}{\gamma}\right) - \frac{(w/c)^2}{\mu^2} \frac{\gamma^2}{\gamma^2-1} K_1(bw/c) \quad (19) \\
 & + \frac{\{(w/c)^2 + \mu^2\}^2}{\mu^2 [\mu^2 + (w/c)^2 (1-1/\gamma^2)]} \frac{w/c}{[\mu^2 + (w/c)^2]^{1/2}} K_1\{b[\mu^2 + (w/c)^2]^{1/2}\} .
 \end{aligned}$$

The first bracket is just that obtained without the form factor and the second represents the changes due to the form factor. Finally if the cross-section expression is formed, a result similar to eq (13b) is found except that the expression in curly brackets is replaced by:

$$\begin{aligned}
 & \ln \gamma \left[\frac{\mu^2 + w^2}{\mu^2 + w^2 (1 - \frac{1}{\gamma^2})} \right]^2 \left\{ 1 + \frac{2}{\gamma^2-1} + \frac{2}{\gamma^2 (1 + \frac{\mu^2}{w^2} - \frac{1}{\gamma^2})} \right\} - \frac{1}{2} \left[\frac{\mu^2 + w^2}{\mu^2 + w^2 (1 - \frac{1}{\gamma^2})} \right]^2 \\
 & - \frac{1}{2} \left[\frac{\mu^2 + w^2}{\mu^2} \right]^2 \left\{ 1 + \frac{(1 - \frac{1}{\gamma^2})^2}{(1 + \frac{\mu^2}{w^2} - \frac{1}{\gamma^2})^2} \left[1 + \right. \right. \\
 & \left. \left. \ln \left(1 + \frac{\mu^2}{w^2} \right) \left(1 - 2 \frac{1 + \frac{\mu^2}{w^2}}{1 + \frac{\mu^2}{w^2} - \frac{1}{\gamma^2}} + 2 \frac{1 + \frac{\mu^2}{w^2}}{\frac{\mu^2}{w^2}} \right) \right] \right\} \quad (20)
 \end{aligned}$$

In the limit $\mu \rightarrow \infty$, it, of course, becomes just that the result in (13b). More important, the $\ln \gamma$ dependence is affected by the form factor inclusion only to the extent that $w^2 \rightarrow w^2 + (w^2/\gamma^2)$; that is, there are changes to order $(\ln \gamma)/\gamma^2$ but not $\ln \gamma$. The terms of order 1 are affected importantly, but, of course, these terms must be discarded in conformity with the omission of the Coulomb and the J_z

terms of eq (7). This calculation with a form factor demonstrates that the $q_T \sim (w/c)/\gamma$ approximation is valid for the leading, $\ln \gamma$, terms, but is not so for the other terms. The question of how much beyond the $\ln \gamma$ term should be kept is, of course, a theological one, since, if it matters, it is all a poor approximation; we have taken the approximate form $\ln \gamma/e$ in eq 13(b), while eq (20) emphasizes the arbitrariness of so definite a choice.

It is to be noted that eq (20) does not provide a high energy cut-off; that is, for $w \rightarrow \infty$, or $(\mu/w) \rightarrow 0$, the $\ln \gamma$ term is unchanged, and the other parts remain innocuously finite. (The finiteness of the $(\mu/w) \rightarrow 0$ limit is not apparent from a glance at eq (20), but, if analyzed in detail it does emerge; it follows more succinctly by returning to the original integral, eq (18) and noting that placing $\mu/w = 0$ leaves a finite integral.) As already noted, for the present problem no additional cut-off is needed since the w -dependence of $\phi_Y(w)$ produces its own cut-off; the decrease of $\phi_Y(w)$ at high w can be traced to the decrease of the necessary momentum that can be supplied by the bound- e^- wave function.

From where does the cut-off come in the case of pair production,⁽¹⁾ e^- and e^+ into continuum states? Certainly, the $\ln w$ dependence of $\phi_Y(w)$ at high w poses an immediate convergence problem. The resolution can be understood in terms of a q_T -dependence quite different from that analyzed above. An important factor in the pair cross-section is that corresponding to the absorption of recoil momentum, q_R , by the Coulomb field,

$$\left(\frac{Ze^2}{q_R}\right)^2; \quad (21)$$

the q_R -dependence has been worked out explicitly⁽³⁾; there it is demonstrated that the important contributions come about, for w large, that is, from the region,

$$cq_R \sim (\text{numerical factor}) \times \frac{(m_e c^2)^2}{w} \sim n \frac{(m_e c^2)^2}{w} \quad (22)$$

and at considerable angles, $\sim \pi/4$ to $\pi/2$, with respect to the photon direction. Then, the q_T will add to the q_R , producing an overall factor proportional to terms of the form

$$\frac{Ze^2}{\left[\frac{1}{c} n \frac{(m_e c^2)^2}{w} + q_T\right]^4}, \quad \frac{Ze^2}{\left[\left(\frac{1}{c} n \frac{m_e c^2}{w}\right)^2 + q_T^2\right]^2}; \quad (23)$$

if we consider only the leading, $\ln \gamma$, term, then it is expected that $q_T \sim (w/c)/\gamma$ and cut-off is reached in the region

$$\frac{w}{\gamma} \sim n \frac{(m_e c^2)^2}{w}. \quad (24)$$

Such a high energy cut-off is not relevant in the bound- e^- creation problem under consideration in this note.

Further Analyses of q_T -Dependences It was pointed out in the paper of Winther and Alder⁽²⁾ that there are transitions in familiar physical systems for which the $(\ln \gamma)$ term is not dominant: L-pole transitions, $L \geq 2$, in systems effectively circumscribed by a dimension, R , whose transition energy is sufficiently small so that $wR \ll 1$. The coherent collective resonant transitions of nuclear physics, other than the "giant" electric dipole, form just such a set;

however, the strength of the electric-dipole would generally dominate and be well given by its $(\ln \gamma)$ term. There are imaginable conditions, as given by special and judicious choice of angular conditions, in which the higher nuclear multipoles are selected and the Weissacker-Williams formalisms are, then, not applicable. These considerations are not of concern in the present pair creation problem for the basic condition, $wR \ll 1$, does not obtain here; thus, since the transition energy is $\sim (2-4)m_e c^2$, and the K-shell orbit dimension is $\sim \lambda_c / \alpha Z$, $wR \sim 3/\alpha Z > 1$. For the considerations of the secondary sweep-off process taken up in the next sections it might be thought to be more nearly relevant, since the photoejection of the created bound electron is dominated by the threshold edge-effect; however for two reasons it is not practically important: 1) $wR \sim [(\alpha Z)^2 m_e c^2 \lambda_c] \sim \alpha Z \sim 1$ for the heavy elements in which any effect is worth considering, and 2) since only the total sweep-off cross-section enters, the electric dipole dominates and so then does the $\ln \gamma$ dependence.

In spite of its promised non-importance to the present problem, this section will be devoted to a brief derivation and analysis of the terms discussed above, which, under certain conditions, will be seen to dominate over those amplified by $(\ln \gamma)$. This is done, in part, out of a duty to completeness, and, in more important part, to make it immediately applicable to the previous considerations. Winther and Alder, in order to fulfill the physically reasonable demands of their nuclear problem, discarded all impact parameters smaller than a

cut-off of order of the nuclear radius. Since no such physical requirement is present in the present pair-creation exercise and none has been used in the previous sections' analyses, a calculation that parallels the previous sections will be written out here.

As a first step, we begin by reconsidering the transverse terms written in eq (8). Without the approximation of $e^{i\vec{q}_T \cdot \vec{\rho}} = 1$, we recalculate

$$\begin{aligned} \int d^2 b \tilde{P}_T(b) &= \int d^2 b \left| \langle B-e, e^+ \left| \int_{-\infty}^{\infty} dt' V(t') \right| 0 \rangle_{T,+} \right|^2 \\ &= \frac{1}{2} \left(4\pi \frac{e}{c} \frac{Z}{p} \right)^2 \left(\frac{w}{v} \right)^2 \int d^2 b \int \frac{d^2 q_T}{(2\pi)^2} \frac{d^2 q_T'}{(2\pi)^2} \langle B-e, e^+ \left| \frac{1}{c} J_{+e} e^{-i\phi} e^{\frac{iw}{vp} z} e^{i\vec{q}_T \cdot \vec{\rho}} e^{-i\vec{q}_T' \cdot \vec{\rho}} \right| 0 \rangle \\ &\quad \langle B-e, e^+ \left| \frac{1}{c} J_{+e} e^{-i\phi'} e^{\frac{iw}{vp} z'} e^{i\vec{q}_T' \cdot \vec{\rho}'} e^{-i\vec{q}_T \cdot \vec{\rho}} \right| 0 \rangle^* \end{aligned} \quad (25)$$

$$\frac{q_T}{[q_T^2 + (w/vp)^2] [q_T^2 + (w/vp)^2 \frac{1}{\gamma^2}]} \frac{q_T}{[q_T'^2 + (w/vp)^2] [q_T'^2 + (w/vp)^2 \frac{1}{\gamma^2}]}.$$

where the tilde, on $\tilde{P}_T(b)$, is to recall the inclusion of the full retardation, and the T-subscript is affixed to emphasize that only the transverse term has been considered. Instead of proceeding as before, having already demonstrated the finiteness of the integrand at small values of b, let us now carry out the b-integration at once to obtain $(2\pi)^2 \delta(\vec{q}_T - \vec{q}_T')$. Then

$$\begin{aligned} \int d^2 b \tilde{P}_T(b) &= (2\alpha Z_p^2) \left(\frac{w}{v} \right)^2 \int d^2 q_T \frac{q_T^2}{[q_T^2 + (w/vp)^2]^2 [q_T^2 + (w/vp)^2 \frac{1}{\gamma^2}]^2} \\ &\quad \left| \langle B-e, e^+ \left| \frac{1}{c} J_{+e} e^{\frac{iw}{vp} z} e^{i\vec{q}_T \cdot \vec{\rho}} \right| 0 \rangle \right|^2; \end{aligned} \quad (26)$$

with the usual expansion of the two-dimensional plane-wave,

$e^{i\vec{q}_T \cdot \vec{\rho}}$, in Bessel functions

$$\int d^2 b \tilde{P}_T(b) = (4\pi\alpha Z_p^2) \left(\frac{w}{V_p}\right)^2 \int d^2 q_T \frac{q_T^3}{[q_T^2 + (w/V_p)^2][q_T^2 + (w/V_p)^2 \frac{1}{\gamma^2}]^2} \times$$

$$\left| \langle B^{-e}, e^+ \left| \frac{1}{c} J_{+e} e^{\frac{i w}{V} z} e^{i n \phi} J_n(q_T \rho) \right| 0 \rangle \right|^2. \quad (27)$$

The q_T -integration can be written in terms of the modified Bessel functions, I_n and K_n :

$$\int d^2 b \tilde{P}_T(b) = 4\pi\alpha Z_p^2 \left(\frac{V_p}{w}\right)^2 \int_n \frac{1}{(1 - \frac{1}{\gamma^2})^2} \theta(\rho' - \rho) \left\{ \frac{1}{2x} \frac{d}{dx} \Big|_{x=1} + \frac{1}{\gamma^2} \frac{1}{2x} \frac{d}{dx} \Big|_{x=1/\gamma} \right.$$

$$\left. + \frac{1 + \frac{1}{\gamma^2}}{1 - \frac{1}{\gamma^2}} \left[\Big|_{x = \frac{1}{\gamma}} - \Big|_{x=1} \right] \right\}$$

$$\times \left\{ \langle B^{-e}, e^+ \left| \frac{1}{c} J_{+e} e^{\frac{i w}{V} z} e^{i n \phi} I_n\left(x \frac{w}{V_p} \rho\right) \right| 0 \rangle \right.$$

$$\left. \langle B^{-e}, e^+ \left| \frac{1}{c} J_{+e} e^{\frac{i w}{V} z'} e^{i n \phi'} K_n\left(x \frac{w}{V_p} \rho'\right) \right| 0 \rangle^* \right\}$$

$$+ (\text{exchange of } I_n, K_n \text{ for region } \rho' < \rho); \quad (28)$$

a somewhat inconsistent notation has been used in depicting the matrix elements in order to avoid a fuller rendition of their detailed structure. For the practical domain, $\gamma \gg 1$, $\gamma \gg \frac{w}{V_p} \rho$,

$$\begin{aligned}
 & \left\{ \frac{1}{2x} \frac{d}{dx} \Big|_{x=1} + \frac{1}{\gamma^2} \frac{1}{2x} \frac{d}{dx} \Big|_{x=\frac{1}{\gamma}} + \frac{1+\frac{1}{\gamma^2}}{1-\frac{1}{\gamma^2}} \left[\Big|_{x=\frac{1}{\gamma}} - \Big|_{x=1} \right] \right\} I_n \left(\frac{x}{\gamma} \right) K_n \left(\frac{x}{\gamma} \right) \\
 &= \frac{1}{2x} \frac{d}{dx} I_n \left(\frac{x}{\gamma} \right) K_n \left(\frac{x}{\gamma} \right) \Big|_{x=1} - \frac{1}{2} \delta_{n,0} \\
 &- \left[\ln \left(\frac{W}{V} \right)^{\rho'} / 2\gamma \right] \delta_{n,0} - I_n \left(\frac{W}{V} \right)^{\rho} K_n \left(\frac{W}{V} \right)^{\rho'} \\
 &+ \frac{1}{2n} \left(\frac{\rho}{\rho'} \right)^n (1 - \delta_{n,0}). \tag{29} \\
 &\gamma_c = 0.577...
 \end{aligned}$$

In the domain, $\frac{W}{V} \rho, \frac{W}{V} \rho' \ll 1$, the leading order terms of this combination are:

$$\begin{aligned}
 & \delta_{n,0} [\ln \gamma - 1] + \delta_{n,1} \left[\frac{1}{8} \left(\frac{W}{V} \right)^{\rho} \left(\frac{W}{V} \right)^{\rho'} \right. \\
 & \left. + \delta_{n>1} \left[\theta \left(\frac{W}{V} r \right)^4 + \theta \left(\frac{W}{V} r \right)^4 \ln \left(\frac{W}{V} r \right) \right] \right], \tag{30}
 \end{aligned}$$

where r in the last bracket stands for some combination of ρ, ρ' with the dimensions of a length. The first term is the $(\ln \gamma/e)$ obtained in the previous sections. The other terms are of the general form of the Winther-Alder terms, but, in fact, will be seen to be two and more orders smaller. Only the $n = 0$ part of the above is important. The leading Winther-Alder terms are not here, but will be found in the Coulomb contribution, which was correctly dropped in seeking the $(\ln \gamma)$ terms.

The analysis of the Coulomb term is similar but simpler:

$$\int d^2b P_c(b) = (4\pi e Z_p)^2 \int \frac{d^2 q_T}{(2\pi)^2} \frac{1}{(q_T^2 + \frac{w^2}{v^2})^2}$$

$$\langle B-e, e^+ | \rho_c e^{\frac{i w}{v} z} e^{i q_T \cdot \vec{\rho}} | 0 \rangle \langle B-e, e^+ | \rho_c e^{\frac{i w}{v} z'} e^{i q_T \cdot \rho'} | 0 \rangle^*$$

$$= 8\pi \alpha Z_p^2 \left(\frac{v}{w}\right)^2 \sum_n \left\{ \theta(\rho' - \rho) \frac{(-1)^n}{2x} \frac{d}{dx} \right\}_{x=1} \quad (31)$$

$$\left[\langle B-e, e^+ | \rho_c e^{\frac{i w}{v} z} e^{i n \phi} \rho I_n\left(\frac{w \rho}{v}\right) | 0 \rangle \langle B-e, e^+ | \rho_c e^{\frac{i w}{v} z'} e^{i n \phi'} \rho K_n\left(\frac{w \rho'}{v}\right) | 0 \rangle^* \right]$$

$$+ (\text{exchange of } I_n, K_n \text{ for region } \rho' < \rho) \rangle.$$

In the domain $\frac{w}{v} \rho, \frac{w}{v} \rho' \ll 1$, the leading terms are

$$\theta(\rho' - \rho) \left\{ -\frac{1}{2x} \frac{d}{dx} \right\}_{x=1} I_n\left(\frac{w}{v} \rho\right) K_n\left(\frac{w}{v} \rho'\right)$$

$$= \left\{ \frac{1}{2} \delta_{n,0} + \delta_{n,1} \left[-\frac{1}{4} \left(\frac{w}{v} \rho\right) \left(\frac{w}{v} \rho'\right) 2n \left(\frac{w}{v} \rho' / 2\right) \right] \right. \quad (32)$$

$$\left. + \delta_{n \geq 2} \left[-\frac{1}{2n} \right] \left(\frac{\rho}{\rho'}\right)^n \left[\frac{1}{n+1} \left(\frac{w}{v} \rho\right)^2 - \frac{1}{4(n-1)} \left(\frac{w}{v} \rho'\right)^2 \right] \theta(\rho' - \rho) \right\}.$$

Comparison with the leading term, that produced by the $n=0$ part of eq(30), is made simpler if it is recalled that in order of magnitude

$$\frac{j}{c} \hat{e} = \left(\frac{w}{c^2}\right) \rho_c e^{i \phi} \rho.$$

Then, if the long wave-length limit is appropriate the contribution of the n^{th} term, $n \geq 2$, compared to the $(\ln \gamma)$ term is in order of magnitude:

$$\frac{1}{\ln \gamma} \left| \frac{\rho_c \left(\frac{w}{v} z \right)^{L-n} e^{i n \phi_p} \frac{w}{v} p}{J_{+1} \left(\frac{w}{v} z \right)^{L-1}} \right|^2 - \frac{1}{\ln \gamma} \left| \frac{J_{+1} \left(\frac{c}{w} \right) e^{-i \phi_p} \left(\frac{w}{v} z \right)^{L-n} e^{i n \phi_p} \frac{w}{v} p}{J_{+1} \left(\frac{w}{v} z \right)^{L-1}} \right|^2$$

$$= \frac{1}{\ln \gamma \left(\frac{wR}{c} \right)^{2(n-1)}}, \quad (33)$$

where $R = Z, p$. Now for a given electric multipole of order L , the highest possible azimuthal n value is just L , and this implies that the transition would prefer that mode by a factor

$$\frac{1}{(\ln \gamma) \left(\frac{wR}{c} \right)^{2(L-1)}} \quad (34)$$

over the Weizsacker-William's contribution.

So far in this section, only the electric multipoles have been considered. However, as will be seen immediately below, a similar analysis of the contributions of the longitudinal current, the J_z term of equation (7), demonstrates for magnetic multipole transitions a similar pattern. Proceeding as before, the contribution of this term is

$$\int d^2 b \tilde{P}_L(b) = (4\pi e Z_p)^2 \int \frac{d^2 q_T}{(2\pi)^2} \frac{q_T^4}{\left(q_T^2 + \frac{w^2}{v^2} \right)^2 \left(q_T^2 + \frac{w^2}{v^2} \frac{1}{\gamma^2} \right)^2} \quad (35)$$

$$\langle B-e, e^+ | J_z e^{\frac{i w}{v} z} \frac{1}{p} e^{i q_T \cdot p} | 0 \rangle \langle B-e, e^+ | J_z e^{\frac{i w}{v} z'} \frac{1}{p} e^{i q_T \cdot p'} | 0 \rangle .$$

Because of the higher power of q_T that appears in the numerator, it is possible to go immediately to the limit $\gamma \rightarrow \infty$ (unlike the transverse case, where the $\ln \gamma$ dependence must be first extracted). Then, the large γ limit is just

$$\int d^2 b P_L(b) = (4\pi e Z_p)^2 \int \frac{d^2 q_T}{(2\pi)^2} \frac{1}{\left(q_T^2 + \frac{w^2}{v^2}\right)^2} \quad (36)$$

$$\chi \langle B-e, e^+ | J_z e^{\frac{i w}{v} z} e^{i q_T \cdot \rho} | 0 \rangle \langle B-e, e^+ | J_z e^{\frac{i w}{v} z} e^{i q_T \cdot \rho} | 0 \rangle.$$

This is just of the form already seen for the Coulomb terms and the results of that calculation can be taken over immediately. If the long wave-limit is appropriate, the contribution of the n^{th} term, $n > 2$, compared to the $(\ln \gamma)$, transverse term, is in order of magnitude

$$\frac{\left| J_z \left(\frac{w}{v} z \right) L^{-n} e^{i n \phi} \rho \left(\frac{w}{v} \rho \right) \right|^2}{\ln \gamma \left| J_{+1} \left(\frac{w}{v} \right) L \right|^2} = \frac{1}{\ln \gamma \left(\frac{w R}{c} \right)^{2(n-1)}} \quad (37)$$

Clearly for $L > 1$ and (wR) sufficiently small, the $\ln \gamma$ enhancement is a poor bargain, and Weizsacker-Williams methods do not result in adequate estimates. However, even for such systems, the $L=1$ transition is adequately given by the Weizsacker-Williams estimates, and, in general, the $L=1$ will dominate the total cross-sections. Then, while a Weizsacker-Williams treatment for all multipoles will include mis-estimates of the higher multipoles, these mis-estimate errors will

amount only to the order of terms neglected in comparison with those weighted by $(\ln \gamma)$. Finally, it is to be emphasized that if the long wave-length limit is not the case, as $wR \sim \theta(1)$ or larger, then the $(\ln \gamma)$ terms dominate for all multiplicities—as can be checked from the unexpanded forms, eqs.(29), (32), since $(I_n K_n) < 1$ and $(e^{1(w/v_p)z} \ln \gamma)) - \ln \gamma > 1$.

It is to be noted that the analyses of this section did not make a separation in terms of a multipole decomposition, but rather mixed all those that contributed to a given z -component of angular momentum. For an order of magnitude estimate this is adequate since, a given n will contain contributions to all multipoles $L \geq n$, but for a given L the dominant relative effect is for the maximum possible n , $n = L$. It should be further confessed that if we had to go beyond an order of magnitude estimate, the polar co-ordinate form would be most unpleasant. Fortunately, there is no need to go further with these terms for the present problem.

Higher Orders—Estimate of Sweep-Off and Damping

As noted earlier on, we were concerned about the possible importance of second order processes. The reasons for concern lie in the numerics; in the next sections, it will be shown that for the case of 100 GeV gold-ions colliding with 100 GeV gold ions, a relative motion characterized by $\gamma = 20,000$, the cross-section for creation of a positron and a K-shell electron is 150 barns; the cross-section for removal of a K-electron carried along by one of the gold-ions is - 55,000 barns.

To answer the question, the effects of the photoejection (called by us the sweep-off) of the created bound electron will be explicitly calculated. Out of the myriad of higher order corrections, we will focus only on those corresponding to the physical sweep-off process and keep only the on-shell contributions. As a first step, consider the next order, taken to be that corresponding to Fig. 1b, and ignoring radiative corrections such as those corresponding to mass or vacuum polarization insertions. Further, only the time-forward-going electron lines are kept. In short, from the entire Greens' function only the pole term will be used:

$$\begin{aligned} & \frac{1}{2\pi} \int dw \left\{ \sum_{e^-} \frac{|e^- \rangle \langle e^-|}{w - E_- + i\eta} + \sum_{e^+} \frac{|e^+ \rangle \langle e^+|}{w + E_+ - i\eta} e^{i\omega(t_1 - t_2)} \right\} \\ & + \frac{1}{2\pi} \int dw \sum_{e^-} |e^- \rangle \langle e^-| (-i\pi\delta(w - E_-)) e^{i\omega(t_1 - t_2)} \quad (38) \\ & = \frac{-i}{2} \sum_{e^-} |e^- \rangle \langle e^-| e^{iE_-(t_1 - t_2)} . \end{aligned}$$

The Coulomb interactions with the target ion are exactly included. With this on-shell, or physical cross-section, description, the remainder of the calculation is very simple. Each interaction now corresponds to a separate factor; further, within each factor, the integrations over the Moller interaction momentum variables proceed just as before and the approximations discussed for the lowest order are valid. Then, each factor will take a form just like that written in eq (9), with two trivial exceptions: (1) the appearance of a factor $(-i/2)$, eq (38), and (2) the matrix element here is that of

photo-ejection. The result for the overall amplitude is then a product of three factors: one representing the first order amplitude, one proportional to the photo-ejection amplitude, and the third to its inverse; the result is:

$$\begin{aligned} (\text{1st order}) \times \left\{ -\frac{1}{4} \frac{\alpha Z_p^2}{n^2} \int \frac{dv}{v} \Phi_{\gamma, \text{photo}} \left(\frac{v}{c} \right)^2 \left| \frac{\gamma}{\gamma^2 - 1} K_1 \left(\frac{bv/c}{\gamma} \right) - \frac{\gamma^2}{\gamma^2 - 1} K_1(bv/c) \right|^2 \right\} \\ = (\text{1st order}) \times (-P_c(b)); \end{aligned} \quad (39)$$

here, $P_c(b)$ is introduced as a short-hand for the curly bracket of eq (39).

Further, it is clear that, using the same on-shell prescription, that the higher orders are also just simple products of the same factors. The result is a simple geometric series that can be summed to:

$$(\text{1st order}) / (1 + P_c(b)), \quad (40)$$

and

$$P(b) = \frac{P_1(b)}{[1 + P_c(b)]^2}, \quad \sigma = \int d^2b \frac{P_1(b)}{[1 + P_c(b)]^2}; \quad (41)$$

$P_1(b)$ is just the first order result, eq (12), that we had before. Before moving ahead to the final integrations, it will be clarifying to consider the magnitudes involved--albeit via simplifying approximations to the impact parameter, b , dependences.

A rough, but useful, approximation, is obtained by noting that the factor

$$\left(\frac{v}{c}\right)^2 \left| \frac{\gamma}{\gamma^2-1} K_1\left(\frac{bv}{c}\right) - \frac{\gamma^2}{\gamma^2-1} K_1\left(\frac{bv}{c}\right) \right|^2 - \frac{1}{b^2} \quad (42a)$$

for

$$\frac{c}{v} < b < \gamma \frac{c}{v}, \quad (42b)$$

and falls off outside this range of b ; for $b < c/v$ the fall-off occurs because of cancellation between the two Bessel functions; for $b > \gamma(c/v)$ it occurs because of the asymptotic d line of the Bessel functions. Now, $P_C(b)$ is dominated by the photo-ejection cross-section, which is well known to peak at threshold, the x-ray edge effect; that is, the important values of v occur near

$$v = (1 \text{ to } 3) \times \frac{1}{2} (\alpha Z_T)^2 m_e c^2,$$

which for the particular case of gold-ions is $\sim (1/3 \text{ to } 1) \times m_e c^2$; the effective range of b is then:

$$(1 \text{ to } 3) \lambda_c < b < (1 \text{ to } 3) \lambda_c \gamma,$$

where λ_c is the Compton wave length. Again for the special example of gold-ions colliding against gold-ions it turns out that

$$\frac{1}{4} \frac{\alpha Z_p^2}{\pi^2} \int \frac{dv}{v} \phi_{\gamma, \text{photon}} = 230 \text{ barns} \sim .15 \lambda_c^2. \quad (43)$$

This immediately establishes that even at the smallest b of the effective range, $\sim \lambda_c$, the effect of the sweep-off process, the factor $|1/(1+P_C(b))|^2$, is small, and its relative importance rapidly decreases, as $[1/(1+.15(\lambda_c/b)^2)]^2$. To estimate the integrated effect it is also necessary to know the b -dependence of $P_1(b)$; this is dominated by the cross-section for e^- -bound, e^+ creation, and as will

be seen later, peaks in the region not far above its threshold, at $\sim 2-4 \pi_e c^2$, so that the effective range of b is roughly $[(1/4)\lambda_c < b < (1/4)\lambda_c \gamma]$. Then, the cross-section with sweep-off, relative to the first order alone, is approximately:

$$\left\{ \int_{\lambda_c/4}^{\lambda_c} \frac{db}{b} + \int_{\lambda_c}^{(\lambda_c/4)\gamma} \frac{db}{b[1+.15/b^2]^2} \right\} / \int_{\lambda_c/4}^{(\lambda_c/4)\gamma} \frac{db}{b} - \left(\frac{\ln \gamma - .15}{\ln \gamma} \right) \quad (44)$$

In other words, the sweep-off correction, $-.15/\ln \gamma$, far from being important, is of the same order as terms discarded in the earlier analyses.

Why is the correction small in spite of the apparently large cross-section? There is a two-fold physical answer. First off, the correct unit for electron processes is $\pi \lambda_c^2$, - 4600 barns; scaled on this basis, the cross-sections do not appear so large. Second, the average probability per se, $P_i(b)$ or $P_o(b)$, is not large; the cross-section becomes sizeable because one sums over a large range of impact parameters, resulting in the slow accumulation of the large logarithm. Since the corrections involve powers of the small probabilities, formed separately for each impact parameter, the correction effect is necessarily small. These results, substantiated by the detailed calculation in the next section, remove the necessity of concern with either the specific sweep-off process evaluated here or self-damping effects, that take a form much like that used for the sweep-off.

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