DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Anisotropic Distribution Function of Minority Tail Ions Generated by Strong Ion-Cyclotron Resonance Heating

C.S. Chang* and P. Colestock
Princeton Plasma Physics Laboratory
Princeton University
Princeton, NJ 08543

PPPL--2622

DE89 013121

The highly anisotropic particle distribution function of minority tail ions driven by ion-cyclotron resonance heating at the fundamental harmonic is calculated in a two-dimensional velocity space. It is assumed that the heating is strong enough to drive most of the resonant ions above the ion-electron critical slowing-down energy. Simple analytic expressions for the tail distribution are obtained for the case when the Doppler effect is sufficiently large to flatten the sharp pitch angle dependence in the bounce averaged quasilinear heating coefficient $D_{\parallel}$, and for the case when $D_{\parallel}$ is assumed to be constant in pitch angle and energy. It is found that a simple constant- $D_{\parallel}$ solution can be used instead of the more complicated sharp- $D_{\parallel}$ solution for many analytic purposes.

*Courant Institute of Mathematical Sciences, New York University, and Korea Institute of Technology

I. Introduction

Plasma heating by electromagnetic waves of Ion-Cyclotron Range of Frequencies (ICRF) is an important issue in both theoretical and experimental aspects of tokamak fusion research. Most of the large size tokamaks have successfully demonstrated the heating of minority resonant ions to the energies well above 1 MeV by fundamental resonance heating of ICRF waves. And the study of the effects of fast ions on plasma behavior, such as transport, instability generation and wall interaction, is becoming an interesting and important topic. The analyses of fast ion effects has not been straightforward because the lowest order distribution function is not Maxwellian. Due to the preferential heating in the perpendicular direction (relative to the magnetic field vector), the distribution function is highly anisotropic as demonstrated numerically in Refs. 1 and 2. A reasonably simple and accurate analytic expression for such tail-distribution functions in two-dimensional velocity space is greatly desired for progress in related analytic theories.

Attempts to calculate such analytic solutions have been reported in the literature. Reference 3 obtained a useful analytic solution, but used an isotropic model. Reference 4 introduced a boundary layer variable to obtain a simple and anisotropic analytic solution for minority tail ions, but an overly crude treatment of the boundary layer variable and pitch-angle scattering operator led to an inaccurate expression for the tail distribution function. Reference 2 obtained a similar solution to that of Ref. 4, but it

used model Coulomb slowing-down and heating operators, and treated the heating effect as a delta-function in pitch angle, resulting in a discontinuous slope in the tail distribution function at the resonance pitch angle.

It is the purpose of the present work to improve the analyses given in Refs. 2 and 4, to obtain more rigorous and accurate expressions for the lowest order particle distribution function of the anisotropic high energy tail ions. The procedure of Ref. 4 will be closely followed, and the solutions will be discussed for a few different resonant broadening cases. In comparing our result with that from the isotropic calculation,³ it should be kept in mind that our result is valid for the tail species only while the isotropic calculation yields the minority distribution for all the energy range. Thus only the tail description part of Ref. 3 should be used for comparison with the present result.

II. Kinetic Formulation

In the presence of the quasilinear ICRF heating operator Q , the lowest order distribution function f of the resonant species is calculated from the drift-kinetic equation

$$\vec{v}_{||} \cdot \nabla f = Cf + Qf,$$

where the vertical drift term has been neglected assuming that the banana width of our tail particles is much smaller than the minor radius. This neglect of vertical drift may restrict the validity of the present analyses to large minor-radius or high magnetic-field tokamaks. The left hand side of

the above drift-kinetic equation is proportional to the bounce frequency, while the right hand side is proportional to the collision frequency. If we notice that the parallel bounce motion of tail particles is much faster than the Coulomb or RF collision rate, we can immediately see from the above equation that the lowest order distribution function must be constant along the magnetic field lines. We will use the notation f_o to denote this lowest order distribution function of the minority tail ions, which is constant along the magnetic field.

Then f_o can be calculated by taking the bounce-average over the guiding center motion:

$$\{C\}f_o + \{Q\}f_o = 0. \quad (1)$$

where $\{x\}$ denotes the bounce average operation

$$\{x\} \equiv \frac{1}{\tau_b L} \int \frac{d\ell}{v_{\parallel}} x, \quad \tau_b \equiv \frac{1}{L} \int \frac{d\ell}{v_{\parallel}},$$

where L is the total connection length for the passing particles, $L \equiv \int d\ell$, and the bounce-average-integral is taken between the turning points of the tail particles. It is well known^{1,2,4} that the perpendicular heating by ICRF resonance forces the turning points of the RF-driven tail particles to approach the zero- $k_{\parallel}$ ICRF-resonance points.

For the tail ions, whose kinetic energy is above the ion-electron critical slowing-down energy, the Coulomb scattering operator is well approximated by

$$C(f_o) = \nu, \frac{1}{v^2} \frac{\partial}{\partial v} v^3 f_o + \frac{\nu_n}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial f_o}{\partial \xi}. \quad (2)$$

where ν_s is the tail ion slowing-down rate due to Coulomb drag on the background electrons, ν_{ti} is the tail ion 90°-scattering rate by the main ions, and $\xi = v_{\parallel}/v$. The bounce average of the Coulomb slowing-down operator, the first operator in Eq. (2), is trivial, since τ_s and the particle speed v are constants of motion.

The bounce-averaged Coulomb pitch-angle scattering operator can be calculated as follows. Using the relation, $1 - \xi^2 = (1 - \xi_m^2)(B/B_m)$, where the subscript m implies that the quantity is evaluated at the minimum Mod-B point, transforming ξ into ξ_m , and performing the bounce average, we obtain for the pitch-angle scattering operator C_{ti}^{ξ}

$$\{C_{ti}^{\xi}\} = \frac{1}{2} \frac{\nu_{ti}}{\tau_b v_{\parallel m}} \frac{\partial}{\partial \xi_m} Y(\xi_m) \frac{\partial}{\partial \xi_m},$$

where

$$Y(\xi_m) = \int \frac{dl}{L} \frac{B_m}{B} (1 - \xi_m^2) \frac{\xi}{\xi_m}.$$

Assuming the variation of B along the magnetic field line to be

$$B = B_0(1 - \lambda \cos \theta),$$

and defining K as

$$K^2 = [2\lambda/(1 - \lambda)] v_{\perp m}^2 / v_{\parallel m}^2, \quad (3)$$

we obtain

$$\xi = \xi_m \sqrt{1 - K^2 \sin^2(\theta/2)}, \quad (4)$$

where θ is a poloidal angle measured from minimum- B plane. The relation $K^2 = 1$ defines the boundary between the trapped and passing particles.

Change of variables from dl/L to $d\theta/2\pi$ and then from $K \sin(\theta/2)$ to $\sin \eta$ yields

$$\tau_b = \frac{1}{\pi v_{\parallel m}} \int_{-\pi/2}^{\pi/2} d\eta \frac{1}{\sqrt{1 - \frac{1}{K^2} \sin^2 \eta}},$$

$$Y(\xi_m) = \frac{1}{\pi K} \int_{-\pi/2}^{\pi/2} d\eta \frac{B_m}{B} \frac{\cos^2 \eta}{\sqrt{1 - \frac{1}{K^2} \sin^2 \eta}}.$$

In the present work, we further simplify the bounce averaged pitch angle scattering operator by assuming a large aspect ratio tokamak and also assuming that the tail ions are deeply trapped ($\xi_m^2 \ll 1$ and $K^2 \gg 1$). The quantities K , τ_b , and Y are easily simplified to

$$K^2 \cong \frac{2\lambda}{1 - \lambda} \frac{1}{\xi_m^2} = \frac{\xi_c^2}{\xi_m^2}, \quad \tau_b \cong \frac{1}{K v_{\parallel m}}, \quad Y(\xi_m) \cong \frac{1}{2K}. \quad (5)$$

and, thus the bounce averaged pitch angle scattering operator becomes

$$\{C_{ii}^{\xi}\} \cong \frac{\nu_{ii}}{4} \frac{1}{\xi_m} \frac{\partial}{\partial \xi_m} \xi_m \frac{\partial}{\partial \xi_m}. \quad (6)$$

The validity of Eq. (5) is limited to the case when the particles are in the deeply trapped regime. Thus, our form for $\{C_{ii}^{\xi}\}$ is not valid for the case when the resonance surface lies near the maximum Mod-B point ($\theta = \pi$), since the location of the banana tips of the high energy resonant particles is near the resonance surfaces.

We now consider the bounce average of ICRF-heating operator

$$Q = \nabla \cdot \bar{\Gamma}_{RF},$$

where $\bar{\Gamma}_{RF}$ is the ICRF-driven particle flow density in velocity space. In the velocity coordinate system described by (v, ξ_m) , we can express $\bar{\Gamma}_{RF}$ as

$$\bar{\Gamma}_{RF} = -\mathbf{D} \cdot \nabla_v f_0.$$

$$= -\hat{v}(D_{vv}\frac{1}{h_1}\frac{\partial f}{\partial v} + D_{v\xi_m}\frac{1}{h_2}\frac{\partial f}{\partial \xi_m}) - \xi_m(D_{\xi_mv}\frac{1}{h_1}\frac{\partial f}{\partial v} + D_{\xi_m\xi_m}\frac{1}{h_2}\frac{\partial f}{\partial \xi_m}),$$

where $\mathbf{D}$ is the RF-driven velocity-space diffusion coefficient tensor whose element is expressed in (v, ξ_m) coordinates, $\hat{v}$ and $\hat{\xi}_m$ are the unit vectors in ∇v and $\nabla \xi_m$ direction, and h_1 , h_2 , and h_3 are the Lamé coefficients derived as (h_3 corresponding to the gyrophase angle)

$$h_1 = 1, \quad h_2 = -\frac{v}{\sqrt{1 - \xi_m^2}}, \quad h_3 = v\sqrt{1 - \xi_m^2}.$$

The different components in the velocity-space diffusion coefficient tensor $\mathbf{D}$ are related to each other through the heating path given by $y = v_{||R} = \text{constant}$, where $v_{||R}$ is the parallel velocity at the resonance location. From the heating relation constrained by the heating path, $\Delta\epsilon = B_R\Delta\mu$, we can easily obtain

$$\Delta v = -\frac{B_R}{B_m} \frac{v^2}{v_{||R}^2} v \xi_m \Delta \xi_m. \quad (7)$$

Using this relation, we derive

$$D_{\xi_m\xi_m} = -\frac{h_2}{h_1} D_{\xi_mv} \frac{\partial \xi_m}{\partial v} \bigg|_y = -\frac{h_2}{h_1} \frac{\eta}{v} D_{\xi_mv},$$

$$D_{\xi_mv} = -\frac{h_2}{h_1} D_{vv} \frac{\partial \xi_m}{\partial v} \bigg|_y = -\frac{h_2}{h_1} \frac{\eta}{v} D_{vv},$$

or

$$D_{\xi_m\xi_m} = -\frac{h_2}{h_1} \frac{\eta}{v} D_{\xi_mv} = \left(\frac{h_2}{h_1}\right)^2 \left(\frac{h_2}{h_1}\right)^2 D_{vv},$$

where

$$\eta \equiv \frac{B_m}{B_R} \frac{y^2}{v^2 \xi_m}.$$

The velocity-space flux $\bar{\Gamma}_{RF}$ is then simplified to

$$\bar{\Gamma}_{RF} = -D_{vv} \left(\hat{v} - \frac{h_2}{h_1} \frac{\eta}{v} \hat{\xi}_m \right) \frac{\partial f_o}{h_1 \partial v} \Big|_y, \quad (8)$$

where the mathematical identity

$$\frac{\partial f}{\partial v} \Big|_y = \frac{\partial f}{\partial v} \Big|_{\xi_m} - \frac{\partial f}{\partial \xi_m} \Big|_v \frac{\partial \xi_m}{\partial v} \Big|_y, \quad (9)$$

has been used. From Eq. (8), we can see that the quasilinear ICRF diffusion tries to drive the particles into a state where f_o is constant along the heating path defined by a constant value of y . By expanding η around the resonance pitch-angle ξ_o , we find

$$\eta = \frac{\xi_m^2 - \xi_o^2}{\xi_m^2} \simeq 2(\xi_m - \xi_o) \ll 1. \quad (10)$$

Thus, the ξ_m -component in Eq. (8) is much smaller than the $\hat{v}$ -component since $\eta \ll 1$ for the highly anisotropic tail ions. However, this term cannot be neglected since the high degree of localization may allow $\partial \eta / \partial \xi_m \sim 1$ in the evaluation of $Q_{RF} = \nabla \cdot \bar{\Gamma}_v$.

The quasilinear ICRF heating operator then becomes, from Eq. (8),

$$\begin{aligned} Q &= \nabla \cdot \bar{\Gamma}_{RF} \\ &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial v} (h_2 h_3 \hat{v} \cdot \bar{\Gamma}_{RF}) + \frac{\partial}{\partial \xi_m} (h_3 h_1 \hat{\xi}_m \cdot \bar{\Gamma}_{RF}) \right] \\ &= -\frac{1}{v^2} \frac{\partial}{\partial v} \Big|_{\xi_m} D_{vv} v^2 \frac{\partial f_o}{\partial v} \Big|_y + \frac{1}{v} \frac{\partial}{\partial \xi_m} \Big|_v D_{vv} \eta \frac{\partial f_o}{\partial v} \Big|_y. \end{aligned}$$

And by taking the bounce average operation, we obtain

$$\{Q\} = - \left[\frac{1}{v^2} \frac{\partial}{\partial v} \Big|_{\xi_m} D_{vv} v^2 \frac{\partial f_o}{\partial v} \Big|_y + \frac{1}{v} \frac{\partial}{\partial \xi_m} \Big|_v D_{vv} \eta \frac{\partial f_o}{\partial v} \Big|_y \right]. \quad (11)$$

where D_b is the bounce-averaged quasilinear velocity space diffusion coefficient

$$D_b \equiv \{D_{vv}\}.$$

A highly localized tail-ion distribution near the resonance pitch-angle can be analyzed by a proper identification of a boundary layer parameter as shown in Ref. 4. ICRF waves, while heating them, drive the resonant ions onto the zero- $k_{\parallel}$ resonance pitch angle, and this heating process is balanced by the Coulomb slowing-down process. Without the effects of pitch angle scattering, all the tail ions will end up at $\xi_m = \xi_o$, but the existence of the pitch angle scattering, though weak, allows the fast ions to spread out somewhat in pitch angle. A rough estimate of the spreading width of the tail particles near the resonance pitch-angle can be obtained by balancing the magnitudes of the Coulomb energy scattering and pitch-angle scattering terms

$$\nu_i \frac{\partial}{\partial v} \cdot \vec{v} f \sim \nu_{ii} \frac{\partial^2 f}{\partial \xi_m^2}.$$

Since $\nu_{ii} \propto v^{-3}$, we can easily see that the localization width $\Delta \xi_m$ satisfies the relation $v^{3/2} \Delta \xi_m \approx \text{constant}$. This observation naturally leads us to the identification of a boundary layer variable

$$\xi_\sigma \equiv \left(\frac{v}{v_c} \right)^{3/2} (\xi_m - \sigma \xi_o), \quad (12)$$

where $\sigma = v_{\parallel}/|v_{\parallel}| = \pm 1$, representing the resonances at ξ_o or $-\xi_o$, and v is normalized to v_c to reflect the physical fact that $\Delta \xi_m \sim 1$ when v is close to the critical ion-electron slowing-down speed. The usefulness of a new

boundary-layer-variable ℓ_σ will be made clear in the following calculation of f_o .

Changing our pitch-angle variable ξ_m into ℓ_σ , and using the localization approximation $\partial(\xi_m f_o)/\partial \ell_\sigma \cong \xi_o \partial f_o / \partial \ell_\sigma$, Eq. (1) can be written as

$$\begin{aligned} \nu_s \frac{1}{v^2} \frac{\partial}{\partial v} \Big|_{\xi_m} v^3 f_o + \frac{\nu_{ti}}{4} \left(\frac{v}{v_c} \right)^3 \frac{\partial^2 f_o}{\partial \ell_\sigma^2} \Big|_v + \frac{1}{v^2} \frac{\partial}{\partial v} \Big|_{\xi_m} D_b v^2 \frac{\partial f_o}{\partial v} \Big|_v \\ - \frac{\sqrt{v}}{v_c^{3/2}} \frac{\partial}{\partial \ell_\sigma} \Big|_v D_b \eta \frac{\partial f_o}{\partial v} \Big|_v = 0. \end{aligned} \quad (13)$$

In consideration of the observation that $f_v \approx \text{constant}$ for $\ell_\sigma = \text{constant}$, we assume that f_o takes the form

$$f_{o\sigma} = V(v) L(\ell_\sigma)$$

with the normalization condition

$$\sum_\sigma \int_{\ell_{\min\sigma}}^{\ell_{\max\sigma}} d\ell_\sigma L(\ell_\sigma) = 1, \quad (14)$$

and

$$\langle \int d^3 v f_o \rangle = n_i, \quad \int d^3 v_m = 2\pi \sum_\sigma \int_0^\infty v_c^{3/2} \sqrt{v} dv \int_{\ell_{\min\sigma}}^{\ell_{\max\sigma}} d\ell_\sigma, \quad (15)$$

where the ℓ_σ -integration is taken along a constant- v contour, $\langle \dots \rangle$ represents flux-surface-average, $\bar{v}_m$ is $\bar{v}$ at the midplane, and the limits of ℓ -integral [$\ell_{\max\sigma} = (v/v_c)^{3/2}(1 - \sigma\xi_o)$, and $\ell_{\min\sigma} = -(v/v_c)^{3/2}(1 + \sigma\xi_o)$] are extended to

$$\ell_{\max\sigma} = -\infty, \quad \ell_{\min\sigma} = \infty.$$

due to the largeness of v/v_e (or equivalently, due to the high degree of localization in pitch angle). The flux-surface-averaged tail density n_t can be obtained from the present velocity representation (v, ξ_m) as follows:

$$\begin{aligned} n_t &= \langle \int d^3v f_o \rangle = \langle 2\pi B \sum_{\sigma} \int \frac{d\epsilon d\mu}{v_{||}} f_o \rangle \\ &= \langle 2\pi B_m \sum_{\sigma} \int \frac{d\epsilon d\mu}{v_{||m}} f_o \frac{B v_{||m}}{B_m v_{||}} \rangle = \int d^3v_m f_o \langle \frac{B \xi_m}{B_m \xi} \rangle. \end{aligned}$$

From the relation between ξ_m and ξ given by Eq. (4), it is straightforward to obtain $\langle B \xi_m / B_m \xi \rangle = (2/\pi K) F(1/K)$ for a large aspect ratio tokamak, where F is the complete elliptic integral. Using $K \gg 1$ for deeply trapped particles, it is easy to see that $\langle B \xi_m / B_m \xi \rangle \simeq 1/K$. For particles turning at the resonance surfaces, we obtain from Eq. (3) that

$$K^2 \simeq \frac{2\lambda}{1-\lambda} \frac{1}{\xi_o^2}. \quad (16)$$

We thus have

$$\langle B \xi_m / B_m \xi \rangle \simeq \xi_o / \xi_c,$$

where $\xi_c = \sqrt{2\lambda/(1-\lambda)}$ is the pitch-angle at the trapped-passing boundary in the velocity space. We then obtain

$$n_t = \frac{\xi_o}{\xi_c} \int d^3v_m f_o.$$

In Eq. (13), the term $\partial f_o / \partial v|_y$ can be changed into $\partial f_o / \partial v|_t$ using the relation

$$\left. \frac{\partial f}{\partial v} \right|_y = \left. \frac{\partial f}{\partial v} \right|_t - \left. \frac{\partial f}{\partial \ell} \right|_v \frac{\partial \ell}{\partial v} \Big|_y,$$

where the second term is small.

A. Constant D_b Approximation

Solving Eq. (13) is not easy because of the added complication that D_b can have a strong dependence in ξ if there is not enough resonance broadening effect,

$$D_b \propto \frac{1}{\sqrt{\xi_m^2 - \xi_s^2 + (\Delta\xi_R)^2}}, \quad (17)$$

where $\Delta\xi_R$ is the amount of resonance broadening in pitch angle. The analysis becomes significantly simplified if we assume that D_b is independent of v and ξ . In this case, the equation for $V(v)$ is obtained by integrating Eq. (13) in ℓ_σ , keeping v constant. The second and fourth terms vanish because of the boundary condition $L(\ell_\sigma)$ and $\partial L/\partial v \rightarrow 0$ as $|\ell_\sigma|$ becomes large, yielding

$$\frac{1}{\tau_s} \frac{1}{v^2} \frac{\partial}{\partial v} v^3 V(v) + \frac{1}{v^2} \frac{\partial}{\partial v} D_b v^2 \frac{\partial}{\partial v} V(v) = 0,$$

which can be changed into the familiar form

$$\frac{1}{\tau_s} \frac{\partial}{\partial v} \cdot \left[\hat{v} \frac{kT_{eff}}{M_t} \frac{\partial}{\partial v} V(v) + \hat{v} v V(v) \right] = 0. \quad (18)$$

where $\tau_s = 1/\nu$, and

$$kT_{eff} \equiv M_t \tau_s D_b. \quad (19)$$

Notice here the resemblance of Eq. (18) to the Coulomb energy scattering operator. The energy scattering part of the RF-collisions takes the role of energy randomization and is balanced by the slowing-down collisions to reach a steady state. The solution of Eq. (18) is well known to be

a Maxwellian.

$$\Gamma_C(v) = \frac{n_t \xi_c / \xi_o}{\pi \Gamma(3/4) v_c^{3/2} v_t^{3/2}} \exp(-v^2/v_t^2), \quad (20)$$

where the subscript "C" is used to imply the constant- D , approximation. Γ is the Gamma-function, $v_t \equiv (2kT_{eff}/M_t)^{1/2}$ is the effective thermal speed of tail particles with the effective temperature T_{eff} , and the normalization condition (15) has been used.

The solution for $L(\ell_\sigma)$ is more difficult to obtain due to the non-orthogonal nature of our coordinates (v, ℓ_σ) . But for the particles well within the boundary layer, $\ell_\sigma \ll 1$, we can relax the non-orthogonality using the relation

$$\begin{aligned} \left. \frac{\partial f_o}{\partial v} \right|_y &= L \left. \frac{\partial V}{\partial v} \right|_y + V \left. \frac{\partial L}{\partial v} \right|_y \\ &= L \left. \frac{\partial V}{\partial v} \right|_y + \left[\frac{3}{2} \frac{\ell_\sigma}{v} + \frac{\partial \xi_m}{\partial v} \right] \left(\frac{v}{v_c} \right)^{3/2} \left. \frac{\partial L}{\partial \ell_\sigma} \right|_{\xi_m}. \end{aligned}$$

The second term in the square bracket can be changed into the same form as the first term using Eqs. (7) and (10), and we have

$$\left. \frac{\partial f_o}{\partial v} \right|_y = L \left. \frac{\partial V}{\partial v} \right|_y + \frac{7}{2} \frac{\ell_\sigma}{v} \left. \frac{\partial L}{\partial \ell_\sigma} \right|_{\xi_m}.$$

The second term is small because of the assumption $\ell_\sigma \ll 1$. Thus, we can treat v and ℓ_σ to be orthogonal for the particles well within the boundary layer,

$$\left. \frac{\partial f_o}{\partial v} \right|_y = L \left. \frac{\partial V}{\partial v} \right|_y \cong L \left. \frac{\partial V}{\partial v} \right|_{\ell_\sigma}, \quad (21)$$

where the second equality is from Eq. (9).

Using Eqs. (19)-(21), we can easily see that the first and third terms in Eq. (13) cancel each other, leaving an equation for $L(\ell_\sigma)$,

$$\frac{\nu_{ti}}{4} \left(\frac{v}{v_c} \right)^3 V \frac{\partial^2 L_\sigma}{\partial \ell_\sigma^2} \bigg|_v \cong \frac{\sqrt{v}}{v_c^{3/2}} \frac{\partial}{\partial \ell_\sigma} \bigg|_v D_b \eta L_\sigma \frac{\partial V}{\partial v} \bigg|_{\ell_\sigma},$$

which can be rearranged into

$$\frac{\partial}{\partial \ell_\sigma} \bigg|_v \left[\frac{v^3 \nu_{ti}}{v_c^3} \frac{T_{eff}}{4} \frac{\partial L}{\partial \ell_\sigma} + D_b \eta L(\ell_\sigma) \right] = 0, \quad (22)$$

where the approximate relation

$$\eta \simeq 2\ell_\sigma \left(\frac{v_c}{v} \right)^{3/2}$$

has been used. A solution to Eq. (22) is again a Maxwellian:

$$L(\ell_\sigma) = A \exp[-(\ell_\sigma/\Delta\ell)^2],$$

where $A = 1/(2\sqrt{\pi}\Delta\ell)$ is the normalization constant given by Eq. (14), and

$$(\Delta\ell)^2 = \frac{v^3 \nu_{ti}}{v_c^3} \frac{kT_{eff}}{D_b 2M_i} = \frac{1}{2} \frac{v^3 \nu_{ti}}{v_c^3 \nu_s} = \frac{4/15}{1 + M_i/M_e}. \quad (23)$$

Here in the above relation, Eq. (19) has been used for the second equality, $\nu_{ti} v^3 = \text{constant}$ has been used for $\Delta\ell = \text{constant}$, and M_i is the background ion mass.

A complete solution is then given by the summation over $\sigma = \pm 1$,

$$f_{o,C} = V_C(v) \sum_{\sigma=\pm 1} L(\ell_\sigma), \quad (24)$$

where an extra subscript "C" is used in $f_{o,C}$ to denote that the solution is from the constant- D_b approximation. Notice here that for $v \ll v_c$, L

becomes a constant independent of l_z , and thus f_s becomes isotropic. However, our f_s in this low energy limit does not yield the correct v -dependence, which is $\exp(-v^2/v_t^2)$ with v_t defined as the thermal speed of the background ion species. This is due to our neglect of the tail-bulk ion energy scattering in the Coulomb operator.

Figure 1a shows a plot of a typical $f_{o,C}$ obtained from the present constant- D_s approximation in $v_{||m} - v_{\perp m}$ space. Hydrogen minority heating in a deuterium plasma is considered with $T_{eff}/E_c = 100$ and the resonance pitch angle = 0.46 at midplane. The comparison in the shape of constant f_o contours with the numerical solution of Ref. 1 can be seen from Fig. 1b. The smoother tips of the rabbit ears in Fig. 1a, than in Fig. 1b, reflect the use of constant- D_s . Figure 2 shows the three-dimensional structure of the present the $f_{o,C}$ -contours (constant- D_s approximation).

B. Velocity Dependent D_s with Doppler-Broadening

We now consider the solution for f_o with more realistic model of D_s . Often in a practical ICRF-heating experiment, the value of $k_{||}$ is not negligibly small and the resonance broadening occurs by the Doppler shift effect for the fast ions localized near the resonance pitch angle. The particles can be resonant with the ICRF-wave at pitch angles other than ξ_o . The amount of Doppler broadening increases as particle energy gets higher since the Doppler broadening is from $k_{||}v_{||}$. And we can easily see from Eq. (17) that the resulting D_s is broadened in ξ_m but may become function of particle

energy, instead.

From the simple estimate, Eq. (12), of the localization width of the fast ions near the resonance pitch angle, we know that the localization width in pitch angle decreases as the particle energy increases. But the amount of resonance broadening in pitch angle increases as the particle energy increases. Therefore, above a certain energy most of the fast ions may be within the broadened resonance layer and D_b cannot be a rapidly varying function of the pitch angle for most of the fast ions. The broadening of the resonance layer can be estimated from the resonance condition $\omega - \Omega = k_{\parallel} v_{\parallel}$. For an ion away from the zero- $k_{\parallel}$ resonance location to be resonant with the wave, it should satisfy

$$\Delta\Omega = k_{\parallel}\Delta v_{\parallel}, \quad (25)$$

where $\Delta\Omega = \Omega_R \Delta B / B_R$ is the difference in Ω between the actual resonance location and the zero- $k_{\parallel}$ resonance location, and $\Delta v_{\parallel}$ is the small parallel velocity at the actual resonance location. The magnitude of $\Delta v_{\parallel}$ can be approximately given by

$$\Delta v_{\parallel} \simeq v \sqrt{\Delta B / B_R}, \quad (26)$$

assuming that the turning point is near B_R . Equation (25) then yields the relation $\Delta\Omega = k_{\parallel} \sqrt{\Delta B / B_R}$ which becomes

$$\sqrt{\Delta B / B_R} \simeq \frac{k_{\parallel} v}{\Omega_R}.$$

By combining this and Eq. (26), it is trivial to obtain the amount of reso-

nance broadening $\Delta\xi_R$ as follows:

$$\Delta\xi_R \simeq \sqrt{\Delta B/B_R} \simeq \frac{k_{\parallel} v}{\Omega_R}. \quad (27)$$

The present fast ions will exist within the resonance broadening layer if $\Delta\xi_m \approx (v_c/v)^{3/2}$ is smaller than $\Delta\xi_R$, or

$$\frac{v^2}{v_c^2} > \left(\frac{\Omega_u}{k_{\parallel} v_c} \right)^{4/5}. \quad (28)$$

For deuterium minority heating in a typical tokamak (we take TFTR experiment here with $k_{\parallel} \approx 0.08 \text{ cm}^{-1}$), we obtain that the kinetic energy of the fast ions needs to be at least five times the critical slowing-down energy in order for the resonance broadening to assure that D_b is a slowly varying function of pitch angle. Since the fast ions in the present analyses are assumed to be at energy much higher than the critical slowing-down energy, it is reasonable to assume here that D_b in TFTR ICRF-experiment with $k_{\parallel} \approx 0.08 \text{ cm}^{-1}$ is not a function of ξ_m (and in many other experimental situations with adequately large $k_{\parallel}$ values). But in this situation, D_b is function of v , instead, as can be seen from Eq. (27): $D_b = D_b^*/v$, where D_b^* is a velocity independent part from Eq. (17) with the broadening given by Eq. (27). Notice here that as the value of $k_{\parallel}$ becomes smaller, the peakedness of D_b toward the resonance pitch angle cannot be avoided.

Thus in the case when the Doppler effect broadens D_b , Eq. (18) becomes

$$\frac{1}{\tau_s} \frac{\partial}{\partial v} \cdot \left[\tilde{v} \frac{\tau_s D_b^*}{v} \frac{\partial}{\partial v} V(v) + \tilde{v} v V(v) \right] = 0, \quad (29)$$

which leads to the solution

$$V_D(v) = \frac{2n_t \xi_c / \xi_0}{3\pi \Gamma(1/2) v_c^{3/2} v_t^{3/2}} \exp(-v^3/v_t^3),$$

where the subscript "D" implies that the Doppler-broadening effect flattens the sharp D_b behavior in ξ_m , and v_t is given by the relation

$$v_t^3 = 3\tau_s D_b^*.$$

The ℓ dependent solution $L(\ell)$ is the same as before, Eq. (23), and we obtain

$$f_{o,D} = \frac{3n_t \xi_s / \xi_o}{2\pi \Gamma(1/2) v_c^{3/2} v_t^{3/2}} \exp(-v^3 / v_t^3) \sum_{\sigma=\pm 1} L(\ell_\sigma). \quad (30)$$

The contour plot of this $f_{o,D}$ is shown in Fig. 3.

C. Inclusion of sharp resonance behavior in D_b

When the Doppler broadening effect is too small to validate the above approximation and no other broadening effect can flatten the sharp structure of D_b in the pitch angle, it is necessary to study the effect of the sharp resonance structure on f_o . In this case, we model the sharp behavior with a delta function and divide D_b into two parts as follows:

$$D_b = \bar{D}_b + \tilde{D}_b \delta(\ell),$$

where $\bar{D}_b$ is a properly defined pitch angle independent part. The δ -function part contribution to the last term in Eq. (13) does not exist because $\delta(\ell) \ell$ is identically zero. Thus in the last term of Eq. (13), D_b simply becomes $\bar{D}_b$.

Since the particles with $|\xi_m| < \xi_o$ are deeply trapped beyond the resonance surface and cannot reach the resonance surface, $D_b = 0$ for those

particles. And thus we need to treat these overly trapped tail ions separately. We first consider the fast ions whose pitch angles are large enough to allow them cross the resonant layers. In this pitch angle domain, $|\xi|_m > \xi_0$, the value of the delta function is zero and thus $D_b = \bar{D}_b$. The following solution can be easily shown to satisfy Eq. (13):

$$f_o = A \exp \left[-\frac{v^2}{v_{eff}^2} \right] \left(e^{-\ell_+^2/(\Delta\ell)^2} + e^{-\ell_-^2/(\Delta\ell)^2} \right),$$

which is identical to the solution in the constant D_b -case except that v_{eff} and $\Delta\ell$ are defined by using $\bar{D}_b$ in place of D_b . The constant A is determined by the normalization condition.

The solution for the non-resonant and sharply resonant part of the tail ions, $|\xi|_m \leq \xi_0$, is more complicated to obtain. In this case, we follow the procedure of Ref. 2 and obtain a model solution by approximating the differential equation (13) as

$$\frac{4 E_t}{3\tau_s} \frac{\partial f_o}{\partial W} + \frac{\nu_{ti}}{4} \left(\frac{v}{v_c} \right)^3 \frac{\partial^2 f_o}{\partial \ell^2} + \bar{D}_b \delta(\ell) K \frac{\partial^2 f_o}{\partial W^2} = 0,$$

where a constant K (of order unity) is introduced to ensure the proper amount of jump in ℓ -derivative at $\ell = 0$, W is the kinetic energy and E_t is the velocity averaged kinetic energy

$$E_t = \int d^3v W f_o,$$

It is easy to show that the following is a solution²:

$$f_o = A \exp \left(-\frac{3W}{4 E_t} \right) \left(e^{-\ell_+/\sigma_\ell} + e^{-\ell_-/\sigma_\ell} \right),$$

$$\sigma_t = \frac{1}{2} \sqrt{\frac{v^2 \nu_{ti}}{v_c^2 \nu_s}} = \frac{\Delta \ell}{\sqrt{2}}.$$

By using the definition $T_{eff} = M_i v_{eff}^2 / 2$, and performing the velocity average, we can easily see that $E_t = (3/4) k T_{eff}$. And the combined solution for f_o for two different regimes in the pitch angle space is given by, after the proper identification of A by normalization condition Eqs. (14 and 15),

$$\begin{aligned} f_{o,S} &= V_C \frac{e^{-\ell_+^2/(\Delta \ell)^2} + e^{-\ell_-^2/(\Delta \ell)^2}}{(\sqrt{2} + \sqrt{\pi}) \Delta \ell}, \quad \text{for } |\xi_m| > \xi_o, \\ &= V_C \frac{e^{-\sqrt{2}\ell_+/\Delta \ell} + e^{-\sqrt{2}\ell_-/\Delta \ell}}{(\sqrt{2} + \sqrt{\pi}) \Delta \ell}, \quad \text{for } |\xi_m| \leq \xi_o, \end{aligned} \quad (31)$$

where the extra subscript "S" represents the sharp resonance model. A plot of this $f_{o,S}$ is shown in Fig. 4. In comparison with $f_{o,C}$, the constant- $D_\parallel$ case (Fig. 1), we can see that Fig. 4 have sharper behavior near the end of the "ears". However, the overall difference between Fig. 1 and Fig. 4 is not significant. Considering that $f_{o,S}$ obtained in this section contains nonanalytic behavior ($f_{o,S}$ will become even more complicated if we try to eliminate the nonanalytic behavior), and also considering the minimal difference between $f_{o,S}$ and $f_{o,C}$, it is expected that the simpler form $f_{o,C}$ obtained from the constant- $D_\parallel$ approximation will suffice many analytic analyses which require a known analytic form of the lowest order non-Maxwellian minority tail distribution function.

III. Discussions and Conclusions

Even though we considered a case with finite $k_{\parallel}$ broadening effect in the present work, the validity of our present analyses is basically limited to the small $k_{\parallel}$ waves. $k_{\parallel}v_{\parallel} \ll \Omega_u$, due to the neglect of direct heating in the $v_{\parallel}$ -direction at the resonance location. As can be seen from the figures, the constant- D_0 approximation yields a reasonably good formula for the lowest order tail distribution function in the case when the value of $k_{\parallel}$ is negligibly small. Attractive features of this $f_{o,C}$ is that the formula is in a very simple mathematical form and it is analytic everywhere. This formula can be used in most of the analytic zero- $k_{\parallel}$ ICRF theories where delicate details of the lowest order distribution function are not needed. For the problems where the inclusion of detailed sharp ξ -dependence is necessary, more complicated form of f_o given as Eq. (31) can be used. In the case when the Doppler broadening effect is large enough to flatten out the sharp ξ -dependence in D_0 [see Eq. (28)], Eq. (30) provides another simple analytic formula for the lowest order distribution.

The isotropic model solution of Ref. 3 could be trivially obtained and shown to be a Maxwellian in energy if we completely neglected the pitch angle scattering parts in Eq. (13), or if we solved for an energy-dependent solution keeping the pitch angle constant at the zero- $k_{\parallel}$ resonance pitch angle($\ell = 0$). However it is interesting to notice here that the more complete anisotropic solutions presented here do not reduce to the isotropic solution of Ref. 3 when integrated over the pitch angle. It is due to the

intrinsic non-Maxwellian nature of the tail solutions. For example, $V(v)$ and $L(\ell)$ together form $f_{o,C}$. $V(v)$ is a Maxwellian in the energy variable but L contains another velocity dependence, making $f_{o,C}$ non-Maxwellian in energy. Thus, the ξ -average becomes

$$\begin{aligned} \int d\xi_m f_o(v, (v/v_c)^{3/2} [\xi_m - \xi_o]) &= \sum_{\sigma} \Delta\ell \int d\ell_{\sigma} (v_c/v)^{-3/2} f_o(v, \ell) \\ &= \Delta\ell \left(\frac{v_c}{v} \right)^{3/2} V(v), \end{aligned}$$

which is not a Maxwellian distribution function.

Therefore the quantity T_{eff} used here is not the temperature (which is well defined for a Maxwellian case only), but merely a convenient parameter simplifying the algebra. And this T_{eff} should not be used in place of the usual temperature parameter. Instead, the average kinetic energy E_t should be used in the description of the present non-Maxwellian plasma. As usual, the average energy E_t is related to the power absorption p_t to the tail species by

$$E_t = \frac{\tau_3 p_t}{2},$$

and p_t is obtained as follows:

$$\begin{aligned} p_t &= \langle \int d^3v \frac{1}{2} M_t v^2 Q f_o \rangle = \int d^3v_m \frac{1}{2} M_t v^2 \langle \frac{B\xi_m}{B_m\xi} Q \rangle f_o \\ &= \frac{\xi_o}{\xi_c} \int d^3v_m \frac{1}{2} M_t v^2 \{Q\}_\delta f, \\ &= \frac{\xi_o}{\xi_c} \int d^3v_m \frac{1}{2} M_t v^2 \left. \frac{1}{v^2} \frac{\partial}{\partial v} \right|_{\xi_m} D_\delta v^2 \left. \frac{\partial f_o}{\partial v} \right|_t \\ &= M_t \frac{\xi_o}{\xi_c} \int d^3v_m v D_\delta \left. \frac{\partial f_o}{\partial v} \right|_t. \end{aligned}$$

where in the third equality, Eqs. (5) and (16) have been used to change r_s into ξ_0 and ξ_c , and in the partial integration process, ξ_m was held constant. In the next partial integration process, ℓ will be held constant with the help of Eqs. (14) and (15), and using the various model of D_b we obtain

$$\begin{aligned} p_t &= \frac{3}{2} D_b M_t n_t, \quad \text{for constant } -D_b \text{ model} \\ &= \frac{\Gamma(1/6)}{2\Gamma(1/2)} \frac{M_t n_t D_b^*}{v_t}, \quad \text{for Doppler model} \\ &= \frac{3}{2} \left[D_b + \frac{2\tilde{D}_b}{\Delta\ell(\sqrt{2} + \sqrt{\pi})} \right] M_t n_t, \quad \text{for sharp } D_b \text{ model.} \end{aligned}$$

The energy-slope of the fast ions which can be observed experimentally may not be simply related to the energy content of the tail species, as was done for the Maxwellian isotropic approximation.

The present tail density n_t is a flux-surface averaged quantity, while most other works used tail density defined at the resonance point in calculating p_t . The evaluation of tail density at the resonance point can be erroneous and model dependent, because the smallness of $v_{||}$ tends to yield a singular effective density at the turning point. And an exact evaluation of the resonant ion density at the resonance point may require a more careful evaluation of particle velocity including *grad-B* drift and gyromotion near the banana tips.

It should be noted here again that the minority distribution function calculated here is for the tail species only. In the case where a significant amount of resonant minority particles are at the energies around and lower than the critical slowing-down energy, these particles may be treated as a

separate species and another form of distribution function should be determined. A unified and accurate analytic solution for the whole resonant minority distribution function is difficult to obtain, but a simplified analysis is under development.

ACKNOWLEDGMENTS

The authors acknowledge Dr. G. W. Hammett of PPPL for valuable discussions and Dr. Kaya Imre of the Courant Institute for help with the graphics program. This work was partially supported by the Korean Science and Engineering Foundation(CSC) and partially by the U.S. Department of Energy under contract Numbers DE-AC02-76-CHO3073 and DE-FG02-86-ER53223.

References

- ¹G. Kerbel and M. McCoy, Phys. Fluids 28, 3629(1985).
- ²G.W. Hammett, Ph.D. Thesis, Princeton University, Plasma Physics Laboratory (1985).
- ³T.H. Stix, Nucl. Fusion 15, 737 (1975).
- ⁴C.S. Chang, Phys. Fluids 28, 3598 (1985).

Figure Captions

Fig. 1. (a) Two-dimensional contour plots of tail ion distribution $f_{o,C}$ in the velocity space with constant D_b assumption: The parallel and perpendicular particle speed is normalized to the ion-electron critical slowing-down speed. Hydrogen minority heating in a deuterium plasma is considered with $T_{eff}/E_c = 100$. (b) Numerical solution of Ref. 1: The contour structure is sharper near the resonance pitch angle due to the smaller $k_{||}$ numbers taken, hence due to the sharp variance of D_b in pitch angle near the resonance pitch angle.

Fig. 2. Three-dimensional structure of the tail distribution function $f_{o,C}$. The input parameters are the same as in Fig. 1.

Fig. 3. Two-dimensional contour plots of $f_{o,D}$ with Doppler-broadening effect. $k_{||} = 0.08 cm^{-1}$ has been used. The input parameters are the same as in Fig. 1.

Fig. 4. Two-dimensional contour plots of $f_{o,S}$ with a sharp variation of D_b in ξ : $\bar{D}_b = \tilde{D}_b$ has been taken. The input parameters are the same as in Fig. 1.

Fig. 1(a)

Fig. 1(b)

Fig. 2

Fig. 3

Fig. 4

EXTERNAL DISTRIBUTION IN ADDITION TO UC-420

Dr. Frank J. Paoloni, Univ of Wollongong, AUSTRALIA
 Prof. M.H. Brennan, Univ Sydney, AUSTRALIA
 Plasma Research Lab., Australian Nat. Univ., AUSTRALIA
 Prof. I.R. Jones, Flinders Univ., AUSTRALIA
 Prof. F. Cap, Inst Theo Phys, AUSTRIA
 Prof. M. Heindler, Institut für Theoretische Physik, AUSTRIA
 M. Goossens, Astronomisch Instituut, BELGIUM
 Ecole Royale Militaire, Lab de Phys Plasmas, BELGIUM
 Commission-European, Dg-XII Fusion Prog, BELGIUM
 Prof. R. Bouclicq, Rijksuniversiteit Gent, BELGIUM
 Dr. P.H. Sakanaka, Instituto Fisica, BRAZIL
 Instituto De Pesquisas Especiais-INPE, BRAZIL
 Documents Office, Atomic Energy of Canada Limited, CANADA
 Dr. M.P. Bachynski, MPB Technologies, Inc., CANADA
 Dr. H.M. Skarsgard, University of Saskatchewan, CANADA
 Dr. H. Barnard, University of British Columbia, CANADA
 Prof. J. Teichmann, Univ. of Montreal, CANADA
 Prof. S.R. Sreenivasan, University of Calgary, CANADA
 Prof. Tudor W. Johnston, INRS-Energie, CANADA
 Dr. Bolton, Centre canadien de fusion magnétique, CANADA
 Dr. C.R. James, Univ. of Alberta, CANADA
 Dr. Peter Lukac, Komenského Univ, CZECHOSLOVAKIA
 The Librarian, Culham Laboratory, ENGLAND
 The Librarian, Rutherford Appleton Laboratory, ENGLAND
 Mrs. S.A. Hutchinson, JET Library, ENGLAND
 C. Mouttet, Lab. de Physique des Milieux Ionisés, FRANCE
 J. Redet, CEN/CADARACHE - Bat 506, FRANCE
 Ms. C. Rinni, Librarian, Univ. of Ioannina, GREECE
 Dr. Tom Mui, Academy Bibliographic Ser., HONG KONG
 Preprint Library, Hungarian Academy of Sciences, HUNGARY
 Dr. B. Das Gupta, Sahe Inst of Nucl. Phys., INDIA
 Dr. P. Kaw, Institute for Plasma Research, INDIA
 Dr. Philip Rosenau, Israel Inst. of Tech, ISRAEL
 Librarian, Int'l Ctr Theo Phys, ITALY
 Prof. G. Rostagni, Istituto Gas Ionizzati Del CNR, ITALY
 Miss Clotilde De Palo, Assoc EURATOM-ENEA, ITALY
 Dr. G. Grosso, Istituto di Fisica del Plasma, ITALY
 Dr. H. Yamato, Toshiba Res & Dev, JAPAN
 Prof. I. Kawakami, Atomic Energy Res. Institute, JAPAN
 Prof. Kyoji Nishikawa, Univ of Hiroshima, JAPAN
 Director, Dept. Large Tokamak Res. JAERI, JAPAN
 Prof. Satoshi Itoh, Kyushu University, JAPAN
 Research Info Center, Nagoya University, JAPAN
 Prof. S. Tanaka, Kyoto University, JAPAN
 Library, Kyoto University, JAPAN
 Prof. Nobuyuki Inoue, University of Tokyo, JAPAN
 S. Mori, JAERI, JAPAN
 H. Jeong, Librarian, Korea Advanced Energy Res Inst, KOREA
 Prof. D.I. Choi, The Korea Adv. Inst of Sci & Tech, KOREA
 Prof. B.S. Liley, University of Waikato, NEW ZEALAND
 Institute of Plasma Physics, PEOPLE'S REPUBLIC OF CHINA
 Librarian, Institute of Phys., PEOPLE'S REPUBLIC OF CHINA
 Library, Tsing Hua University, PEOPLE'S REPUBLIC OF CHINA
 Z. Li, Southwest Inst. Physics, PEOPLE'S REPUBLIC OF CHINA
 Prof. J.A.C. Cabral, Inst Superior Tecnico, PORTUGAL
 Dr. Octavian Petrus, AL I CUZA University, ROMANIA
 Dr. Jam de Villiers, Fusion Studies, AEC, SO AFRICA
 Prof. M.A. Heilberg, University of Natal, SO AFRICA
 C.I.E.M.A.T., Fusion Div. Library, SPAIN
 Dr. Lennart Stenflo, University of UMEA, SWEDEN
 Library, Royal Institute of Tech, SWEDEN
 Prof. Hans Wilhelmson, Chalmers Univ of Tech, SWEDEN
 Centre Phys des Plasmas, Ecole Polytech Fed, SWITZERLAND
 Bibliotheek, Fon-Inst Voor Plasma-Fysica, THE NETHERLANDS
 Metin Durgut, Middle East Technical University, TURKEY
 Dr. D.D. Ryutov, Siberian Acad Sci, USSR
 Dr. G.A. Eliseev, Kurchatov Institute, USSR
 Dr. V.A. Glukhikh, Inst Electrophysical Apparatus, USSR
 Prof. D.S. Padichenko, Inst. of Phys. & Tech. USSR
 Dr. L.M. Kovrizhnykh, Institute of Gen. Physics, USSR
 Nuclear Res. Establishment, Julich Ltd., W. GERMANY
 Bibliothek, Inst. für Plasmaforschung, W. GERMANY
 Dr. K. Schindler, Ruhr-Universität Bochum, W. GERMANY
 ASDEX Reading Rm, c/o Wagner, IPP/Max-Planck, W. GERMANY
 Librarian, Max-Planck Institut, W. GERMANY
 Prof. R.K. Janev, Inst of Phys, YUGOSLAVIA