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Turbulence and Self-Consistent Fields in Plasmas

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ABSTRACT

This paper is concerned with the role of self-consistency of the electric field in 1-D plasma turbulence. We first show that in the non-self consistent electric field problem excellent agreement is found between numerical experiments and quasilinear theory whenever the imposed electric field Fourier components have random phase. A discrepancy is exhibited between quasilinear prediction and numerical simulations in the self-consistent electric field case. This discrepancy is explained by the creation of a long correlation time of the electric field resulting from a strong wave-particle interaction. A comparison is made between quasilinear and renormalized propagator theories, and the Dupree "Clump" theory. These three theories are found to be self-contradictory in the regime of strong wave-particle interaction because they make an a priori quasi-gaussian assumption for the electric field. The Direct Interaction Approximation is then shown to be a closure taking into account in a self-consistent way non-gaussian effects.

1. INTRODUCTION

This paper is concerned with the role of self-consistent electromagnetic fields in the theory of turbulent plasmas. In recent years dramatic progress has occurred in our understanding of the motion of charged particles in prescribed electromagnetic fields: We have learned that there are thresholds for the onset of chaotic motion which are governed by conditions such as resonance (or island) overlap; the chaotic motion is characterized by diffusion of particles in velocity and by the exponential divergence of initially nearby trajectories. Above the stochastic threshold particles can wander all over and indeed their trajectories appear to completely fill certain regions of phase space. The detailed studies of intrinsic stochasticity have concentrated on the prescribed field problem.

In this paper we will try to make a preliminary assessment of the importance of the self-consistency of the electric fields in the turbulent or chaotic behavior of plasmas. We will first review in Section 2 some numerical studies of prescribed field models in parameter regimes well above the stochastic threshold. In certain cases it is known that, even for a single realization of the imposed field, a statistical theory on the level of the familiar quasilinear theory provides an accurate description of the diffusion in velocity space when compared to numerical solutions. It is less well-known that the exponential separation of orbits is likewise accurately predicted by a statistical theory of the two-point correlation function. In Section 3 we review the results of statistical theories which involve the ensemble average over many realizations of the imposed fields; in particular the imposed electrical fields are considered to be quasigaussian random variables, in a sense which will be defined in Section 2. Usually this property is realized by assuming random phases for each mode. We examine the conditions under which the predictions of such a statistical theory are able to describe the properties of a single realization such as a numerical simulation run.

In Section 4 we will review some aspects of the self-consistent field case. Self-consistent numerical simulations in the quasilinear parameter regime are not in agreement with the imposed field statistical theories. The reason is the following: the self-consistent Maxwell equations are nonlinear and impose phase correlations between modes. This invalidates the random phase or quasigaussian fluctuation assumption and introduces new correlation times into the problem; this is shown in an iterative calculation in Section 5. The imposed field statistical theories cannot be made self-consistent a posteriori because they neglect important nongaussian correlations.

In a sense (to be discussed) the assumption that E is a gaussian random variable is an assumption of maximum chaos. Since self-consistency invalidates this assumption then a self-consistent system must have less than maximum chaos. Self-consistency implies a kind of self-organization of the chaos.

The formulation of a self-consistent, statistical description of a turbulent plasma is being pursued with renewed vigor at the present time. An approach that shows considerable promise involves the application of the strong turbulence approximation known as the direct interaction approximation (DIA) to Vlasov turbulence. This approximation incorporates certain nongaussian correlations into a statistical theory that retains the basic conservation laws and other realizability properties. In Section 5 we will examine certain features of this theory and make some comparisons with the imposed field theories.

2. REVIEW OF INTRINSIC STOCHASTICITY STUDIES IN PRESCRIBED FIELDS

The imposed field studies seek to establish the properties of single particle trajectories in prescribed electric and magnetic fields. Here we will restrict our considerations to one-dimensional electrostatic models with electric fields of the form

$$E(x,t) = \sum_{n=1}^{N_m} \sum_k |E_k| \exp(i\theta_k) \expi[kx - \omega_n(k)t] \quad (2.1)$$

We consider a finite system of length L so that the k modes are discrete and we allow the possibility of a set of frequencies $\omega_n(k)$ associated with each mode, where $1 \leq n \leq N_m$. For simplicity we will write most of our equations explicitly for the case $N_m = 1$ and $\omega_1(k) = \omega(k)$. Newton's equations for the motion of a particle of charge q and mass m in this field are of course:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \frac{q}{m} E(x,t) = \frac{q}{m} \sum_k |E_k| \expi[kx - \omega(k)t + \theta_k] \end{aligned} \quad (2.2)$$

and are highly nonlinear. We first focus on a single resonance associated with a given mode k_0 ; Newton's equation for a particle interacting with this single mode can be written

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \frac{q}{m} E_{k_0} \sin[k_0 x - \omega(k_0)t] \end{aligned} \quad (2.3)$$

It is convenient to transform to the coordinates X, V in the wave frame:

$$\begin{aligned} x &= v_\phi(k_0)t + X \\ v &= v_\phi(k_0) + V \end{aligned} \quad (2.4)$$

where $v_\phi(k_0) = \omega(k_0)/k_0$ is the wave phase-velocity. In this frame the equations of motion are

$$\begin{aligned} \dot{X} &= V \\ \dot{V} &= \frac{q}{m} E_{k_0} \sin k_0 X \end{aligned} \quad (2.5)$$

These are the well-known equations for the nonlinear pendulum whose solution is exactly known. The phase space is divided into trapping and passing regions as shown in Fig. 1 by the separatrix whose equation is

$$V = \pm \left[2 \frac{g}{m} \frac{|E_{k_0}|}{k_0} \right]^{\frac{1}{2}} \cos \frac{k_0 X}{2} \quad (2.6)$$

The "trapping half width" of the islands of the separatrix is

$$\Delta V_h(k_0) = \sqrt{2 \frac{g}{m} \frac{|E_{k_0}|}{k_0}} \quad (2.7)$$

In the case of many modes we can label each resonance by its phase velocity and trapping half-width as if it were an isolated resonance. The distance in v between the phase velocities of the isolated resonances is

$$\begin{aligned} \delta v &= v_\phi(k_{n+1}) - v_\phi(k_n) \\ &= \frac{\partial}{\partial k} v_\phi(k) \times \frac{2\pi}{L} \end{aligned} \quad (2.8)$$

where $k_n = 2\pi n/L$. For very small amplitudes the resonances, at a first approximation, can be considered as isolated and we have the picture of superimposed resonances shown in Fig. 2.

The work of Chirikov¹ showed that it was useful to define the stochasticity parameter k_s :

$$k_s = \frac{\Delta V_h(k_n) + \Delta V_h(k_{n+1})}{\delta v} \quad (2.9)$$

He showed that for $k_s \ll 1$ the motion is regular (except in small disconnected regions of phase space) but for $k_s > 1$ it is chaotic in the phase space domain for which the velocity belongs to the interval of phase velocities of the resonances. As an example we display the results of Escande and Doveil² for the case of two well-separated resonances with the equations of motion

$$\dot{x} = v$$

$$\dot{v} = M_1 \sin x + M_2 \sin k(x-t)$$

where $v_\phi(k) = 1$ and $v_\phi(0) = 0$. They computed the orbits $x(t)$, $v(t)$ as a function of t from some initial conditions x_0 , v_0 at $t = 0$. The results are summarized in Fig. 3 for the case $k_s \ll 1$. The primary resonances are distorted as evidenced by the deviation of the separatrices from the sinusoidal shape. In addition, new secondary resonances appear. The trajectories near the separatrices become fuzzy forming a stochastic layer. The important result is that there still exists definite passing trajectories and these isolate the phase space in the sense that a particle cannot travel from one set of trapping islands to another. Thus the motion is stable and $|v(t) - v_0|$ is bounded so a particle can't wander far in velocity space; thus there is no diffusion. As k_s is increased (but $k_s \lesssim 1$) the width of the stochastic layer grows as does the size of the secondary resonances. There exists a value of k_s (near unity) where all the stochastic layers join together at which point the system undergoes a gross

stochastic instability in which particles can wander over the phase space between the resonances. Chirikov originally postulated that the stochastic threshold occurred at $k_s = 1$. More recent work^{3,2} has shown that the precise value of the threshold is somewhat less the unity. In the following we will assume $k_s \gg 1$, well above the chaotic threshold. We will not attempt a precise definition of chaos here; suffice it to say for our purposes, the chaotic regime is characterized by diffusion of particles in velocity and exponential divergence of nearby orbits. The former is a very practical consequence of chaos the latter is a more theoretical consequence related to the rate at which information is lost by the system as characterized by the so-called Kolmogorov entropy.

There have been many numerical studies of these quantities for various models of the imposed field of the form in equation (2.2) which appear to fall into three classes:

- i) The first class considers only a single wavenumber but associates an infinite number of frequencies (harmonics) with this wave number (i.e., $\omega_n(k) = n\omega_0$ and $N_m = \infty$). This class includes the famous Chirikov-Taylor mapping and has been investigated by Rechester and White⁴ and others.⁵
- ii) A second class of models has an infinite (or large) number of modes k each with an infinite number of harmonics of a fundamental frequency (i.e., $\omega_n(k) = n\omega(k)$ and $N_m = \infty$).

Such models have been investigated by Rechester, Rosenbluth and White,^{6,7} Molvig et al.⁸

- iii) The class of models closest to our interests have an infinite (or large) spectrum of k values each with a definite frequency $\omega(k)$ (i.e., $N_m = 1$ and $\omega_1(k) = \omega(k)$). Simulations of this case have been carried out by Flynn⁹ and Doveil.¹⁰ In these numerical calculations values of $|E_k|$ for Eq. 2.2 were assigned and the phases θ_k were randomly chosen between 0 and 2π with no correlation between different values of k . This is usually called the Random Phase Approximation (RPA). This procedure generates a stochastic process which is asymptotically equivalent to a gaussian process in the limit of a large number of modes.¹¹ S. P. Gary and D. Montgomery,¹² K. N. Graham and J. A. Fejer¹³ also did similar calculations, though they used somewhat different processes than random phases in order to generate a quasigaussian electric field. In the following, for simplicity, we will consider in the imposed electric field case only the models with random phase Fourier components. The single particle trajectories $x(x_0, v_0, t)$ and $v(x_0, v_0, t)$ were computed numerically from initial conditions x_0, v_0 at $t = 0$. Velocity diffusion was studied by computing the spatial average over the initial positions of the squared velocity deviation

$$\overline{\Delta v^2(t)} \equiv \overline{(v(x_0, v_0, t) - v_0)^2}^{x_0} \quad (2.10)$$

$$\overline{\psi}^{x_0} \equiv \frac{1}{L} \int_0^L \psi(x_0) dx_0$$

Results from these numerical calculations showed that:

i) Δv^2 was proportional to t (except for very short times). The observed velocity diffusion coefficient in the numerical experiments can thus be defined as

$$D_{\text{exp}} (\text{def}) = \frac{1}{2t} \overline{\Delta v^2(t)} \quad (2.11)$$

ii) The observed diffusion coefficient was in good agreement with the prediction of quasilinear theory

$$D_{\text{exp}} = D^{\text{Q.L.}} \quad (2.12)$$

where

$$D^{\text{Q.L.}} = \frac{g}{m} \sum_k \sum_n |E_k^n|^2 \int_0^t d\tau \exp i \omega_n(k) - kv \tau \quad (2.13)$$

The quasilinear theory is a statistical theory, averaged over many realizations of the phases, θ_k , which are assumed to be random. We will discuss statistical theories in the next section.

Similar good agreement between the observed diffusion coefficient and the predictions of quasilinear theory was found in the work of Rechester, Rosenbluth and White⁷ as it can be seen in Fig. 4. For their model the quasilinear prediction is $D^{\text{QL}} = \varepsilon^2 M/4$ where ε is the amplitude of the Fourier components and M is the number of harmonics (Nm in our notation).

Rechester, Rosenbluth and White also computed the so-called "Kolmogorov entropy" defined as

$$h = \lim_{t \rightarrow \infty} \frac{1}{t} \lim_{d_0 \rightarrow 0} \ln \left[\frac{d(t)}{d_0} \right] \quad (2.14)$$

where d is the "distance" between two nearby orbits. We can take this to be

$$d(t) = \left[k_+^2 (x_1 - x_2)^2 + \frac{(v_1 - v_2)^2}{v_+^2} \right]^{1/2} \quad (2.15)$$

where $x_1 = x(t, x_{01}, v_{01})$, $v_1 = v(t, x_{01}, v_{01})$ and $x_2 = x(t, x_{02}, v_{02})$, $v_2 = v(t, x_{02}, v_{02})$ corresponding to different but nearby initial conditions at $t = 0$; $d_0 = d(t = 0)$. The normalization constants in this expression are $v_+ = \frac{1}{2}(v_1 + v_2)$ and $k_+ = k(v_+)$; $k(v)$ is the solution of the implicit equation $\omega(k(v)) = k(v)v$. Positive values of h are interpreted to be a signature of chaotic behavior. Roughly speaking, in physical terms, this definition implies that for short times $d(t) \approx d_0 \exp ht$ and this implies a loss of memory of initial conditions and a sufficient condition for mixing and ergodicity.

The results of the numerical experiments of Rechester et al.,^{6,7} are shown in figure 5. They showed that h_{exp} exists and is independent of initial conditions if d_0 is sufficiently small and obtained excellent agreement with the

prediction $h^{Q.L.}$ obtained from a statistical theory assuming a quasigaussian electric field. This theory, which is discussed in the next section provides an explicit relation of $h^{Q.L.}$ to the quasilinear diffusion coefficient

$$h^{Q.L.} = 0.36 \left(k_+^2 D^{Q.L.} \right)^{1/3} \quad (2.16)$$

The numerical experiments showed that to excellent accuracy

$$h_{\text{exp}} = h^{Q.L.} = 0.36 \left(k_+^2 D^{Q.L.} \right)^{1/3} \quad (2.17)$$

3. STATISTICAL TURBULENCE THEORIES FOR PRESCRIBED RANDOM FIELDS

We next generalize our considerations to a system of many-charged particles moving in electrostatic fields. If the fields are prescribed, we have a system of independent particles whose trajectories have the properties discussed in Section 2. The many particle situation is conveniently described by the one particle phase space distribution function $f(x, v, t)$ which evolves after a time t from some initial distribution $f_0(x, v)$ at $t = 0$ according to the formula

$$f(x, v, t) = \int dx_0 \int dv_0 \delta[x - x(t, x_0, v_0)] \delta[v - v(t, x_0, v_0)] f_0(x, v) \quad (3.1)$$

where $x(t, x_0, v_0)$ and $v(t, x_0, v_0)$ are just the position and velocity discussed in Section 2. The phase-space density evolves according to the Liouville equation:

$$\left[\partial_t + v \partial_x + \frac{q}{m} E(x, t) \partial_v \right] f(x, v, t) = 0 \quad (3.2)$$

which expresses the conservation of density along phase-space trajectories. In the self-consistent case we can still use Eq. 3.2 in which $E(x, t)$ is the self-consistent field determined by Poisson's equation 4.1.¹⁴

In the prescribed field problem we assume $E(x, t)$ is a random variable over an ensemble each realization of which is given by an expression of the form (2.1). This is the so-called stochastic acceleration problem based on the linear stochastic equation (3.2). This problem has received much attention.¹⁵ We give here the results of this analysis in the quasilinear regime which involves the following assumptions

- i) $k_s \gg 1$
- ii) $h^{-1} \equiv \tau_{p.m} \ll \tau_{<f>}$ (3.3)
- iii) quasigaussian approximation for electric field fluctuations, e.g., phase θ_k random and uncorrelated.

Condition i) simply assumes that the system is well above the stochastic threshold. In condition iii), the quasigaussian approximation is defined as the case in which the correlation functions of any order are characterized by a unique correlation length. More precisely, throughout all this article, we name quasigaussian any random process for which an irreducible correlation function of any order is not zero only if all the arguments lie inside a same interval of length ℓ_c , where ℓ_c is the correlation length of the two points correlation function.^c This condition can be realized by the random phase approximation.

In condition ii) we have defined the phase mixing time $\tau_{p.m.}$ as the inverse of the Kolmogorov entropy; $\tau_{p.m.}$ is a measure of the time over which information of initial conditions is lost. This time, $\tau_{p.m.}$, must be much less than the time of evolution of the mean distribution function, $\tau_{<f>}$. We can estimate the latter as

$$\tau_{<f>} \simeq \frac{(\Delta v_f)^2}{D^{Q.L.}} \quad (3.4)$$

where Δv_f is a measure of the width in velocity space for which waves interact resonantly with particles and $D^{Q.L.}$ the quasilinear diffusion coefficient of Eq. 2.13. Note that $D^{Q.L.} \simeq (q/m)^2 \langle E^2 \rangle \tau_c$ where $\tau_c = [\Delta k(v - v_g)]^{-1}$ is the correlation time of the electric field fluctuations as seen by a typical resonant particle; v_g is the group particle of the wave with wave number $k(v)$. We can then write

$$\tau_{<f>}/\tau_c = (\omega_b \tau_c)^{-4} \text{ where } \omega_b = \frac{q}{m} \bar{k} \langle E^2 \rangle^{1/2}$$

is often called the effective bounce frequency; i.e., the frequency of a particle trapped in a monochromatic wave of amplitude $\langle E^2 \rangle^{1/2}$ and wave number $\bar{k}$. With these definitions it is found that condition ii) can also be written as

$$(\omega_b \tau_c)^4 \ll 1 \quad (3.5)$$

Under these conditions it is known¹⁶ that the equation of evolution of the ensemble averaged or mean distribution $\langle f \rangle$ is

$$\partial_t \langle f \rangle = \partial_v D^{Q.L.}(v) \partial_v \langle f \rangle \quad (3.6)$$

where $D^{Q.L.}$ is given in 2.13. The ensemble average is over many realizations of the electric fields. We assume statistical homogeneity, i.e. $\langle E(x, t) \rangle = 0$ and $f_0(x, v) = f_0(v)$.

An equation for the two-phase point, one time correlation function

$$\langle \delta f(x, v, t) \delta f(x', v', t') \rangle, \quad \delta f(x, v, t) = f - \langle f \rangle,$$

can also be derived in this limit (Dupree¹⁷). In the limit of small separation in velocity space $|k(v_+)v_-| \ll \sup [(k^2 D^{Q.L.})^{1/3}, \gamma_k]$, and $|x_-/v_+| \ll \inf [(k^2(v_+)D^{Q.L.})^{-1/3}, \gamma_k^{-1}]$, this equation can be written as:

$$\begin{aligned} \left[\partial_t + v_- \partial_{x_-} - (D^{Q.L.} - D_{12}(x_-)) \partial_{v_-}^2 \right] \langle \delta f(x, v, t) \delta f(x', v', t) \rangle \\ = 2D_{12}(x_-) [\partial_{v_+} \langle f(v_+, t) \rangle]^2 \end{aligned} \quad (3.7)$$

where D_{12} is the relative diffusion coefficient

$$D_{12}(x_-) = \left(\frac{g}{m}\right)^2 \sum_k \sum_n |E_k|^2 \text{Re} G_{k, \omega_n(k)}(v_+) \cos kx_- \quad (3.8)$$

These equations are expressed in relative coordinates: $v_- = v - v'$, $x_- = x - x'$ and centrix coordinates: $v_+ = \frac{v+v'}{2}$ assuming spatial homogeneity. The propagator $G_{k, \omega}(v)$ to be used in 3.7 is the usual renormalized propagator given by:

$$G_{k, \omega}(v) = \int_0^\infty \exp(i(\omega - kv)\tau) \exp\left(-\frac{(k^2(v)D)\tau^3}{3}\right) d\tau \quad (3.9)$$

Dupree¹⁷ was the first to show that equation (3.7) predicts that initially nearby trajectories will diverge exponentially in time. He considered the source free solution to eqn. (3.7), $G_2(x_-, v_-; v_+, t)$, and generated a closed set of equations for the moments $\langle x_-(t)^n v_-(t)^m \rangle \equiv \int dx_- \int dv_- x_-^n v_-^m G_2$. The solution of the resulting equations showed that $\langle x_-(t)^2 \rangle$, $\langle v_-(t)^2 \rangle$ increased from their initial values as $\exp([4k_+^2 D^{Q.L.}]^{1/3} t)$. A more precise contact between the dynamical definition has been made by Rechester, Rosenbluth and White.⁶ Since the quantity h as defined by Eq. 2.14 is numerically observed to be independent of initial conditions, these authors argue that it is a statistical quantity. More explicitly they invoke ergodic properties which permit them to replace the time average

$$\frac{1}{t} \int_0^t \frac{\partial}{\partial t'} \ln \frac{d(t')}{d_0} dt' = \frac{1}{t} \ln \frac{d(t)}{d_0}$$

by the statistical average $\frac{\partial}{\partial t} \langle \ln \frac{d(t)}{d_0} \rangle$. By writing the quasilinear equation corresponding to the two-point, one-time correlation function they find:

$$h^{Q.L.} = \frac{\partial}{\partial t} \langle \ln \frac{d(t)}{d_0} \rangle = 0.36 (k_+^2 D^{Q.L.})^{1/3} \quad (3.10)$$

These authors observed an excellent agreement between the statistical prediction $h^{Q.L.}$ and the experimental value h_{exp} obtained from numerical integration of the orbits (Fig. 5).

Thus there is excellent agreement between the predictions of the quasilinear statistical theories for velocity diffusion (i.e., $D_{\text{exp}} \approx D^{Q.L.}$) and the corresponding predictions for the Kolmogorov entropy (i.e., $h_{\text{exp}} \approx h^{Q.L.}$) in the case of prescribed quasigaussian random fields when the conditions (3.3) for the validity of quasilinear theory are satisfied. In Section 4 we will show there is no corresponding agreement between standard statistical theory and the results of certain self-consistent computer simulations.

We complete this section by pointing out the following question: why the results of an ensemble averaged statistical theory can accurately describe the results of a single numerical experiment; i.e., a single realization of the ensemble. The answer is really not obvious as it is sometimes considered to be;

the statistical quantities $D^{Q.L.}$ and $h^{Q.L.}$ clearly do not depend on the phases, but the results D_{exp} and h_{exp} of one-numerical experiment with a given assignment of the phases could in principle depend on the phases. The simulations described above are essentially spatially homogeneous and explicitly calculated only spatial averages. In particular, the spatially averaged phase space distribution $f(v,t)$ is defined as

$$\bar{f} = \frac{1}{L} \int_0^L dx f(x,v,t) \quad .$$

Pesme and Brisset¹⁸ have considered the following situation: for each realization the mode amplitudes $|E_k|$ form a smooth function of k and the phases θ_k are randomly distributed and independent of each other. For such a given realization they have shown that $\bar{f}$ obeys the quasilinear equation (3.5) with $\langle f \rangle$ replaced by $\bar{f}$ if the conditions (3.3) are satisfied. More precisely they have shown that the corrections to this equation are statistically small under these conditions. The condition $k_s \gg 1$ is easily shown to be equivalent to the following condition on the length of the system:

$$L/v \gg h^{-1} \quad \text{or} \quad L \gg \ell_{p.m} \equiv v \tau_{p.m.} \quad (3.11)$$

In this case during the transit time of a particle of velocity v through the system it sees many independent partitions of the system which are equivalent to independent realizations. This is because on one hand, after a time h^{-1} the particle has lost all memory of its initial conditions. On the other hand, the correlation length of the electric field fluctuations is $\ell_c \approx 1/\Delta k$ and when the conditions 3.3 are satisfied, we have $\ell_c \ll \ell_{p.m}$ so that the electric field in each partition is independent. Therefore spatial averaging is then equivalent to ensemble averaging.

4. COMPARISON OF SELF-CONSISTENT NUMERICAL SIMULATIONS WITH STANDARD QUASILINEAR THEORY

Here we will review the simulations of the bump on tail instability in a one-dimensional, one-component plasma which were carried out and analyzed by Adam, Laval and Pesme.¹⁹ The conditions of these simulations were those commonly thought to be in the regime of validity of quasilinear theory.²⁰ In fact, the simulations did not follow the quasilinear predictions and this failure is now attributed to the breakdown of the random phase assumption in a self-consistent plasma

The fact that self-consistency results in qualitatively and quantitatively different behavior than in the imposed field case has also been verified in a rather different physical situation by the simulations of Bezzerides, Gitomer and Forslund.²¹ These authors investigated the production of hot electrons in resonance absorption of light. Detailed consideration of this case is not possible here and we refer the reader to this work.

In the self-consistent problem the electric field is, of course, determined by the charge density of all the particles in the system. Poisson's equation is written as:

$$\nabla \cdot E(x,t) = 4\pi q \int dv f(x,v,t) - 4\pi q N \quad (4.1)$$

where f is a smooth function¹⁴ and N is the ion density. The Vlasov-Poisson system of equations thus becomes a nonlinear field theory. In the self-consistent case the field $E(x,t)$ cannot be specified, but evolves dynamically from the initial conditions ($f_0(x,v) = f(x,v,t=0)$).

The bump on tail instability develops from the initial conditions shown in Fig. 6a. Here is shown the initial spatial averaged distribution $f(v,t=0) = f_{k=0}(v,t=0)$. The phases of $f_k(v,t=0)$ are randomly chosen from one mode to the next. The fourier amplitudes $E_k(t=0)$ for $k \neq 0$ are computed from Poisson's equation, i.e., $E_k(t=0) = (4\pi q/ik) \int dv f_k(v,t=0)$ but are not shown. The initial amplitudes $|E_k(t=0)|$ are shown schematically in Fig. 6.a.

It is well-known that the linear dispersion relation derived from $\epsilon_0(k,\omega) = 0$ where $\epsilon_0(k\omega)$ is the linear dielectric function predicts the familiar Landau growth rate for Langmuir waves provided $\gamma_k^{Q.L.}/\omega_p \ll 1$:

$$\frac{\gamma_k^{Q.L.}}{\omega_k} = \frac{\pi \omega_p^2}{2Nk|k|} \left[1 - \frac{k}{\omega_k} \frac{d\omega_k}{dk} \right] \left(\frac{\partial \bar{f}}{\partial v} \right)_{v=\omega_k/k} \quad (4.2)$$

Particles with velocities v near the phase velocity $v_\phi(k) = \omega(k)/k$ of the wave see essentially a static field and interact strongly with the wave.

If $(\partial_v \bar{f})_{v=v_\phi} > 0$ then the waves grow because there is energy transfer from

particles to the wave. For the initial condition on $\bar{f}$ chosen by Adam et al., there is a band of wave numbers whose growth rates ($\gamma_k^{Q.L.} > 0$) are very close in value and for short times a broad spectrum of waves grows up as shown in Fig. 6b. At later times (Fig. 6c) the spectrum becomes noisy and uneven but still broad and begins to cause diffusion of particles in velocity which tends to flatten out $\bar{f}$ in the unstable region of velocities. At longer times (Fig. 6d) becomes essentially flat in this region, the waves cease to grow and a broad but noisy quasistationary spectrum of waves results. Except for very short times the spectrum has grown to a level for which $k_s \gg 1$ and the condition for stochasticity expected for an imposed field is exceeded. Also, throughout the simulation the condition for quasilinear theory $\hbar^{-1} = \tau_{p.m.} \ll \tau_{\bar{f}}$ are well satisfied.

The comparison between the numerical results and the quasilinear prediction for the growth rate for a typical unstable mode is shown in Fig. 7. The numerical results for the ensemble average over ten simulation runs (solid line) lie significantly above the predicted quasilinear growth rate (dashed line).

Now recall the usual quasilinear equation for the time development of the mean intensity in mode k :

$$\partial_t \langle |E_k|^2 \rangle = 2 \gamma_k^{Q.L.} \langle |E_k|^2 \rangle + S_k \quad (4.3)$$

In the usual quasilinear theory the source term S_k which arises from mode-mode coupling effects is argued to be negligible when the conditions (3.3) are fulfilled. The coupled equations (4.3) with $S_k = 0$ and (3.5) constitute the usual quasilinear theory. Together they conserve particle number, momentum, and energy.

The simulations appear to show that the growth rate which should appear in (4.3) is greater than $\gamma_k^{Q.L.}$; in this case equation (3.5) must also be altered (e.g., $D \neq D^{Q.L.}$) in order to maintain conservation of energy. Therefore, we infer from the simulations that

$$\gamma_{exp}(k) \neq \gamma_k^{Q.L.}(k)$$

$$D_{exp} \neq D^{Q.L.}$$

Since of the conditions (3.3) for the validity of quasilinear theory both i) ($k_s \gg 1$) and ii) ($\hbar^{-1} \ll \tau_{cf}$) are satisfied during the simulation we must conclude that iii) the quasigaussian fluctuation assumption is not valid in the self-consistent field case.

This statement is not surprising in itself since we know that the self-consistent Poisson equation is nonlinear. As a result nongaussian electric field fluctuations will evolve from initially gaussian electric field fluctuations.

5. SELF-CONSISTENT STATISTICAL THEORIES OF PLASMA TURBULENCE

5.1 Introduction

The understanding of the self-consistent statistical description of Vlasov turbulence is in a new stage of development today and the study of this problem will certainly be the subject of future research in the nonlinear physics of plasmas.

In the prescribed electric field problem we were concerned with the motion of particles in random forces. The property for the electric field of being prescribed did not appear explicitly in the derivation of the equations of evolution for $\langle f \rangle$ and $\langle \delta f \delta f \rangle$. These equations (eq. 3.6 and 3.7) actually reduce to two Fokker-Planck equations and they can be easily derived by computing the friction and diffusion coefficients corresponding to the motion of a particle in the random field. This is the reason why it has been thought for a long time that the statistical description of the particle motion in self-consistent Vlasov turbulence was the same as in the prescribed electric field case. We have, however, to keep in mind that a basic validity condition for quasilinear theory - as well as for the Fokker-Planck equation - is the quasigaussian property of the electric field. In the stochastic acceleration problem this property was always imposed, e.g., through random phases, but in the Vlasov-Poisson case it is necessary to check the validity of this hypothesis.

We consider explicitly a self-consistent Vlasov turbulence situation for which the conditions 3.3i and 3.3ii are satisfied.

We assume also the usual weak turbulence hypothesis that assigns N_m well-defined frequencies $\omega_m(k)$ to a wave number k . These frequencies have to be computed self-consistently through a nonlinear dispersion relation. For simplicity we will take $N_m = 1$. All the quantities such as k_s , ω_b , τ_c are now computed through the instantaneous values of the self-consistent electric field. Inequality 3.3i) is easily satisfied in unstable situations. Inequality 3.3ii) is usually considered as the validity condition of quasilinear theory and is fulfilled, for instance, in the warm beam-plasma instability.

We define as quasigaussian theories the statistical descriptions of Vlasov turbulence which assume a priori that the self-consistent electric field has quasigaussian properties. In this section we will first present the usual argument given for justifying this assumption. We will then show that the quasigaussian theories are self-contradictory in the regimes where the non-linearity plays a role. The reason for discrepancy will be exhibited through a perturbative calculation: the statistical properties of the self-consistent electric field are not characterized by a unique correlation length $\ell_c = 1/\Delta k$ but also by the phase-mixing length $\ell_{p.m.} = v\tau_{p.m.}$.

We close this Section by describing some of the features of the direct interaction approximation (DIA) which is a renormalized theory which takes into account certain nongaussian effects.

5.2 The quasigaussian hypothesis

In the earlier derivations of quasilinear theory the random-phase assumption was usually implicitly made.^{22,23} The underlying reason has been made explicit by O'Neil²⁴ as follows:

Let us assume on physical grounds that well above the stochastic threshold the electric field is itself turbulent and is characterized by an autocorrelation length ℓ_c^{eff} . In a finite box of length L , the Fourier components of the electric field are defined as:

$$E_k(t) = \frac{1}{2\pi} \int_0^L dx E(x,t) \exp-ikx \quad \text{with } k = \frac{n2\pi}{L}.$$

If we now divide the interval $[0,L]$ into segments of length ℓ_0 where $\ell_c^{eff} \ll \ell_0 \ll L$, the electric field in each segment is independent of the field in other segments. Therefore E_k can also be written as

$$E_k = \frac{1}{2\pi} \sum_{j=1}^N \int_{(j-1)\ell_0}^{j\ell_0} dx E(x,t) \exp-ikx \quad \text{with } N = L/\ell_0.$$

Thus E_k appears to be the sum of many independent variables and by the central limit theorem E_k behaves as a Gaussian variable in the limit L/ℓ_0 very large.

Let us emphasize that the only hypothesis in this demonstration is the existence of an autocorrelation length l_c^{eff} and that seems quite reasonable in a turbulent situation.

5.3 The quasigaussian theories

On the basis of the previous section, the quasigaussian theories assume a priori that the last condition iii) of 3.3 is satisfied - i.e., the electric field behaves as a quasigaussian process.

The scheme of these theories is the following:

- 1) Two Fokker-Planck equations describe the evolution of $\langle f \rangle$ and $\langle \delta f \delta f' \rangle$, namely the quasilinear equation 3.6 and the Dupree equation 3.7. The friction and diffusion coefficients involve the spectral density $\langle |E_k|^2 \rangle$.
- 2) The spectrum $\langle |E_k|^2 \rangle$ is then computed by writing Poisson's equation as:

$$k^2 \langle |E_k|^2 \rangle = (4\pi)^2 q^2 \int dv dv' \langle \delta f \delta f' \rangle_k \quad (5.1)$$

We thus obtain a closed set of three equations for the three unknowns $\langle f \rangle$, $\langle \delta f \delta f' \rangle$, and $\langle |E_k|^2 \rangle$. However, it is necessary to go beyond this level in order to compute the irreducible correlation functions of the electric field of higher order than 2. This scheme would be consistent only if these correlation functions satisfy the definition of a quasigaussian process given in §2.

An indirect demonstration of the inconsistency of this scheme in the regime $(k^2 D)^{1/3} \gg \gamma_k^{\text{Q.L.}}$ has been given by Adam, Laval and Pesme.²⁵ They solved the Dupree equation 3.7 and by inserting its solution into Poisson's equation 5.1; they found a growth rate different from the quasilinear prediction. This result, combined with quasilinear equation 3.6 does not allow energy conservation. These authors conclude that the quasilinear theory is inconsistent in one dimension and in the regime $(k^2 D^{\text{Q.L.}})^{1/3} \gg \gamma_k^{\text{Q.L.}}$ and showed that the contribution of the 4-point irreducible correlation function is not negligible, in contradiction to the quasigaussian hypothesis.

Even though the quasigaussian scheme appears to be invalid, the solution of the Dupree equation 3.7 exhibits certain features worth being mentioned. In the limit of inequalities 3.3f) and ii), and for $|k(v_+) v_-| \ll \sup (\gamma_k^{\text{Q.L.}}, (k^2 D^{\text{Q.L.}})^{1/3})$, and $|x_-/v_+| \ll \inf [(k(v_+)^2 D^{\text{Q.L.}})^{-1/3}, \gamma_k^{-1}]$, equation 3.7 becomes:

$$\begin{aligned} \left[\partial_t + v_- \partial_{x_-} - 2D^{\text{Q.L.}}(v_+) \partial_{v_-}^2 \right] \langle \delta f \delta f' \rangle &= \\ 2D^{\text{Q.L.}}(v_+) \cos(k(v_+)x_-) \left[\partial_{v_+} \langle f \rangle \right]^2 & \\ - 2D^{\text{Q.L.}}(v_+) \cos(k(v_+)x_-) \partial_{v_-}^2 \langle \delta f \delta f' \rangle & \end{aligned} \quad (5.2)$$

The last term of the R.H.S. of Eq. 5.2 is called the mode-coupling term since it couples together the k and $k-k(v_+)$, $k + k(v_+)$ Fourier components of $\langle \delta f \delta f' \rangle$.

The ratio $R = (k^2 D^{Q.L.})^{1/3} / \gamma_k^{Q.L.}$ turns out to be a measure of the nonlinearity and can be considered as a Reynolds number. For $R \ll 1$ the mode-coupling term is negligible and the quasilinear results are recovered. For $R \gg 1$ the mode coupling terms can be shown to be of the same order of magnitude as the other terms. The mode coupling terms give a contribution for $v \approx v' \approx \omega_k/k$ and generate harmonics coupled together through the resonant particles. As a result, the correlation function $\langle \delta f \delta f' \rangle$ is found to have the following convenient harmonic expansion

$$\begin{aligned} & \langle \delta f(x, v, t) \delta f(x', v', t') \rangle \\ &= \sum_n g_n(v_-) \exp \left[i n k(v_+)(x-x') \right] \exp \left[-i n \omega_{|k(v_+)|} (t-t') \right] \end{aligned} \quad (5.3)$$

In the regime $R \gg 1$, where the nonlinearity plays a role, the correlation function is thus the sum of an infinite number of harmonics coupled together. When inserted into Poisson equations the contribution coming from this harmonic coupling appears as a source term radiating coherently. We can, therefore, say that the nonlinearity reinforces the coherence of the solution. As a consequence, Adam, Laval, and Pesme indeed have found an increase of the growth rate compared to the quasilinear result. This feature is generalized in a self-consistent way in the DIA.

The Dupree "clump theory" likewise predicts an enhanced radiation due to macroscopic granulation of the phase space density - a sort of enhanced discreteness noise due only to nonlinear processes in the continuous Vlasov approximation. The Dupree theory applies to a "strong" turbulence limit in which no definite normal mode frequency exists for a given k . Therefore $k(v) = \omega_{k(v)}/v$ is not defined in this limit.

The results of Adam, Laval and Pesme apply in the limit of quasilinear ordering (eq. 3.31 and 3.31i) when $\gamma_k/\omega_k \ll 1$ so that a definite frequency is associated with a definite k . Particles of velocity v interact strongly with waves in the spectrum of wave number $k(v)$ such that particles tend to be periodically correlated with the separation $2\pi/k(v)$. Physically, the particles are more likely to be found near the troughs of these resonant waves.

5.3 The quasis gaussian hypothesis revisited

In this subsection we will show in a perturbative way how the mode-coupling terms generate a nongaussian electric field. We will follow essentially the arguments of Laval and Pesme.²⁶

Let $\bar{f}_0(v, t)$ the space-averaged distribution function and define δf as $\delta f = f(x, v, t) - f_0(v)$. Following Kadomtsev, the equation for δf can be written in Fourier space as:

$$\delta f_{k, \omega} = g_{k\omega}^0 \left[E_k f_0 \right]_{\omega} + g_{k\omega}^0 \sum_{k' \neq 0} \int d\omega' E_{k-k', \omega-\omega'} f_{k', \omega'}$$

where $g_{kw}^0 = \frac{(q/m)}{\gamma[\omega - kv + i\alpha]} \frac{\partial}{\partial v}$,

in which α provides a contour prescription for v -integration.

Equation 5.5 can be solved iteratively beginning with $\delta f_{k,w}^{(1)} = g_{kw}^0 [E_k f_0]_w$ in a functional series in powers of E . This series is then inserted into Poisson's equation $ikE_{kw} = 4\pi q \int \delta f_{kw}(v) dv$ and yields a nonlinear equation for E_{kw} of the following form:

$$\varepsilon(k, \omega) E_{kw} = \mu_2(k, \omega) E E + \mu_3(k, \omega) E E E + \dots \quad (5.6)$$

where ε is the quasilinear dispersion relation:

$$\varepsilon(k, \omega) = 1 + i \frac{4\pi q}{k} \int dv g_{kw}^0 \bar{f}_0 \quad (5.7)$$

and

$$\mu_2(k, \omega) E E = \sum_{k'} \int d\omega' V_{kw, k'\omega'} E_{k'\omega'} E_{k-k', \omega-\omega'} \quad (5.8a)$$

$$\mu_3(k, \omega) E E E = \sum_{k'} \sum_{k''} \iint d\omega' d\omega'' V_{kw, k'\omega', k''\omega''} E_{k'\omega'} E_{k''\omega''} E_{k-k'-k'', \omega-\omega'-\omega''} \quad (5.8b)$$

with

$$V_{kw, k'\omega'} = \frac{q}{k \varepsilon_0} \int g_{kw}^0 g_{k-k', \omega-\omega'}^0 \bar{f}_0 dv \quad (5.9)$$

$$V_{kw, k'\omega', k''\omega''} = \frac{q}{k \varepsilon_0} \int g_{kw}^0 g_{k-k', \omega-\omega'}^0 g_{k-k'-k'', \omega-\omega'-\omega''}^0 \bar{f}_0 dv$$

Equation 5.7 can be solved iteratively by setting

$$E = E^{(1)} + E^{(2)} + E^{(3)} \dots \quad (5.10)$$

The lowest order gives $\varepsilon E^{(1)} = 0$. This is the approximation of quasilinear theory in which the imaginary part of the root of $\varepsilon(k, \omega)$ determines the quasilinear growth rate $\gamma_k^{Q.L.}$; for simplicity in the following we set $\gamma_k^{Q.L.} = \gamma_k$. We have $E_k^{(1)}(t) = E_k(0) \exp[-i\omega_k t + \gamma_k t]$. We assume $E_k(0)$ as a random phased field. The mode-coupling terms will generate nongaussian correlations:

$$\varepsilon E^{(2)} = \mu_2 E^{(1)} E^{(1)} \quad (5.11a)$$

$$\varepsilon E^{(3)} = \mu_2 E^{(2)} E^{(1)} + \mu_2 E^{(1)} E^{(2)} + \mu_3 E^{(1)} E^{(1)} E^{(1)} \quad (5.11b)$$

We now consider the 4-point correlation function $\langle E_{k_1+k_2+k_3} E_{-k_1} E_{-k_2} E_{-k_3} \rangle$.

On substitution of the expansion (5.10) for each E_{k_i} we find first of all the usual

terms $\langle E_{\Sigma k_i}^{(1)} E_{-k_1}^{(1)} E_{-k_2}^{(1)} E_{-k_3}^{(1)} \rangle$ which, because of the gaussian assumption for $E^{(1)}$ can be cast into a sum of $\langle E^{(1)} E^{(1)} \rangle \langle E^{(1)} E^{(1)} \rangle$. Let us now compute the

nongaussian contributions. At the lowest order we obtain the sum of four terms

like $\langle E_{k_i}^{(3)} E_{k_j}^{(1)} E_{k_k}^{(1)} E_{k_l}^{(1)} \rangle$ with $\Sigma k_m = 0$. The terms $E^{(2)}$ are indeed smaller by a factor γ_k/ω_{pe} since $\varepsilon(\omega, k)$ in Eq. 5.11a has to be taken at frequency $\omega = 0$

or $\pm 2\omega_{pe}$ and cannot be made small.

From Eq. 5.11b we find

$$E_k^{(3)}(t) = \sum_{k'} \sum_{k''} \frac{V_{k\Sigma\Omega} \frac{k' \Omega_{k'}}{\varepsilon(k, \Sigma\Omega)} \frac{k'' \Omega_{k''}}{\varepsilon(k, \Sigma\Omega)}} E_{k'}^{(1)}(t) E_{k''}^{(1)}(t) E_{k-k'-k''}^{(1)}(t) \quad (5.12)$$

where $\Omega_k = \omega_k + i\gamma_k$ and $\Sigma\Omega = \Omega_{k'} + \Omega_{k''} + \Omega_{k-k'-k''}$. After substitution of this expression into $\langle E^{(3)} E^{(1)} E^{(1)} E^{(1)} \rangle$ and use of the gaussian assumption for $E^{(1)}$, we obtain the following expression:

$$\begin{aligned} & \langle E_{k_1+k_2+k_3}^{(3)}(t_1) E_{-k_1}^{(1)}(t_2) E_{-k_2}^{(1)}(t_3) E_{-k_3}^{(1)}(t_4) \rangle \\ &= \left(\frac{g}{m}\right)^2 \langle |E_{k_1}^{(1)}(0)|^2 \rangle \langle |E_{k_2}^{(1)}(0)|^2 \rangle \langle |E_{k_3}^{(1)}(0)|^2 \rangle \\ & \quad \exp(-i\omega_{k_1}(t_1-t_2)) \exp(-i\omega_{k_2}(t_1-t_3)) \exp(-i\omega_{k_3}(t_1-t_4)) \\ & \quad \exp \gamma_{k_1}(t_1+t_2) \exp \gamma_{k_2}(t_1+t_3) \exp \gamma_{k_3}(t_1+t_4) \\ & \quad \cdot \left\{ \frac{2k_1^2}{\left[(2k_1+k_2+k_3) \frac{\omega_{k_1}}{k_1} - 2iv \right]^3 \left[(k_1+k_2) \frac{\omega_{k_1}}{k_1} - iv \right]} \right\} \\ & \quad + \text{permutations } (k_1, k_2, k_3) \quad , \text{ with } v = \gamma_{k_1} \end{aligned} \quad (5.13)$$

Here the matrix element V has been explicitly evaluated using 5.8b. The dielectric function has been taken to be on resonance, i.e., $\omega_{k_1} + \omega_{k_2} + \omega_{k_3} \approx \omega_{k_1+k_2+k_3}$ which implies that frequencies and wave numbers have not all the same sign. We have also assumed that the dispersion of the roots ω_k is weak so that $\varepsilon(\Sigma\Omega) \approx (\frac{\partial \varepsilon}{\partial \omega}) 2i\gamma_k^{Q.L.}$. The expression 5.13 enables us to compute the contribution of the irreducible 4-point correlation function in the evaluation of the mode amplitudes. At the lowest order we have

$$i \frac{\partial \varepsilon}{\partial \omega_k} \left[\partial_t - 2 \gamma_k^{Q.L.} \right] \langle |E_k(t)|^2 \rangle = \langle E_{-k}^{(3)} \mu_3(\Omega_k - k) E^{(1)} E^{(1)} E^{(1)} \rangle + \langle E_k^{(3)} \mu_3(\Omega_{-k} - k) E^{(1)} E^{(1)} E^{(1)} \rangle \quad (5.14)$$

which leads to the result:

$$\langle |E_k|^2(t) \rangle = \langle |E_k^{(1)}(t)|^2 \rangle \left[1 + \left[0.24(k^2 D)^{1/3} / \gamma_k \right]^6 \right] \quad (5.15)$$

The modification of the spectrum as given by the expression inside the bracket of Eq. 5.15 shows again that the mode coupling terms lead to an increase of the energy transfer from resonant particles towards waves. This result is strictly valid only for $\gamma_k^{Q.L.} \gg (k^2 D)^{1/3}$ but it clearly shows that the irreducible 4-point correlation function gives a non-negligible contribution as soon as $(k^2 D)^{1/3} \gtrsim \gamma_k^{Q.L.}$ which invalidates the quasigaussian assumption.

Let us consider now the apparent paradox with O'Neill's argument.²⁴ By Fourier transform of Eq. 5.13 we obtain the expression in real space of the 4-point correlation function

$$\begin{aligned} & \langle E^{(3)}(x_1, t_1) E^{(1)}(x_2, t_2) E^{(1)}(x_3, t_3) E^{(1)}(x_4, t_4) \rangle = \\ & \sum_{k_1, k_2, k_3} \langle E_{k_1+k_2+k_3}^{(3)}(t_1) E_{-k_1}^{(1)}(t_2) E_{-k_2}^{(1)}(t_3) E_{-k_3}^{(1)}(t_4) \rangle \\ & \exp i k_1(x_1 - x_2) \exp i k_2(x_1 - x_3) \exp i k_3(x_1 - x_4) \end{aligned} \quad (5.16)$$

If the expression in the curly brackets of the R.H.S. of Eq. 5.13 was a smooth function of k_1 , k_2 and k_3 , then this correlation function would be essentially a product of three two-point correlation functions of the form

$$\phi(\zeta, \tau) = \Sigma \langle |E_k(0)|^2 \rangle \exp -i[\omega_k \tau - k\zeta]$$

For correlations along the unperturbed orbits $\zeta = v\tau$, this has the usual correlation time $\tau_c = |\Delta k(v - v_g)|^{-1}$ discussed previously.

On the contrary, however, the expression in curly brackets of the R.H.S. of Eq. 5.13 is a peaked function for $v|k_1+k_2| \lesssim v$. In the regime $R = (k^2 D)^{1/3} / \gamma_k \gg 1$, Eq. 5.15 shows that the series expansion in power of $E^{(1)}$ is not convergent. The free propagator g^0 has then to be replaced by the renormalized propagator g^r

$$g_{\omega, k}^r = (-q/m) \int_0^\infty d\tau \exp i(\omega - kv)\tau \exp - \left[\frac{(k^2 D^{Q.L.}) \tau^3}{3} \right] \frac{\partial}{\partial v} \quad (5.17)$$

In the estimation of order of magnitudes it is sufficient to replace $v = \gamma_k^{Q.L.}$ by $v = (k^2 D^{Q.L.})^{1/3}$. The nongaussian correlation function is, therefore, characterized by the time $(k^2 D^{Q.L.})^{-1/3}$ which is of the order the inverse of Kolmogorov entropy $h^{Q.L.} = 0.36(k^2 D^{Q.L.})^{1/3}$ encountered earlier. Since $(k^2 D)^{-1/3} \gg \tau_c$ we see that self-consistency introduces a tendency toward self-organization of the turbulent system.

We thus see that O'Neil's argument still holds if ℓ_c^{eff} is no longer the usual autocorrelation length $\ell_c = (\Delta k)^{-1}$ but is replaced by $\ell_{p.m} = v \tau_{p.m}$. However, by redefining the correlation time in this manner $\tau_c^{eff} = (k^2 D)^{-1/3}$ we would obtain a convergence parameter $\epsilon^{eff} = (\omega_b \tau_c^{eff})^4 = \frac{1}{[\omega_b \tau_c]} 4/3 \gg 1$. A more detailed diagrammatic expansion leads to the conclusion that in the regime $(k^2 D)^{1/3} \gg \gamma_k$ all the irreducible correlation functions contribute at the same order of magnitude as the quasilinear term. This result invalidates the theories which consist of a simple renormalization of the propagator such as developed by Rudakov and Tsytovich, Choi and Horton.²⁷ In the regime defined by 3.3i) and ii) they lead to the quasilinear result since they don't contain the mode coupling term of Eq. 5.2. It can be shown²⁸ that a partial summation of the neglected irreducible correlation functions leads to the Dupree equation 3.7. Therefore, we may summarize the different levels of renormalization:

1. WEAK TURBULENCE THEORY - This makes the usual quasilinear prediction but suffers from the defect of secular divergences arising in the neglected irreducible correlation functions.

These secularities are removed by renormalization of the propagator which leads to:

2. SIMPLY RENORMALIZED THEORIES - These again make the usual quasilinear prediction but still suffer from non-negligible contributions from neglected irreducible correlation functions.

A partial summation of the irreducible correlation functions leads to:

3. DUPREE'S THEORY - This theory predicts a zero order modification of the quasilinear result coming from mode coupling terms. However, this theory considers only part of the irreducible correlation functions and is not properly self-consistent.

5.4 SELF-CONSISTENT STATISTICAL DESCRIPTIONS AND THE DIA

It appears from the considerations above that most familiar statistical theories of Vlasov turbulence until recently, have been quasigaussian theories in the sense that they explicitly or implicitly make the assumption of quasigaussian electric field correlations.

A self-consistent statistical description of Vlasov turbulence must take into account the relationship between perturbations in the averaged phase space distribution $\langle f_1 \rangle \equiv \langle f(v_1, x_1, t_1) \rangle$ and perturbations in the mean electric field $\langle E_1 \rangle = \langle E(x_1, t_1) \rangle$ as well as the relationship between the fluctuations in these quantities: $\delta f_1 = f_1 - \langle f_1 \rangle$ and $\delta E_1 = E_1 - \langle E_1 \rangle$. These relationships are, of course, imposed by Poisson's eqn. (4.1). For the discussion here we follow closely the results of reference 29.

From the Vlasov eqn. (32) the well-known equations for the mean distribution function is easily found:

$$\left[\partial_{t_1} + v_1 \partial_{x_1} + \frac{q}{m} \langle E_1 \rangle \partial_{v_1} \right] \langle f_1 \rangle = -\frac{q}{m} \partial_{v_1} \langle \delta E_1 \delta f_1 \rangle + \eta_1 \quad (5.18)$$

This couples $\langle f \rangle$ to the fluctuation correlation function $\langle \delta E \delta f \rangle$. The averaged Poisson equation is

$$\nabla_1 \cdot \langle E_1 \rangle = 4\pi q \int dv_1 \langle f_1 \rangle = 4\pi q N \quad (5.19)$$

We have added an artificial phase space source density $\eta_1 = \eta(x_1, v_1, t_1)$ which we can consider as a probe of the system. Suppose for $\eta = 0$ the system of equations has a well-defined solution $\langle f_1 \rangle$, $\langle E_1 \rangle$; we then add a local phase space source perturbation $\delta\eta$ which changes $\langle f_1 \rangle$ to $\langle f_1 \rangle + \delta\langle f_1 \rangle$ and $\langle E_1 \rangle$ to $\langle E_1 \rangle + \delta\langle E_1 \rangle$.

The increments δ are functional differentials such as those used in the calculus of variations of continuous fields. A relationship between the increments $\delta\langle f \rangle$, $\delta\langle E \rangle$ and $\delta\langle \eta \rangle$ is readily found from eq. (5.18) to be of the form

$$G_{12}^{-1} \delta\langle f_2 \rangle + \frac{q}{m} \partial_{v_1} \Gamma_{12} \delta\langle E_2 \rangle = \delta\eta_1 \quad (5.20)$$

By taking the functional differential of eqn. (5.18) it is easy to identify the coefficients as

$$G_{12}^{-1} = (\partial_{t_1} + v_1 \partial_{x_1} + \frac{q}{m} \langle E_1 \rangle \partial_{v_1}) \delta(1-2) + \frac{q}{m} \partial_{v_1} \frac{\delta\langle \delta E_1 \delta f_1 \rangle}{\delta\langle f_2 \rangle} \quad (5.21)$$

$$\Gamma_{12} = \langle f_1 \rangle \delta(1-2) + \frac{\delta\langle \delta E_1 \delta f_1 \rangle}{\delta\langle E_2 \rangle} \quad (5.22)$$

To simplify writing the equations we are using a summation convention where, for example, $G_{12}^{-1} \delta\langle f_2 \rangle \equiv \int dv_2 \int dx_2 \int dt_2 G_{12}^{-1} \delta\langle f_2 \rangle$, etc. and $\delta(1-2) \equiv \delta(v_1 - v_2) \delta(x_1 - x_2) \delta(t_1 - t_2)$. The last terms on the right hand side of eqns. (5.21) and (5.22) contain functional derivatives of the correlation function $\langle \delta E_1 \delta f_1 \rangle$ with

respect to changes in the mean values $\langle f \rangle$ and $\langle E \rangle$, respectively. These functional derivatives completely describe the renormalization of the response functions G_{12} and Γ_{12} .

The response function G_{12} is called the "quasiparticle propagator"; without renormalization it is clear from eqn. (5.21) that G_{12} simply describes particle propagation along unperturbed orbits (in the mean field $\langle E_1 \rangle$ which is zero in a homogeneous system.) The renormalization term takes into account the self-consistent response of the fluctuating turbulent "background" through which the particle propagates. In an imposed fluctuation limit this propagator reduces to the familiar Dupree renormalized propagator given in eqn. 3.9.

The response function $\partial_{v_1} \Gamma_{12}$ is called the "quasiparticle potential source."

It takes into account the effective source density $\delta\eta$ which must arise when there is a change in the mean electrostatic field $\delta\langle E \rangle$ in order to keep $\langle f \rangle$ constant. Without renormalization, this term is simply proportional to $\partial_{v_1} \langle f_1 \rangle$.

The renormalization of this quantity is not familiar but was actually considered by Kadomstev³⁰ in Section III of his 1965 book (equation 21a). (Kadomstev's renormalized theory has some elements in common with the DIA, but differs in several important respects related to self-consistency.)

We next consider the fluctuations. By subtracting eq. (5.18) from eqn. (3.2) and noting the definitions (5.21) and (5.22) we can derive the exact equation for δf_1

$$\delta f_1 = \overbrace{(q/m) G_{12} \partial_{v_2} \Gamma_{23} \delta E_3}^{\delta f_1^{\text{coh}}} + \overbrace{(q/m) G_{12} \partial_{v_2} \left[\delta f_1 \delta E_1 - \langle \delta f_1 \delta E_1 \rangle - \frac{\delta \langle \delta f_1 \delta E_1 \rangle}{\delta \langle f \rangle} \delta f_2 - \frac{\delta \langle \delta f_1 \delta E_1 \rangle}{\delta \langle E_2 \rangle} \delta E_2 \right]}^{\delta f_1^{\text{inc}}} \quad (5.23)$$

We have indicated that δf_1 can be separated into a coherent part proportional to the electrostatic fluctuation δE and an incoherent part. In a spatially homogeneous plasma it is easy to see that the Fourier component δf_k^{coh} is proportional to δE_k . For the incoherent part terms like $\delta f_1 \delta E_1$ involve Fourier convolutions such as $\delta f_{k-k'}, \delta E_{k'}$ (with $\delta E_{k=0} = 0$) and thus involves the coupling of different Fourier modes.

If the expression (5.23) is inserted into Poisson's equation $\nabla_1 \cdot \delta E_1 = 4\pi q \int dv_1 [\delta f_1^{\text{coh}} + \delta f_1^{\text{inc}}]$ one can rearrange the coherent contribution to write

$$\nabla_1 \cdot \epsilon_{12} \delta E_2 = 4\pi q \int dv_1 \delta f_1^{\text{inc}} \quad (5.24)$$

where ϵ_{12} is the dielectric operator

$$\epsilon_{12} = \delta(t_1 - t_2) \delta(x_1 - x_2) + \frac{4\pi q^2}{m} \nabla_1 \cdot \int dv_3 \int dx_3 \frac{\delta(t_1 - t_3)}{|x_1 - x_3|} G_{34} \partial_{v_4} \Gamma_{42} \quad (5.25)$$

The coherent part has produces the susceptibility function in the dielectric operator. For a homogeneous stationary system, the Fourier coefficient of ϵ can be written

$$\epsilon(k, \omega) = 1 + (4\pi q^2 / mk^2) \int dv_3 \int dv_4 G_{v_3 v_4}(k, \omega) ik \cdot \partial_{v_4} \Gamma_{v_4}(k, \omega)$$

We see from eqn. (5.24) that the charge density associated with δf^{inc} acts as an incoherent source for the electrostatic fluctuation. For example, the two-point electrostatic correlation function can be written as

$$\langle \delta E_1 \delta E_{1'} \rangle = \partial_{x_1} (\nabla_1^2 \epsilon_{12})^{-1} (4\pi q)^2 \int dv_2 \int dv'_2 \langle \delta f_2^{inc} \delta f_{2'}^{inc} \rangle \cdot \partial_{x'_1} (\nabla_{1'}^2 \epsilon_{1'2'}) \quad (5.26a)$$

in terms of the inverse dielectric operator. Or

$$\langle |\delta E|^2 \rangle_{k\omega} = \frac{(4\pi q)^2}{k^2 |\epsilon(k\omega)|^2} \int dv \int dv' \langle \delta f_v^{inc} \delta f_{v'}^{inc} \rangle \quad (5.26b)$$

for the spectral intensity in a homogeneous, stationary case.

The two-point correlation function for the phase space distribution can likewise be written as

$$\begin{aligned} \langle \delta f_1 \delta f_{1'} \rangle &= (q^2 / m^2) G_{12} \partial_{v_2} \Gamma_{23} G_{1'2'} \partial_{v'_2} \Gamma_{2'3'} \langle \delta E_3 \delta E_{3'} \rangle \\ &+ \left[\delta(1-2) \delta(1'-2') - \delta(1-2) P_{1'2'} - P_{12} \delta(1'-2') \right] \langle \delta f_2^{inc} \delta f_{2'}^{inc} \rangle \end{aligned} \quad (5.27)$$

where P_{12} is a "polarization" operator

$$P_{12} = (4\pi q^2 / m) G_{13} \partial_{v_3} \Gamma_{34} (\nabla_4^2 \epsilon_{42})^{-1} \quad (5.28)$$

The first term on the right side is the correlation $\langle \delta f_1^{coh} \delta f_{1'}^{coh} \rangle$ while the first term in square brackets arises from $\langle \delta f_1^{inc} \delta f_{1'}^{inc} \rangle$. The second and third term in square brackets arise respectively from $\langle \delta f_1^{inc} \delta f_{1'}^{coh} \rangle$ and $\langle \delta f_1^{coh} \delta f_{1'}^{inc} \rangle$. In writing these terms in terms of the operator P we have used Poisson's equation in the form of eq. (5.24) to eliminate δE in terms of δf^{inc} . By doing this we have expressed correlation functions of the form $\langle \delta E \delta E \delta f \rangle$ in terms of $\langle \delta f_1^{inc} \delta f_{1'}^{inc} \rangle$ which can be seen from the definition of δf^{inc} , eq. 5.23, to involve 4-point and higher correlation functions. In the original quasilinear theory the latter correlation function is assumed to be of order $\langle \delta E^2 \rangle^2$ and therefore negligible. This estimate does not take into account the small denominators arising from wave-particle interactions and the large dielectric response $\epsilon \approx 0$

discussed in Section 5.3. The vanishing of the cross correlation terms is made explicitly in the theory of Dupree (ref. 17 eqn. (21)).

The equations written in this section up to this point are formally exact. We see that the system is not closed since the correlation function $\langle \delta f_1^{\text{inc}} \delta f_2^{\text{inc}} \rangle$ involves three-point and four-point correlation functions.

An approximation which closes these equations called the "Direct Interaction Approximation," DIA, has attracted renewed interest. This type of theory, first applied by Kraichnan to Navier-Stokes turbulence,³¹ treats certain non-gaussian correlations in a nonperturbative way. This approach was reasonably successful for moderate Reynold's numbers but failed to accurately describe the inertial range for high Reynold's numbers. The failure was due to the inaccurate treatment of the convection of short scale turbulence by the long scale motion. The DIA was written down for the Vlasov-Poisson system by Orszag and Kraichnan³² in 1967. This work received very little attention until recently when DuBois and Espedal³³ showed how to "factor" the theory into single (quasi) particle and collective responses. This exposed the existence in the DIA of new self-consistent fluctuation (or nonlinear polarization) contributions which were not found in the older quasigaussian imposed field theories. The new terms are proportional to the dielectric response ϵ^{-1} such as the terms in the polarization operator in eqn. (5.27); various estimates have shown these terms to be of the same order as the imposed field terms. The DIA based theory was investigated in detail by Krommes and coworkers³⁴ who also considered application to the guiding center model and to drift wave turbulence.³⁵

Here we will list the approximations to the set of exact equations listed above which are equivalent to the DIA:

1. The incoherent correlation function $\langle \delta f_1^{\text{inc}} \delta f_1^{\text{inc}} \rangle$ is computed to terms of quadratic order in the correlations using a quasigaussian approximation for the four-point correlation function

$$\begin{aligned} \langle \delta f_1^{\text{inc}} \delta f_1^{\text{inc}} \rangle &\cong (q^2/m^2) G_{12} \partial_{v_2} G_{1,2} \partial_{v_2} \langle [\delta f_2 \delta E_2 - \langle \delta f_2 \delta E_2 \rangle] [\delta f_2 \delta E_2 - \langle \delta f_2 \delta E_2 \rangle] \rangle \\ &\cong q^2/m^2 G_{12} \partial_{v_2} G_{1,2} \partial_{v_2} [\langle \delta E_2 \delta E_2 \rangle \langle \delta f_2 \delta f_2 \rangle + \langle \delta E_2 \delta f_2 \rangle \langle \delta f_2 \delta E_2 \rangle] \end{aligned} \quad (5.29)$$

(Note that since eq. (5.30) contains the renormalized propagator G it is not the same as the simple gaussian closure of the hierarchy as used for example in the notorious quasinormal theory.)

2. The inverse propagator G^{-1} and the response function Γ are computed to lowest (linear) order in the correlation functions.

To carry out the last step the functional derivatives of $\langle \delta E \delta f \rangle$ in eqs. (5.21) and (5.22) must be evaluated to this order. This can be accomplished

by using an exact relation connecting the fluctuations which is easily derived from eq. (5.23):

$$\langle \delta f_1 \delta E_1 \rangle = \frac{q}{m} G_{12} \partial_{v_2} \Gamma_{23} \langle \delta E_3 \delta E_1 \rangle + 4\pi q \partial_{x_1} (\nabla_{1, \varepsilon_{1,2}}^2)^{-1} \langle \delta f_1^{\text{inc}} \delta f_2^{\text{inc}} \rangle \quad (5.31)$$

This equation can also be used for the right-hand side of eq. (5.18) of the mean distribution. In computing the required functional derivatives of $\langle \delta f \delta E \rangle$ for self-consistent fluctuations it is necessary to consider the functional dependence on $\langle f \rangle$ or $\langle E \rangle$ of the dielectric response, ε^{-1} , which enters explicitly in the last term of eq. (5.31) and also in $\langle \delta E \delta E \rangle$ as shown explicitly in eq.

(5.26). We can give here only the result of this calculation for G_{12}^{-1} for a homogeneous, stationary system:

$$G_{vv'}^{-1}(k, \omega) = -i(\omega - k \cdot v) \delta(v - v') + \frac{q^2}{m^2} \partial_v \cdot \int dk' \int d\omega' \left\{ \langle \delta E \delta E \rangle_{k', \omega'} \right. \\ \left. \left[G_{vv'}(k - k', \omega - \omega') + \int d\bar{v} P_v(k - k', \omega - \omega') G_{v\bar{v}}(k - k', \omega - \omega') \right] \right. \\ \left. + 4\pi q \frac{\langle \delta f_v \delta E \rangle_{k', \omega'}}{(k - k')^2 \varepsilon(k - k', \omega - \omega')} \int d\bar{v} G_{v\bar{v}}(k - k', \omega - \omega') \right\} \cdot \partial_{v'} \quad (5.32)$$

The last two terms in curly brackets are self-consistent fluctuation terms which arise from the functional dependence discussed above and do not occur in the quasis Gaussian field theories. These self-consistent fluctuation contributions have been estimated in various cases to be as important as the remaining term which appears in the quasis Gaussian field theories and produces the renormalized propagator in eq. 5.1. In the quasilinear ordering of $\tau_{p.m.} \ll \tau_{\langle f \rangle}$ and $k_s \gg 1$ for a one-dimensional one-component plasma we have been able to show that the self-consistent fluctuation terms cannot be neglected. A similar equation results for the renormalized $\Gamma_v(k, \omega)$ which we will not discuss here. Further results are given in reference 29.

If the DIA approximation of eqn. (5.30) for $\langle \delta f^{\text{inc}} \delta f^{\text{inc}} \rangle$ is inserted into eq. (5.27) we obtain the following equation for the two-point correlation function

$$\langle \delta f_1 \delta f_1 \rangle \cdot (q^2/m^2) \left[G_{12} G_{1'2'} - \overbrace{P_{13} G_{32} G_{1'2'} - G_{12} P_{1'3'} G_{3'2'}}^{\text{s.c.}} \right] \\ \cdot \partial_{v_2} \partial_{v_2'} \left[\langle \delta E_2 \delta E_2' \rangle \langle \delta f_2 \delta f_2' \rangle + \overbrace{\langle \delta E_2 \delta f_2' \rangle \langle \delta f_2 \delta E_2' \rangle}^{\text{s.c.}} \right] \\ = (q^2/m^2) G_{12} \partial_{v_2} \Gamma_{23} G_{1'2'} \partial_{v_2'} \Gamma_{2'3'} \langle \delta E_3 \delta E_3' \rangle \quad (5.34)$$

When the terms labelled s.c. (which arise from the $\langle \delta f^{\text{coh}} \delta f^{\text{inc}} \rangle$ cross correlations) are dropped in this expression and only the imposed field renormalization terms in G are retained, this equation can be shown to reduce to the Dupree equation for $\langle \delta f(x,v,t) \delta f(x'v't) \rangle$ given in equation (3.7). This approximation is valid only for quasis Gaussian imposed electric field fluctuations; we can see explicitly in this case that the remaining contribution from $\langle \delta f_1^{\text{inc}} \delta f_1^{\text{inc}} \rangle$ in eqn. (5.34); i.e., $(q^2/m^2) G_{12} G_{1,2} \delta v_2 \delta v_2 \langle \delta f_2 \delta f_2 \rangle \langle \delta E_2 \delta E_2 \rangle$ reduces to the mutual diffusion (or mode coupling) term of equations (3.7) or (5.2). It is this term which produces the finite Kolmogorov entropy, h , discussed in Section 3. Thus there is a direct connection in this case between a finite level for the incoherent noise function $\langle \delta f_1^{\text{inc}} \delta f_1^{\text{inc}} \rangle$ (sometimes called the nonlinear noise) and the intrinsic stochastic behavior of the system.

The reduction of these DIA equations to a calculable form is very difficult even in the limit of quasilinear ordering. The harmonic expansion of eq. (5.3) still appears to be useful in the DIA. Using this technique we have been able to show that none of the self-consistent terms can be dropped even in quasilinear ordering when $R \gg 1$. Only the equations for the harmonic components $n = 0, \pm 1$ and 2 are significantly changed by the self-consistent terms. Work is still in progress in deducing the consequences of the DIA in the quasilinear limit.

The DIA is the natural first approximation in the renormalized perturbation theories of Kraichnan, Martin, Siggia and Ross (MSR)³⁶ and others. The general conditions for the convergence of the MSR expansion are completely unknown; probably only convergence in the sense of an asymptotic expansion is expected at best. It appears to be a possible, but extremely tedious, task to examine the size of the leading vertex corrections (in the sense of MSR) to the DIA. There is some hope that in quasilinear ordering these vertex corrections may be negligible. This hope is based on the following points: i) In the imposed field case the DIA theory becomes exactly quasilinear theory (QLT) in quasilinear ordering; the corrections in this case are of order $(\tau_{p.m.}/\tau_{cf}) \ll 1$. ii) In the self-consistent field case the DIA differs from QLT in quasilinear ordering; the differences due to nonlinear polarization effects are the same order as the QLT terms. iii) If self-consistency also does not change the order of the vertex corrections, the corrections to the DIA will also be of order $(\tau_{p.m.}/\tau_{cf})$ as in i).³⁷

To make a point of contact between this formalism and the results of Adam, Laval and Pesme, we write an equation for the time evolution of $\langle |\delta E|_k^2 \rangle$:

$$\partial_t \langle |\delta E|_k^2 \rangle = 2 \hat{\gamma}_k \langle |\delta E|_k^2 \rangle + S_k$$

where

$$S_k = (4\pi q)^2 k^{-2} \int dv \int dv' \langle \delta f^{\text{inc}} \delta f^{\text{inc}} \rangle_{k, \omega=\omega(k)} (\partial \epsilon / \partial \omega_k)$$

This can be shown to be an exact consequence of eq. (5.26a) provided $\hat{\gamma}_k \ll \hat{\omega}_k$ where $\hat{\omega}_k + i\hat{\gamma}_k$ is the zero of the dielectric function in the complex ω plane: $\epsilon(k, \hat{\omega}_k + i\hat{\gamma}_k) = 0$. In the limit of quasilinear ordering ($\tau_{p.m.} \ll \tau_{cf}$, $k_3 \gg 1$)

it is found that $\hat{\omega}_k$ and $\hat{\gamma}_k$ are given by their linear values: $\hat{\omega}_k^2 = \omega_p^2 + 3k^2 v_e^2$, $\hat{\gamma}_k = \gamma_k^{Q.L.}$. In conventional quasilinear theory it is assumed that S_k arises from mode-mode coupling effects of order $\langle |\delta E_k|^2 \rangle^2$ and is thus negligible.

However, to account for the results of Adam, Laval and Pesme discussed in Section 4 it is necessary that to leading order

$$S_k = (\gamma_k^{\text{exp}} - \hat{\gamma}_k^{Q.L.}) \langle |\delta E_k|^2 \rangle$$

where γ_k^{exp} is the observed growth rate. It can be verified by iteration of the eq. (5.34) for $\langle \delta f_1 \delta f_1 \rangle$ in powers of $\langle |\delta E|^2 \rangle$ that the mode-mode contributions, including nonlinear polarization effects, are indeed of order $\langle |\delta E_k|^2 \rangle$ provided $\gamma_k < (k^2 D^{Q.L.})^{1/3}$.

Conclusion

In the first part of this paper we have reviewed the results of intrinsic stochasticity theories in the case where the imposed electric field has random phase Fourier components. Whenever the mode amplitudes are well above the threshold for intrinsic stochasticity, an excellent agreement has been demonstrated between the numerical experiments and the quasilinear predictions for the diffusion coefficient and the Kolmogorov entropy.

In the second part we have considered the Vlasov-Poisson turbulence in which the electric field is self-consistently related to the particles motion through Poisson's equation. We have first observed a striking difference compared with the imposed electric field case: the numerical simulations exhibit a significant discrepancy with the quasilinear prediction. The origin of this discrepancy has then been shown to be a consequence of a strong enhancement of the wave-particle interaction. As a result, the statistical properties of the self-consistent electric field are characterized by a second correlation time $(k^2 D)^{-1/3}$ much longer than the usual correlation time τ_c . Self-consistency leads thus to a sort of self-organization of the turbulence. Dupree's theory now appears to be a partial summation of the irreducible correlation functions of the electric field which are neglected in the simply renormalized theories. It has, however, a remaining lack of consistency intrinsic to all quasigaussian theories. We have then displayed the characteristic features of DIA applied to Vlasov turbulence and we have shown how this closure takes into account in a self-consistent way the strong wave-particle interaction and the resulting nongaussian properties of the electric field.

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37. The condition $\tau_{p,m} \ll \tau_{cf}$ appears to be only a restriction on the level of electric field fluctuations whereas the condition for smallness of the vertex corrections to the DIA must also involve some restriction on $\langle \delta f^2 \rangle$. The fluctuation level $\langle \delta f^2 \rangle$ is asymptotically in time proportional to $\langle \delta E^2 \rangle$ and both are bounded. Convergence would seem possible for sufficiently small and smooth initial fluctuations $\langle \delta f^2 \rangle$.

Figure Captions

- Fig. 1: Separatrix in phase-space for a single resonance.
- Fig. 2: Approximate resonance domains in phase space for several resonances such that $k_s \ll 1$.
- Fig. 3: Distortion of primary resonances and appearance of secondary resonances for $k_s = 0.4$ (from Escande et al., Ref. 2).
- Fig. 4: Comparison between numerical experiments (dots ϕ) and the quasilinear prediction (solid line) for the value of the diffusion coefficient (from Rechester et al., Ref. 2).
- Fig. 5: Comparison between numerical experiments (dots ϕ) and the quasilinear prediction (solid line) for the value of the Kolmogorov entropy (from Rechester et al., Ref. 6, 7).
- Fig. 6: Time evolution of the space-averaged distribution function $\bar{f}(v,t)$ and of the electric field mode amplitude $|E_k|(t)$, a. initial conditions, b. earlier stage: growth of $|E_k|$ and no modification of f , c. nonlinear saturation stage: plateau function of f , d. stationary final state.
- Fig. 7: Growth rates versus phase velocity: the dashed line is the quasilinear prediction, the dotted curve is obtained by numerical time derivation of the ensemble average of the mode energy and the solid curve is a least square fit of the preceding one (from Adam et al., Ref. 19).

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig . 5

6. a

6. b

6. c

6. d

Fig. 6

Fig . 7