MASTER

UCRL--53718

DE86 009556

Betatron Motion and Optical Deflection in Free Electron Lasers

T. J. Karr

Manuscript date: October 6, 1985

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

LAWRENCE LIVERMORE NATIONAL LABORATORY
University of California • Livermore, California • 94550

Available from: National Technical Information Service • U.S. Department of Commerce
5285 Port Royal Road • Springfield, VA 22161 • A02 • (Microfiche A01)

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

CONTENTS

Abstract	1
Definition of the Problem	2
Notation	4
Transverse Mode Expansions	9
Propagation of the Field Solution	11
Field Equation with Betatron Oscillation	18
Solution by Perturbation Theory	19
Fundamental Mode Input	23
Field Centroids	24
Averaging Over Fluctuations	27
Free Electron Laser Amplifier	30
References	47

BETATRON MOTION AND OPTICAL DEFLECTION IN FREE ELECTRON LASERS

ABSTRACT

Transverse displacement of the electron beam at input to the wiggler induces betatron oscillation, imparting a sinusoidal transverse displacement to the gain medium seen by the optical beam. As the light propagates down this "rippled" medium it is guided by the dispersion, thereby acquiring some transverse sinusoidal displacement. At output from the wiggler the optical beam in general has acquired some transverse component of velocity, that is, it propagates off-axis. As the e-beam displacement fluctuates statistically with every pulse, the optical output direction jitters around the nominal axis.

The jitter is estimated with a model that includes off-axis optical propagation through a "rippled" gain medium but neglects feedback of the off-axis light on the e-beam. The model is solved in perturbation theory. The jitter is calculated at the final aperture of the optical system beyond the wiggler. The result is (normalized to the final aperture diffraction)

$$\frac{\text{rms focused jitter}}{\text{diffraction width}} \approx |n_i| \frac{w}{\lambda} \frac{\lambda_\beta}{r_b} \frac{a}{r_b} \Delta(L) ,$$

where

- r_b = e-beam radius,
- w = optical beam radius ($1/e^2$ intensity),
- λ = optical wavelength,
- n_i = imaginary optical index on-axis in e-beam,
- λ_β = betatron wavelength,
- a = rms e-beam transverse displacement,
- L = wiggler length, and
- $\Delta(L)$ = a damped oscillating function of L .

For the proposed Paladin free electron laser at $\lambda = 10 \mu\text{m}$ on the Advanced Test Accelerator (ATA), this implies

$$\frac{\text{jitter}}{\text{diffraction}} < 0.25 \frac{a}{r_b} \quad (\text{Paladin}) .$$

DEFINITION OF THE PROBLEM

The wiggler B-field will be focusing in the transverse plane, and each electron will oscillate in the effective confining potential. This potential is nearly harmonic, with anharmonic terms $\approx 10^{-2} \times$ harmonic terms. Thus all electrons, regardless of their initial position in the beam, will betatron oscillate in z with the same wavelength λ_β , and the oscillations will stay in phase for $\approx 100 \lambda_\beta$. A single electron oscillates with an amplitude/phase determined by the electron's initial position in phase space at the wiggler entrance.

If the entire electron beam is displaced from the phase space origin, then the e-beam current density will betatron oscillate as it transports down the wiggler. This is shown in Fig. 1. Initial beam displacements can arise from misalignment of the accelerator/transport with respect to the wiggler or from pulse-to-pulse fluctuations of the beam distribution in phase space. An emittance selector aligned to the wiggler can clip the e-beam but cannot erase the effect of a displaced, nonuniform, phase space distribution. Therefore, an inherent feature of the free electron laser (FEL) is the sinusoidal betatron oscillation of the whole e-beam in the wiggler.

The shape of the electron gain medium is expected to affect the generation, amplification and propagation of light in the wiggler. In particular, electron betatron oscillation may cause light to propagate off the wiggler axis. If this occurs at the wiggler exit, then the optical beam has a static misalignment and/or dynamic jitter with respect to the wiggler axis. If off-axis light propagation occurs inside the wiggler, then the optical beam may spill outside the interaction region, with effects on gain and beam quality. The impact of off-axis e-beam propagation on gain has been discussed before,¹ but the optical propagation effects were not analyzed.

The interaction of photons and electrons in the FEL is nonlinear, nonlocal, and anisotropic. To make some progress in estimating effects of the betatron oscillation, I make some simplifying assumptions:

(A) Small signal exponential gain. This assumption neglects gain saturation, rapid changes in bunching, and all other nonlinear effects of the light on the e-beam. In particular, it neglects instabilities driven by nonlinear feedback between optical and electron-beam transverse oscillations.

(B) Intensity gain $\gamma(\vec{r}, z)$ and phase dispersion $\chi(\vec{r}, z)$ are proportional to current density at $\vec{r}, z$, with the coefficient of proportionality assumed to be constant. This assumption neglects $\left\langle \frac{e^{i\psi}}{\gamma} \right\rangle$ fluctuations in $\vec{r}, z$ (which average out over $z \ll \lambda_\beta$) and neglects the angular dependence of FEL gain/dispersion¹ (which is small for angles $\ll \frac{1}{\gamma \sqrt{N_w \beta}}$, with $N_w \beta$ = number of wiggler periods in a betatron wave).

(C) Monochromatic input light at frequency ω , with electric field $\equiv \vec{E}(\vec{r}, z) e^{-i\omega t}$.

(D) Gain and dispersion are slowly varying,

$$|\nabla \chi| \ll k \chi$$

$$|\nabla \gamma| \ll k \gamma .$$

Assumptions (A) and (B) imply the following:

(A') The FEL wiggler is a linear optical medium with complex refractive index.

(B') The medium is distributed in (r, z) as a simple sinusoid transverse displacement of the input current density $J(r, t)$.

The mathematical model I solve is specified by (A'), (B'), (C), (D), which is slightly more general than (A)-(D). FEL nonlinearities that only modify the betatron frequency, or lead to a stable superposition of frequencies, are consistent with the model and could be analyzed in it.

With these assumptions, Maxwell's equation for the optical E field is (in MKS units)

$$(\nabla^2 + \omega^2 \mu \epsilon) \vec{E} = -\nabla \left(\frac{1}{\epsilon} \vec{E} \cdot \nabla \epsilon \right) - \left(\frac{1}{\mu} \nabla \mu \right) \times \nabla \times \vec{E} \approx 0 , \quad (1)$$

where

$$\sqrt{\mu \epsilon} = \frac{1}{c} (1 + \chi) - \frac{i}{2\omega} \gamma \text{ inside the wiggler, and } = 0 \text{ outside.} \quad (2)$$

The problem of estimating optical effects is now reduced to solving Eq. (1) for the E-field with a fixed χ and γ , and propagating this E to a boundary or to the far-field.

NOTATION

$(\vec{r}, z)$ = transverse, longitudinal space coordinates

H_m = mth Hermite polynomial²

$E_{mn}^h(\vec{r}, z)$ = homogeneous Gaussian mode

$$= \frac{w^h(0)}{w^h(z)} H_m \left(\frac{\sqrt{2} x}{w^h(z)} \right) H_n \left(\frac{\sqrt{2} y}{w^h(z)} \right) \exp \left[-r^2/Q^{h2}(z) \right. \\ \left. + i \frac{\omega}{c} z - i(m+n+1) \tan^{-1} (z/z_R) \right]$$

$R^h(z)$ = radius of curvature

$$= z (1 + z^2/z_R^2)$$

$w^h(z)$ = spot $1/e^2$ intensity radius ("spot size")

$$= w^h(0) \sqrt{1 + z^2/z_R^2}$$

k = vacuum wave number

$$= \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

z_R = Rayleigh range

$$= \frac{\pi w^h{}^2}{\lambda}$$

θ_d = diffraction angle (half-cone)

$$= \frac{\lambda}{\pi w^h(0)}$$

$$\frac{1}{Q^{h2}(z)} = \frac{-i\omega}{c} \frac{1}{2R^h(z)} + \frac{1}{w^{h2}(z)}$$

$\gamma(\vec{r}, z)$ = gain profile (dimensions length^{-1})

γ_0 = on-axis gain profile (without betatron oscillation)

$\chi(\vec{r}, z)$ = dispersion profile (dimensionless)

χ_0 = on-axis dispersion profile (without betatron oscillation)

$n(\vec{r}, z)$ = complex refractive index

$$= 1 + \chi - \frac{i\gamma}{2k}$$

$$n_o = 1 + \chi_o - \frac{i\gamma_o}{2k} = n_r + in_i$$

$E_{mn}(\vec{r}, z)$ = quadratic Gaussian mode or Gauss-Hermite function

$$= H_m \left(\frac{\sqrt{2} x}{Q} \right) H_n \left(\frac{\sqrt{2} y}{Q} \right) \exp \left[\frac{-r^2}{Q^2} + ik_{mn} z \right]$$

$|m, n\rangle$ = normalized transverse quadratic Gaussian mode

$$= \frac{Q^2}{2} \left[\pi m! n! 2^{m+n} \right]^{-1/2} E_{mn} \exp [-ik_{mn} z]$$

$$\frac{4}{Q^4} = \frac{2k^2}{r_b^2} (n_o - 1) = \frac{2k^2}{r_b^2} \left(\chi_o - \frac{i\gamma_o}{2k} \right)$$

$$\frac{1}{Q^2} = \frac{-i k}{2R} + \frac{1}{w^2}$$

R = radius of curvature

w = spot $1/e^2$ intensity radius ("spot size")

d = spot diameter = $2w$

k_{mn} = mode-dependent complex wave number

$$= \left\{ k^2 + 2k^2 (n_o - 1) - 2(m+n+1) \left[\frac{2k^2}{r_b^2} (n_o - 1) \right]^{1/2} \right\}^{1/2}$$

$$\equiv k N_{mn} - \frac{i}{2} \Gamma_{mn}$$

N_{mn} = mode-dependent real refractive index

Γ_{mn} = mode-dependent gain

D = final optical aperture diameter

θ_o = diffraction limit of aperture diameter D (half-cone)

$$= \frac{2\lambda}{\pi D}$$

$\vec{f}_c$ = wiggler output optical-beam centroid direction

$\vec{\theta}_c$ = far field optical-beam centroid direction

$\vec{h}_c$ = near-field optical-beam centroid displacement

r_b = e-beam radius

$\vec{r}_c(z)$ = e-beam centroid

$$= [x_c \sin(k_\beta z + \delta_x), y_c \sin(k_\beta z + \delta_y)]$$

$$\vec{r}_c' = \frac{d\vec{r}_c}{dz}$$

λ_w = wiggler period

k_w = wiggler wave number

$$= \frac{2\pi}{\lambda_w}$$

λ_β = betatron wavelength

k_β = betatron wave number

$$= \frac{2\pi}{\lambda_\beta}$$

$N_{w\beta}$ = number of wiggler periods in $\lambda_\beta = \lambda_\beta / \lambda_w$

$(\vec{\xi}, \zeta)$ = dimensionless complex coordinates

$$= \frac{\sqrt{2}}{Q} (\vec{r}, z)$$

$\vec{\rho}_c$ = dimensionless complex centroid

$$= \frac{\sqrt{2}}{Q} \vec{r}_c$$

(A): $\vec{\xi}_0$ = complex initial displacement

$$\equiv -i \frac{\sqrt{2}}{Q} \begin{bmatrix} x_c e^{i\delta_x}, y_c e^{i\delta_y} \end{bmatrix}$$

$$\vec{\xi}_0^* \equiv i \frac{\sqrt{2}}{Q} \begin{bmatrix} x_c e^{-i\delta_x}, y_c e^{-i\delta_y} \end{bmatrix}$$

κ = dimensionless complex vacuum wave number

$$= \frac{Q}{\sqrt{2}} \kappa$$

ν_β = dimensionless complex betatron frequency

$$= \frac{Q}{\sqrt{2}} \frac{k_\beta}{2\pi}$$

$\hat{e}_1, \hat{e}_2$ = basis vectors in the complex $\vec{\xi}$ space

$$k_o = k_{oo}$$

$$\Delta k = k_{10} - k_{oo} \approx \frac{i}{R} - \frac{1}{z_R}$$

α_i = ith complex optical scale factor

$$= Q_{i-1}/Q_i$$

$$\alpha = \prod_{i=1}^n \alpha_i = Q/Q_n$$

$$\vec{\eta} = \alpha \vec{\xi} = \frac{\sqrt{2}}{Q_n} \vec{r}$$

β_i = ith complex field factor

$$= e^{-i \tan^{-1}(z/z_{R \ i-1})}$$

$$\beta = \prod_{i=1}^n \beta_i$$

E_{wig} = electric field in wiggler or at exit

E_{far} = electric field at focal plane

$\mathcal{E}_{mn}(\vec{\xi}, \zeta)$ = perturbed mode with betatron oscillation

s = complex betatron coupling constant

$$\approx k_{mn} k_\beta Q^2$$

A = dimensionless deflection coefficient

$$= 2 w_{rb} k k_\beta$$

$\bar{h}_c$ = rms near-field centroid displacement

$\bar{\theta}_c$ = rms far-field centroid angle

$\bar{f}_c$ = rms exit centroid angle

$\Delta(z,f)$ = damped oscillating function

L = wiggler length

a = rms initial e-beam displacement

ω_p = plasma frequency

a_w = normalized wiggler vector potential

$$= eA_w/mc^2$$

a_s = normalized radiation vector potential

I = e-beam current

J_o = on-axis current density

γ = normalized electron kinetic energy

I_A = Alfvén current

ψ = electron phase = $k_w z + kz - \omega t$

F = bucket fractional half-height⁷

$$= 2 \sqrt{\frac{a_w a_s}{1 + a_w^2}} \cdot \Gamma(\psi_r)$$

B = e-beam brightness = $2I/(\text{emittance})^2$

G_a = optical output angle jitter gain

G_d = optical intensity displacement jitter gain

TRANSVERSE MODE EXPANSIONS

In some special cases the field equation [Eq. (1)] can be solved in closed form as a superposition of normal modes. There are several cases relevant to our problem.

The homogeneous Gaussian modes solve Eq. (1) for gain $\gamma = 0$, under the approximation $\lambda \ll w^h(0)$ (usually true in practice).³ The homogeneous Gaussian modes represent approximately the optical field outside the e-beam and are referred to as modes of free space. The beam can be focused down to a waist, size $w^h(0)$. Beyond the waist it expands asymptotically to a cone with half-angle θ_d . The E_{mn}^h at fixed z are similar to quantum-mechanical harmonic oscillator wave functions with an extra phase term for spherical wave-front curvature. They are orthogonal with real norm, and form a complete set.

The quadratic Gaussian modes solve Eq. (1) for a parabolic gain profile,

$$\gamma = \gamma_0 \left(1 - \frac{r^2}{r_b^2} \right) \quad (3)$$

under the approximation $\gamma_0 \ll k$ which drops all γ_0^2 terms.⁴ The charge density into the wiggler nominally has a parabolic profile (uniform density in phase space through a emittance selector with elliptic hole in phase space). Thus, the quadratic Gaussian modes represent (approximately) the small signal optical field in the e-beam for perfect on-axis propagation (no betatron oscillation). For $\gamma_0 > 0$ (gain) we get $R > 0$ (diverging beam). The tendency of the beam to spread by diffraction is counteracted by the gain or dispersion guiding. For a beam with size w the effects cancel, and the beam will propagate down the wiggler without change in transverse structure. The net gain and dispersion depend on the transverse mode,

$$N_{mn} - 1 \approx \chi - (m+n+1) \frac{2}{k^2 w^2} \quad , \quad (4a)$$

$$\Gamma_{mn} \approx \gamma_0 - (m+n+1) \frac{2}{R} \quad , \quad (4b)$$

so higher order modes have less gain.

The E_{mn} at fixed z are analytic continuations in r of exact quantum mechanical harmonic oscillator functions. They are orthogonal with complex norm, i.e.,

$$\int E_{mn}(\vec{r}, z) E_{pq}^*(\vec{r}, z) d\vec{r} = \exp[2ik_{mn}z] \frac{Q^2}{2} \pi m!n! 2^{m+n} \delta_{mp} \delta_{nq}, \quad (5)$$

and of course form a complete set. At any fixed z -plane, an arbitrary field can be expanded as a superposition of $|mn\rangle$ modes. The expansion coefficients are z -dependent. The $|mn\rangle$ are a good choice of modes if the expansion is simple or has a clear physical interpretation. $|mn\rangle$ are natural basis vectors for a perturbation theory of the oscillating gain medium, because in the limit of no betatron oscillation a physically important FEL solution is a pure $|mn\rangle$ mode, which propagates self-similarly down the wiggler.

The FEL parabolic gain profile Eq. (3), even under ideal conditions, is correct only for $r \leq r_b$; outside the e-beam $\gamma = 0$. We should use quadratic modes for $r < r_b$ and homogeneous modes for $r > r_b$, and adjust coefficients to fit the boundary conditions at $r = r_b$. In the practical cases of interest, $w \approx r_b$. Since most of the optical energy is contained within the e-beam, we make the (almost certainly unnecessary) simplification of using quadratic modes for all r (corresponding to continuing Eq. (3) to $\gamma < 0$). It is straightforward to extend the analysis to external free space modes.

PROPAGATION OF THE FIELD SOLUTION

Before attacking the field equation of a betatron-oscillating gain medium it is useful to consider the other part of the problem--how to propagate the solution in free space outside the e-beam or wiggler. The field solution can be simplified if the propagation effects of specific functions are recognized.

The optical system relays the wiggler radiation field E_{wig} to the target plane. The target plane is the conjugate image plane to some plane z_0 near the wiggler. If z_0 is the end of the wiggler ($z_0 = L$) then the target field E_{far} is the real image of the wiggler field solution $E_{wig}(z_0)$. The far-field interpretation is trivial in this special case. For example, the off-axis centroid displacement of $E_{wig}(z_0)$ [normalized to the radius of $E_{wig}(z_0)$] equals the off-axis centroid angle of E_{far} (normalized to the E_{far} angular radius). In general the conjugate plane z_0 is inside the wiggler, and E_{far} is the image of a virtual field E_{vir} that transforms into $E_{wig}(L)$ under free space propagation from z_0 to L . We need the transformation from $E_{wig}(L)$ to E_{far} or, equivalently, to $E_{vir}(z_0)$.

The general optical system after the wiggler is shown schematically in Fig. 1. We assume the propagation is unobscured and unvignetted. At wiggler exit we assume (with complete generality) that the field is a quadratic mode E_{mn} with complex argument ξ , parameter Q , spot diameter $d=2w$. This E_{mn} is a superposition of homogeneous modes E_{pq}^h , $p \leq m$, $q \leq n$, all with $Q^h = Q$:

$$E_{mn} = \sum_{\substack{p \leq m \\ q \leq n}} a_{pm} \left(\frac{w}{Q}\right) a_{qn} \left(\frac{w}{Q}\right) E_{pq}^h$$

(the superposition accounts for the difference between $\vec{\xi}$ and $2\sqrt{2} \vec{r}/d$ inside each Hermite polynomial). The problem reduces to propagating free space modes through the optical system. In the i th stage the beam propagates in vacuum for distance L_i , expanding to spot diameter D_i ; it then is focused by an infinite nonabsorbing "thin lens" with focal length f_i . The beam leaves the i th optic with radius of curvature R_i and Rayleigh range z_{Ri} .

Off-axis Gaussian beams in systems such as shown in Fig. 2 propagate along ray trajectories.⁵ The evolution of transverse beam properties and

Figure 1. Betatron oscillation of the current density in a wiggler.

d, D_i = beam diameter
 R, R_i = radius of curvature
 L_i = propagation distance ($i-1 \rightarrow i$)
 f_i = focal length
 Q, Q_i = complex Gaussian parameter

Figure 2. Schematic optical system outside the wiggler.

the propagation of the beam center follow the "ABCD law."⁴ A free space mode is described by the complex parameter Q . Vacuum propagation over L_i transforms

$$Q_{i-1} \rightarrow Q_i^{(-)} = \left(1 + \frac{2iL_i}{Q_{i-1}^2 k} \right)^{1/2} Q_{i-1} ,$$

which includes the beam expansion $D_{i-1} \rightarrow D_i$. Focusing by optic i transforms

$$Q_i^{(-)} \rightarrow Q_i = \left(1 - \frac{L_i}{f_i} + \frac{ik Q_{i-1}^2}{2f_i} \right)^{-1/2} Q_i^{(-)} .$$

The combined effect is

$$Q_{i-1} \rightarrow Q_i = \frac{\left(1 + \frac{2iL_i}{Q_{i-1}^2 k}\right)^{1/2}}{\left(1 - \frac{L_i}{f_i} + \frac{ik Q_{i-1}^2}{2f_i}\right)^{1/2}} Q_{i-1} \equiv \left(\frac{1}{\alpha_i}\right) Q_{i-1} \quad (6a)$$

and

$$D_{i-1} \rightarrow D_i \quad (6b)$$

The m th mode also picks up a complex factor β_i^m , where

$$\beta_i = e^{-i \tan^{-1} (L_i / z_{Ri-1})}.$$

Therefore, the effect of the complete optical train through optic n is to transform the quadratic mode

$$H_m(\xi_1) H_n(\xi_2) \exp [-\xi^{2/2}] \rightarrow$$

$$E_{near} = \frac{d}{D} \sum_{\substack{p \leq m \\ q \leq n}} a_{pm} \left(\frac{w}{Q}\right) \beta^p a_{qn} \left(\frac{w}{Q}\right) \beta^q H_p\left(\frac{2\sqrt{2}x}{D}\right) H_q\left(\frac{2\sqrt{2}y}{D}\right) \exp [-\alpha^2 \xi^{2/2}] \quad (7a)$$

with

$$\alpha = \prod_{i=1}^n \alpha_i = \frac{Q}{Q_n} \quad (7b)$$

and

$$\beta = \prod_{i=1}^n \beta_i \quad (7c)$$

We are usually interested in the E field at the target point. This can be approximated by Kirchhoff's scalar diffraction theory. Focusing by the final optic to distance R_n exactly cancels the quadratic (Fresnel) terms in the phase factor of the Kirchhoff-Fresnel integral.⁶ Neglecting phase terms above quadratic, we are left with the linear (Fraunhofer) theory of a

beam collimated at n . The collimated beam has $Q_n = \frac{D}{2}$, $R_n = \infty$. From Eq. (7) the collimated near field leaving the final optic has

$$\exp [-\alpha^2 \xi^2 / 2] = \exp [-4r^2 / D^2] \quad .$$

The Fraunhofer far field is (neglecting an overall phase)

$$E_{\text{far}}(\vec{r} = \vec{\theta}_z, z) \approx \int d\vec{s} E_{\text{near}}(\vec{s}) \exp [-ik\vec{\theta} \cdot \vec{s}] \quad , \quad (8)$$

that is, $E_{\text{far}} \approx$ Fourier transform of E_{near} . Gaussian modes have the simple Fourier transform property

$$\int d\vec{s} H_m(\vec{s}) \exp [-s^2 / 2 - i\vec{k} \cdot \vec{s}] = \sqrt{2\pi} (-i)^m H_m(\vec{k}) \exp [-k^2 / 2] \quad . \quad (9)$$

Therefore, the effect of unobscured, unvignetted propagation from the wiggler exit to the focal plane of optic n is (up to a normalization and a mode-independent phase) to transform the wiggler field from

$$E_{\text{wig}} = H_m(\xi_1) H_n(\xi_2) \exp [-\xi^2 / 2] \quad (10a)$$

into

$$E_{\text{far}}(\vec{\theta}) \approx \sum_{\substack{p \leq m \\ q \leq n}} a_{pm} (-i\beta)^p a_{qn} (-i\beta)^q H_p\left(\frac{\sqrt{2} \theta_1}{\theta_o}\right) H_q\left(\frac{\sqrt{2} \theta_2}{\theta_o}\right) \exp [-\theta^2 / \theta_o^2] \quad . \quad (10b)$$

The field at any other point (after optic n) can be obtained by Fresnel propagation of this far-field result.

Equation (10) has a simple physical interpretation. The matrix $[a_{pq}]$ transforms $E_{\text{wig}}(L) \rightarrow E_{\text{vir}}(z_o)$, independent of the optical system. The factor $(\beta)^p$ is the accumulated phase shift of $E_{\text{vir}} \rightarrow E_{\text{near}}$ through the optical system and is sensitive to the detailed geometry of the optics. Final propagation to E_{far} brings another $(-i)^p$ phase shift.

For many FEL optical schemes the phase shifts are especially simple. Typically, $L_1 \gg z_R$, so $\beta_1 \approx -i$. Often, D_i ($i = 1, \dots, n$) is large, so $L_i \ll z_{Ri-1}$ ($i = 2, \dots, n$) and $\beta_i \approx 1$ ($i = 2, \dots, n$) (the beam is imaged from optic $i - 1$ to optic i). Then, $-i\beta \approx \pm 1$ and, therefore,

$$E_{wig} = E_{mn} \rightarrow E_{far} \approx H_m \left(\frac{\pm w}{Q} \frac{\sqrt{2}\theta_1}{\theta_0} \right) H_n \left(\frac{\pm w}{Q_0} \frac{\sqrt{2}\theta_2}{\theta_0} \right) \exp [-\theta^2/\theta_0^2] \quad (11)$$

In this case the optical system perfectly images E_{vir} onto the target. The relationship between wiggler and far field is shown in Fig. 3. In a real system, finite optics and finite propagation distances produce imperfect imaging of the wiggler to far field.

(a)

(b)

Figure 3. Relationship between near- and far-field beam motion: (a) Near-field beam is tilted, with waist on axis; (b) near-field beam is untilted, with waist off axis.

The beam location in near and far field is of interest. The beam centroids are

wiggler field:

$$\vec{h}_c = \int \vec{r} |E_{wig}(\vec{r})|^2 d\vec{r} / \int |E_{wig}(\vec{r})|^2 d\vec{r} \quad (12a)$$

far field:

$$\vec{\theta}_c = \int \vec{\theta} |E_{far}(\vec{\theta})|^2 d\vec{\theta} / \int |E_{far}(\vec{\theta})|^2 d\vec{\theta} \quad (12b)$$

With perfect imaging [Eq. (11)] $h_c/w = \theta_c/\theta_o$. With real imaging [Eq. (10)] we expect the on-axis beam spread $\theta > \theta_o$, because of large-angle diffraction from finite apertures and paths. We expect that real imaging will produce $\theta_c/\theta < h_c/w$. Henceforth, we assume perfect imaging [Eq. (11)] to estimate the upper bound of θ_c .

In perturbation theory

$$E_{wig} = |mn\rangle + \sum_{pq} C_{pq} |pq\rangle \text{ with } C_{pq} \ll 1.$$

Modes $|pq\rangle$ contributing to h_c and θ_c are identified by opposite odd/even symmetry from $|mn\rangle$.

Another quantity of interest is the direction of the beam exiting the wiggler. This can be found from the Fraunhofer far field of E_{wig} ,

$$E_{wig} = H_m(\xi_1) H_n(\xi_2) \exp[-\xi^2/2]$$

$$\rightarrow E_1(\vec{\theta}) \approx \sum_{\substack{p \leq m \\ q \leq n}} (-i)^{p+q} a_{pm} \left(\frac{w}{Q}\right) a_{qn} \left(\frac{w}{Q}\right) H_p\left(\frac{\sqrt{2}\theta_1}{\theta_d}\right) H_q\left(\frac{\sqrt{2}\theta_2}{\theta_d}\right) \exp[-\theta^2/\theta_d^2] \quad (13)$$

The exit angle centroid is

$$\vec{f}_c = \int \vec{\theta} |E_1(\vec{\theta})|^2 d\vec{\theta} / \int |E_1(\vec{\theta})|^2 d\vec{\theta} \quad (14)$$

FIELD EQUATION WITH BETATRON OSCILLATION

By assumption (B') above, the gain with betatron motion is

$$\gamma(\vec{r}, z) = \gamma_0 \left\{ 1 - \frac{[\vec{r} - \vec{r}_c(z)]^2}{r_b^2} \right\}, \quad (15)$$

and similarly for the dispersion $\chi(r, z)$. The beam centroid initial displacement (x_c, y_c) and phase (δ_x, δ_y) are arbitrary constants. In dimensionless complex coordinates ξ, ζ the field equation is

$$\left\{ \nabla_\xi^2 + \frac{d^2}{d\zeta^2} + \kappa^2 + \frac{2r_b^2}{Q^2} - (\xi - \vec{r}_c(\zeta))^2 \right\} E(\xi, \zeta) = 0. \quad (16)$$

Fourier transforming E in ζ ,

$$E(\xi, \zeta) = \int d\nu \exp [2\pi i \nu \zeta] E(\xi, \nu)$$

and using the complex initial displacements we can write Eq. (14) as

$$\left\{ \nabla_\xi^2 + \left[\frac{2r_b^2}{Q^2} + \kappa^2 - (2\pi\nu)^2 \right] - \xi^2 \right\} E(\xi, \nu) \quad (17a)$$

$$- \frac{1}{2} \vec{\xi}_0 \cdot \vec{\xi}_0^\dagger E(\xi, \nu) \quad (17b)$$

$$+ \vec{\xi} \cdot \vec{\xi}_0 E(\xi, \nu - \nu_\beta) + \vec{\xi} \cdot \vec{\xi}_0^\dagger E(\xi, \nu + \nu_\beta) \quad (17c)$$

$$- \frac{1}{4} \vec{\xi}_0 \cdot \vec{\xi}_0 E(\xi, \nu - 2\nu_\beta) - \frac{1}{4} \vec{\xi}_0^\dagger \cdot \vec{\xi}_0^\dagger E(\nu + 2\nu_\beta) = 0. \quad (17d)$$

This form of the field equation shows the mode coupling expected from betatron motion--Eq. (17) couples quadratic modes differing by $\pm 1, \pm 2$ and generates $\pm k_\beta$ sidebands. A single mode cannot propagate in the oscillating gain medium. If the field at the wiggler input is a single mode, then additional modes must grow as z advances down the wiggler. The additional modes describe the off-axis energy and power flow.

SOLUTION BY PERTURBATION THEORY

We can find approximate solutions of Eq. (17) by perturbation theory.

Assume solution $E(\xi, v)$ is an asymptotic series in initial beam displacement $\xi_0, \xi_0^\dagger$:

$$E(\xi, v) = E^{(0)}(\xi, v) + \xi_0 E^{(1)}(\xi, v) + \xi_0^\dagger E^{\dagger(1)}(\xi, v) + \xi_0 \xi_0 E^{(2)}(\xi, v) + \xi_0^\dagger \xi_0^\dagger E^{\dagger(2)}(\xi, v) + \xi_0 \xi_0^\dagger E^{(2)}(\xi, v) + O(\xi_0^3) \quad (18)$$

$E^{(1)}$ is a vector and $E^{(2)}$ a tensor in ξ -space. Substitute Eq. (18) into Eq. (17) and equate orders of ξ_0 to find the equations for $E^{(0)}, E^{(1)}, E^{(2)}$:

$$[\hat{H} + \Omega(v)] E^{(0)}(v) = 0 \quad , \quad (19a)$$

where

$$\hat{H} \equiv \nabla_\xi^2 - \xi^2 \quad (19b)$$

and

$$\Omega(v) \equiv \kappa^2 + \frac{2r_b^2}{Q^2} - 4\pi^2 v^2 \quad , \quad (19c)$$

$$[\hat{H} + \Omega(v + v_\beta)] E_i^{(1)}(v + v_\beta) + \xi_i E^{(0)}(v) = 0 \quad , \quad (20a)$$

$$[\hat{H} + \Omega(v - v_\beta)] E_i^{\dagger(1)}(v - v_\beta) + \xi_i E^{(0)}(v) = 0 \quad , \quad (20b)$$

$$[\hat{H} + \Omega(v + 2v_\beta)] E_{ij}^{(2)}(v + 2v_\beta) + \frac{1}{2} [\xi_i E_j^{(1)}(v + v_\beta) + \xi_j E_i^{(1)}(v + v_\beta)] - \frac{1}{4} \delta_{ij} E^{(0)}(v) = 0 \quad , \quad (21a)$$

$$\begin{aligned}
[\hat{H} + \Omega (\nu - 2\nu_\beta)] E_{ij}^{(2)} (\nu - 2\nu_\beta) + \frac{1}{2} [\xi_i E_j^{(1)} (\nu - \nu_\beta) \\
+ \xi_j E_i^{(1)} (\nu - \nu_\beta)] - \frac{1}{4} \delta_{ij} E^{(0)}(\nu) = 0 \quad , \quad (21b)
\end{aligned}$$

and

$$\begin{aligned}
[\hat{H} + \Omega(\nu)] E_{ij}^{(2)}(\nu) + \xi_i E_j^{(1)}(\nu - \nu_\beta) \\
+ \xi_j E_i^{(1)}(\nu + \nu_\beta) - \frac{1}{2} \delta_{ij} E^{(0)}(\nu) = 0 \quad . \quad (21c)
\end{aligned}$$

Equations (19)-(21) are written in a form emphasizing the correspondence with quantum mechanics. The unperturbed "Hamiltonian" operator is $\hat{H}$, with "energy" eigenvalue Ω , and the perturbing "interaction" is $\tilde{\xi}$.

Elementary solutions are constructed from a single quadratic mode,

$$E^{(0)} = |m, n\rangle \frac{d\Omega}{d\nu} \delta [\Omega - 2(m+n+1)] \quad (22a)$$

with ν and k_{mn} determined through the δ -function,

$$\Omega(\nu) = 2(m+n+1) \quad (22b)$$

and

$$k_{mn} = 2\pi \frac{\sqrt{2}}{Q} \nu \quad . \quad (23)$$

The resulting solution, Eq. (18), is defined as the perturbed mode $\mathcal{E}_{mn}(\vec{\xi}, \zeta)$. Equations (20) and (21) can be solved for $E^{(1)}$, $E^{(2)}$,

$E^{(2)}$ as a superposition of $|pq\rangle$ by inverting $\hat{H}$ with resolvent

$$\frac{1}{\hat{H} + \Omega} = \sum_{pq} \frac{|p, q\rangle \langle p, q|}{\Omega - 2(p+q+1)} \quad , \quad (24)$$

where the complex norm [Eq. (5)] is implied. The interaction term in Eqs. (20) and (21) inverts by the commutator

$$\begin{aligned}
\left[\frac{1}{\hat{H} + \Omega}, \hat{\xi} \right] &= \sum_{p,q} \frac{\sqrt{2}}{[\Omega - 2(p+q+2)][\Omega - 2(p+q+1)]} \\
&\times \left\{ \hat{e}_1 \sqrt{p+1} [|p+1, q\rangle \langle p, q| - |p, q\rangle \langle p+1, q|] \right. \\
&\left. + \hat{e}_2 \sqrt{q+1} [|p, q+1\rangle \langle p, q| - |p, q\rangle \langle p, q+1|] \right\}. \quad (25)
\end{aligned}$$

Equation (20) has one commutator, so $E^{(1)}$ contains modes $|pq\rangle$ with $p+q = m+n \pm 1$; Eq. (21) has two commutators, so $E^{(2)}$, $E^{(2)}$ contain modes with $p+q = m+n, m+n \pm 2$

We want to estimate the far-field centroid $\vec{\theta}_c$ of the perturbed mode. By Eqs. (12) and (18) we only need $E^{(1)}$. To $\mathcal{O}(\vec{\xi}_0)$, the perturbed mode solution is

$$\begin{aligned}
\exp[-ik_{mn}z] \mathcal{E}_{mn}(\vec{\xi}, z) &= |mn\rangle \\
&+ \frac{1}{2} \left[\frac{\xi_{01}}{2+s} \exp[ik_{\beta}z] + \frac{\xi_{01}^{\dagger}}{2-s} \exp[-ik_{\beta}z] \right] |m+1, n\rangle \\
&- \frac{1}{2} \left[\frac{\xi_{01}}{2-s} \exp[ik_{\beta}z] + \frac{\xi_{01}^{\dagger}}{2+s} \exp[-ik_{\beta}z] \right] |m-1, n\rangle \\
&+ \frac{1}{2} \left[\frac{\xi_{02}}{2+s} \exp[ik_{\beta}z] + \frac{\xi_{02}^{\dagger}}{2-s} \exp[-ik_{\beta}z] \right] |m, n+1\rangle \\
&- \frac{1}{2} \left[\frac{\xi_{02}}{2-s} \exp[ik_{\beta}z] + \frac{\xi_{02}^{\dagger}}{2+s} \exp[-ik_{\beta}z] \right] |m, n-1\rangle \quad (26)
\end{aligned}$$

where the coupling constant is

$$s \equiv \Omega(v) - \Omega(v + v_{\beta}) \approx 8\pi^2 v v_{\beta} = k_{mn} k_{\beta} Q^2. \quad (27)$$

Large s is weak coupling, small s is strong coupling. The singularity at $s = \pm 2$ is real; the asymptotic series approximation [Eq. (18)] breaks down there. Off the singularity, a solution to arbitrarily high order in ξ_0

can be found by this method (with an arbitrarily large effort of algebra). At the singularity, the field solution does not have a power series expansion, so a "small" betatron oscillation ξ_0 may have a "large" optical effect.

The general field solution in a betatron-oscillating gain medium is a superposition of perturbed modes $\mathcal{E}_{pq}$. The field anywhere in the wiggler is

$$\mathcal{E}(\vec{\xi}, z) = \sum_{pq} c_{pq} \mathcal{E}_{pq}(\vec{\xi}, z) \quad (28)$$

where c_{pq} are chosen to give

$$\mathcal{E}(\xi, 0) = \sum_{pq} c_{pq} \mathcal{E}_{pq}(\vec{\xi}, 0) , \quad (29)$$

a specified initial transverse field at wiggler input. Note that the $\mathcal{E}_{pq}$ are not orthogonal in the complex norm [Eq. (5)], so the c_{pq} cannot be found by a simple projection $\langle \mathcal{E}_{pq} | \mathcal{E} \rangle$.

An important case is when the field entering the wiggler is $|0,0\rangle$. Then the appropriate superposition, to $\mathcal{O}(\xi_0)$, is

$$|0,0\rangle = \mathcal{E}_{\text{wig}}(\xi,0) - \mathcal{E}_{0,0}(\vec{\xi},0) - \frac{1}{2} \left(\frac{\xi_{01}}{2+s} + \frac{\xi_{01}^\dagger}{2-s} \right) \mathcal{E}_{1,0}(\vec{\xi},0) - \frac{1}{2} \left(\frac{\xi_{02}}{2+s} + \frac{\xi_{02}^\dagger}{2-s} \right) \mathcal{E}_{0,1}(\vec{\xi},0) \quad (30)$$

Each $\mathcal{E}_{mn}$ in Eq. (30) transforms to $z > 0$ according to Eq. (26). Therefore, to $\mathcal{O}(\xi_0)$ the (unnormalized) field in the wiggler is

$$E_{\text{wig}}(\xi,z) \exp[-ik_0 z] \simeq \exp[-\xi^2/2] \left[1 + \left(\exp[ik_\beta z] - \exp[i\Delta kz] \right) \frac{\vec{\xi}_0^\dagger \cdot \vec{\xi}}{2+s} + \left(\exp[-ik_\beta z] - \exp[-i\Delta kz] \right) \frac{\vec{\xi}_0^\dagger \cdot \vec{\xi}}{2-s} \right] \quad (31)$$

or, in terms of the centroid oscillation,

$$E_{\text{wig}}(\vec{\xi},z) \simeq \exp[-\xi^2/2 + ik_0 z] \quad .$$

$$\left(1 + \frac{2}{s^2 - 4} \vec{\xi} \cdot \left\{ (-i)s \left[\vec{\rho}_c \left(z + \frac{\lambda_\beta}{4} \right) - \vec{\rho}_c \left(\frac{\lambda_\beta}{4} \right) \exp[i\Delta kz] \right] - 2 \left[\vec{\rho}_c(z) - \vec{\rho}_c(0) \exp[i\Delta kz] \right] \right\} \right) \quad (32)$$

Equation (32) is our desired result for the optical field in a betatron-oscillating gain medium. The resulting far field is

$$E_{\text{far}}(\vec{\theta}) \simeq \exp[-\theta^2/\theta_0^2] \cdot \left(1 \pm \frac{2d\vec{r}^\dagger}{(s^2 - 4) Q^2 \theta_0} \left\{ (-i)s \left[\vec{r}_c \left(L + \frac{\lambda_\beta}{4} \right) - \vec{r}_c \left(\frac{\lambda_\beta}{4} \right) \exp[i\Delta kL] \right] - 2 \left[\vec{r}_c(L) - \vec{r}_c(0) \exp[i\Delta kL] \right] \right\} \right) \quad (33)$$

FIELD CENTROIDS

The near-field centroid is found by substituting Eq. (32) in (12):

$$\begin{aligned} \vec{h}_c \approx \mathcal{N}_{\text{near}}^{-1} \int d\vec{r} \exp [-2r^2/w^2] \vec{r} \vec{r} \\ \cdot 2\text{Re} \left\{ \frac{2}{s^2 - 4} \frac{2(-i)}{Q^2} \left[s \left(\vec{r}_c(L + \lambda_\beta/4) - \vec{r}_c(\lambda_\beta/4) \exp [i\Delta k L] \right) \right. \right. \\ \left. \left. - 2i \left(\vec{r}_c(L) - \vec{r}_c(0) \exp [i\Delta k L] \right) \right] \right\}, \end{aligned} \quad (34)$$

where

$$\mathcal{N}_{\text{near}} = \int d\vec{r} \exp [-2r^2/w^2] = \frac{\pi}{2} w^2, \quad (35)$$

normalizes $E_{ig}(\vec{\xi}, 0)$. The projection tensor in Eq. (34) is

$$\mathcal{N}_{\text{near}}^{-1} \int d\vec{r} \exp [-2r^2/w^2] \vec{r} \vec{r} = \frac{w^2}{4} \mathbb{I}. \quad (36)$$

The scale factor $s/Q^2 \approx k k_\beta$ (real), so introducing the dimensionless deflection coefficient $A = 2wkr_b k_\beta$ the centroid is

$$\begin{aligned} \frac{\theta_c}{\theta_0} = \frac{\vec{h}_c}{w} \approx A \text{Im} \left\{ \frac{1}{s^2 - 4} \cdot \frac{1}{r_b} \left[\left(\vec{r}_c(L + \lambda_\beta/4) - \vec{r}_c(\lambda_\beta/4) \exp [i\Delta k L] \right) \right. \right. \\ \left. \left. - \frac{2i}{s} \left(\vec{r}_c(L) - \vec{r}_c(0) \exp [i\Delta k L] \right) \right] \right\}. \end{aligned} \quad (37)$$

Similarly, the exit angle centroid is

$$\begin{aligned} \frac{\vec{f}_c}{\theta_d} \approx -A \text{Re} \left\{ \frac{1}{s^2 - 4} \cdot \frac{1}{r_b} \left[\left(\vec{r}_c(L + \lambda_\beta/4) - \vec{r}_c(\lambda_\beta/4) \exp [i\Delta k L] \right) \right. \right. \\ \left. \left. - \frac{2i}{s} \left(\vec{r}_c(L) - \vec{r}_c(0) \exp [i\Delta k L] \right) \right] \right\}. \end{aligned} \quad (38)$$

Equations (37) and (38) are the normalized, instantaneous, optical beam deflections as a function of wiggler and gain medium parameters, to first order in the betatron motion $\vec{r}_c$.

The following are some limiting approximations.

$|s| \gg 1$:

$$\begin{aligned} \frac{\vec{f}_c}{\theta_d} &\approx -d \operatorname{Re} \left\{ \frac{1}{k_o k_\beta Q^4} \left[\vec{r}_c (L + \lambda_\beta/4) - \vec{r}_c (\lambda_\beta/4) \exp [i\Delta k L] \right] \right\} \\ &= - \frac{wk}{k_\beta r_b^2} \operatorname{Re} \left\{ \frac{k(n_o - 1)}{k_o} \left[\vec{r}_c (L + \lambda_\beta/4) - \vec{r}_c (\lambda_\beta/4) \exp [i\Delta k L] \right] \right\}. \end{aligned} \quad (39)$$

From Eq. (4) we see that

$$\frac{k_o}{k} \approx n_o + \frac{\Delta k}{k}, \quad (40)$$

so

$$\begin{aligned} \frac{k(n_o - 1)}{k_o} &\approx (n_o - 1) (n_o^* - \frac{1}{kz_R} - \frac{i}{kR}) \\ &\approx n_o - 1 \end{aligned} \quad (41)$$

if

$$|n_r - 1|, |n_i| \ll 1 \quad \text{and} \quad \lambda \ll z_R, R.$$

Therefore, for $|s| \gg 1$

$$\begin{aligned} \frac{\vec{f}_c}{\theta_d} &\approx - \frac{wk}{r_b k_\beta} \cdot \chi_o \cdot \frac{1}{r_b} \cdot \left\{ \vec{r}_c (L + \lambda_\beta/4) \right. \\ &\quad \left. - e^{-L/R} \vec{r}_c (\lambda_\beta/4) \left[\cos(L/z_R) - \frac{\gamma_o}{2k\chi_o} \sin(L/z_R) \right] \right\} \end{aligned} \quad (42)$$

and

$$\frac{\vec{\theta}_c}{\theta_o} = \frac{\vec{h}_c}{w} \approx - \frac{wk}{r_b k_B} \cdot \frac{\gamma_o}{2k} \cdot \frac{1}{r_b} \cdot \left\{ \vec{r}_c(L + \lambda_B/4) - e^{-L/R} \vec{r}_c(\lambda_B/4) \left[\cos(L/z_R) - \frac{2k\chi_o}{\gamma_o} \sin(L/z_R) \right] \right\}. \quad (43)$$

$|s| \ll 1$:

$$\begin{aligned} \frac{\vec{f}_c}{\theta_o} &\approx -wRe \left\{ \frac{i}{2Q} \left[\vec{r}_c(L) - \vec{r}_c(0) \exp[i\Delta kL] \right] \right\} \\ &\approx - \frac{wr_b k}{2R} \cdot \frac{1}{r_b} \cdot \left\{ \vec{r}_c(L) - e^{-L/R} \vec{r}_c(0) \left[\cos(L/z_R) + \frac{R}{z_R} \sin(L/z_R) \right] \right\} \end{aligned} \quad (44)$$

and

$$\frac{\vec{\theta}_c}{\theta_d} = \frac{\vec{h}_c}{w} \approx \frac{wr_b}{w^2} \cdot \frac{1}{r_b} \cdot \left\{ \vec{r}_c(L) - e^{-L/R} \vec{r}_c(0) \left[\cos(L/z_R) - \frac{z_R}{R} \sin(L/z_R) \right] \right\}. \quad (46)$$

AVERAGING OVER FLUCTUATIONS

The centroid angle, Eqs. (38), (42), and (44), is the instantaneous optical deflection when the e-beam enters the wiggler with initial conditions $r_c(0)$. If the initial conditions vary from pulse to pulse, the appropriate observable is a statistical average of θ_c . To estimate the average optical deflection, we choose the following simple model for the distribution function:

$$(i) \langle x_c(0) x_c'(0) \rangle = \langle y_c(0) y_c'(0) \rangle = 0 \quad (47)$$

$$(ii) \langle x_c(0) y_c(0) \rangle = \langle x_c'(0) y_c'(0) \rangle = 0 \quad (48)$$

$$(iii) \langle x_c^2(0) \rangle = \frac{1}{k_\beta^2} \langle x_c'^2(0) \rangle$$

$$\langle y_c^2(0) \rangle = \frac{1}{k_\beta^2} \langle y_c'^2(0) \rangle \quad . \quad (49)$$

$$(iv) \langle |\vec{r}_c(0)|^2 \rangle = a^2 \quad . \quad (50)$$

The e-beam betatron motion can be expressed in terms of initial displacement and velocity:

$$\vec{r}_c(z) = \vec{r}_c(0) \cos k_\beta z + \frac{1}{k_\beta} \vec{r}_c'(0) \sin k_\beta z \quad . \quad (51)$$

To simplify the notation, define

$$\frac{1}{s^2 - 4} = u = u_r + i u_i$$

$$\frac{-2i}{s(s^2 - 4)} = v = v_r + i v_i \quad . \quad (52)$$

Then we can express the exit angle centroid as

$$\frac{\vec{f}_c}{\theta_d} \approx -A \cdot \frac{1}{r_b} \cdot \left\{ u_r \left[\vec{r}_c(L + \lambda_\beta/4) - e^{-L/R} \vec{r}_c(\lambda_\beta/4) \left(\cos(L/z_R) + \frac{u_i}{u_r} \sin(L/z_R) \right) \right] \right. \\ \left. + v_r \left[\vec{r}_c(L) - e^{-L/R} \vec{r}_c(0) \left(\cos(L/z_R) + \frac{v_i}{v_r} \sin(L/z_R) \right) \right] \right\} . \quad (53)$$

Then the rms deflection is

$$\frac{\bar{f}_c}{\theta_d} \approx A |u_r| \frac{a}{r_b} \left[\Delta^2 \left(L, \frac{u_i}{u_r} \right) + \left(\frac{v_r}{u_r} \right)^2 \Delta^2 \left(L, \frac{v_i}{v_r} \right) \right. \\ \left. + \left(\frac{v_r}{u_r} \right) \left(\frac{v_i}{v_r} - \frac{u_i}{u_r} \right) e^{-L/R} \sin(L/z_R) \sin(k_\beta L) \right]^{1/2} , \quad (54)$$

where

$$\Delta(L, f) = \left\{ \left[1 - e^{-L/R} \left(\cos(L/z_R) + f \sin(L/z_R) \right) \right]^2 \right. \\ \left. + 2e^{-L/R} (1 - \cos k_\beta L) \left[\cos(L/z_R) + f \sin(L/z_R) \right] \right\} \quad (55)$$

is a damped oscillating function of L . Eq. 54 is the average normalized optical deflection at the wiggler exit. Similarly, the average normalized far field deflection is

$$\frac{\bar{\theta}_c}{\theta_o} = \frac{\bar{h}_c}{w} \approx A |u_i| \frac{a}{r_b} \left[\Delta^2 \left(L, \frac{-u_r}{u_i} \right) + \left(\frac{v_i}{u_i} \right)^2 \Delta^2 \left(L, \frac{-v_r}{v_i} \right) \right. \\ \left. + \left(\frac{v_i}{u_i} \right) \left(\frac{u_r}{u_i} - \frac{v_r}{v_i} \right) e^{-L/R} \sin(L/z_R) \sin(k_\beta L) \right]^{1/2} . \quad (56)$$

The limiting cases are

$$|s| \gg 1:$$

$$Au \rightarrow \frac{2wr_b}{kk_\beta Q^4} \approx \frac{wk}{r_b k_\beta} (n_o - 1) \gg Av , \quad (57)$$

$$\frac{\bar{f}_c}{\theta_d} \rightarrow (n_r - 1) \frac{w}{\lambda} \frac{\lambda_\beta}{r_b} \frac{a}{r_b} \Delta \left(L, \frac{-\gamma_o}{2k\chi_o} \right), \quad (58)$$

$$\frac{\bar{\theta}_c}{\theta_o} = \frac{\bar{h}_c}{w} \rightarrow n_i \frac{w}{\lambda} \frac{\lambda_\beta}{r_b} \frac{a}{r_b} \Delta \left(L, \frac{2k\chi_o}{\gamma_o} \right). \quad (59)$$

$$|s| \ll 1:$$

$$A_v \rightarrow \frac{-iwr_b}{Q^2} \approx \frac{-wkr_b}{2R} - \frac{iwr_b}{w^2} \gg A_u. \quad (60)$$

$$\frac{\bar{f}_c}{\theta_d} \rightarrow \pi \frac{w}{\lambda} \frac{r_b}{R} \frac{a}{r_b} \Delta \left(L, \frac{R}{z_R} \right), \quad (61)$$

$$\frac{\bar{\theta}_c}{\theta_o} = \frac{\bar{h}_c}{w} \rightarrow \frac{r_b}{w} \frac{a}{r_b} \Delta \left(L, -\frac{z_R}{R} \right). \quad (62)$$

Suppose $\bar{r}_c(0)$ has a Gaussian distribution after averaging over all $\bar{r}_c'(0)$ [and likewise $\bar{r}_c'(0)$ Gaussian after averaging over $r_c(0)$]. Then the average far-field intensity of the unobscured, unvignetted optical beam will have its $1/e^2$ half-angle increased (from the zero-deflection case) by

$$\theta_o \rightarrow \theta_o \sqrt{1 + 2 \left(\frac{\bar{\theta}_c}{\theta_o} \right)^2} \quad (63a)$$

and its average on-axis intensity will be reduced by

$$\frac{1}{1 + 2 \left(\frac{\bar{\theta}_c}{\theta_o} \right)^2}. \quad (63b)$$

FREE ELECTRON LASER AMPLIFIER

The results we have shown, Eqs. (37), (38), (54) and (56), are valid for any linear optical medium with sinusoidal displacement. To apply the results to the FEL we select appropriate values for n_o . In the standard model⁷

$$n_o - 1 = \frac{\omega^2}{2\omega^2} \frac{a_w^2}{a_s} \frac{\langle e^{-i\psi} \rangle}{\gamma} \quad (64a)$$

$$\approx \frac{2a_w^3 (1 + a_w^2)}{k_w F^2 \gamma^5} \frac{\langle e^{-i\psi} \rangle}{\gamma} \sqrt{\frac{I}{I_A}} \sqrt{\frac{B}{I_A}}, \quad (64b)$$

for a matched beam with parabolic current density, radius r_b , and for F

$$\text{sufficiently small that } \frac{\langle e^{-i\psi} \rangle}{\gamma} \approx \frac{1}{\gamma} \langle e^{-i\psi} \rangle.$$

In a tapered wiggler with high gain the optical index changes in z . If exact resonance is maintained during tapering,

$$n_o - 1 \propto \frac{a_w^3 \langle e^{-i\psi} \rangle}{F^2 \gamma^3}. \quad (64c)$$

The properties of Gaussian quadratic modes in the FEL are derivable from n_o , Eq. (64). Specifically,

$$\frac{1}{z_R} - \frac{i}{R} \approx \frac{\sqrt{2} a_w^2 (1 + a_w^2)^{1/2} \sqrt{\langle e^{-i\psi} \rangle}}{F \gamma^{5/2}} \sqrt{\frac{B}{I_A}} \quad (65a)$$

$$\propto \frac{a_w^2 \sqrt{\langle e^{-i\psi} \rangle}}{\sqrt{k_w} F \gamma^{3/2}} \quad \text{with tapering,} \quad (65b)$$

$$w \approx \frac{(1 + a_w^2)^{1/4} F^{1/2} \gamma^{1/4}}{\sqrt{2} k_w^{1/2} a_w \sqrt{\text{Re} \sqrt{\langle e^{-i\psi} \rangle}}} \left(\frac{I_A}{B} \right)^{1/4} \quad (66a)$$

$$\propto \frac{F^{1/2} \gamma^{3/4}}{k_w^{1/4} a_w \sqrt{\text{Re} \sqrt{\langle e^{-i\psi} \rangle}}} \quad \text{with tapering,} \quad (66b)$$

and

$$s \approx \frac{k_w F \gamma^{3/2}}{a_w (1 + a_w^2)^{1/2} \sqrt{\langle e^{-i\psi} \rangle}} \sqrt{\frac{I_A}{B}} \quad (67a)$$

$$\propto \frac{F \gamma^{1/2} k_w^{1/2}}{a_w \sqrt{\langle e^{-i\psi} \rangle}} \quad \text{with tapering.} \quad (67b)$$

Figures 4, 5, and 6 show χ_0 , $d = 2w$ and s for a range of energy and bucket height.

Figure 4. Contour plot of $\chi_0 = \text{Re}(n_0 - 1)$ for $a_w = 1$, $\lambda_w = 8$ cm, $B/I_A = 100/\text{cm}^2\text{-rad}^2$, $I/I_A = 0.2$, $\langle \cos \psi \rangle = \langle \sin \psi \rangle = 0.5$. Contours are labeled by $\log \chi_0$. "Bucket half-height" $= 2 \sqrt{a_w a_s / (1 + a_w^2)}$.

Figure 5. Contour plot of optical beam diameter d for the same parameters as Fig. 4. Contours are labeled by centimeters.

Figure 6. Contour plot of coupling constant $|s|$ for the same parameters as Fig. 4.

Self-consistency of the model requires that $w < r_b$, so that the fictitious negative χ_0 , γ_0 region outside r_b does not affect the results. For the matched parabolic beam,

$$r_b = \sqrt{\frac{2}{k_w a_w}} \left(\frac{I}{B} \right)^{1/4}, \quad (68)$$

so

$$\frac{w}{r_b} \approx \frac{(1 + a_w^2)^{1/4} F^{1/2} \gamma^{1/4}}{2 a_w^{1/2} \sqrt{\text{Re} \sqrt{e^{-i\psi}}}} \left(\frac{I_A}{I} \right)^{1/4} \quad (69a)$$

$$\propto \frac{k_w^{1/4} F^{1/2} \gamma^{3/4}}{2 a_w^{1/2} \sqrt{\text{Re} \sqrt{e^{-i\psi}}}} \quad \text{under tapering.} \quad (69b)$$

This is plotted in Fig. 7. For sufficiently high energy, large buckets, low fields or low currents, the ratio > 1 and the model is not valid. However,

Figure 7. Contour plot of w/r_b for the same parameters as Fig. 4.

many FELs fall in the parameter range where $w/r_b < 1$, i.e., most of the light is confined inside the e-beam. The model should quantitatively describe optical guiding and beam deflection for untapered wigglers in this regime. The model should also illustrate qualitative behavior for tapered wigglers if the tapering is not too fast in z , that is, for

$$\frac{da_w}{dz} \ll \frac{1}{z_R}, \frac{1}{R} \quad (70)$$

Our model is in agreement with two- and three-dimensional particle simulations of FEL amplifiers⁸ on the following points:

(a) The 0,0 Gaussian mode has the highest gain. With a long wiggler under axisymmetric equilibrium, only TEM_{00} output will be seen.

(b) The optical beam has a diverging phase front ($R > 0$) through the wiggler.

(c) The optical beam is guided by the dispersion χ_0 and can propagate through long wigglers with $L \gg z_R$.

There is good quantitative agreement with specific wiggler designs, discussed below.

A sample case for a 10- μm FEL amplifier is shown in Table 1. The e-beam parameters of the analytical model were selected from predictions of the proposed Paladin FEL at ATA by the two-dimensional simulation FRED. The agreement between FRED and the model for optical beam characteristics is good. Analytic predictions of the effect of betatron oscillations are shown in Fig. 8. The near- and far-field optical beams betatron oscillate approximately $\pi/2$ out of phase with the electron beam. The optical oscillations are not pure sinusoids but also contain exponential damping of the initial conditions according to Eqs. (37) and (38). The maximum far-field optical deflection is ≈ 0.25 times the maximum e-beam deflection.

A sample case for a 1- μm FEL amplifier is shown in Table 2. The agreement between FRED and the analytic model is excellent. Analytic predictions of the betatron effects are shown in Fig. 9. Qualitatively, the optical motion is similar to Fig. 8 because both cases have $|s| \gg 1$ and are described by Eqs. (42) and (43). The maximum far-field deflection is 0.15 times the maximum e-beam deflection.

These cases illustrate the general features of optical deflection, Eqs. (37) and (38). At $z = 0$ the light is on-axis [the initial condition of Eqs. (30)]. The light betatron oscillates with period λ_B . In the weak

Table 1. 10- μ m free electron laser (FEL).

	2-d simulation at $z = 10$ m	Analytic model
a_w	1.0	1.0
B/I_A	$29.4/\text{cm}^2\text{-rad}^2$	$30/\text{cm}^2\text{-rad}^2$
I/I_A	0.175	0.2
λ_w	8 cm	8 cm
r_b	0.395 cm	0.45 cm
γ	90	90
F	0.033	0.028
$\langle \cos \psi \rangle$	0.5	0.5
$\langle \sin \psi \rangle$	0.4	0.4
<hr/>		
w	0.39 cm	0.40 cm
z_R	4.9 m	5.2 m
R	12.9 m	12.5 m

Table 2. 1- μ m free electron laser (FEL).

	3-d simulation at $z = 55$ m	Analytic model
a_w	2.0	2.0
B/I_A	$117.6/\text{cm}^2\text{-rad}^2$	$100/\text{cm}^2\text{-rad}^2$
I/I_A	0.175	0.2
λ_w	8 cm	8 cm
r_b	0.186 cm	0.239 cm
γ	420	400
F	0.037	0.028
$\langle \cos \psi \rangle$	0.5	0.5
$\langle \sin \psi \rangle$	0.4	0.4
<hr/>		
w	0.239 cm	0.275 cm
z_R	17.9 m	18 m
R	52.7 m	55 m

Figure 8. Centroid betatron oscillations for the 10-μm FEL in Table 1; (a) f_c for $\delta_x = 0$, (b) h_c for $\delta_x = 0$, (c) f_c for $\delta_x = \pi/2$, (d) h_c for $\delta_x = \pi/2$.

Figure 8. (Continued)

Figure 9. Centroid betatron oscillations for the 1-μm FEL in Table 2; (a) f_c for $\delta_x = 0$, (b) h_c for $\delta_x = 0$, (c) f_c for $\delta_x = \pi/2$, (d) h_c for $\delta_x = \pi/2$.

Figure 9. (Continued)

coupling limit (s large), the light is $\pi/2$ out of phase with the e-beam oscillation. In strong coupling (s small), the relative phase is sensitive to s . Electron-beam initial conditions affect the light for $z \approx R$, then damp out.

The average deflection under fluctuating initial conditions is given by Eqs. (54) and (56). For long wigglers ($L \gg R$), initial conditions are damped out by $\Delta(L) \rightarrow 1$ and only the e-beam conditions at z affect the light. In this case

$$\frac{\bar{f}_c}{\theta_d} \rightarrow A |u_r| \sqrt{1 + \left(\frac{v_r}{u_r}\right)^2} \frac{a}{r_b} . \quad (71)$$

Examination of Eq. (38) shows for $L \gg R$ the oscillation amplitude is

$$\frac{f_c(\max)}{\theta_d} \rightarrow A |u_r| \sqrt{1 + \left(\frac{v_r}{u_r}\right)^2} \frac{r_c(\max)}{r_b} . \quad (72)$$

The exit angle optical jitter gain :

$$G_a = A |u_r| \sqrt{1 + \left(\frac{v_r}{u_r}\right)^2} \quad (73)$$

is shown in Figs. 10 and 11 for two typical cases. High- γ , large bucket wigglers have $G_a < 1$. The region of steeply increasing G_a is near the crossover between weak and strong coupling, where $|s-2| \ll 1$.

Similarly, the displacement jitter gain

$$G_d = A |u_i| \sqrt{1 + \left(\frac{v_i}{u_i}\right)^2} , \quad (74)$$

$$\frac{\bar{\theta}_c}{\theta_o} = \frac{\bar{h}_c}{w} + G_d \cdot \frac{a}{r_b} , \quad (75)$$

and

$$\frac{h_c(\max)}{w} \rightarrow G_d \cdot \frac{r_c(\max)}{r_b} . \quad (76)$$

Figure 10. (a) Output angle jitter gain G_a for the same parameters as Fig. 4; (b), (c) contours of G_a .

Figure 10. (Continued)

Figure 11. (a) Output angle jitter gain G_a for $a_w = 2$, all other parameters same as Fig. 10; (b), (c) contours of G_a .

Figure 11. (Continued)

G_d is shown in Figs. 12 and 13. A comparison of Figs. 10 and 11 or 12 and 13 shows that increasing a_w moves the high-gain cliff out toward higher γ and larger fields. That is, in the "low jitter" region the jitter gain decreases as a_w decreases.

Figure 12. (a) Beam displacement jitter gain G_d for the same parameters as Fig. 10; (b), (c) contours of G_d .

Figure 12. (Continued)

Figure 13. (a) Beam displacement jitter gain G_d for $a_w = 2$, all other parameters same as Fig. 12; (b), (c) contours of G_d .

Figure 13. (Continued)

REFERENCES

1. W. B. Colson, G. Dattoli, and F. Ciocci, "The Angular Gain Spectrum of Free Electron Lasers," Phys. Rev. A **31**, 828 (1985).
2. Milton Abramowitz and Irene Stegun, Eds. Handbook of Mathematical Functions (Dover, 1965), Ch. 22.
3. H. Kogelnik and T. Li, Appl. Opt. **5**, 1550 (1966).
4. H. Kogelnik, "On the Propagation of Gaussian Beams of Light Through Lenslike Media Including Those with a Loss and Gain Variation," Appl Opt. **4**, 1562 (1965).
5. L. W. Casperson, "Gaussian Light Beams in Inhomogeneous Media," Appl. Opt. **12**, 2434 (1973).
6. Max Born and Emil Wolf, Principles of Optics, Third Ed. (Pergamon, 1964), Ch. 8.
7. N. Kroll, P. Morton, and M. Rosenbluth, "Free-Electron Lasers with Variable Parameter Wigglers," IEEE QE-17, 1436 (1981).
8. E. T. Scharlemann, private communication.

MAE/cw