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**VARIATIONAL-MOMENT METHOD
FOR COMPUTING
MAGNETOHYDRODYNAMIC EQUILIBRIA**

by
L. L. LAO

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VARIATIONAL-MOMENT METHOD
FOR COMPUTING MAGNETOHYDRODYNAMIC EQUILIBRIA*

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ABSTRACT

A fast yet accurate method to compute magnetohydrodynamic equilibria is provided by the variational-moment method, which is similar to the classical Rayleigh-Ritz-Galerkin approximation. The equilibrium solution sought is decomposed into a spectral representation. The partial differential equations describing the equilibrium are then recast into their equivalent variational form and systematically reduced to an optimum finite set of coupled ordinary differential equations. An appropriate spectral decomposition can make the series representing the solution converge rapidly and hence substantially reduces the amount of computational time involved. The moment method was developed first to compute fixed-boundary inverse equilibria in axisymmetric toroidal geometry, and was demonstrated to be both efficient and accurate. The method since has been generalized to calculate free-boundary axisymmetric equilibria, to include toroidal plasma rotation and

pressure anisotropy, and to treat three-dimensional toroidal geometry. In all these formulations, the flux surfaces are assumed to be smooth and nested so that the solutions can be decomposed in Fourier series in inverse coordinates. These recent developments and the advantages and limitations of the moment method are reviewed. The use of alternate coordinates for decomposition is discussed.

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CONTENTS

ABSTRACT	iii
1. INTRODUCTION	1-1
2. BASIC ELEMENTS OF THE VARIATIONAL MOMENT METHOD	2-1
3. SPECTRAL REPRESENTATION OF MAGNETIC SURFACES	3-1
4. 2-D AXISYMMETRIC TOROIDAL EQUILIBRIA	4-1
4.1. Isotropic Stationary Equilibria	4-1
4.2. Anisotropic Static Equilibria	4-8
4.3. Free Boundary Equilibria	4-10
5. 3-D TOROIDAL EQUILIBRIA	5-1
6. ADVANTAGES AND LIMITATIONS OF THE MOMENT METHOD	6-1
7. ACKNOWLEDGMENT	7-1
8. REFERENCES	8-1

FIGURES

3-1. Coordinate system	3-2
4-1. Magnetic surface contours of a fixed boundary ISX-B equilibrium	4-5
4-2. Various ISX-B equilibrium profiles	4-7
4-3. Magnetic surface contours of a free boundary D-III equilibrium	4-13

FIGURES (Continued)

4-4. Various D-III equilibrium profiles	4-15
5-1. Magnetic surface contours of a 3-D fixed boundary $\ell = 2$ stellarator equilibrium	5-8

TABLES

4-1. Equilibrium parameters for a fixed boundary ISX-B equilibrium	4-6
4-2. Comparisons of aspect ratio, elongation, and safety factor at the magnetic axis and the required CPU time on the Cray computer for a fixed boundary ISX-B equilibrium using different numbers of amplitude functions	4-8
4-3. Equilibrium parameters of a free boundary D-III equilibrium	4-14
5-1. Equilibrium parameters for a 3-D fixed boundary $\ell = 2$ stellarator equilibrium	5-9

1. INTRODUCTION

For many applications in plasma physics, it is essential to have a computationally fast, yet sufficiently accurate, method to calculate magnetohydrodynamic (MHD) equilibria. This is particularly true for data analysis in tokamaks where the two-dimensional (2-D) quasi-linear Grad-Shafranov equation has to be solved repeatedly to obtain the toroidal current density profile that best fits the experimentally measured external magnetic data [1], and for exploring new non-axisymmetric three-dimensional (3-D) toroidal configurations where determination of MHD equilibria is computationally expensive. The variational moment method has been developed to provide an efficient computational tool for computing MHD equilibria in 2-D and 3-D toroidal geometry with nested and smooth flux surfaces. It was developed first to compute fixed boundary isotropic static equilibria in 2-D axisymmetric toroidal geometry [2,3], and was demonstrated to be both efficient and accurate [2]. It has since become an important computational tool for data analysis of high beta non-circular tokamak plasmas [4]. The success has prompted the generalizations of the moment method to calculate free boundary axisymmetric equilibria [5,6], to include toroidal plasma rotation and pressure anisotropy [7], and to treat 3-D toroidal geometry [8,9,10]. In this paper, the elements of the variational moment method, its advantages and limitations, and these recent developments are reviewed.

In Section 2, the basic elements of the variational moment method are summarized. The spectral representation of magnetic surfaces is discussed in Section 3. In Section 4, the applications of the moment method to compute 2-D axisymmetric equilibria are reviewed. In Section 5, the generalizations to compute 3-D toroidal equilibria are reviewed. Finally, in Section 6, the advantages and the limitations of the moment method are discussed.

2. BASIC ELEMENTS OF THE VARIATIONAL MOMENT METHOD

The variational moment method is similar to the classical Rayleigh-Ritz-Galerkin approximation [11] and consists of two major elements. First, the unknown solution to a set of partial differential equations is reduced to a finite number of unknown functions of one variable by decomposing in terms of a set of basis functions. An appropriate choice of basis functions for decomposition can make the series representing the solution converge rapidly and hence substantially reduces the amount of computational time involved. Secondly, the ordinary differential equations describing the expansion coefficients of the unknown solution are obtained systematically by first recasting the original set of partial differential equations into an equivalent variational form. The stationary condition of this variational form then yields a finite number of coupled ordinary differential equations which best describe the expansion coefficients.

Since the partial differential equations describing MHD equilibria are non-linear, direct substitution of the expansion series into the partial differential equations will in general yield more equations than the number of unknown expansion coefficients. The use of a variational principle is essential to ensure the optimum truncation procedure. In the case of MHD equilibria where the magnetic surfaces form a set of nested and smooth tori around a single magnetic axis, a useful set of independent coordinates is

the flux coordinates based on the magnetic surfaces, which consist of a radial and two angular coordinates. The equilibrium solution can then be conveniently represented in Fourier series in these angular variables. All present formulations of the moment method are limited to this choice of representation and have not considered the possible appearances of magnetic islands and separatrixes. In this case, the problem of determining MHD equilibria is equivalent to that of finding the geometry of the magnetic surfaces and a periodic stream function. The spectral representation of magnetic surfaces is considered in the next section.

3. SPECTRAL REPRESENTATION OF MAGNETIC SURFACES

In considering a 3-D toroidal system with nested and smooth magnetic surfaces, it is convenient to adapt the flux coordinates (ρ, θ, ζ) as independent coordinates where ρ is a radial variable that labels the magnetic surfaces and increases monotonically away from the magnetic axis, θ is a poloidal angle that increases by 2π the short way around the torus, and ζ is a toroidal angle that increases by a multiple of 2π the long way around the torus (Fig. 3-1). The magnetic surfaces may then be conveniently represented in parametric form using the cylindrical coordinates $R(\rho, \theta, \zeta)$, $\phi(\rho, \theta, \zeta)$, and $Z(\rho, \theta, \zeta)$, which are now dependent variables. The angular coordinates θ and ζ are arbitrary except for the requirement that the Jacobian of the transformation from (R, ϕ, Z) to (ρ, θ, ζ) coordinates

$$\begin{aligned} \tau &= \frac{\partial(R, \phi, Z)}{\partial(\rho, \theta, \zeta)} , \\ &= \phi_\rho(Z_\theta R_\zeta - R_\theta Z_\zeta) + \phi_\theta(Z_\zeta R_\rho - R_\zeta Z_\rho) \\ &\quad + \phi_\zeta(Z_\rho R_\theta - R_\rho Z_\theta) , \end{aligned} \tag{3.1}$$

is finite. Here, the subscripts ρ , θ , and ζ denote differentiation with respect to these variables. The elements of the metric tensor characterizing the transformation are given by

$$g_{\mu\nu} = R_\mu R_\nu + R^2 \phi_\mu \phi_\nu + Z_\mu Z_\nu , \tag{3.2}$$

Fig. 3-1. Coordinate system.

where $\mu, \nu \in (\rho, \theta, \zeta)$. Exploiting the periodicity in the angles θ and ζ , R and Z may be represented in double Fourier series in θ and ζ as follow

$$R(\rho, \theta, \zeta) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} [R_{mn}(\rho) \cos(m\theta - n\zeta) + \tilde{R}_{mn}(\rho) \sin(m\theta - n\zeta)] \quad , \quad (3.3)$$

$$Z(\rho, \theta, \zeta) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} [Z_{mn}(\rho) \sin(m\theta - n\zeta) + \tilde{Z}_{mn}(\rho) \cos(m\theta - n\zeta)] \quad . \quad (3.4)$$

For 2-D axisymmetric systems, ϕ is an ignorable coordinate and the angle ζ may be conveniently chosen to be ϕ . R and Z may then be represented as

$$R(\rho, \theta) = \sum_{m=0}^{\infty} [R_m(\rho) \cos m\theta + \tilde{R}_m(\rho) \sin m\theta] \quad , \quad (3.5)$$

$$Z(\rho, \theta) = \sum_{m=0}^{\infty} [Z_m(\rho) \sin m\theta + \tilde{Z}_m(\rho) \cos m\theta] \quad . \quad (3.6)$$

Equations (3.5) and (3.6) are reminiscent of the pioneering work by Greene et al. [13], who introduced the notion of transformation to the inverse flux coordinates and representing the magnetic surfaces in Fourier series. For systems possessing up-down symmetry [$R(\rho, \theta) = R(\rho, -\theta)$ and $Z(\rho, \theta) = -Z(\rho, -\theta)$], the terms $\tilde{R}_m$ and $\tilde{Z}_m$ are absent and a particular convenient representation for R and Z is given in Refs. 2 and 12,

$$R(\rho, \theta) = R_0(\rho) - R_1(\rho) \cos \theta + \sum_{m=2}^{\infty} R_m(\rho) \cos m\theta \quad , \quad (3.7)$$

$$Z(\rho, \theta) = E(\rho) \sum_{m=1}^{\infty} R_m(\rho) \sin m\theta \quad , \quad (3.8)$$

where the poloidal angle θ has been chosen such that $Z_m = ER_m$. The amplitudes $R_0(\rho)$, $R_1(\rho)$, and $Z_1(\rho)$ describe the shift, the minor radius, and the ellipticity of the magnetic surfaces, respectively, whereas the amplitudes $R_2(\rho)$ and $Z_2(\rho)$ describe the triangularity of the flux surfaces. A few amplitudes ($m < 3$) in the Fourier series are generally sufficient to

describe many equilibrium shapes such as those in the Doublet III (D-III) and the Impurity Study Experiment (ISX-B).

For 3-D non-axisymmetric systems, since most of those under consideration form periodic sectors toroidally, a convenient choice of ζ is given by $\zeta = N_t \phi$ where N_t is the number of periodic sectors. The expressions for the Jacobian τ and the metric elements $g_{\mu\nu}$ then simplify considerably. For systems possessing reflection symmetry [$R(\rho, \theta, \zeta) = R(\rho, -\theta, -\zeta)$ and $Z(\rho, \theta, \zeta) = -Z(\rho, -\theta, -\zeta)$], the terms $\tilde{R}_{mn}$ and $\tilde{Z}_{mn}$ are absent. For example, magnetic surfaces that consist of poloidally rotating ellipses which have reflection symmetry about the plane $\phi = 0$ may be represented as

$$R(\rho, \theta, \zeta) = R_{00}(\rho) - R_{11}^+(\rho) \cos(\theta + \zeta) + R_{11}^-(\rho) \cos(\theta - \zeta) \quad , \quad (3.9)$$

$$Z(\rho, \theta, \zeta) = R_{11}^+(\rho) \sin(\theta + \zeta) + R_{11}^-(\rho) \sin(\theta - \zeta) \quad , \quad (3.10)$$

where the amplitudes R_{11}^+ and R_{11}^- are related to the elongation $E(\rho)$ and the minor radius ρ of the ellipses through

$$R_{11}^+(\rho) = \frac{E(\rho) + 1}{2} \rho \quad , \quad (3.11)$$

$$R_{11}^- = \frac{E(\rho) - 1}{2} \rho \quad . \quad (3.12)$$

Generally, the number of amplitude functions necessary to accurately describe an equilibrium depends on the shape of the boundary magnetic surface in a fixed boundary problem and on the currents in the external shaping

coils in a free boundary problem. Thus, the choice of θ and ζ in a fixed boundary problem should be guided by the requirement that they result in the least number of amplitude functions necessary to describe the boundary magnetic surface. In either problem, the magnetic surfaces near the magnetic axis in a given $\phi = \text{constant}$ plane can be shown to form concentric ellipses centered at the magnetic axis which can be adequately described using a few amplitude functions. This will at least ensure the rapid convergence of the Fourier series of R and Z near the magnetic axis when ϕ is ignorable or when the non-axisymmetric perturbations are small.

Finally, either of the angles θ or ζ may be chosen so that the magnetic field lines are straight in the (θ, ζ) planes. Although the MHD equilibrium equations appear in a simpler form in this representation, this choice of θ or ζ generally conflicts with the geometric requirement of optimum representation of magnetic surfaces [9,10].

4. 2-D AXISYMMETRIC TOROIDAL EQUILIBRIA

4.1. ISOTROPIC STATIONARY EQUILIBRIA

In the presence of axisymmetry and only toroidal rotation, the stationary MHD equilibrium equations may be reduced to a single second order quasi-linear partial differential equation which has the familiar form of the inverse Grad-Shafranov equation [the Grad-Shafranov equation in (ρ, θ) coordinates] [7]

$$\begin{aligned} G_\rho &\equiv \frac{\Psi'(\rho)}{16\pi^3 R^2} \left[\Delta^* \Psi(R, Z) + \frac{8\pi^2 R}{c} J_p \right] , \\ &= 0 , \end{aligned} \tag{4.1}$$

where Ψ is the poloidal magnetic flux enclosed between the flux surface ρ and the axis of symmetry,

$$\begin{aligned} \Delta^* \Psi(R, Z) &\equiv R^2 \bar{\nabla} \cdot \frac{\bar{\nabla} \Psi}{R^2} , \\ &= \frac{R^2}{\sqrt{g}} \left\{ \frac{\partial}{\partial \rho} \left[\frac{g_{\theta\theta}}{\sqrt{g}} \Psi'(\rho) \right] - \frac{\partial}{\partial \theta} \left[\frac{g_{\rho\theta}}{\sqrt{g}} \right] \Psi'(\rho) \right\} , \end{aligned} \tag{4.2}$$

$\sqrt{g} = R\tau$, J_p is the toroidal component of the current density

$$J_p = \frac{2\pi c R}{\Psi'(\rho)} \left[\frac{\partial}{\partial \rho} P(\rho, R) + \frac{F F'(\rho)}{\pi c^2 R^2} \right] , \tag{4.3}$$

F is the poloidal current enclosed between the axis of symmetry and the magnetic surface of interest and P is the plasma pressure

$$P(\rho, R) = \tilde{P}(\rho) \exp\left[\frac{\omega_t^2(\rho)}{2T(\rho)} (R^2 - R_C^2)\right] \quad (4.4)$$

Here, ω_t is the plasma toroidal rotation frequency, T has the physical interpretation of plasma temperature and has been assumed to be constant on a magnetic surface, and $\tilde{P}(\rho) = P(\rho, R_C)$. Throughout this paper, the prime superscript on a function denotes differentiation with respect to its argument; $F'(\rho) = dF/d\rho$, etc.

In the absence of toroidal rotation $\omega_t = 0$, Eq. (4.1) reduces to the inverse Grad-Shafranov equation. It may be used to determine either $R(\rho, \theta)$ or $Z(\rho, \theta)$. The other then is determined by the specification of the poloidal angle θ . The generalized Grad-Shafranov equation (4.1) with the fixed boundary condition $\rho(\text{boundary}) = \text{constant}$ may be shown to be equivalent to the stationary condition of the integral

$$Q = \int_0^a \int_0^{2\pi} d\rho d\theta \sqrt{g} \left[\frac{B_p^2}{8\pi} - \frac{B_t^2}{8\pi} - P(\rho, R) \right] \quad (4.5)$$

although subject to the fixed boundary constraint $\delta\rho(\text{boundary}) = 0$ [7].

Here, B_p and B_t are the poloidal and the toroidal components of the magnetic field

$$B_p = \frac{|\Psi'(\rho)|}{2\pi} \left(\frac{g_{\theta\theta}}{g} \right)^{1/2}, \quad (4.6)$$

$$B_t = \frac{2F(\rho)}{cR}, \quad (4.7)$$

and a is the boundary surface label. The ordinary differential equations describing the Fourier amplitudes $R_m(\rho)$, $\tilde{R}_m(\rho)$, $Z_m(\rho)$, and $\tilde{Z}_m(\rho)$ may be obtained by requiring that Q be stationary with respect to the variations of these amplitudes. They may be shown to be moments of the generalized inverse Grad-Shafranov equation (4.1) with basis functions $M_{R_m} = RZ_\theta \cos m\theta$, $M_{\tilde{R}_m} = RZ_\theta \sin m\theta$, $M_{Z_m} = RR_\theta \sin m\theta$, $M_{\tilde{Z}_m} = RR_\theta \cos m\theta$, or their combinations, depending on the choice of the poloidal angle θ [2]. For example, if the magnetic surfaces possess up-down symmetry and R and Z are represented as truncated Fourier series

$$R(\rho, \theta) = \sum_{m=0}^{M_R} R_m(\rho) \cos m\theta, \quad (4.8)$$

$$Z(\rho, \theta) = \sum_{m=1}^{M_Z} Z_m(\rho) \sin m\theta, \quad (4.9)$$

the moment equations describing R_m and Z_m are

$$\langle M_{R_m} G_\rho \rangle = 0 \quad ; \quad m = 0, \dots, M_R \quad , \quad (4.10)$$

$$\langle M_{Z_m} G_\rho \rangle = 0 \quad ; \quad m = 1, \dots, M_Z \quad , \quad (4.11)$$

where $\langle \dots \rangle = \int_0^{2\pi} d\theta / 2\pi \dots$. Note that in the limit $M_R \rightarrow \infty$ and $M_Z \rightarrow \infty$, either of the sequences of the moment equations (4.10) or (4.11) converges to the equation $G_\rho = 0$. When M_R and M_Z are finite, the moment equations (4.10) and (4.11) are independent from each other because of the non-linear harmonic couplings. This provides an additional flexibility in choosing θ by requiring both R_m and Z_m to satisfy the moment equations (4.10) and (4.11).

The moment equations form a set of coupled second order quasi-linear ordinary differential equations. To solve the equations, four driving functions must be specified in the toroidal rotation case and two in the static case ($\omega_t = 0$). In addition, each equation must be implemented with two boundary conditions. The analytical property of the amplitude functions near the magnetic axis provides one boundary condition, and in the fixed boundary case, the shape of the boundary magnetic surface provides the other. The moment equations may be solved numerically using existing ordinary differential equation solvers [14,15].

In Fig. 4-1, the magnetic surface contours of an isotropic static ISX-B equilibrium computed using the moment method are compared against those computed using a more exact 2-D code [2]. The solid lines represent the moment

Fig. 4-1. Magnetic surface contours of a fixed boundary ISX-B equilibrium. The solid contours are computed from the moment method, and the dashed contours (which are barely distinguishable) are obtained from a standard 2-D code.

calculation and the dashed lines represent the 2-D calculation. Also shown in Fig. 4-1 are contours of constant poloidal angle θ . The parameters for this case are given in Table 4-1. Identical driving functions and boundary surface shape are used in both calculations. In Fig. 4-2, the shift $\tilde{S}(x)$,

TABLE 4-1
EQUILIBRIUM PARAMETERS FOR A
FIXED BOUNDARY ISX-B EQUILIBRIUM

$R_0(a)$	91.9 cm
a	27.0 cm
$R_2(a)$	1.1 cm
$E(a)$	1.1
B_{t0}	12.9 kG
$I(a)$	220 kA
$\bar{\beta}_t$	4%
$\bar{\beta}_p$	3.3

elongation $\tilde{E}(x)$, triangularity $\tilde{D}(x)$, the differential magnetic surface volume $V'(x)$, the safety factor $q(x)$, and the poloidal magnetic flux $\Psi(x)$ are compared for the moment and the 2-D solutions, where $x = \rho/a$. In the moment calculation, R and Z are represented in Fourier series as given in Eqs. (3.7) and (3.8) and four amplitude functions (R_0 , R_1 , E , R_2) are used. In this case, the moment calculation is faster computationally than the 2-D calculation by at least a factor of 10 while reproducing the same equilibrium with a degree of accuracy sufficient for most purposes.

The amount of computational time required for the moment method to find an equilibrium depends mainly on the number of the amplitude functions being determined and the relative error desired. The convergent property of the

Fig. 4-2. Various ISX-B equilibrium profiles. Solid curves are obtained from the moment method and dashed curves from a standard 2-D code.

moment solution is illustrated in Table 4-2 where the aspect ratio of the magnetic axis $A_m = R_0(0)/a$, the elongation at the magnetic axis E_m , the safety factor at the magnetic axis q_m , and the CPU time on the CRAY computer

TABLE 4-2
COMPARISONS OF ASPECT RATIO, ELONGATION, AND SAFETY FACTOR
AT THE MAGNETIC AXIS, A_m , E_m , AND q_m , AND THE REQUIRED CPU TIME
ON THE CRAY COMPUTER FOR A FIXED BOUNDARY ISX-B EQUILIBRIUM
USING DIFFERENT NUMBERS OF AMPLITUDE FUNCTIONS

N	A_m	E_m	q_m	CPU Time (CRAY)
R_0, R_1, E	3.767	1.220	1.330	65 ms
$R_0, R_1, E,$ and R_2	3.769	1.214	1.321	101 ms
$R_0, R_1, E,$ R_2 , and R_3	3.769	1.216	1.324	258 ms

required to achieve a relative error of less than 10^{-3} are compared for the above equilibrium using different number of amplitude functions N . It is seen that the CPU time required scales approximately as N^2 .

4.2. ANISOTROPIC STATIC EQUILIBRIA

In the absence of flows, the anisotropic MHD equilibrium equations may be reduced for the axisymmetric case to the following pair of equations [16]

$$G_\rho \equiv \frac{\sigma \Psi'(\rho)}{16\pi^3 R^2} \left[\Delta^* \Psi(R, Z) + \frac{8\pi^2 R}{c} J_p \right] = 0, \quad (4.12)$$

$$\frac{\partial P_\parallel(\rho, B)}{\partial B} = \frac{P_\parallel(\rho, B) - P_\perp(\rho, B)}{B}, \quad (4.13)$$

where σ is the anisotropy factor

$$\sigma = 1 + \frac{4\pi(P_{\perp} - P_{\parallel})}{B^2} , \quad (4.14)$$

J_p is the toroidal component of the current density

$$J_p = \frac{2\pi cR}{\sigma \Psi'(\rho)} \left[\frac{\partial}{\partial \rho} P_{\parallel}(\rho, B) + \frac{\tilde{F}\tilde{F}'(\rho)}{\sigma \pi c^2 R^2} \right] + \frac{c \Psi'(\rho)}{8\pi^2 R \sigma} \bar{\nabla} \sigma \cdot \bar{\nabla} \rho , \quad (4.15)$$

$$\tilde{F}(\rho) = \sigma F . \quad (4.16)$$

Note that unlike the isotropic pressure case, the poloidal current F is not constant on a magnetic surface. Equation (4.12) has the familiar form of the Grad-Shafranov equation and describes force balance in the radial direction whereas Eq. (4.13) describes force balance along a magnetic field line. However, unlike the Grad-Shafranov equation which is elliptic, Eq. (4.12) remains elliptic only if [17]

$$\sigma > 0 , \quad (4.17)$$

$$\sigma B_t^2 + B_p^2 \left[1 + \frac{1}{B} \frac{\partial}{\partial B} P_{\perp}(\rho, B) \right] > 0 . \quad (4.18)$$

In mirror geometry, $B_t = 0$ and Eq. (4.14) reduces to the mirror stability criterion

$$1 + \frac{1}{B} \frac{\partial P_{\perp}(\rho, B)}{\partial B} > 0 . \quad (4.19)$$

For present day tokamaks, the pressure anisotropy and the toroidal beta values are small and conditions (4.17) and (4.18) are readily satisfied.

Equation (4.12) with the fixed boundary condition $\rho(\text{boundary}) = \text{constant}$ may be shown to be equivalent to the stationary condition of the integral

$$Q = \int_0^a \int_0^{2\pi} d\rho d\theta \sqrt{g} \left[\sigma \left(\frac{B_p^2}{8\pi} - \frac{B_t^2}{8\pi} \right) - \frac{P_{\parallel}(\rho, B) + P_{\perp}(\rho, B)}{2} \right] , \quad (4.20)$$

although subject to the fixed boundary constraint $\delta\rho(\text{boundary}) = 0$ and the conditions (4.13) and (4.16) [7]. The ordinary differential equations describing the Fourier amplitudes R_m , $\tilde{R}_m$, Z_m , $\tilde{Z}_m$ may be obtained by taking variations of Q and shown to be moments of the inverse anisotropic Grad-Shafranov equation (4.12) with basis functions M_{R_m} , $M_{\tilde{R}_m}$, M_{Z_m} , or $M_{\tilde{Z}_m}$ depending again on the choice of the poloidal angle θ . The moment equations form a set of coupled second order ordinary differential equations and may be solved using existing ordinary equilibrium equation solvers.

4.3. FREE BOUNDARY EQUILIBRIA

In many physical situations of interest, the shape of the outermost boundary surface is not known, but rather, must be determined self-consistently from the limiter location and the currents in, or the poloidal flux values at, the external shaping coils. Then, the fixed outer boundary

conditions implementing the moment equations are replaced by the following set of approximate boundary conditions [5]

$$\langle T_x \{ \tilde{\Psi}[\bar{r}(a, \theta)] - \Psi(a) \} \rangle = 0 \quad , \quad (4.21)$$

where $\bar{r} = (R, Z)$,

$$\begin{aligned} \tilde{\Psi}[\bar{r}(a, \theta)] = & \sum_{n=1}^{N_e} G[\bar{r}(a, \theta)/\bar{r}_{en}] I_{en} \\ & + \frac{I(a)}{\langle \frac{g_{\theta\theta}}{\sqrt{g}} \rangle \big|_a} \int_0^{2\pi} \frac{d\theta'}{2\pi} \frac{g_{\theta\theta}(a, \theta')}{\sqrt{g}} G[\bar{r}(a, \theta)/\bar{r}'(a, \theta')] \quad , \end{aligned} \quad (4.22)$$

$I(a)$ is the toroidal current carried by the plasma, I_{en} and $\bar{r}_{en}$ are the current strength and the position of the n external current filament, and $G(\bar{r}/\bar{r}')$ is the Green's function for a unit toroidal current source. Here, T_x are the basis functions. The exact forms of T_x depend again on the choice of θ and are combinations of $T_{R_m} = u_R \cos m\theta$, $T_{\tilde{R}_m} = u_R \sin m\theta$, $T_{Z_m} = u_Z \sin m\theta$, and $T_{\tilde{Z}_m} = u_Z \cos m\theta$ where

$$u_R = \Psi'(\rho) R Z_\theta / \sqrt{g} \quad , \quad (4.23)$$

$$u_Z = -\Psi'(\rho) R R_\theta / \sqrt{g} \quad . \quad (4.24)$$

The boundary surface label a and the poloidal magnetic flux at a , $\Psi(a)$, are determined from the position of the poloidal limiter contour and the choice of the magnetic surface label ρ . For example, if the Fourier amplitude $R_1(\rho)$ is used as a flux surface label, a is then determined from the

geometric condition that the boundary magnetic surface must be tangential to the limiter contour. $\Psi(\rho)$, and hence $\Psi(a)$, are then determined using the moment equation and the outer boundary condition describing $R_1(\rho)$ [5].

Alternatively, the outer boundary conditions implementing the moment equations may be obtained by minimizing the least squares difference between $\tilde{\Psi}[\bar{r}(a, \theta)]$ and $\Psi(a)$ [6], although this seems to be less economical computationally.

In Fig. 4-3, the magnetic surface contours of a free boundary isotropic static D-III equilibrium computed using the moment method are compared against those computed using a more exact 2-D code [5]. The solid lines represent the moment calculation and the dashed lines represent the 2-D calculation. Also shown in Fig. 4-3 are contours of constant poloidal angle θ and the poloidal limiter contour. The parameters for this case are given in Table 4-3. Identical driving functions, limiter contour, and poloidal flux values at the external shaping coils are used in both calculations. Again, the shift $\tilde{S}(x)$, elongation $\tilde{E}(x)$, triangularity $\tilde{D}(x)$, the differential magnetic surface volume $V'(x)$, the safety factor $q(x)$, and the poloidal magnetic flux $\chi(x) = |\Psi(x) - \Psi(0)|$ are compared in Fig. 4-4. In the moment calculation the magnetic surfaces are taken to have the form

$$R(\rho, \theta) = R_0(\rho) - \rho \cos\theta + R_2(\rho) \cos 2\theta \quad , \quad (4.25)$$

$$\begin{aligned} Z(\rho, \theta) = & \tilde{Z}_0(\rho) + \tilde{Z}_1(\rho) \cos\theta + \tilde{Z}_2(\rho) \cos 2\theta \\ & + E(\rho)[\rho \sin\theta + R_2(\rho) \sin 2\theta] \quad . \end{aligned} \quad (4.26)$$

Fig. 4-3. Magnetic surface contours of a free boundary D-III equilibrium. The solid contours are computed from the moment method, and the dashed contours are obtained from a standard 2-D code.

In this case, the moment calculation is faster computationally than the 2-D calculation by about a factor of 10.

TABLE 4-3
EQUILIBRIUM PARAMETERS OF A
FREE BOUNDARY D-III EQUILIBRIUM

$q(0)$	0.95
$I(a)$	345 kA
B_{t0}	6.19 kG
$\bar{\beta}_t$	4.0%
$\bar{\beta}_p$	0.73

Similar to the fixed boundary case, the amount of computational time required for the moment method depends mainly on the number of amplitude functions being determined and the relative error desired. A typical free boundary moment calculation with a relative error of 10^{-3} requires approximately 0.5 to 0.7 sec of CRAY computer time for four amplitude functions (R_0, R_1, R_2, E) and approximately 1.5 to 1.8 sec of CRAY time for seven amplitude functions ($R_0, R_1, R_2, E, \tilde{Z}_0, \tilde{Z}_1, \tilde{Z}_2$).

Fig. 4-4. Various D-III equilibrium profiles. Solid curves are obtained from the moment method and the dashed curves from a standard 2-D code.

5. 3-D TOROIDAL EQUILIBRIA

In a 3-D toroidal geometry with nested magnetic surfaces, the isotropic static MHD equilibrium equations may be reduced to the following pair of coupled partial differential equations [8,9,10]

$$\begin{aligned}
 G_\rho &\equiv \hat{G} \cdot \bar{e}_\rho \\
 &= \frac{B^\theta}{4\pi} \left[\frac{\partial}{\partial \rho} (g_{\theta\theta} B^\theta + g_{\theta\zeta} B^\zeta) - \frac{\partial}{\partial \theta} (g_{\rho\theta} B^\theta + g_{\rho\zeta} B^\zeta) \right] \\
 &\quad + \frac{B^\zeta}{4\pi} \left[\frac{\partial}{\partial \rho} (g_{\theta\zeta} B^\theta + g_{\zeta\zeta} B^\zeta) - \frac{\partial}{\partial \zeta} (g_{\rho\theta} B^\theta + g_{\rho\zeta} B^\zeta) \right] + p'(\rho) \\
 &= 0 \quad , \quad (5.1)
 \end{aligned}$$

and

$$\begin{aligned}
 J^\rho &\equiv \bar{J} \cdot \bar{\nabla}_\rho \\
 &= \frac{c}{4\pi\sqrt{g}} \left[\frac{\partial}{\partial \theta} (g_{\theta\zeta} B^\theta + g_{\zeta\zeta} B^\zeta) - \frac{\partial}{\partial \zeta} (g_{\theta\theta} B^\theta + g_{\theta\zeta} B^\zeta) \right] \quad , \quad (5.2)
 \end{aligned}$$

where $\hat{G} \equiv \bar{\nabla}P - \bar{J} \times \bar{B}/c$ is the equilibrium operator, $\bar{J}$ is the current density, $\bar{e}_\mu \equiv \partial \bar{r} / \partial \mu$, $\mu \in (\rho, \theta, \zeta)$, are the covariant basis vectors, B^θ and B^ζ are the contravariant components of the magnetic field $\bar{B}$

$$\bar{B} = B^\theta \bar{e}_\theta + B^\zeta \bar{e}_\zeta \quad , \quad (5.3)$$

$$B^\theta \equiv \bar{\mathbf{B}} \cdot \bar{\nabla} \theta = \frac{\chi'(\rho) - \Lambda_\zeta}{2\pi\sqrt{g}} \quad , \quad (5.4)$$

$$B^\zeta \equiv \bar{\mathbf{B}} \cdot \bar{\nabla} \zeta = \frac{\Phi'(\rho) + \Lambda_\theta}{2\pi\sqrt{g}} \quad , \quad (5.5)$$

$\chi(\rho)$ is the poloidal flux per 2π radian of ζ enclosed by the magnetic surface ρ , $\Phi(\rho)$ is the toroidal flux enclosed by the magnetic surface ρ , and $\Lambda(\rho, \theta, \zeta)$ is a stream function which is periodic in θ and ζ . Equation (5.1) describes radial force balance and in the presence of axisymmetry it reduces to the inverse Grad-Shafranov equation. Equation (5.2) describes force balance in the two angular directions and in the axisymmetric case, it may be solved analytically to yield the expression for the toroidal magnetic field $B_t = 2F(\rho)/cR$ given in Eq. (4.7). These two equations may be used to determine two of the four dependent variables R , ϕ , Z , and Λ . The remaining two are determined by the choices of the angles θ and ζ . The characteristics of these coupled equations have been derived and examined in Ref. 10.

The stream function Λ may be transformed away by choosing a particular θ or ζ [8] so that the magnetic field lines are straight in the (θ, ζ) planes. However, this choice of θ or ζ has the disadvantage that the Fourier series representation of R and Z converges slowly [9,10] as discussed in Section 3. Secondly, although Eq. (5.2) is linear in Λ , it is nonlinear in R or Z . Thus, it is advantageous to retain Λ in the expression for the magnetic field $\bar{\mathbf{B}}$ and regard Eq. (5.1) as one defining R or Z and Eq. (5.2)

as one defining Λ . Since Λ is periodic in θ and ζ , it may be represented in a double Fourier series in θ and ζ as

$$\Lambda(\rho, \theta, \zeta) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} [\Lambda_{mn}(\rho) \sin(m\theta - n\zeta) + \tilde{\Lambda}_{mn}(\rho) \cos(m\theta - n\zeta)] \quad (5.6)$$

For magnetic surfaces possessing reflection symmetry, the terms $\tilde{\Lambda}_{mn}$ are absent.

Equations (5.1) and (5.2) with the fixed boundary condition $\rho(\text{boundary}) = \text{constant}$ may be shown to be equivalent to the stationary condition of the total energy in the system [18]

$$Q = \int_0^a \int_0^{2\pi} \int_0^{2\pi} \rho d\rho d\theta d\zeta \sqrt{g} \left(\frac{B^2}{8\pi} + \frac{p}{\gamma - 1} \right) \quad (5.7)$$

although subject to the fixed boundary constraint $\delta\rho(\text{boundary}) = 0$, the magnetic flux conservation condition (5.3), and the adiabatic pressure constraint

$$p(\rho) = N_m(\rho) [V'(\rho)]^{-\gamma} \quad (5.8)$$

where γ is a constant which has the physical interpretation of the ratio of the plasma specific heats and $N_m(\rho)$ is a prescribed function of ρ which is related to the plasma mass contained in the flux surface ρ . In the limit $\gamma \rightarrow 0$, the pressure $p(\rho)$ becomes one of the two prescribed functions and

the energy principle reduces to the form originally given in Ref. 19. The moment equations describing R_{mn} , $\tilde{R}_{mn}$, Z_{mn} , $\tilde{Z}_{mn}$, Λ_{mn} , and $\tilde{\Lambda}_{mn}$ may be obtained by taking variations of Q . They may be shown to be combinations of moments of G_R and G_Z and moments of J^ρ with basis functions $C_{mn} = \sqrt{g} \cos(m\theta - n\zeta)$ and $S_{mn} = \sqrt{g} \sin(m\theta - n\zeta)$. Here, G_R and G_Z are the projections of the equilibrium operator $\hat{G}$ in the R and the Z directions, respectively,

$$\begin{aligned} G_R &\equiv \hat{G} \cdot \bar{\nabla} R \\ &= \rho_R G_\rho + \frac{J^\rho}{2\pi c} (\theta_R \phi' - \zeta_R \chi') \quad , \end{aligned} \quad (5.9)$$

$$\begin{aligned} G_Z &\equiv \hat{G} \cdot \bar{\nabla} Z \\ &= \rho_Z G_\rho + \frac{J^\rho}{2\pi c} (\theta_Z \phi' - \zeta_Z \chi') \quad , \end{aligned} \quad (5.10)$$

and

$$\begin{bmatrix} \rho_R & \rho_\phi & \rho_Z \\ \theta_R & \theta_\phi & \theta_Z \\ \zeta_R & \zeta_\phi & \zeta_Z \end{bmatrix} = \begin{bmatrix} R_\rho & R_\theta & R_\zeta \\ \phi_\rho & \phi_\theta & \phi_\zeta \\ Z_\rho & Z_\theta & Z_\zeta \end{bmatrix}^{-1} \quad . \quad (5.11)$$

For example, if the magnetic surfaces possess reflection symmetry and R , Z , and Λ are represented as truncated Fourier series

$$R(\rho, \theta, \zeta) = \sum_{n=-N_R}^{N_R} \sum_{m=0}^{M_R} R_{mn}(\rho) \cos(m\theta - n\zeta) \quad , \quad (5.12)$$

$$Z(\rho, \theta, \zeta) = \sum_{n=-N_Z}^{N_Z} \sum_{m=0}^{M_Z} Z_{mn}(\rho) \sin(m\theta - n\zeta) , \quad (5.13)$$

and

$$\Lambda(\rho, \theta, \zeta) = \sum_{n=-N_\Lambda}^{N_\Lambda} \sum_{m=0}^{M_\Lambda} \Lambda_{mn}(\rho) \sin(m\theta - n\zeta) , \quad (5.14)$$

then the moment equations describing R_{mn} , Z_{mn} , and Λ_{mn} are, respectively,

$$\begin{aligned} \langle\langle C_{mn} G_R \rangle\rangle &= 0 ; & m &= 0, \dots, M_R , \\ & & n &= -N_R, \dots, N_R , \end{aligned} \quad (5.15)$$

$$\begin{aligned} \langle\langle S_{mn} G_Z \rangle\rangle &= 0 ; & m &= 0, \dots, M_Z , \\ & & n &= -N_Z, \dots, N_Z , \end{aligned} \quad (5.16)$$

and

$$\begin{aligned} \langle\langle S_{mn} J^\rho \rangle\rangle &= 0 ; & m &= 0, \dots, M_\Lambda , \\ & & n &= -N_\Lambda, \dots, N_\Lambda , \end{aligned} \quad (5.17)$$

where

$$\langle\langle \dots \rangle\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\zeta}{2\pi} \dots . \quad (5.18)$$

Note that in the limit M_R , N_R , M_Z , N_Z , M_Λ , and $N_\Lambda \rightarrow \infty$, the sequences of the ordinary differential equations (5.17) describing Λ_{mn} converge to the partial differential equation $J^\rho = 0$ given in Eq. (5.2) and either of the sequences of the ordinary differential equations (5.15) describing R_{mn} , or

(5.16) describing Z_{mn} , converges to the partial differential equation $G_\rho = 0$ given in Eq. (5.1). However, when M_R , N_R , M_Z , and N_Z are finite, the two sequences of ordinary differential equations (5.15) and (5.16) are independent from each other. This provides an additional flexibility in the choices of the two angular variables. For example, θ may be chosen implicitly by representing R and Z as truncated Fourier series given in Eqs. (5.12) and (5.13) and requiring both R_{mn} and Z_{mn} to satisfy Eqs. (5.15) and (5.16). Here, reflection symmetry of the magnetic surfaces has been assumed.

The moment equations describing the Fourier harmonics of R and Z [Eqs. (5.15) and (5.16)] form a set of coupled second order quasi-linear ordinary differential equations, whereas the moment equations describing the Fourier harmonics of Λ [Eq. (5.17)] form a set of linear algebraic equations in Λ_{mn} at each magnetic surface ρ . The ρ derivatives of the Fourier harmonics of Λ , which appear in the ordinary differential equations defining the Fourier harmonics of R and Z , may be obtained by taking the ρ derivatives of these algebraic equations. To solve the moment equations, two driving functions must be specified. In addition, each of the moment equations defining the Fourier harmonics of R and Z must be implemented with two boundary conditions. As in the 2-D case, the analytical property of the amplitude functions near the magnetic axis provides one boundary condition, and in the fixed boundary case considered here, the shape of the boundary magnetic surface provides the other. These equations have been solved numerically using several methods [8,9,10]. For example, they may be solved using

existing ordinary differential equation solvers by first rewriting as a set of first order ordinary differential equations [10]. Alternatively, they may be solved using an iteration scheme inherent in the energy integral given in Eq. (5.7) by introducing an artificial time variable and converting the ordinary differential equations into partial differential equations [9].

In Fig. 5-1, the magnetic surface contours of an $\ell = 2$ stellarator equilibrium in four different ζ planes computed using the moment method are shown [10]. Also shown are contours of constant poloidal angle θ . The equilibrium parameters for this case are given in Table 5-1. The magnetic surfaces and the stream function Λ are taken to have the forms

$$\begin{aligned} R(\rho, \theta, \zeta) = & R_{00}(\rho) + R_{10}(\rho) \cos\theta + R_{11}^+(\rho) \cos(\theta + \zeta) \\ & + R_{11}^-(\rho) \cos(\theta - \zeta) + R_{01}(\rho) \cos\zeta \quad , \end{aligned} \quad (5.19)$$

$$\begin{aligned} Z(\rho, \theta, \zeta) = & -R_{10}(\rho) \sin\theta + Z_{11}^+(\rho) \sin(\theta + \zeta) \\ & + Z_{11}^-(\rho) \sin(\theta - \zeta) + Z_{01}(\rho) \sin\zeta \quad , \end{aligned} \quad (5.20)$$

$$\begin{aligned} \Lambda(\rho, \theta, \zeta) = & \Lambda_{10}(\rho) \sin\theta + \Lambda_{11}^+(\rho) \sin(\theta + \zeta) \\ & + \Lambda_{11}^-(\rho) \sin(\theta - \zeta) + \Lambda_{01}(\rho) \sin\zeta \quad . \end{aligned} \quad (5.22)$$

Note that the angle θ is defined implicitly by Eqs. (5.19) and (5.20), $\zeta = N_t \phi$, where $N_t = 2$, and $\rho = \phi'(\rho)$. The mass function $N_m(\rho)$ and the rotational transform per period $\mu(\rho) \equiv \chi'(\rho)/\phi'(\rho)$ are used as driving functions.

Fig. 5-1. Magnetic surface contours of a 3-D fixed boundary $\iota = 2$ stellarator equilibrium.

TABLE 5-1
EQUILIBRIUM PARAMETERS FOR A 3-D
FIXED BOUNDARY $\ell = 2$ STELLARATOR EQUILIBRIUM

$R_{00}(a)/a$	4.00
$R_{10}(a)/a$	-1.00
$R_{11}^+(a)/a$	0.05
$R_{11}^-(a)/a$	0.35
$R_{01}(a)/a$	0.00
$Z_{11}^+(a)/a$	-0.05
$Z_{11}^-(a)/a$	0.25
$Z_{01}(a)/a$	0.00
β_{t0}	0.03
$\mu(0)$	0.50
$\mu(a)$	0.53

Similar to the 2-D case, the amount of computational time required to find a 3-D equilibrium depends mainly on the number of amplitude functions used and the relative error desired. In this case, to achieve a relative error of less than 10^{-3} (with a residue less than 10^{-7}), about 280 sec of VAX-11/780 computer CPU time is required [10].

6. ADVANTAGES AND LIMITATIONS OF THE MOMENT METHOD

The main advantage of the variational moment method is that it allows efficient computation of MHD equilibria by using symmetries in the equilibrium geometry to decompose the unknown solution into rapidly converging series. Thus, an appropriate choice of basis functions for decomposition is essential to the accuracy and the efficiency of the algorithm. Secondly, a variational method is used to derive an optimum set of ordinary differential equations which can be used to accurately determine the expansion coefficients of the unknown solution. This allows a systematic and efficient treatment of the closure problem that arises when a finite series is used to approximate a solution. Finally, by reducing the original partial differential equations to a finite set of ordinary differential equations, existing well-established ordinary differential equation solvers and techniques can be used readily to find numerical solutions to these equations.

On the other hand, the moment algorithm will cease to be effective when the equilibrium geometry is sufficiently complicated so that the series approximation to the unknown solution does not converge or only converges slowly. Secondly, in all present formulations of the moment method, the magnetic surfaces are assumed to form nested and smooth tori around a single magnetic axis so that the equilibrium solution may be represented as Fourier series in angles based on these surfaces. The appearances of magnetic

islands and separatrixes are explicitly excluded from the formulations. In the 2-D axisymmetric case, this restriction on separatrixes may be relaxed by using polar coordinates, for example, rather than the flux coordinates as independent variables. The poloidal magnetic flux ψ then becomes the unknown dependent variable and may be expanded in Fourier series in the polar angle. This, in principle, will allow a treatment of the separatrixes, although the convergence of this series representation of ψ and the desirability of this scheme remain to be studied.

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