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**Lectures on the
Practical Solution of
Differential Equations**

Lawrence Dresner

OPERATED BY
UNION CARBIDE CORPORATION
FOR THE UNITED STATES
DEPARTMENT OF ENERGY

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LECTURES ON THE PRACTICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Lawrence Dresner

Date Published - November 1979

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ABSTRACT

This report comprises lectures on the practical solution of ordinary and partial differential equations given in the In-Hours Continuing Education Program for Scientific and Technical Personnel at Oak Ridge National Laboratory.

INTRODUCTION

Almost all physical laws are expressed mathematically as ordinary or partial differential equations. When we encounter differential equations, most of us are able to deal with them only if they can be readily integrated in terms of simple, well-known functions. Many ordinary and partial differential equations arising in technical practice do not have simple solutions, especially nonlinear equations. Such equations represent obstacles that frustrate progress in important technological problems.

There exists a substantial arsenal of methods for the approximate solution of differential equations. The information offered here will explain a group of methods that I have found very useful in my own work. These methods are characterized by relative computational simplicity, wide generality, and sufficient accuracy for most technical purposes. Mastery of these methods will enable the average worker to deal effectively with many of the ordinary and partial differential equations encountered in practice. Furthermore, such mastery will serve as an introduction to additional study by those interested in further improving their skills.

The explanatory material will be accompanied by detailed examples drawn from the fields of neutron transport, mass transfer, gas dynamics, and superconductivity. In all cases the emphasis will be on the attainment of practical results.

1. CURVE FITTING METHODS

One of the most common problems encountered in the attempt to solve a differential equation is that although we cannot find an "exact" solution, we nevertheless know what the solution "looks like." In this situation we may try curve fitting by choosing a trial function which has the right shape and contains some arbitrary parameters and determine the arbitrary parameters by some criterion of best fit. Below are five useful criteria and the rationale behind each.

Collocation

The trial function is required to satisfy the differential equation at several isolated points. As many points are selected as needed to determine the arbitrary constants. The selection of the points is arbitrary, and the resulting accuracy will be different for various selections.

Weighted Residual Methods

When the trial function is substituted in the differential equation, the right-hand side is not zero. The value of the right-hand side is called the residual ϵ . The arbitrary constants in the trial function are chosen to make the residual small. In one method, that of the least squared residual, the mean squared residual is made as small as possible; i.e., we minimize $\int_a^b \epsilon^2 dx$. In another method, Galerkin's method, we choose as the trial function a linear combination $C_1\phi_1 + C_2\phi_2 + \dots + C_k\phi_k$ of some mutually orthogonal functions ϕ_i . The coefficients C_i are chosen to make the residual ϵ orthogonal to $\phi_1, \phi_2, \dots, \phi_k$. The rationale of Galerkin's method is that while one despairs of solving the differential equation exactly (which would require the complete space of orthogonal functions $\phi_i, i = 1, 2, \dots, \infty$), one can solve it in the finite subspace defined by the basis $\phi_1, \phi_2, \dots, \phi_k$.

Both least squared residual and Galerkin's method are somewhat arbitrary. In computing the mean squared residual, an arbitrary weighting factor may be introduced. In Galerkin's method, an arbitrary weighting factor may be introduced into the definition of the inner product, changing the ϕ_i .

Variational Method

In the variational method, a functional is found which has an extremum (minimum, maximum, or inflection) for the solution of the differential equation. The trial function is inserted in the functional, and the functional is evaluated in terms of the arbitrary constants. The latter are then chosen to make the functional an extremum.

Because the variational method has more uses than just providing a criterion of best fit, a separate section is devoted to it (Sect. 3).

Integral Method

The rationale of the integral method is to seek to reduce the information content of the differential equation. The differential equation is too stringent a condition for the trial function to satisfy. The information in the condition can be reduced by integrating it over the interval of interest. It can also be multiplied by functions of the independent and dependent variables and then integrated. As many integral relations are created as there are arbitrary constants in the trial function. Then these constants are chosen so that the integral relations are satisfied when the trial function is substituted in them. This method, too, is arbitrary insofar as there is a free choice of the integral relations to use.

The integral method has particular advantage when one wishes to use piecewise continuous but nondifferentiable trial functions, e.g., step functions. An example of this will be given later.

Example 1

Begin with an example of Collatz's.¹

$$\frac{d^2y}{dx^2} + (1 + x^2)y + 1 = 0 . \quad (1a)$$

$$y(-1) = y(1) = 0 . \quad (1b)$$

This equation describes the variation with position x of the bending moment y of a strut of nonuniform flexural rigidity $[v(1 + x^2)^{-1}]$ under a uniformly distributed transverse load and an axial compression (see Fig. 1). The trial function chosen is

$$y = (a + bx^2)(1 - x^2) , \quad (2)$$

which represents the most general, even, fourth-order polynomial that vanishes at $x = \pm 1$.

Collocation. If Eq. (2) satisfies Eq. (1a) at $x = 0$ and $x = 1/2$, by substitution

$$a - 2b = 1 \quad (x = 0) \quad (3a)$$

$$68a + 49b = 64 \quad (x = 1/2) , \quad (3b)$$

from which it follows that

$$a = \frac{177}{185} , \quad b = \frac{-4}{185} . \quad (3c)$$

In Table I, column 3, the percentage error of trial function (2) is shown with the values of a and b . In columns 4 and 5 are percentage errors using two other pairs of points for collocation.

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Fig. 1. Nonuniform strut under a uniformly distributed transverse load and an axial compression.

Table I. Comparison of exact and approximate solutions of Eq. (1) by various curve fitting methods

		Collocation (%)			Least squared residual (%)		Galerkin (variational) (%)	Integral (%)	Collocation (%)	
		x = 0	x = 0	x = 1/2						
		x = 1/2	x = 1	x = 1	wt = 1	wt = $x^{-1/2}$			x = 0.1	x = 0.7
		a=0.956757	0.857143	1.015464	a=0.932718	a=0.912399	a=0.933443	a=0.929658	0.936110	a=0.922393
Exact		b=-0.0216216	-0.0714286	-0.103093	b=-0.0681805	b=-0.0608491	b=-0.0543274	b=-0.0638783	-0.0681571	b=-0.0410131
x	y									
0.0	0.932054	2.7	-8.7	8.9	0.071	-2.2	0.15	-0.26	0.44	-1.0
0.1	0.922394	2.7	-8.8	8.9	0.035	-2.2	0.13	-0.29	0.40	-1.1
0.2	0.893420	2.7	-8.9	8.7	-0.070	-2.3	0.067	-0.38	0.29	-1.1
0.3	0.845158	2.8	-9.2	8.3	-0.23	-2.4	-0.021	-0.52	0.13	-1.1
0.4	0.777693	3.0	-9.5	7.9	-0.44	-2.6	-12	-0.69	-0.067	-1.1
0.5	0.691223	3.2	-9.8	7.4	-0.65	-2.7	-0.19	-0.87	-0.28	-1.0
0.6	0.586136	3.6	-10.2	6.8	-0.84	-2.8	-0.21	-1.0	-0.47	-0.90
0.7	0.463112	4.2	-10.5	6.3	-0.97	-2.9	-0.14	-1.1	-0.59	-0.64
0.8	0.323231	5.0	-10.7	5.7	-0.99	-2.8	-0.090	-1.0	-0.60	-0.19
0.9	0.168104	6.2	-10.7	5.3	-0.83	-2.5	0.53	-0.78	-0.44	0.50
1.0	0.000000									

Least squared residual. Inserting Eq. (2) into Eq. (1a), the residual is

$$\epsilon = (1 - a + 2b) - 11bx^2 - ax^4 - bx^6 . \quad (4)$$

A tedious calculation shows that

$$\int_0^1 \epsilon^2 dx = \frac{68}{45} a^2 + \frac{63404}{4095} b^2 + \frac{7096}{1155} ab - \frac{12}{5} a - \frac{76}{21} b + 1 . \quad (5)$$

To minimize this expression, one can differentiate partially with respect to a and b

$$a: \frac{136}{45} a + \frac{7096}{1155} b = \frac{12}{5} \quad (6a)$$

$$b: \frac{7096}{1155} a + \frac{126808}{4095} b = \frac{76}{21} , \quad (6b)$$

from which it follows that

$$a = 0.932718 , \quad b = -0.0681806 . \quad (7)$$

The percentage error can be found in column 6 of Table I. It is less than 1%, and in the neighborhood of $x = 0$ it is less than 0.1%.

Lest I imply the unwarranted conclusion that the least squares method is better than collocation, I have repeated the computations using $x^{-1/2}$ as weight in the calculation of the mean squared residual. Then

$$2 \int_0^1 \epsilon^2 x^{-1/2} dx = 4 \left(\frac{196}{153} a^2 + \frac{480884}{49725} b^2 + \frac{7768}{4095} ab - \frac{20}{9} a - \frac{36}{65} b + 1 \right) , \quad (8)$$

which is a minimum when

$$a = 0.912399, \quad b = -0.0608491. \quad (9)$$

The percentage error is shown in column 7 of Table I.

Galerkin method. Basic functions here are even polynomials that vanish at ± 1 and are orthogonal on the interval $(-1, +1)$. Direct calculation shows the first two to be

$$P_2(x) = 1 - x^2 \quad (10a)$$

$$P_4(x) = 1 - 8x^2 + 7x^4. \quad (10b)$$

The quantities a and b are chosen so that the residual in Eq. (4) is orthogonal to P_2 and P_4 . This means we are solving the differential equation in the two-dimensional vector space spanned by P_2 and P_4 . Thus

$$\epsilon \perp P_2: \quad 228a + 52b = 210 \quad (11a)$$

$$\epsilon \perp P_4: \quad 1496a + 8696b = 924, \quad (11b)$$

from which it follows that $a = 0.933443$ and $b = -0.0543274$. The percentage error is shown in column 8 of Table I.

Variational method. Equations (1a) and (1b) represent the Euler equations for those functions which make the functional

$$H(y) = \frac{1}{2} \int_{-1}^{+1} \left[(1 + x^2)y^2 + 2y - \left(\frac{dy}{dx} \right)^2 \right] dx \quad (12)$$

a maximum:

$$\delta H = \int_{-1}^{+1} [(1 + x^2)y\delta y + \delta y - y'(\delta y)'] dx \quad (13a)$$

$$= \int_{-1}^{+1} [(1 + x^2)y + y'' + 1]\delta y dx - y'\delta y \Big|_{-1}^{+1} = 0 . \quad (13b)$$

Inserting trial function (2) into function (12) yields

$$-H = (2508a^2 + 1324b^2 + 1144ab - 4620a - 924b)/3465 . \quad (14)$$

Maximizing H by differentiating partially with respect to a and b results in

$$a: 5016a + 1144b = 4620 \quad (15a)$$

$$b: 1144a + 2648b = 924 . \quad (15b)$$

Then $a = 0.933443$ and $b = -0.0543274$, the same values as obtained in the Galerkin method. This is no coincidence.

Insofar as choosing the best member of a family of trial functions is concerned, the variational method has no advantage over Galerkin's method, and the search for a variational principle has little value. But if the functional H represents a quantity of interest, then the variational method allows it to be estimated with far better accuracy than that of the trial function. If y^* is the exact solution of Eqs. (1a) and (1b), and if $y = y^* + \delta y$, then $H(y) = H(y^* + \delta y) = H(y^*) + 0 (\overline{\delta y}^2)$ because the first-order term vanishes as shown in Eqs. (13a) and (13b). Equation (14) shows $H(y) = 0.615052$ for the best values of a and b given in Eqs. (15a) and (15b). If one sets $y = y^*$ in Eq. (12), integrates the last term by parts, and uses Eqs. (1a) and (1b), one finds that

$$H(y^*) = \frac{1}{2} \int_{-1}^{+1} y^* dx = \int_0^1 y^* dx = \langle y^* \rangle . \quad (16)$$

Calculation by Simpson's rule using the exact entries in Table I gives $H(y^*) = 0.615099$. The error in using the estimate $H(y)$ for $H(y^*)$ is thus 7.7×10^{-5} compared with an error $\sim 1-2 \times 10^{-3}$ in the estimate of y^* itself. Hence, had one been interested in the average value of y , the variational principle would have supplied it with far better accuracy than it did the individual values of y themselves.

Integral method. If Eq. (1a) is integrated from 0 to 1,

$$y'(1) + \int_0^1 (1 + x^2)y dx + 1 = 0 . \quad (17)$$

If one multiplies by x and integrates, one finds after a partial integration that

$$y'(1) + y(0) + \int_0^1 x(1 + x^2)y dx + \frac{1}{2} = 0 . \quad (18)$$

If one multiplies by y and integrates, one finds after partial integration that

$$\int_0^1 [(1 + x^2)y^2 - (y')^2 + y] dx = 0 . \quad (19)$$

If one substitutes trial function (2) into trial functions (17) and (18), one obtains

$$126a + 190b = 105 \quad (20a)$$

$$16a + 45b = 12 , \quad (20b)$$

from which it follows that $a = 489/526 = 0.929658$ and $b = -84/1315 = -0.0638783$. The percentage error is shown in column 9 of Table I.

If one substitutes trial function (2) into trial function (19), one gets

$$2508a^2 + 1324b^2 + 1144ab - 2310a - 462b = 0 . \quad (21)$$

From Eqs. (21) and (20a) [i.e., from relations (17) and (19)] we get $a = 0.936110$ and $b = -0.0681571$. The corresponding percentage errors are shown in column 10 of Table I.

Table I seems to imply that collocation is less accurate than the other methods. However, the accuracy depends strongly on the choice of points at which the differential equation is to be satisfied. If one chooses $x = 0.1$ and 0.7 , one finds that the residual ϵ given in Eq. (4) vanishes for $a = 0.922393$ and $b = -0.0410131$. The percentage errors are shown in column 11 of Table I. These errors are much smaller than those in columns 3, 4, and 5.

The situation is summed up in a review by Finlayson and Scriven² where they express their agreement with the following words of Crandall: "The variation between results obtained by applying different criteria to the same trial family . . . is much less significant than the variations that can result from the choice of different trial families." The choice of criterion is usually made on the basis of the difficulty of the calculations it involves. Generally speaking, the choice of trial function and criterion are strongly related, as the next example shows.

Example 2

The second example is the strongly nonlinear equation

$$(m + 1) \frac{d^2(y^m)}{dx^2} + mx \frac{dy}{dx} - y = 0 \quad (22)$$

where m is a number ~ 30 . Equation (22) arises in the calculation of the current distribution among the filaments of a multifilamentary superconductor. On the basis of the critical state model (which is correct in the limit $m \rightarrow \infty$) the solution should look like the curve in Fig. 2. Furthermore, the boundary conditions that determine the desired solution of Eq. (22) turn out to be

$$\left. \frac{d(y^m)}{dx} \right|_{x=x_0} = 0 \quad (23a)$$

$$\int_0^{x_0} y dx = \text{a prescribed constant} \quad (23b)$$

where x_0 is the first root of y .

This is indeed an unusual differential equation with unusual boundary conditions. The bit of knowledge that the solution is almost rectangular is crucial. The first impulse is to approximate the solution with a rectangle as shown in Fig. 3. But for such a trial function, $dy/dx = d^2(y^m)/dx^2 = 0$ except at $x = x_0$, where both derivatives are infinite. This creates insurmountable difficulties for the methods of collocation and weighted residuals, but not for the integral method, which seems uniquely suited to the system in Eqs. (22) and (23).

If Eq. (22) is multiplied by x and integrated from 0 to x_0 , integrating the first and second terms by parts, then

$$(m+1)y^m(0) = (2m+1) \int_0^{x_0} xy dx. \quad (24)$$

Inserting the rectangular trial function results in

$$x_0 = \left(\frac{2m+2}{2m+1} \right)^{1/2} y^{(m-1)/2}(0), \quad (25a)$$

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Fig. 2. Sketch of the shape of the solution of Eq. (22) being sought.

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Fig. 3. Rectangular approximation to the solution sketched in Fig. 2.

from which it follows that

$$\int_0^{x_0} y dx = \left(\frac{2m+2}{2m+1} \right)^{1/2} y^{(m+1)/2} (0) . \quad (25b)$$

How do Eqs. (25a) and (25b) compare with exact solutions? It is enough to compare for the special case $y(0) = 1$. For, the differential equation (22) is invariant to the group of transformations

$$\left. \begin{aligned} y &= \lambda y^* \\ x &= \lambda^{(m-1)/2} x^* \end{aligned} \right\} 0 > \lambda < \infty . \quad (26a)$$

$$(26b)$$

In other words, if Eqs. (26a) and (26b) are substituted into Eq. (22), Eq. (22) again becomes the differential equation for y^* in terms of x^* . So too is Eq. (23a) invariant. Finally,

$$\int_0^{x_0} y dx = \lambda^{(m+1)/2} \int_0^{x_0^*} y^* dx^* . \quad (27)$$

If $y^*(0) = 1$, then by elimination of λ from Eqs. (26) and (27) one gets

$$x_0 = y^{(m-1)/2}(0) \cdot x_0^* \quad (28a)$$

$$\int_0^{x_0} y dx = y^{(m+1)/2}(0) \cdot \int_0^{x_0^*} y^* dx^* . \quad (28b)$$

Direct numerical integration shows that $x_0^* = (m+1)/m$ and $\int_0^{x_0^*} y^* dx^* = 1$. When $m = 30$, $x_0^* = 1.0333$ and $\int_0^{x_0^*} y^* dx^* = 1$, compared with the approximate value 1.0082 for both from Eqs. (25a) and (25b).

The estimates given in Eq. (25) can be improved as follows. If one integrates Eq. (22) from 0 to x_0 , integrating the second term by parts, one gets

$$m y^{m-1}(0) \left(\frac{dy}{dx} \right)_{x=0} = - \int_0^{x_0} y dx . \quad (29)$$

$$\left(\frac{dy^*}{dx^*} \right)_{x^*=0} = - \frac{\int_0^{x_0^*} y^* dx^*}{m} . \quad (30)$$

While small, $(dy^*/dx^*)_{x^*=0}$ is not zero, which means the rectangular trial function would be better replaced by a trapezoid, as in Fig. 4. The approximate slope is then

$$s \equiv \left(\frac{dy^*}{dx^*} \right)_{x^*=0} = - \frac{1}{m} \left(\frac{2m+2}{2m+1} \right)^{1/2} . \quad (31)$$

Inserting the trapezoidal trial function in Eq. (24) yields

$$x_0^* = \left(\frac{2m+2}{2m+1} \right)^{1/2} \left(1 - \frac{2}{3m} \cdot \frac{2m+2}{2m+1} \right)^{-1/2} \quad (32a)$$

and

$$\int_0^{x_0^*} y^* dx^* = x_0^* \left[1 - \frac{1}{2m} \left(\frac{2m+2}{2m+1} \right)^{1/2} x_0^* \right] . \quad (32b)$$

For $m = 30$, $x_0^* = 1.0197$ and $\int_0^{x_0^*} y^* dx^* = 1.0023$, a slight improvement in accuracy.

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Fig. 4. Trapezoidal approximation to the solution sketched in Fig. 2. (The slope of the nearly horizontal line is exaggerated.)

Example 3: Van der Pol's equation

Van der Pol's equation is

$$\frac{d^2x}{dt^2} + \epsilon(x^2 - 1) \frac{dx}{dt} + x = 0 , \quad (33)$$

which is considered here only for values of $\epsilon \ll 1$. Interpreting the terms on the left-hand side as an acceleration, a damping force, and a restoring force, respectively, yields some qualitative inkling of the behavior of the solutions. The restoring force is always directed oppositely to the acceleration; hence in the absence of damping ($\epsilon = 0$), Eq. (33) describes simple harmonic motion (e.g., that of an oscillator or pendulum). When $\epsilon \neq 0$, are there periodic solutions to Eq. (33)? When $\epsilon = 0$, one can multiply Eq. (33) by $\dot{x} \equiv dx/dt$ and find

$$\frac{d}{dt} \left(\frac{\dot{x}^2}{2} + \frac{x^2}{2} \right) = 0 , \quad (34)$$

which indicates that $\dot{x}^2 + x^2$ is constant for any orbit of the oscillator. This is nothing more than the law of conservation of energy. The same thing performed on Eq. (33) yields

$$\frac{d}{dt} \left(\frac{\dot{x}^2}{2} + \frac{x^2}{2} \right) + \epsilon(x^2 - 1)\dot{x}^2 = 0 . \quad (35)$$

If there is a periodic solution, and it is integrated with respect to t over a full period (for which $\dot{x}^2/2 + x^2/2$ returns to its initial value), the results are

$$\oint (x^2 - 1) \dot{x} dx = 0 . \quad (36)$$

Equation (36) can form the basis of an approach by the integral method to the solution of Eq. (33). The simplest trial function is $x = a \cos t$, which solves Eq. (33) for all a when $\epsilon = 0$. Then Eq. (36) becomes

$$\int_0^{2\pi} (a^2 \cos^2 t - 1) \sin^2 t dt = 0 , \quad (37)$$

from which it follows that

$$a^2 - 4 = 0 \quad \text{or} \quad a = 2 . \quad (38)$$

A convenient way to compare this periodic solution with the exact periodic solution is to plot it in the plane whose coordinates are x and $\dot{x}$. This plane is called the phase plane and is of great value in the study of equations such as Eq. (33). Because t does not appear in the differential equation explicitly, one can set $y = dx/dt$ and write Eq. (33) as

$$\frac{dy}{dt} = -\epsilon(x^2 - 1)y - x , \quad (39)$$

so that by dividing by $y = dx/dt$

$$\frac{dy}{dx} = -\epsilon(x^2 - 1) - \frac{x}{y} . \quad (40a)$$

Now t has been completely eliminated from the differential equation. Equation (40a) can be written as

$$ydy + xdx = -\epsilon(x^2 - 1)ydx . \quad (40b)$$

Integrating around a closed curve in the x, y plane, we get Eq. (36) again, which leads to the solution

$$\left. \begin{array}{l} x = 2 \cos t \\ y = 2 \sin t \end{array} \right\} \text{ therefore } x^2 + y^2 = 4 , \quad (41)$$

i.e., a circle of radius 2 in the x,y plane. Shown in Fig. 5 is a circle of radius 2 as well as some exact points for $\epsilon = 0.10, 0.25$, and 1.00.

The periodic solution which has been approximated is a "limit cycle," that is to say, all trajectories starting inside it spiral out to it, and all trajectories starting outside it spiral in to it.

Although there is voluminous literature about the Van der Pol equation, it will not be discussed any further here.

Example 4: Blasius's equation

Blasius's equation arises in the study of boundary layer development along a semi-infinite flat plate. (See Fig. 6.)

$$f''' + ff'' = 0 \quad (42a)$$

$$f(0) = 0 , \quad f'(0) = 0 , \quad f'(\infty) = 2 . \quad (42b)$$

The term f is related to the stream function of the boundary layer flow. In order to calculate the frictional drag over a length x of the plate, the derivative $(\partial V_x / \partial y)_{y=0}$ is needed, which is directly related to $f''(0)$. Hence our main goal is to estimate $f''(0)$.

From the boundary conditions in Eq. (42b) and also from its interpretation as a stream function, it is apparent that $f(x)$ must look like Fig. 7. It proves convenient to introduce the function

$$g = f - 2x + a , \quad (43)$$

which resembles Fig. 8 and satisfies the equations

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Fig. 5. Exact and approximate solutions to Van der Pol's equation.

Fig. 6. Sketch of the boundary layer developing along a semi-infinite plate.

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Fig. 7. Sketch of the solution of Eqs. (42a) and (42b).

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Fig. 8. Sketch of the function $g(x)$ defined in Eq. (43).

$$g''' + g'' (g + 2x - a) = 0 \quad (44a)$$

$$g(0) = a, \quad g'(0) = -2, \quad g'(\infty) = 0 \quad (44b)$$

and which also vanishes at $x = \infty$.

An unusual and troublesome feature of this problem is that we are trying to find the second derivative of the solution of the differential equation. It is possible to find a very good fit to a given function by an approximating function, and yet get a very bad estimate of the higher derivatives by differentiating the approximating function. While the reader may be prepared to concede this point, I take the liberty of further illustrating its seriousness.

Shown in Fig. 9 is a plot of the exact values of $g(x)$ calculated numerically. Shown also is a curve of the function

$$g(x) = a \exp \left(-\frac{2x}{a} - bx^2 \right) \quad (45)$$

plotted for the values $a = 1.7193$ and $b = 0.375$, for which $g'(0) = -2$ and $g(\infty) = 0$. The value $g(0) = 1.7193$ agrees with the numerical solution to five figures, and Eq. (45) agrees with the numerical results to about 1% or better over the whole range from 0 to 1. Evidently, Eq. (45) fits the solution to Eq. (44a) very well indeed. Direct calculation from Eq. (45) shows that $f''(0) = g''(0) = 4/a - 2ab = 1.037$, whereas the exact value of $f''(0) = g''(0)$ is 1.3282.

This problem can be overcome by an application of the integral method. If Eq. (44a) is integrated over x from 0 to ∞ , integrating the second term by parts gives

$$g''(0) = 2g(0) - \int_0^\infty (g')^2 dx. \quad (46)$$

The value of $g''(0)$, which Eq. (46) provides if a close fit to g is known, should be much more accurate than differentiating the close fit.

Fig. 9. Exact and approximate values of $g(x)$.

Evaluation of the integral in Eq. (46) by Simpson's rule gives $\int_0^{\infty} (g')^2 dx = 2.1128$ for the approximation (45) with $a = 1.7193$ and $b = 0.375$. Thus $g''(0) = 2 \times 1.7193 - 2.1128 = 1.3258$, which is within 0.2% of the exact result, confirming the expectation that Eq. (46) is the path to accurate estimates of $g''(0)$ from approximations to $g(x)$.

Next, I show how one might have proceeded in the absence of knowledge of the numerical solution. Using Eq. (42a) and the equations derived from it through repeated differentiation to calculate the derivatives of f at $x = 0$, one finds that the next nonvanishing derivative after the second is the fifth, and the next after that is the eighth. Thus for small x ,

$$g = a - 2x + bx^2 . \quad (47)$$

Equation (47) cannot be used directly as a trial function because it eventually becomes negative owing to elimination of the higher-order terms. Using the exponential of the right-hand side to avoid this problem, one gets a trial function of the form Eq. (45).

Determining the constants a and b requires a combination of the collocation and the integral method. The procedure outlined below was chosen to minimize the computational labor. At $x = 0$, Eq. (45) gives

$$g(0) = a ; \quad g'(0) = -2 \quad (48a)$$

$$g''(0) = \frac{4}{a} - 2ab \quad (48b)$$

$$g'''(0) = 12b - \frac{8}{a^2} . \quad (48c)$$

If Eq. (45) is to satisfy Eq. (44a) at $x = 0$, $g'''(0)$ must equal 0. Thus

$$b = \frac{2}{3a^2} . \quad (49a)$$

Equation (45) then becomes

$$g(x) = a \exp \left[-2\left(\frac{x}{a}\right) - \frac{2}{3} \left(\frac{x}{a}\right)^2 \right] . \quad (49b)$$

The constant a is determined as follows. Multiplying Eq. (44a) by $g + 2x - a$ and integrating twice by parts yields

$$\int_0^{\infty} g'(g' + 2)(g' + 2x - a)dx = -1 , \quad (50)$$

Substituting Eq. (49b) into Eq. (50) results in

$$a^2 = - \left[\int_0^{\infty} h'(h' + 2)(h + 2z - 1)dz \right]^{-1} \quad (51a)$$

where

$$h(z) = \exp \left[-2z \left(1 + \frac{z}{3} \right) \right] . \quad (51b)$$

Evaluation of Eq. (51a) with Simpson's rule gives $a = 1.4678$. Use of Eq. (49b) with this value of a in Eq. (46) gives $g''(0) = 1.2533$ after another numerical integration, within 6% of the exact result. The trial function Eq. (49b) is shown in Fig. 9.

Summary and Conclusions

The examples used to illustrate the curve fitting methods (especially examples 2, 3, and 4) were chosen to convince the reader that an imaginative and flexible approach can often clear away formidable, even apparently insuperable difficulties. The kind of analysis presented here often provides a sufficient answer in a practical sense. Even if it does not, it is a valuable prelude to a more detailed analysis; the results may warn us of errors and also suggest useful approximations.

2. GROUP THEORETIC METHODS

In the study of Blasius's equation, I presented some numbers for comparison with the approximate solution. These numbers were obtained by a numerical integration of the differential equation. Examination of Eqs. (42a) and (42b) indicates lack of one boundary condition at $x = 0$, e.g., $f''(0)$, so that the integration cannot begin there. If one guesses a value of $f''(0)$ and then integrates Eq. (42a) starting at $x = 0$, there is no guarantee that the result will be $f'(\infty) = 2$. It appears necessary to undertake a tedious process of trial and error to find the solution satisfying the two-point boundary conditions in Eq. (42b).

In general this would be true, but in the case of Blasius's equation trial and error can be avoided by exploiting the invariance of Eq. (42a) under the transformation group

$$\left. \begin{array}{l} f = \lambda f_1 \\ x = x_1/\lambda \end{array} \right\} 0 < \lambda < \infty. \quad (52)$$

This means that if $f_1(x_1)$ is a solution of Eq. (42a), then so is the new function $f(x)$ obtained from $f_1(x_1)$ by replacing the value f_1 of the function with $\lambda f_1 = f$ and the argument x_1 at which this value occurs with $x_1/\lambda = x$. (See Fig. 10.)

If the derivative of f_1 at x_1 is df_1/dx_1 , the derivative of f at x is $\lambda^2(df_1/dx_1)$. For if f_1 differs by df_1 at two points that are dx_1 apart, then f will differ by $df = \lambda df_1$ at two corresponding points that are $dx = dx_1/\lambda$ apart. Similarly, $d^2f/dx^2 = \lambda^3(d^2f_1/dx_1^2)$ and $d^3f/dx^3 = \lambda^4(d^3f_1/dx_1^3)$.

If one chooses $f_1(0) = 0$, $f_1'(0) = 0$, and $f_1''(0) = A$ and finds $f_1'(\infty) = B$ by numerical integration, then $f(0) = 0$, $f'(0) = 0$, $f''(0) = \lambda^3 A$, and $f'(\infty) = \lambda^2 B$. A numerical integration started with $A = 1$ gives $B = 1.6554$. Thus, $\lambda = 1.0992$ and $f''(0) = \lambda^3 = 1.3280$ (1.3282 exact).

To sum up then, I have reduced a two-point boundary value problem to a one-point initial value problem by making use of the fact that all the pertinent integral curves are members of a self-similar family. By "self-similar" I mean that any member of the family can be obtained from

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Fig. 10. Illustration of the relation between a curve $f_1(x_1)$ and its image $f(x)$ under the group of Eq. (52).

any other by changes of scale of the abscissa and ordinate given by Eq. (52). The use of such self-similarity (or group invariance, as it is also called) in analyzing and solving differential equations was first suggested by the Norwegian mathematician Sophus Lie about 100 years ago. This section is devoted to explaining what can be learned about both ordinary and partial differential equations from their group invariance.

Lie's theory concerns invariance of differential equations to any group of transformations. However, in practice, the most commonly occurring kind of group by far is the stretching or affine group

$$\left. \begin{aligned} x' &= \lambda^\alpha x \\ y' &= \lambda^\beta y \end{aligned} \right\} \quad 0 < \lambda < \infty . \quad (53)$$

(Notice that no generality is lost if $\alpha = 1$.) This section is restricted to groups of the kind in Eq. (53) in order to simplify the proofs to their barest essentials. In any case, Lie's theory of differential equations is not terribly involved, and the reader can find a lucid, general account of it in Cohen's book.³

First-Order Ordinary Differential Equations: Integrating Factors

Often in technical practice differential equations of the following form are encountered:

$$M(x,y)dx + N(x,y)dy = 0 . \quad (54)$$

The solution of such an equation is a one-parameter family of curves

$$\phi(x,y) = C . \quad (55)$$

In order to find the function $\phi(x,y)$, one must integrate Eq. (54). However, Eq. (54) is not in general a perfect differential, though it can always be made one by multiplying by some suitable function $F(x,y)$ of x and y . For, according to Eq. (55)

$$\phi_x(x,y)dx + \phi_y(x,y)dy = 0 \quad (56)$$

so that

$$\frac{\phi_x}{M} = \frac{\phi_y}{N} \equiv F(x,y) . \quad (57)$$

The last equality *defines* the function F . Then

$$\begin{aligned} \phi_x &= F \cdot M \\ \phi_y &= F \cdot N \end{aligned} \quad (58)$$

so that multiplication of Eq. (54) by F gives Eq. (56), a perfect differential. Therefore, F is an integrating factor.

The value of F can be found from the invariance of the differential equation (54) to Eq. (53) as follows. The invariance of the differential equation to Eq. (53) means that Eq. (53) carries each member of the family of Eq. (55) into some other member according to the value of λ . That is to say, when the point x,y traces out the curve $\phi(x,y) = C$, its image x',y' traces out another curve of the family $\phi(x',y') = C(\lambda)$. Thus (taking $\alpha = 1$)

$$\phi(\lambda x, \lambda^\beta y) = C(\lambda) . \quad (59)$$

If Eq. (59) is differentiated with respect to λ , and λ set equal to 1, then

$$x\phi_x + \beta y\phi_y = (dC/d\lambda)_{\lambda=1} = \text{a constant} . \quad (60)$$

Substituting Eq. (58) into Eq. (60) gives

$$F(xM + \beta yN) = \text{a constant} \quad (61a)$$

so that

$$F = \text{constant}/(xM + \beta yN) \quad (61b)$$

is an integrating factor for Eq. (54).

Example 1

The following differential equation arose in the computation of the velocity of propagation of normal zones in a superconductor cooled by boiling helium

$$aydy + (x - by)dx = 0 \quad \text{where } a \text{ and } b \text{ are constants .} \quad (62)$$

It is obvious that it is invariant to Eq. (53) with $\alpha = \beta = 1$. Thus an integrating factor is

$$F(x,y) = [ay^2 + x(x - by)]^{-1} . \quad (63)$$

Therefore,

$$\frac{\partial \phi}{\partial y} = \frac{ay}{ay^2 + x(x - by)} , \quad (64)$$

from which it follows that⁴

$$\phi = \frac{1}{2} \ln[ay^2 + x(x - by)] + \frac{b}{\sqrt{4a^2 - b^2}} \arctan \left(\frac{2ay/x - b}{\sqrt{4a^2 - b^2}} \right) + \text{an arbitrary function of } x . \quad (65)$$

Differentiation of Eq. (65) with respect to x shows that

$$\frac{\partial \phi}{\partial x} = \frac{x - by}{ay^2 + x(x - by)} \quad (66)$$

only if the arbitrary function of x is 0. Hence, Eq. (65) with no arbitrary function of x gives the solution of Eq. (62).

Example 2: Kamke⁵ #1.126

$$\left. \begin{array}{l} xy' = yf(xy) \\ yf(xy)dx - xdy = 0 \end{array} \right\} \quad f = \text{an arbitrary function of one variable .} \quad (67)$$

Equation (67) is invariant to Eq. (53) with $\alpha = 1$ and $\beta = -1$. An integrating factor is then $[xyf(xy) + xy]^{-1}$. Then

$$\frac{\partial \phi}{\partial x} = \frac{yf(xy)}{xy[f(xy) + 1]} \quad (68)$$

or

$$\phi = \int_0^{xy} \frac{f(u)du}{u[f(u) + 1]} + H(y) \quad (69)$$

where H is, as yet, an undetermined function of y .
Then

$$\frac{\partial \phi}{\partial y} = \frac{xf(xy)}{xy[f(xy) + 1]} + H'(y) \quad (70)$$

Comparing Eq. (70) with Eq. (67) shows that

$$H'(y) = -\frac{1}{y} \quad (71)$$

so that

$$H(y) = -\ln y \quad (72)$$

Thus the general solution of Eq. (67) becomes

$$\ln y = \int_C^{xy} \frac{f(u)du}{u[1 + f(u)]} \quad (73)$$

where C is an arbitrary constant.

Remark: Formula (61) for the integrating factor can easily be extended to groups other than affine groups. If Eq. (53) is replaced by

$$\begin{aligned} x' &= X(x, y; \lambda) \\ y' &= Y(x, y; \lambda) \end{aligned} \quad \text{with } X = x \text{ and } Y = y \text{ when } \lambda = 1, \quad (74)$$

then Eq. (59) becomes

$$\phi[X(x,y;\lambda), Y(x,y;\lambda)] = C(\lambda) \quad (75)$$

and Eq. (60) becomes

$$\xi\phi_x + \eta\phi_y = \left(\frac{dC}{d\lambda}\right)_{\lambda=1} \quad (76)$$

where

$$\left. \begin{aligned} \xi(x,y) &\equiv \left(\frac{\partial X}{\partial \lambda}\right)_{\lambda=1} \\ \eta(x,y) &\equiv \left(\frac{\partial Y}{\partial \lambda}\right)_{\lambda=1} \end{aligned} \right\} . \quad (77)$$

Then Eq. (61b) becomes

$$F = \text{constant}/(\xi M + \eta N) . \quad (78)$$

Example 3: Kamke⁵ #1.344

$$\left. \begin{aligned} (\ln y + 2x - 1)y' &= 2y \\ 2ydx - (\ln y + 2x - 1)dy &= 0 \end{aligned} \right\} . \quad (79)$$

A group which leaves Eq. (79) invariant is

$$\left. \begin{aligned} y' &= \lambda y \\ x' &= x - \frac{1}{2} \ln \lambda \end{aligned} \right\} \quad \begin{aligned} y' &= y \\ x' &= x \end{aligned} \quad \text{when } \lambda = 1 . \quad (80)$$

Thus, $\xi = -1/2$, $\eta = y$, and an integrating factor is

$$-(\xi M + \eta N)^{-1} = (y \ln y + 2xy)^{-1} . \quad (81)$$

Then

$$\frac{\partial \phi}{\partial x} = \frac{2}{2x + \ln y} \quad (82)$$

so that

$$\phi = \ln(2x + \ln y) + G(y) . \quad (83)$$

Then

$$\frac{\partial \phi}{\partial y} = \frac{1}{2xy + y \ln y} + G'(y) . \quad (84)$$

From Eq. (79) it follows that $G'(y) = -1/y$ or $G(y) = -\ln y$.

Then, from Eq. (83) one finds that

$$2x + \ln y = Cy \quad (C = e^\phi) . \quad (85)$$

Substitution of Eq. (85) into Eq. (79) shows that Eq. (85) solves Eq. (79) for $C \leq 0$ as well as for $C > 0$.

Separation of Variables

An alternative to introducing an integrating factor in first-order equations is to separate variables. Here, too, group invariance can help.

In general, to separate variables one must introduce "canonical" coordinates, i.e., transform to coordinates in which the group becomes a translation group. But in the case of an affine (stretching) group, it is sufficient to replace the dependent variable by a *group invariant*, $u = y/x^\beta$. For if $dy/dx = f(x, y)$,

$$\frac{du}{dx} = \frac{f}{x^\beta} - \beta \frac{y}{x^{\beta+1}} = \frac{1}{x} \left(\frac{f}{x^{\beta-1}} - \beta u \right) .$$

Because

$$f(\lambda x, \lambda^\beta y) = \lambda^{\beta-1} f(x, y) ,$$

$$x \frac{\partial f}{\partial x} + \beta y \frac{\partial f}{\partial y} = (\beta - 1) f .$$

The characteristic equation of this partial differential equation is

$$\frac{dx}{x} = \frac{dy}{\beta y} = \frac{df}{(\beta - 1)f} ,$$

so that

$$\frac{f}{x^{\beta-1}} = G\left(\frac{y}{x^\beta}\right) = G(u) .$$

Therefore,

$$\frac{dx}{x} = \frac{du}{G(u) - \beta u} .$$

Separatrices

Figure 11 is a plot of individual solutions of the family of Eq. (85). It appears that there are two kinds of solutions — those for which x has a minimum ($C > 0$) and those for which x does not ($C < 0$). When x has a minimum, $dx/dy = 0$. From Eq. (85) it then follows that the locus of the minima is given by

$$\left. \begin{aligned} y &= C^{-1} \\ x &= \frac{1}{2}(1 + \ln C) \end{aligned} \right\} \text{ or } y = e^{1-2x} . \quad (86)$$

This curve is shown dashed in Fig. 11. From the parametric representation of the locus of the minima, it would appear that minima can only occur if $C > 0$; otherwise the equations for x and y give imaginary values. Thus the curve for $C = 0$, namely $y = e^{-2x}$ (shown heavy in Fig. 11), separates the two kinds of solutions, i.e., those with a minimum value of x and those which have no minimum.

A curve, such as the one for which $C = 0$, that separates two different families of solutions is called a separatrix. Separatrices have some interesting properties — one of the most important is that they may be determined from the group invariance of the differential equation

Fig. 11. The family of integral curves given by Eq. (85).

purely algebraically, i.e., without solving (integrating) the differential equation.

To see this, first note that the separatrix is invariant to the transformation group; that is, it is the locus of the images of its own points. For invariance of the differential equation to a transformation group means that a plot of the integral curves, such as Fig. 11, will be the same after transformation as it was before. Individual curves may be mapped onto others, but the family of images and the family of original curves must be the same. This means that the separatrix must be in the same place after mapping as before, i.e., it must be invariant to the mapping. This is easy to verify in the case of the integral curves in Eq. (85). Under the transformation of Eq. (80), the curve $2x' + \ln y' = Cy'$ maps into the curve $2x + \ln y = (\lambda C)y$. Thus, curves with positive C transform among themselves, curves with negative C transform among themselves, and the curve for which $C = 0$ is its own image.

If a curve L is its own image under the general transformation in Eq. (74), then for small $d\lambda = \lambda - 1$, the image of the point (x, y) , the point $(x', y') = (x + \xi d\lambda, y + \eta d\lambda)$, must also be on L . (See Fig. 12.) If $d\lambda$ is small enough, (x', y') is close enough to (x, y) to approximate

$$\frac{dy}{dx} = \frac{y' - y}{x' - x} = \frac{\eta}{\xi}. \quad (87)$$

Solutions of Eq. (87) are curves that are invariant to the transformation group in Eq. (74).

Because the separatrix lies infinitely close to solutions of the differential equation, it, too, must be a solution. Thus it may be determined by substituting Eq. (87) for dy/dx into the differential equation. The result is an algebraic equation for the separatrix.

In Example 3, $\eta/\xi = dy/dx = -2y$. Substituting this into Eq. (79), $2y(\ln y + 2x) = 0$ from which either $y = 0$ or $y = e^{-2x}$. The latter curve is the separatrix; the former is an invariant curve of Eq. (80) that lies close to solutions of Eq. (79) at any x for a sufficiently large negative C .

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Fig. 12. Sketch to aid in the calculation of invariant curves.

Separatrices have practical value when they are the solution of the differential equation that one is looking for, a circumstance not at all rare in technical practice. The procedure outlined above may also be used to find singular solutions when they exist.

Second-Order Ordinary Differential Equations

Consider the second-order differential equation

$$y'' = f(x, y, y') \quad (88)$$

that is invariant to the affine (stretching) group of Eq. (53) with $\alpha = 1$.

Invariance means that

$$f(\lambda x, \lambda^\beta y, \lambda^{\beta-1} y') = \lambda^{\beta-2} f(x, y, y') \quad (89)$$

for any x, y and y' . Lie has shown how to find new variables $u(x, y)$ and $v(x, y, y')$ such that in terms of u and v , Eq. (88) becomes an equation of first order.

Lie suggests that u and v must be invariant to the group in Eq. (53), i.e.,

$$\left. \begin{aligned} u(x, y) &= u(\lambda x, \lambda^\beta y) \\ v(x, y, y') &= v(\lambda x, \lambda^\beta y, \lambda^{\beta-1} y') \end{aligned} \right\} . \quad (90)$$

A satisfactory choice for the group in Eq. (53) is

$$\left. \begin{aligned} u &= \frac{y}{x^\beta} \\ v &= \frac{y'}{x^{\beta-1}} \end{aligned} \right\} . \quad (91)$$

From Eq. (91) it follows that

$$\frac{dv}{dx} = \frac{y''}{x^{\beta-1}} - (\beta - 1) \frac{y'}{x^\beta} = \frac{f(x,y,y')}{x^{\beta-1}} - (\beta - 1) \frac{y'}{x^\beta} \quad (92a)$$

$$\frac{du}{dx} = \frac{y'}{x^\beta} - \beta \frac{y}{x^{\beta+1}}. \quad (92b)$$

By dividing, we get

$$\frac{dv}{du} = \frac{f(x,y,y')/x^{\beta-2} - (\beta - 1)v}{v - \beta u}. \quad (93)$$

Now it turns out that $f(x,y,y')/x^{\beta-2}$ is a function of u and v alone. For if Eq. (89) is differentiated with respect to λ and λ set equal to 1,

$$x \frac{\partial f}{\partial x} + \beta y \frac{\partial f}{\partial y} + (\beta - 1)y' \frac{\partial f}{\partial y'} = (\beta - 2)f. \quad (94)$$

The characteristic equation of this first-order linear partial differential equation is

$$\frac{dx}{x} = \frac{dy}{\beta y} = \frac{dy'}{(\beta - 1)y'} = \frac{df}{(\beta - 2)f}. \quad (95)$$

The general solution of Eq. (94) is obtained by equating one integral of Eq. (95) to an arbitrary function of two others. Three integrals are $y/x^\beta = u$, $y'/x^{\beta-1} = v$, and $f/x^{\beta-2}$. Thus, Eq. (89), through Eqs. (94) and (95), means that $f/x^{\beta-2} = G(u,v)$, some function of u and v . Equation (88) has been reduced to Eq. (93), a first-order differential equation in u and v only, thus lowering the order of the differential equation by one. While this proof holds only for affine groups, Lie's more general proof shows the theorem to be true for all groups.

This accomplishment is tempered somewhat by noting that there are still two integrations to perform to get y as a function of x : one to get v as a function of u and a second to solve the resulting first-order differential equation for y' (because v is a function of x, y and y').

Example 1: Emden-Fowler equation of order 5

The Emden-Fowler equation

$$\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + y^n = 0 \quad (96)$$

arises in the study of the equilibrium mass distribution in a gas cloud with the adiabatic exponent $(n + 1)/n$ held together by gravitational forces. The desired solutions are regular (nonsingular) and have a regular derivative at the origin (because y is the gravitational potential). Equation (96) is invariant to the affine group in Eq. (53) with $\alpha = 1$ and $\beta = 2/(1 - n) = -1/2$ when $n = 5$. Then $u = x^{1/2}y$ and $v = x^{3/2}y'$ are invariants. Then

$$\frac{dv}{du} = -\frac{2u^5 + v}{u + 2v}, \quad (97)$$

which is of first order as expected. Equation (97) is integrable and gives

$$3uv + 3v^2 + u^6 = \text{constant}. \quad (98)$$

In terms of x and y , Eq. (98) becomes

$$3x^2yy' + 3x^3(y')^2 + x^3y^6 = \text{constant} = 0; \quad (99)$$

because y and y' are regular when $x = 0$, the constant in Eq. (99) must vanish.

Equation (99) is also invariant to Eq. (53) with $\alpha = 1$ and $\beta = -1/2$. Because this group is a stretching group, the invariant $w = u^2 = xy^2$ can be introduced as a new dependent variable and the variables can be separated.

After some calculation, one finds

$$\frac{dx}{x} = \frac{\sqrt{3}}{2} \frac{dw}{w(3/4 - w^2)^{1/2}} . \quad (100)$$

Using the trigonometric substitution $w = \sqrt{3} \sin \theta/2$, we can integrate Eq. (100) and after some computation find

$$w = \frac{3ax}{x^2 + 3a^2} , \quad a = \text{constant} \quad (101a)$$

or

$$y = \left(\frac{3a}{x^2 + 3a^2} \right)^{1/2} , \quad a = \text{constant} . \quad (101b)$$

Example 2: Bessel's equation

Not every differential equation will be so easily integrable as the Emden-Fowler equation for $n = 5$. Nevertheless, Lie's methods yield much useful information by what I call phase-plane analysis. As an example, I take Bessel's equation of order zero because its solutions, though not expressible in terms of elementary functions, are rich in properties:

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = 0 . \quad (102)$$

I shall be primarily interested in the solution of Eq. (102) that is regular at the origin. Equation (102) is invariant to the group

$$\left. \begin{array}{l} y' = \lambda y \\ x' = x \end{array} \right\} . \quad (103)$$

Invariants of the group in Eq. (103) are

$$\left. \begin{array}{l} u = x \\ v = \frac{1}{y} \frac{dy}{dx} \end{array} \right\} . \quad (104)$$

From Eq. (104) it follows that

$$\frac{dv}{du} = -1 - \frac{v}{u} - v^2 . \quad (105)$$

This first-order differential equation is not integrable in terms of elementary functions, but one can nonetheless learn much about the solutions of Eq. (102) from studying it.

Shown in Fig. 13 is part of the direction field of Eq. (105). The curve C, $u = -v/(1 + v^2)$, is the locus for which $dv/du = 0$. On the line $u = 0$ (v -axis), $dv/du = \pm\infty$. Finally, on the line $v = 0$, $dv/du = -1$. The slope dv/du changes sign only at points at which $dv/du = 0$ or $\pm\infty$. The slope is positive where noted in Fig. 13 and negative elsewhere. Finally, when u is finite and v is very large, dv/du is large.

Solutions which cross the u -axis must look like the curve S_1 . Solutions which cross the curve C look like S_2 . Integral curves cannot cross both C and the u -axis, so there must be a separatrix between the two families, namely, curve S that passes through the origin.

S is the only possible curve for which y and dy/dx can remain finite when $u = x \rightarrow 0$. Near the origin in the u, v plane, one can neglect the term v^2 compared with 1 in Eq. (105). Then Eq. (105) becomes

$$\frac{dv}{du} = -1 - \frac{v}{u} \quad v, u \ll 1 . \quad (106)$$

This is invariant to $v' = \lambda v$, $u' = \lambda u$ so that the separatrix is given by

$$\frac{v}{u} = -1/2 \quad v, u \ll 1 . \quad (107)$$

In terms of x and y , Eq. (107) reads

$$\frac{1}{y} \frac{dy}{dx} = -x/2 \quad (108)$$

which integrates to give

$$y = \text{constant} \times e^{-x^2/4} = \text{constant} \times (1 - x^2/4 + \dots) \quad (109)$$

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Fig. 13. The direction field of Eq. (105), which arises from Bessel's equation.

near the origin. Thus the separatrix S corresponds to solutions regular at $x = 0$.

As the magnitude of v increases ($v \rightarrow \pm\infty$) on an integral curve such as S_1 , v^2 becomes $\gg v/u$ and 1. Thus asymptotically, $dv/du = -v^2$ or $v = (u - a)^{-1}$ where a is an integration constant. Thus $dy/y = dx/(x - a)$ or $y = b(x - a)$, where b is another integration constant. Thus each singularity of v corresponds to a zero of y . The first zero of $J_0(x)$ thus corresponds to the first singularity of the separatrix S .

After reaching the first singularity on S , J_0 jumps to the curve S_3 . Each successive singularity in the u, v plane corresponds to a root of $J_0(x)$ [or any other integral curve of Eq. (102)].

When u is large, but v is not, i.e., between zeroes and for large x , Eq. (105) can be written as

$$\frac{dv}{du} = -1 - v^2 - \frac{v}{u} = -1 - \left(v + \frac{1}{2u}\right)^2 + \frac{1}{4u^2} \quad (110)$$

or, neglecting terms of order $1/u^2$,

$$\frac{d}{du} \left(v + \frac{1}{2u}\right) = -1 - \left(v + \frac{1}{2u}\right)^2. \quad (111)$$

Equation (111) can be integrated to give

$$v = -\tan(u - a) - \frac{1}{2u}, \quad a = \text{constant}, \quad (112)$$

from which it follows at once that the asymptotic form of all solutions of Eq. (102) is

$$y = bx^{-1/2} \cos(x - a), \quad b = \text{constant}. \quad (113)$$

Figure 14 summarizes what has been learned about $J_0(x)$.

It is even possible to get some information on the nature of the solutions of Eq. (102) singular at the origin. The curves like S_1 or S_2 near $u = 0$, v^2 , $v/u \gg 1$ so that

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Fig. 14. Drawing summarizing what has been learned about $J_0(x)$, the Bessel function of order zero.

$$\frac{dv}{du} = -\frac{v}{u} - v^2 \quad (114)$$

or

$$\frac{d(uv)}{(uv)^2} + \frac{du}{u} = 0 . \quad (115)$$

The solution of Eq. (115) is

$$v = \frac{1}{u \ln(u/a)} , \quad a = \text{constant} . \quad (116)$$

The solution of Eq. (116) in terms of x and y is

$$y = b \ln(x/a) , \quad b = \text{constant} . \quad (117)$$

Thus all solutions except those corresponding to the separatrix S have logarithmic singularities near the origin.

Analysis of this type can be applied as well to nonlinear equations and is an extremely useful, if not essential, prelude to more extensive calculation.

Partial Differential Equations

Example 1: Diffusion equation

Consider the diffusion equation

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} \quad (118)$$

as a representative of the partial differential equations that occur in technical practice. Equation (118) is invariant to the group in Eq. (119).

$$\left. \begin{aligned} c' &= \lambda^{2\alpha} c \\ t' &= \lambda^2 t \\ x' &= \lambda x \end{aligned} \right\} . \quad (119)$$

Are there solutions of Eq. (118) that are invariant to the group in Eq. (119)? If there are, they must be of the form $F(x,t,c) = 0$ where the function F is invariant to Eq. (119)

$$F(\lambda x, \lambda^2 t, \lambda^{2\alpha} c) = F(x,t,c) . \quad (120)$$

Differentiating with respect to λ and setting $\lambda = 1$, one finds

$$xF_x + 2tF_t = 2\alpha cF_c = 0 . \quad (121)$$

The characteristic equation for Eq. (121) is

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{dc}{2\alpha c} \quad (122)$$

which has the two integrals x^2/t and c/t^α . These integrals are also invariants of the group in Eq. (119). Hence, the most general form for F is

$$F(x,t,c) = G\left(\frac{x^2}{t}, \frac{c}{t^\alpha}\right) \quad (123)$$

where G is an as yet undetermined function. An invariant solution of the form $0 = F = G$ can then be written as

$$\frac{c}{t^\alpha} = g\left(\frac{x^2}{t}\right) \quad (124)$$

by solving $G = 0$ for c/t^α .

Because g is a function of *one variable only*, substitution of Eq. (124) into the partial differential equation (118) results in an *ordinary* differential equation for g . In other words, if invariants of the group are chosen as new variables, the partial differential equation reduces to an ordinary differential equation.

To test this, I calculate

$$\frac{\partial c}{\partial t} = \alpha t^{\alpha-1} g - t^{\alpha} \frac{x^2}{t^2} g' \quad (125a)$$

$$\frac{\partial c}{\partial x} = t^{\alpha} \left(\frac{2x}{t} \right) g' \quad (125b)$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{2}{t} t^{\alpha} g' + t^{\alpha} \left(\frac{2x}{t} \right)^2 g'' \quad (125c)$$

If one equates Eqs. (125a) and (125c) and divides by $t^{\alpha-1}$, one finds

$$4Zg'' + (2 + Z)g' - \alpha g = 0 \quad (126)$$

where Z is an abbreviation for the variable x^2/t . All solutions of Eq. (118) that are invariant to Eq. (119) must be the form of Eq. (124), where g solves the ordinary differential equation (126). The group invariance helps indicate the right combinations of variables to put into Eq. (124).

What about the value of α ? The value of α depends on the boundary conditions. First, consider the problem of a half space initially at zero temperature (c) suddenly at $t = 0$ put in contact (at $x = 0$) with another half space whose temperature is always $c = 1$. The boundary conditions for this problem are

$$c = 0 \quad \text{at } t = 0 \text{ for } x > 0 \quad (127a)$$

$$c = 1 \quad \text{at } x = 0 \text{ for } t > 0 \quad (127b)$$

$$c = 0 \quad \text{as } x \rightarrow \infty \text{ for } t > 0 \quad (127c)$$

Equations (127b) and (124) show that α must be zero and $g(0) = 1$. Then Eqs. (127a) and (127c) both become $g(\infty) = 0$. In this case, three boundary conditions for c collapse to just two for g . *This will not be true in general. If not, the solution of Eq. (118) sought is simply not of the form of Eq. (124), i.e., not invariant to the group in Eq. (119).* With $\alpha = 0$, $g(0) = 1$, and $g(\infty) = 0$, one can solve Eq. (126) easily to find

$$g = \operatorname{erfc} \left(\sqrt{\frac{Z}{4}} \right) \equiv \frac{2}{\sqrt{\pi}} \int_{\sqrt{Z}/4}^{\infty} e^{-u^2} du \quad (128a)$$

or

$$C(x,t) = \operatorname{erfc} \left(\frac{x}{\sqrt{4t}} \right). \quad (128b)$$

Second, consider the problem of an instantaneous unit heat pulse delivered at the point $x = 0$ at the time $t = 0$. By conservation of energy, $\int_{-\infty}^{\infty} c dx = 1$ for all times $t > 0$.

Now

$$1/2 = \int_0^{\infty} c dx = \int_0^{\infty} t^{\alpha} g dx = \int_0^{\infty} g(Z) \frac{dZ}{2\sqrt{Z}} \quad \text{only if } \alpha = -1/2. \quad (129)$$

If $\alpha = -1/2$ the integral can be independent of time; if not, it is not. The solution of Eq. (126) for $\alpha = -1/2$ which vanishes as $Z = x^2/t \rightarrow \infty$ is a constant $\times \exp(-Z/4)$. In Eq. (129) the constant equals $1/2\sqrt{\pi}$ so that

$$C(x,t) = \frac{\exp \left(-\frac{x^2}{4t} \right)}{2\sqrt{\pi t}}. \quad (130)$$

There is a multitude of physical problems described by partial differential equations that can be reduced to ordinary differential equations by exploiting their group invariance. I have encountered practical problems of this sort involving the spreading of pressure waves from explosions, the buildup of the filtered species at the membrane filter in water desalting, the distribution of current in a multifilamentary superconductor, and the motion of a shock-loaded elastic membrane.

There are many more applications that others have pointed out in hydrodynamics, plasma physics, nonlinear diffusion, solid mechanics, boundary layer flow, etc. It is safe to say that the method just explained is one of the most important and widely applicable methods available for the solution of partial differential equations.

Example 2: Steady-state concentration polarization in a slit, entrance region

In the entrance region shown in Fig. 15, the excess salt concentration at the membrane is governed by the partial differential equation

$$\frac{\partial^2 c}{\partial y^2} = y \frac{\partial c}{\partial x} \quad (131)$$

with the boundary conditions

$$\frac{\partial c}{\partial y} = -1 \text{ at } y = 0 \text{ for } x > 0 \quad (132a)$$

$$c(x = 0, y) = 0 \text{ for } y > 0 \quad (132b)$$

$$c(x, y = \infty) = 0 \text{ for } x > 0. \quad (132c)$$

Equation (131) is invariant to the group

$$\begin{aligned} c' &= \lambda^\alpha c \\ y' &= \lambda y \\ x' &= \lambda^3 x, \end{aligned} \quad (133)$$

the invariants of which are y^3/x and $c/x^{\alpha/3}$. In order to satisfy Eq. (132a), i.e., that Eq. (132a) should also be invariant to Eq. (133), α must equal 1. Then

$$c = x^{1/3} g\left(\frac{y^3}{x}\right). \quad (134)$$

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Fig. 15. Sketch explaining concentration polarization in steady-state reverse osmosis: entrance region.

The conditions in Eqs. (132b) and (132c) are both consistent with $g(\infty) = 0$. Substituting Eq. (134) into Eq. (131) gives, after some rearrangement, the ordinary differential equation

$$27Zg'' + 3(6 + Z)g' - g = 0 . \quad (135)$$

Because $\partial c / \partial y = 3y^2x^{-2/3}g' = 3Z^{2/3}g'$, the boundary condition in Eq. (132a) becomes

$$\lim_{Z \rightarrow 0} (Z^{2/3}g') = -\frac{1}{3} . \quad (136)$$

Equation (135) with the boundary conditions of Eqs. (136) and (137) can be solved in terms of confluent hypergeometric functions. This will not be done here because of the labor involved, but the essential points of the solution have already been made, namely

(1) substitution of the solution of Eq. (134) made up out of the group invariants reduces the partial differential equation to an ordinary differential equation;

(2) the value of the remaining constant α is determined by one of the boundary conditions;

(3) the remaining two boundary conditions for the partial differential equation collapse to one for the ordinary differential equation; and

(4) the functional form of Eq. (134), which follows from the group invariance, already tells us that the excess concentration at the wall varies as the one-third power of distance down the channel, without our having to solve any differential equations at all.

Example 3: Motion of a shock-loaded elastic ribbon

The equation of motion of the struck membrane (shown in Fig. 16) is

$$\mu y_{tt} = p + \sigma y_{xx} \quad (137)$$

Fig. 16. Sketch showing deformation of shock-loaded elastic membrane.

where μ is the mass per unit area, p is the shock pressure, and σ is the surface tension in the stretched part of the membrane. Furthermore,

$$\sigma = \frac{Y}{L} \left[\int_0^L (1 + y_x^2)^{1/2} dx - L \right] \quad (138a)$$

$$= \frac{Y}{2L} \int_0^L y_x^2 dx . \quad (138b)$$

For special units of mass, length, and time in which $\mu = Y = L = 1$,

$$y_{tt} = p + \frac{1}{2} y_{xx} \int_0^1 y_x^2 dx . \quad (139)$$

In the early stages of motion, the waves propagating in from the edges have not yet reached the center, but have only traveled a distance $A(t)$ from the edges. The central portions have traveled a distance $(1/2) pt^2$ uniformly upwards. Hence, if

$$y = \frac{1}{2} pt^2 - Z , \quad (140)$$

then

$$Z_{tt} = Z_{xx} \int_0^{A(t)} Z_x^2 dx \quad 0 < Z < A(t) \quad (141)$$

and

$$Z[A(t), t] = 0 \quad (142a)$$

$$Z(0, t) = \frac{1}{2} pt^2 \quad (142b)$$

$$Z(x, 0) = 0 . \quad (142c)$$

This integrodifferential equation and associated boundary conditions are most unusual, and Example 3 was chosen for that reason. Equation (141) is invariant to the group

$$\left. \begin{aligned} Z' &= \lambda^\beta Z \\ t' &= \lambda^\alpha t \\ x' &= \lambda x \end{aligned} \right\} \quad \text{with } \alpha + \beta = \frac{3}{2}. \quad (143)$$

Two invariants of Eq. (143) are $Z/t^{1/\alpha}$ and $x/t^{1/\alpha}$ so that

$$Z = t^{\beta/\alpha} F\left(\frac{x}{t^{1/\alpha}}\right) \quad (144)$$

where F is a function yet to be determined. In order to satisfy Eq. (142b), $\beta/\alpha = 2$. Therefore $\alpha = 1/2$ and $\beta = 1$. Then Eq. (144) becomes

$$Z = t^2 F\left(\frac{x}{t^2}\right). \quad (145)$$

In order to satisfy Eq. (142a), the following must be true:

$$A(t) = Bt^2, \quad B = \text{a constant as yet undetermined} \quad (146a)$$

$$F(B) = 0. \quad (146b)$$

Substituting Eq. (145) into Eq. (141) yields

$$4w^2 F'' - 2wF' + 2F = F'' \cdot \int_0^B (F')^2 dw \quad w = \frac{x}{t^2}. \quad (147)$$

Now $\int_0^B (F')^2 dw = \text{a constant}$; call it $4C^2$. Thus

$$4(w^2 - C^2)F'' - 2wF' + 2F = 0. \quad (148)$$

In order to avoid a sharp crease in the membrane at $x = A(t) = Bt^2$, i.e., at $w = x^2/t = B$, $F'(B)$ must equal zero. Furthermore, $F(B) = 0$. The value and derivative of a nontrivial solution can only vanish simultaneously at a *singular* point. (Consider, for example, the solution $y = x^{3/2}$ of the second-order differential equation $4x^2y'' = 3y$ at $x = 0$.) Thus $C = B$, so that Eq. (148) becomes

$$4(w^2 - B^2)F'' - 2wF' + 2F = 0 \quad (149a)$$

$$F(B) = F'(B) = 0 \quad (149b)$$

$$\int_0^B (F')^2 dw = 4B^2 \quad (149c)$$

$$F(0) = \frac{1}{2}p. \quad (149d)$$

Equations (149) can be solved without difficulty and all the boundary conditions satisfied by setting $F = wG$. The calculations, which are straightforward and somewhat tedious, shed no more light on the use of group invariance to solve complex partial differential equations.

Summary and Conclusions

Not every differential equation is invariant to an easily detected one-parameter group, so that group theoretic methods are not of universal application. On the other hand, it very often happens that differential equations met in practice are invariant to a group. Then the powerful methods of this section can be used, usually with tremendous impact. The application of group invariance to partial differential equations is far and away the most important application.

3. VARIATIONAL METHODS

In Sect. 1, the variational method was used as a criterion of best fit in the application of curve fitting methods. It was pointed out that while the variational method has no special advantage as a criterion of best fit, if the functional being extremized represents a quantity of interest, it can be estimated with far better accuracy than the accuracy of the trial function itself. This application of the variational method is an important one and is especially well adapted to solving physical problems involving simple partial differential equations in not-so-simple geometries.

Lately, both Noble and Arthurs have shown how to obtain two "complementary" variational principles, one of which bounds the quantity of interest from above, the other from below. This method, described in detail by Arthurs, allows one to get inclusive bounds for quantities of interest and hence to know the maximum error in a variational estimate.

The application of variational methods is not restricted to eigenvalue problems, but the method is powerful when used to estimate eigenvalues as will be shown later.

We have already seen an example of the application of the variational method to an ordinary differential equation in Sect. 1 (Collatz's equation). We turn therefore to an application involving a partial differential equation.

Example 1: Resistance of a Slab of Varying Thickness

Consider the slab shown in Fig. 17; let its thickness in the z direction be the unit of length. Let the two ends of the slab ($x = 0$ and $x = L$) be maintained at different potentials $\phi_0 > \phi_L$. A current then flows. How big is this current, or (in different words) what is the electrical resistance of the slab to current flow in the x direction?

The formulation of the differential equations for this problem is easy.

$$\nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{E} = 0, \quad \vec{J} = \sigma \vec{E} \quad (150)$$

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Fig. 17. Cross section of the resistive slab of varying thickness.

where $\vec{E}$ is the electric vector, $\vec{J}$ is the current density, and σ is the conductivity. Because $\vec{E}$ is both irrotational (zero curl) and solenoidal (zero divergence), two different potentials can be introduced, a scalar potential or a vector potential. (In this problem, these correspond to the electric potential and the stream function, respectively, as shown below.)

Let us begin with the scalar potential.

$$\vec{E} = \nabla\phi \quad (\nabla \times \vec{E} = 0 \text{ identically}) \quad (151a)$$

$$\nabla^2\phi = 0 \quad (\nabla \cdot \vec{E} = 0) \quad (151b)$$

The power dissipated in the slab is

$$\int_V (J^2/\sigma) dV = \int_V \sigma (\nabla\phi)^2 dV.$$

This power can also be represented by $(\phi_0 - \phi_L)^2/R$, where R is the resistance of the slab. Thus

$$(\sigma R)^{-1} = \frac{\int_V (\nabla\phi)^2 dV}{(\phi_0 - \phi_L)^2} = \int_V (\nabla\phi)^2 dV \quad \text{if } \phi_0 = 1 \quad \text{and} \quad \phi_L = 0. \quad (152)$$

For the boundary conditions pertaining in this problem, the functional $\int_V (\nabla\phi)^2 dV$ is stationary (has an extremum) for solutions of Eq. (151b). The boundary conditions are

$$\phi(0, y) = \phi_0 = 1 \quad (153a)$$

$$\phi(L, y) = \phi_L = 0 \quad (153b)$$

$$\left(\frac{\partial\phi}{\partial n}\right)_S = \frac{\vec{J} \cdot \vec{n}}{\sigma} = 0 \text{ on the lateral surface.} \quad (153c)$$

Suppose $\phi = \phi_* + \epsilon\eta$ where ϕ_* is the exact solution, ϵ is a small parameter, and $\epsilon\eta$ is a small deviation of the trial function ϕ from ϕ_* . Then

$$\int_V (\nabla\phi)^2 dV = \int_V (\nabla\phi_*)^2 dV + 2\epsilon \int_V \nabla\phi_* \cdot \nabla\eta dV + \epsilon^2 \int_V (\nabla\eta)^2 dV. \quad (154)$$

It can be shown that the second term vanishes by integration of the vector identity $\nabla \cdot (\eta \nabla \phi_*) = \nabla\eta \cdot \nabla\phi_* + \eta \nabla^2 \phi_*$ over V . The last term vanishes because $\nabla^2 \phi_* = 0$, and the left-hand side gives

$$\int_V \nabla \cdot (\eta \nabla \phi_*) dV = \iint_{\text{surface of } V} \eta \left(\frac{\partial \phi_*}{\partial n} \right) ds \quad (155)$$

which vanishes because (1) $\partial\phi_*/\partial n = 0$ on the lateral surface S , and (2) η vanishes at $x = 0$ and $x = L$, if one restricts himself to trial functions satisfying the boundary conditions (153a) and (153b). This means if $(\sigma R)^{-1}$ is estimated with a trial function the error of which is of order ϵ , the estimate has an error of order ϵ^2 . In this case, the estimate is also an upper bound to $(\sigma R)^{-1}$.

If the variation of thickness with distance x is not too great, the equipotential surfaces are nearly planes of constant x . Assume a trial function $\phi = \phi(x)$ with $\phi(0) = 1$ and $\phi(L) = 0$. Note that ϕ does not, and need not, satisfy Eq. (153c). Then

$$\int_V (\nabla\phi)^2 dV = 2 \int_0^L \left(\frac{d\phi}{dx} \right)^2 Y(x) dx \quad (156)$$

where $Y(x)$ is the half-thickness of the slab at x . From Eq. (154) it is clear that $\int_V (\nabla\phi)^2 dV$ is a minimum for the exact solution ϕ_* ; hence, $\phi(x)$ minimizes the right-hand side of Eq. (156). If $\phi = \phi_m(x) + \epsilon\eta(x)$,

$$\int_0^L \left(\frac{d\phi}{dx} \right)^2 Y(x) dx = \int_0^L \left[\left(\frac{d\phi_m}{dx} \right)^2 Y(x) + 2\epsilon \frac{d\phi_m}{dx} \frac{d\eta}{dx} Y(x) + \epsilon^2 \left(\frac{d\eta}{dx} \right)^2 Y(x) \right] dx . \quad (157)$$

The solution ϕ_m should make the linear term in ϵ vanish for arbitrary η .

Now

$$\int_0^L \frac{d\phi_m}{dx} \frac{d\eta}{dx} Y dx = \frac{d\phi_m}{dx} Y \eta \Big|_0^L - \int_0^L \frac{d}{dx} \left(\frac{d\phi_m}{dx} Y \right) \eta dx . \quad (158)$$

The integrated term vanishes because $\eta(L) = \eta(0) = 0$. The last term will vanish for arbitrary η if

$$\frac{d}{dx} \left(\frac{d\phi_m}{dx} Y \right) = 0 \quad (159a)$$

or

$$\frac{d\phi_m}{dx} = \frac{C}{Y} , \quad C = \text{constant of integration.} \quad (159b)$$

C is determined by the requirement that $\phi_m(0) = 1$ and $\phi_m(L) = 0$.

$$1 = C \int_0^L \frac{dx}{Y} . \quad (160)$$

Substituting Eq. (159b) into Eq. (156) gives

$$(\sigma R)^{-1} = \int_V (\nabla^2 \phi) dV = 2 \int_0^L \frac{C^2}{Y} dx = \frac{2}{\int_0^L dx/Y} , \quad (161a)$$

which can be rewritten as

$$R = \rho \int_0^L \frac{dx}{2Y}, \quad \rho = \sigma^{-1} = \text{resistivity} . \quad (161b)$$

Equation (161b) is a variational estimate and therefore is expected to be rather accurate. It has an interesting physical interpretation — namely, it is the overall resistance of disks of thickness dx and lateral dimensions $1 \times 2Y$ connected in series.

Pólya⁶ has called the method employed here the method of assigned level lines (or assigned level surfaces). In this method the equipotential (level) surfaces are chosen, and the value of the potential on each level surface is determined by the variational principle. *This has the effect of replacing a partial with an ordinary differential equation.*

Vector Potential

Let me introduce the vector potential

$$\vec{E} = \nabla \times \vec{A} \quad (\nabla \cdot \vec{E} = 0 \text{ identically}) \quad (162a)$$

$$\nabla \times (\nabla \times \vec{A}) = 0 \quad (\nabla \times \vec{E} = 0) . \quad (162b)$$

Because neither the vector potential nor any other physical quantity can depend on the z -coordinate, no generality is lost taking $\vec{A} = [0, 0, A(x, y)]$:

$$E_x = \frac{\partial A_z}{\partial y} \quad (163a)$$

$$E_y = - \frac{\partial A_z}{\partial x} \quad (163b)$$

$$E_z = 0 . \quad (163c)$$

Then $\nabla \cdot \vec{A} = 0$ so that Eq. (162b) can be replaced by

$$\nabla^2 \vec{A} = 0 . \quad (164)$$

The physical interpretation of A as the stream function is necessary to get the boundary conditions.

$$\begin{aligned} \int_0^P \vec{J} \cdot \vec{ds} &= \int_0^P (-J_y dx + J_x dy) = \sigma \int_0^P \left(\frac{\partial A_z}{\partial x} dx + \frac{\partial A_z}{\partial y} dy \right) \\ &= \sigma [A(P) - A(0)] . \end{aligned} \quad (165)$$

If $A = 0$ on the x -axis, then $\sigma A(P)$ is the current flowing through any arc between the x -axis and the point P (see Fig. 18). Thus on the outer surface $\dot{S}$, $A = I/2\sigma$

$$A(0) = 0 \quad (166a)$$

$$A(S) = \frac{I}{2\sigma} . \quad (166b)$$

The total power dissipation is

$$I^2 R = \int_V \sigma E^2 dV = \sigma \int_V (\nabla A)^2 dV . \quad (167)$$

As before, the term on the extreme right-hand side gives an estimate of $I^2 R$ with an error of second order. Moreover it is an upper bound. Because Eq. (152) gives an upper bound to R^{-1} , it gives a lower bound to R itself. Thus the methods of the scalar and vector potentials give inclusive bounds for R .

To check that Eq. (167) really gives a stationary estimate, one can set $A = A_* + \epsilon \eta$ and find

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Fig. 18. Sketch to clarify the physical interpretation of the stream function.

$$\int_V (\nabla A)^2 dV = \int_V (\nabla A_*)^2 dV + 2\epsilon \int_V \nabla A_* \cdot \nabla \eta dV + \epsilon^2 \int_V (\nabla \eta)^2 dV . \quad (168)$$

Now

$$\int_V \nabla A_* \cdot \nabla \eta dV = \int_S \eta \frac{\partial A_*}{\partial n} ds - \int_V \eta \nabla^2 A_* dV . \quad (169)$$

The last term is 0 because $\nabla^2 A_* = 0$. The middle term is 0 because (1) $\eta = 0$ on S if we restrict ourselves to trial functions which equal $I/2\sigma$ on S and (2) by assumption the straight side sections in Fig. 17 are equipotentials ($0 = E_y = -\partial A/\partial x = -\partial A/\partial n$). Thus the term linear in ϵ in Eq. (168) vanishes.

Take the streamlines of current (lines of constant A) to be the curves

$$y = \lambda Y(x) \quad 0 < \lambda < 1 . \quad (170)$$

Each curve is thus labelled with a value of λ and $A = A(\lambda)$ only. I now introduce λ and x as new coordinates. Because they are not cartesian, the components of the metric tensor will have to be calculated in order to calculate

$$\int_V (\nabla A)^2 dV :$$

$$\overline{ds}^2 = \overline{dx}^2 + \overline{dy}^2 = \overline{dx}^2 + (Yd\lambda + \lambda Y'dx)^2$$

$$= \overline{dx}^2 [1 + \lambda^2 (Y')^2] + 2\lambda Y Y' dx d\lambda + Y^2 \overline{d\lambda}^2 .$$

Therefore

$$g_{xx} = 1 + \lambda^2 (Y')^2$$

$$g_{\lambda\lambda} = Y^2$$

$$g_{x\lambda} = g_{\lambda x} = \lambda Y Y'$$

$$g = \det(g_{ij}) = Y^2$$

$$g^{\lambda\lambda} = \frac{[1 + \lambda^2 (Y')^2]}{Y^2}$$

$$(\nabla A)^2 = g^{ij} \frac{\partial A}{\partial x^i} \frac{\partial A}{\partial x^j} = g^{\lambda\lambda} \left(\frac{dA}{d\lambda} \right)^2$$

$$dydx = \sqrt{g} d\lambda dx = Y d\lambda dx .$$

Therefore

$$\int_V (\nabla A)^2 dV = 2 \int_0^L dx \int_0^1 d\lambda \cdot \frac{1 + \lambda^2 (Y')^2}{Y} \left(\frac{dA}{d\lambda} \right)^2 . \quad (171)$$

To make Eq. (171) stationary

$$\frac{d}{d\lambda} \left(\frac{1 + \lambda^2 (Y')^2}{Y} \frac{dA}{d\lambda} \right) = 0 \quad (172a)$$

$$A(0) = 0, \quad A(1) = \frac{I}{2\sigma} . \quad (172b)$$

Therefore

$$\frac{dA}{d\lambda} = \frac{CY}{1 + \lambda^2 (Y')^2} ; \quad C = \text{constant of integration} \quad (172c)$$

$$\frac{I}{2\sigma} = C \int_0^1 \frac{Y d\lambda}{1 + \lambda^2 (Y')^2} . \quad (172d)$$

Now

$$\begin{aligned} \frac{I^2 R}{\sigma} &= \int_V (\nabla A)^2 dV = 2 \int_0^L dx \int_0^1 d\lambda \frac{C^2 Y}{1 + \lambda^2 (Y')^2} \\ &= \frac{I^2}{2\sigma^2} \int_0^L dx \left[\int_0^1 \frac{Y d\lambda}{1 + \lambda^2 (Y')^2} \right]^{-1} \end{aligned}$$

so that after carrying out the λ integration,

$$\sigma R = \int_0^L \frac{dx}{2Y} \frac{Y'}{\arctan Y'} . \quad (173)$$

Finally, then

$$\int_0^L \frac{dx}{2Y} \frac{Y'}{\arctan Y'} > \sigma R > \int_0^L \frac{dx}{2Y} \quad (174)$$

Shown below is the fraction $Y'/\arctan Y'$ for various values of Y' .

Y'	0.1	0.2	0.3	0.4	0.5	0.75	1.0
$\frac{Y'}{\arctan Y'}$	1.003	1.013	1.029	1.051	1.078	1.165	1.273

From the closeness of these bounds it is apparent that high accuracy is attainable when the extremized function represents the physical quantity of interest.

Method of Noble and Arthurs

The use of the scalar and vector potentials in its present form, though it serves us well in the problem just explained, is not sufficiently general for widespread use. Noble⁷ has shown a systematic way to get upper and lower variational bounds for the same quantity. Arthurs⁸ has elaborated Noble's method in a very fine monograph.

The key to Noble's method is the formulation of the problem in the Hamiltonian form. Suppose we start with an ordinary differential equation which is the Euler-Lagrange equation of a Lagrangean $L(q, \dot{q})$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \quad , \quad (175)$$

The momentum conjugate to q is $p = \partial L / \partial \dot{q}$ and the Hamiltonian of the system is $H = p\dot{q} - L$. In terms of H , the variational principle which leads to Eq. (175), the principle of least action, requires the functional

$$\int_a^b L dt = \int_a^b (p\dot{q} - H) dt \quad (176)$$

to be stationary for solutions of Eq. (175).

Noble's work begins by considering variations away from the true solution of Eq. (175)

$$p = p_* + \epsilon \eta$$

$$\epsilon \ll 1 \quad .$$

$$q = q_* + \epsilon \zeta$$

Then

$$\begin{aligned}
 \int_a^b L dt &= \int_a^b (p_* + \epsilon \eta) (\dot{q}_* + \epsilon \dot{\zeta}) dt - \int_a^b H(p_* + \epsilon \eta, q_* + \epsilon \zeta) dt \\
 &= \int_a^b L_* dt + \epsilon \cdot \int_a^b (\overset{\textcircled{1}}{\eta \dot{q}_*} + \overset{\textcircled{2}}{\dot{\zeta} p_*} - \overset{\textcircled{2}}{\zeta H_{q_*}} - \overset{\textcircled{1}}{\eta H_{p_*}}) dt \\
 &\quad + \frac{\epsilon^2}{2} \cdot \int_a^b (2\eta \dot{\zeta} - \zeta^2 H_{qq_*} - 2\eta \zeta H_{pq_*} - \eta^2 H_{pp_*}) dt, \quad (177)
 \end{aligned}$$

where subscripts denote partial differentiation. Now

$$\int_a^b \dot{\zeta} p_* dt = p_* \zeta \Big|_a^b - \int_a^b \zeta \dot{p}_* dt.$$

If $p_* \zeta \Big|_a^b = 0$, then the terms marked 2 cancel and the term linear in ϵ vanishes.

I have just proved the classical principle of least action for Hamilton's equations. Noble's idea is to require the trial functions p and q to satisfy one or the other Hamilton equation.

Suppose $\dot{q} \equiv H_p$. Then

$$\dot{q}_* + \epsilon \dot{\zeta} = H_p(q_* + \epsilon \zeta, p_* + \epsilon \eta) = H_{p_*} + \epsilon (H_{pq_*} \zeta + H_{pp_*} \eta)$$

so that

$$2\eta \dot{\zeta} = 2H_{pq_*} \eta \zeta + 2H_{pp_*} \eta^2.$$

Thus, the coefficient of $\epsilon^2/2$ in the third term in Eq. (177) is

$$\int_a^b (\eta^2 H_{pp_*} - \zeta^2 H_{qq_*}) dt. \quad (178)$$

Suppose instead $\dot{p} = -H_q$. Then

$$\dot{p}_* + \epsilon \dot{\eta} = -H_q(q_* + \epsilon \zeta, p_* + \epsilon \eta) = -H_{q_*} - \epsilon(H_{qq_*} \zeta + H_{pq_*} \eta) .$$

Now

$$\int_a^b 2\eta \dot{\zeta} dt = 2\eta \zeta \Big|_a^b - \int_a^b 2\zeta \dot{\eta} dt .$$

If $2\eta \zeta \Big|_a^b = 0$, then in the last term in Eq. (177) the coefficient of $\epsilon^2/2$ is

$$\int_a^b (\zeta^2 H_{qq_*} - \eta^2 H_{pp_*}) dt . \quad (179)$$

Noble observed that Eqs. (178) and (179) are exact negatives of one another; *thus one of the Hamilton equations gives an upper limit to the action $\int_a^b L dt$ and the other gives a lower limit.*

Do not forget that the trial functions must satisfy the conditions $p_* \zeta \Big|_a^b = 0$ and $2\eta \zeta \Big|_a^b = 0$; i.e., the condition $p \zeta \Big|_a^b = 0$. This could be satisfied, for example, if by the terms of the problem $q(a) = q_a$ and $q(b) = q_b$ so that $\zeta(a) = \zeta(b) = 0$. This means one would have to solve the differential equation $\dot{q} = H_p$ with the boundary conditions $q(a) = q_a$, $q(b) = q_b$.

Example 2: Eddy Current Heating of a Conducting Cylinder

The following equation arises in calculating the heating of a long conducting cylinder C by a slow ramped magnetic field parallel to the axis of the cylinder. (See Fig. 19; the shape of the cylinder is arbitrary.)

$$\nabla^2 \phi = -1; \quad \phi(C) = 0 . \quad (180)$$

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Fig. 19. A long conducting cylinder C in a slow ramped magnetic field parallel to its axis.

The quantity desired is

$$\int_A (\nabla\phi)^2 dV , \quad (181)$$

which is related to the instantaneous eddy current power being dissipated in the cylinder.

A Lagrangean $L(\phi, \nabla\phi)$ for the differential equation (180) is

$$L = \phi - \frac{1}{2} (\nabla\phi)^2 . \quad (182)$$

The value of the action integral for the solution ϕ_* of Eq. (180) is

$$\begin{aligned} A &= \int_A \left[\phi_* - \frac{1}{2} (\nabla\phi_*)^2 \right] dV = \int_A \left[\phi_* \underbrace{(-\nabla^2\phi_*)}_{=+1} - \frac{1}{2} (\nabla\phi_*)^2 \right] dV \\ &= \int_A \frac{1}{2} (\nabla\phi_*)^2 dV . \end{aligned}$$

Thus the physical quantity of interest is related to the value of the action integral, which is the setting in which Noble's method is intended to be used.

The momentum conjugate to ϕ is

$$\vec{p} = \frac{\partial L}{\partial(\nabla\phi)} = \nabla\phi . \quad (183)$$

The Hamiltonian is then

$$H = \vec{p} \cdot \nabla\phi - L \quad (184a)$$

$$= \frac{-p^2}{2} - \phi , \quad (184b)$$

and the Hamiltonian equations are

$$\nabla \phi = -\vec{p} \quad (185a)$$

$$\nabla \cdot \vec{p} = 1. \quad (185b)$$

If Eq. (185a) is first satisfied, then the action A can be written

$$A_1 = \int_A \left[\phi - \frac{1}{2} (\nabla \phi)^2 \right] dV. \quad (186)$$

To verify that A is stationary and to find its second variation, set

$$\phi = \phi_* + \epsilon \zeta$$

$$A_1 = A_* + \epsilon \int_A (\zeta - \nabla \phi_* \cdot \nabla \zeta) dV - \frac{\epsilon^2}{2} \int_V (\nabla \zeta)^2 dV.$$

Now

$$\int_A (\zeta - \nabla \phi_* \cdot \nabla \zeta) = - \int_A (\zeta \nabla^2 \phi_* + \nabla \phi_* \cdot \nabla \zeta) dV = - \int_C \zeta \nabla \phi_* \cdot \vec{ds}$$

so that the first-order term will vanish if $\zeta(C) = 0$, i.e., if the trial functions in Eq. (186) are restricted to those for which $\phi(C) = 0$.

Now satisfy Eq. (185b). Because this constraint involves only $\vec{p}$, ϕ can be chosen arbitrarily. Then

$$\begin{aligned} A &= \int_A \left(\vec{p} \cdot \nabla \phi + \frac{\vec{p}^2}{2} + \phi \right) dV \\ &= \int_A \left(\vec{p} \cdot \nabla \phi + \underbrace{\phi \nabla \cdot \vec{p}}_{=1} + \frac{\vec{p}^2}{2} \right) dV \\ &= \int_A \frac{\vec{p}^2}{2} dV + \int_C \phi \vec{p} \cdot \vec{ds}. \end{aligned}$$

If $\phi = \phi_*$ is chosen, the action integral can be written as

$$A_2 = \int_A \frac{p^2}{2} dV .$$

Then, setting $\vec{p} = \vec{p}_* + \epsilon \vec{\eta}$,

$$A_2 = A_* + \epsilon \int_A \vec{p}_* \cdot \vec{\eta} dV + \frac{\epsilon^2}{2} \int_A \eta^2 dV .$$

Now, $p_* = \nabla \phi_*$ so that

$$\begin{aligned} \int_A \vec{p}_* \cdot \vec{\eta} dV &= \int_A (\nabla \phi_*) \cdot \vec{\eta} = \int_A [\nabla \cdot (\phi_* \vec{\eta}) - \phi_* \nabla \cdot \vec{\eta}] dV \\ &= \int_C \phi_* \vec{\eta} \cdot \vec{ds} - \int_V \phi_* \nabla \cdot \vec{\eta} dV = 0 \end{aligned}$$

because (1) $\phi_*(C) = 0$ and (2) $\nabla \cdot \vec{\eta} = 0$ because $\nabla \cdot p_* = \nabla \cdot p = 1$.

Finally, then

$$\int_V \frac{p^2}{2} dV > A_* > \int_V \left[\phi - \frac{1}{2} (\nabla \phi)^2 \right] dV \quad (187)$$

where the trial functions are constrained by $\nabla \cdot \vec{p} = 1$ and $\phi(C) = 0$. Computations may be facilitated by setting $\vec{p} = -\nabla \psi$, in which case the left-hand side becomes $(1/2) \int_V (\nabla \psi)^2 dV$ with $\nabla^2 \psi = -1$ but otherwise arbitrary.

Variational Calculation of Eigenvalues

Eigenvalues can often be calculated by the variational method because they can be represented as the extreme values of an appropriate functional (e.g., Rayleigh's principle). Eigenvalue problems have a

rich literature, but I shall only touch on one simple, useful technique recommended by Collatz for estimating the first eigenvalue of problems of the form

$$H\phi = \lambda G\phi \quad (188)$$

where G and H are positive-definite, self-adjoint differential operators, λ is a number (eigenvalue), and ϕ is a function (eigenfunction). By positive-definite I mean that the inner product

$$(u, Hu) = \int_a^b uHu dx > 0 \quad (189)$$

for any arbitrary $u(x)$ which satisfies the boundary conditions associated with the differential equation (188). By self-adjoint I mean

$$(u, Hv) - (v, Hu) \equiv \int_a^b (uHv - vHu) dx = 0 \quad (190)$$

for any u and v satisfying the boundary conditions.

Example 1

Consider the problem $\nabla^2\phi + \lambda\phi = 0$ or $-\nabla^2\phi = \lambda\phi$, $\phi = 0$ on S , the surface of V . Then

$$\begin{aligned} (u, Hu) &= \int_V u(-\nabla^2 u) dV = \int_V (\nabla u)^2 dV - \cancel{\int_S u \nabla u \cdot \vec{ds}} > 0 \quad (\text{positive-definite}) \\ (u, Hv) &= \int_V u(-\nabla^2 v) dV = \int_V \nabla u \cdot \nabla v dV - \cancel{\int_S u \nabla v \cdot \vec{ds}} \\ &= \int_V v(-\nabla^2 u) dV + \cancel{\int_S v \nabla u \cdot \vec{ds}} = (v, Hu) \quad (\text{self-adjoint}) . \end{aligned}$$

In general, there is more than one eigenvalue λ and more than one eigenfunction ϕ . The eigenvalues of positive-definite operators are all positive: for $\lambda_k = (\phi_k, H\phi_k) / (\phi_k, G\phi_k) > 0$.

If the eigenvalues are discrete and nondegenerate (as I shall assume to simplify the proofs), there is then a lowest one λ_1 .

Rayleigh's quotient

The variational calculation of eigenvalues is based in one way or another on Rayleigh's quotient

$$R(u) = \frac{(u, Hu)}{(u, Gu)} . \quad (191)$$

Rayleigh's quotient is stationary (an extremum) near eigenfunctions of Eq. (188), and its extreme values are the eigenvalues. If we set $u = u_0 + \epsilon \delta u$, then $R(u) = R(u_0) + \epsilon \delta R + O(\epsilon^2)$ where

$$\delta R = 2 \frac{(\delta u, Hu)(u, Gu) - (u, Hu)(\delta u, Gu)}{(u, Gu)^2} . \quad (192)$$

In obtaining Eq. (192) I have used the self-adjointness of H and G . If $u = \phi_k$, an eigenfunction, then $\delta R = 0$. Conversely, if $\delta R = 0$, u can only equal ϕ_k because δu is entirely arbitrary (except for satisfying the boundary conditions). If $(\delta u, Hu) = 0$ then $(\delta u, Gu) = 0$. Therefore, Hu and Gu are both perpendicular to the subspace complementary to Hu . Therefore, Hu and Gu are proportional; i.e., $Hu = \lambda Gu$. Thus $u = \phi_k$.

When H and G are positive-definite, and the eigenvalues are all positive, the extrema of Rayleigh's quotient are minima. This is clear for the lowest eigenvalue because $R(u)$ can never have a value $< \lambda_1$. If it did, it would have to have a minimum value between λ_1 and 0 [$R(u) > 0$ for all u]. However, then there would be an eigenvalue between λ_1 and 0, contradicting the hypothesis that λ_1 is the lowest eigenvalue. If $(u, G\phi_1) = 0$ is a further constraint on the function u , then $R(u)$ cannot

equal λ_1 because that would require $u = \phi_1$, and $(\phi_1, G\phi_1) > 0$. Then $R(u) > \lambda_1$. By an argument exactly similar to the one above, the minimum of R under the constraint $(u, G\phi_1) = 0$ is λ_2 and is achieved when $u = \phi_2$. Note that $(\phi_2, G\phi_1) = 0$ as required by the classical elementary proof.

Example 2

First take an elementary example, namely, the problem

$$-y'' = \lambda y$$

$$y(-1) = y(1) = 0$$

which has the exact solution $y = \cos(x\pi/2)$. Thus $\lambda = \pi^2/4$. Now $G = 1$ and $H = -d^2/dx^2$. H is self-adjoint (with the given boundary conditions)

$$\begin{aligned} (u, Hv) &= \int_{-1}^{+1} u \left(-\frac{d^2v}{dx^2} \right) dx = u \left(-\frac{dv}{dx} \right) \Big|_{-1}^{+1} + \int_{-1}^{+1} \frac{du}{dx} \frac{dv}{dx} dx \\ &= v \frac{du}{dx} \Big|_{-1}^{+1} - \int_{-1}^{+1} v \frac{d^2u}{dx^2} dx = (v, Hu) . \end{aligned}$$

Furthermore, because

$$(u, Hu) = \int_{-1}^{+1} u \left(-\frac{d^2u}{dx^2} \right) dx = u \left(-\frac{du}{dx} \right) \Big|_{-1}^{+1} + \int_{-1}^{+1} \left(\frac{du}{dx} \right)^2 dx ,$$

H is positive-definite. Then

$$R(y) = \frac{\int_{-1}^{+1} y \left(-\frac{d^2y}{dx^2} \right) dx}{\int_{-1}^{+1} y^2 dx} = \frac{\int_{-1}^{+1} \left(\frac{dy}{dx} \right)^2 dx}{\int_{-1}^{+1} y^2 dx} .$$

The simplest trial function is $y = 1 - x^2$. (Notice that normalization does not matter because Rayleigh's quotient is homogeneous.) Then $R = 5/2 > \pi^2/4 = 2.4674$, with an error of 1.3%. The next, more complicated trial function is $y = 1 - ax^2 + (a - 1)x^4$. Then

$$R(y) = 3 \frac{11a^2 - 36a + 60}{2a^2 - 16a + 56}.$$

R has a minimum at $a = 1.220750$ equal to 2.467437 . $[(\pi^2/4) = 2.467401$; fractional error $= 1.5 \times 10^{-5}]$.

Collatz's method for a lower bound

The use of Rayleigh's quotient gives only an upper bound to the lowest eigenvalue. Collatz describes a method which is effective in getting a lower bound to the lowest eigenvalue. The method is based on the iterative method of solving the eigenvalue problem.

Schwarz constants and Schwarz quotients. Suppose we define a series of iterated functions f_k .

$$Hf_{k+1} = Gf_k. \quad (193)$$

Successive iterates approach the "shape" of ϕ_1 . One can see this heuristically if the eigenfunction expansion theorem is assumed: if $f_k = \sum a_n \phi_n$, then $f_{k+1} = \sum a_n \phi_n / \lambda_n$. [The proof is $Hf_{k+1} = \sum (a_n / \lambda_n) (H\phi_n) = \sum (a_n / \lambda_n) (\lambda_n G\phi_n) = G \sum a_n \phi_n = Gf_k$.] One would therefore also expect the series of Rayleigh quotients

$$\frac{(f_k, Hf_k)}{(f_k, Gf_k)} = \frac{(f_k, Gf_{k-1})}{(f_k, Gf_k)}$$

to decrease towards the limit λ_1 . The quantities

$$S_{i+j} = (f_i, Hf_j) = (f_i, Gf_{j-1}) \quad (194)$$

are called Schwarz constants, and the Rayleigh quotients S_{2k}/S_{2k+1} are called Schwarz quotients.

To show that the Schwarz quotients decrease, use the positive-definiteness of H and G . If $u = S_{2k+1}f_k - S_{2k}f_{k+1}$, then $0 < (u, Hu) = S_{2k}^2 S_{2k+2} - S_{2k+1}^2 S_{2k}$, so that

$$\frac{S_{2k}}{S_{2k+1}} > \frac{S_{2k+1}}{S_{2k+2}}. \quad (195a)$$

Similarly $(u, Gu) > 0$ gives

$$\frac{S_{2k-1}}{S_{2k}} > \frac{S_{2k}}{S_{2k+1}}. \quad (195b)$$

Therefore

$$\frac{S_2}{S_3} > \frac{S_3}{S_4} \dots > \lambda_1. \quad (196)$$

The convergence of the sequence of Schwarz quotients to λ_1 is not of great practical use because even in some simple examples, the convergence of the sequence is slow. If in Example 2 $f_1 = 1 - x^2$, then $f_2 = (5 - 6x^2 + x^4)/12$. Then $S_2 = 8/3$, $S_3 = 16/15$, $S_4 = 77/180$, $S_2/S_3 = 2.50$, and $S_3/S_4 = 2.49351$. The error in the first quotient is 1.3%; that in the second quotient is 1.06%, a small reduction. On the other hand, choosing a fourth-order polynomial with an arbitrary parameter and minimizing the Rayleigh quotient with respect to the parameter improves the fractional error by nearly three orders of magnitude, as has been seen.

Collatz's method. Collatz's method begins with a starting function for the iterative method that is orthogonal to λ_1 .

$$g_1 = f_1 - \phi_1 \frac{\overbrace{(f_1, G\phi_1)}^{\equiv C}}{(\phi_1, G\phi_1)} . \quad (197)$$

Now

$$Hg_2 = Gg_1 = Gf_1 - CG\phi_1 .$$

$$= Hf_2 - C \frac{H\phi_1}{\lambda_1}$$

or

$$g_2 = f_2 - \frac{C\phi_1}{\lambda_1} .$$

Then

$$g_{k+1} = f_{k+1} - \frac{C\phi_1}{\lambda_1} . \quad (198)$$

The important thing about the iterates g_k is that they are all orthogonal to ϕ_1 . For

$$(\phi_1, Gf_{k+1}) = (f_{k+1}, G\phi_1)$$

$$= \frac{(f_{k+1}, H\phi_1)}{\lambda_1}$$

$$= \frac{(\phi_1, Hf_{k+1})}{\lambda_1}$$

$$= \frac{(\phi_1, Gf_k)}{\lambda_1}$$

$$\begin{aligned}
&= \frac{(f_k, G\phi_1)}{\lambda_1} \\
&= \dots \\
&= \frac{(\phi_1, Gf_1)}{\lambda_1^k} .
\end{aligned}$$

Therefore,

$$(\phi_1, Gg_{k+1}) = (\phi_1, Gf_{k+1}) - \frac{C}{\lambda_1^k} (\phi_1, G\phi_1) = \frac{(f_1, G\phi_1)}{\lambda_1^k} - \frac{(f_1, G\phi_1)}{\lambda_1^k} = 0 .$$

Then the series of Schwarz quotients based on the g_k converges from above to λ_2 . Now the Schwarz constants T_k based on the g_k are related to the S_k by

$$\begin{aligned}
T_{k+1} &= (g_1, Hg_k) = (g_1, Gg_{k-1}) \\
&= \left(f_1 - C\phi_1, G \left[f_{k-1} - \frac{C}{\lambda_1^{k-2}} \phi_1 \right] \right) \\
&= S_{k+1} - C(\phi_1, Gf_{k-1}) - \frac{C}{\lambda_1^{k-2}} (f_1, G\phi_1) + \frac{C^2}{\lambda_1^{k-2}} (\phi_1, G\phi_1) \\
&= S_{k+1} - C^2 \frac{(\phi_1, G\phi_1)}{\lambda_1^{k-2}} - \frac{C^2}{\lambda_1^{k-2}} (\phi_1, G\phi_1) + \frac{C^2}{\lambda_1^{k-2}} (\phi_1, G\phi_1) .
\end{aligned}$$

Therefore,

$$T_k - \lambda_1 T_{k+1} = S_k - \lambda_1 S_{k+1} . \quad (199)$$

Then

$$\frac{T_{k-1} - \lambda_1 T_k}{T_k - \lambda_1 T_{k+1}} = \frac{S_{k-1} - \lambda_1 S_k}{S_k - \lambda_1 S_{k+1}} . \quad (200)$$

Defining the Schwarz quotients,

$$\frac{S_{k-1}}{S_k} = \mu_k, \quad \frac{T_{k-1}}{T_k} = \nu_k ,$$

Eq. (200) becomes

$$\frac{\mu_k - \lambda_1}{\mu_{k+1} - \lambda_1} \mu_{k+1} = \frac{\nu_k - \lambda_1}{\nu_{k+1} - \lambda_1} \nu_{k+1} > \nu_{k+1} > \lambda_2 , \quad (201)$$

which finally gives

$$\lambda_1 > \frac{\lambda_2 - \mu_k}{\lambda_2 - \mu_{k+1}} \cdot \mu_{k+1} = \mu_{k+1} - \frac{\mu_k - \mu_{k+1}}{(\lambda_2/\mu_{k+1}) - 1} . \quad (202)$$

The inequality is strengthened if λ_2 is replaced by a lower bound ℓ_2 .

The bound will be a close one if μ_k and μ_{k+1} are close.

A simple variation of Collatz's method can be summarized as follows

1. Guess f_2 .
2. Calculate f_1 : $Hf_2 = Gf_1$. Usually G is a much simpler operator than H , often only a multiplicative function or constant. The computation of f_1 then involves only differentiation. Furthermore, if one traces through the proofs one finds that f_1 need only obey a sufficient number of boundary conditions so that $(f_1, Gf_1) > 0$ and $(f_1, Gu) = (u, Gf_1)$ for all admissible u (i.e., u obeying the boundary conditions).

3. Calculate

$$S_3 = (f_1, Gf_1)$$

$$S_4 = (f_2, Gf_1)$$

$$S_5 = (f_2, Gf_2)$$

$$\mu_4 = \frac{S_3}{S_4}$$

$$\mu_5 = \frac{S_4}{S_5}.$$

4. Then, $\mu_5 > \lambda_1 > \mu_5 - \frac{\mu_4 - \mu_5}{(\ell_2/\mu_5) - 1}$ where $\ell_2 \leq \lambda_2$.

So far I have not explained how to get a close lower limit ℓ_2 to the second eigenvalue λ_2 . The easiest way is through the following comparison theorem. If there are two eigenvalue problems $H^{(1)}\phi = \lambda G^{(1)}\phi$ and $H^{(2)}\phi = \lambda G^{(2)}\phi$ so related that the Rayleigh quotient of the first $R^{(1)}(u) = (u, H^{(1)}u)/(u, G^{(1)}u)$ is always less than or equal to the Rayleigh quotient of the second $R^{(2)}(u)$ for all u , then $\lambda_k^{(1)} \leq \lambda_k^{(2)}$, $k = 1, 2, \dots$. In other words, one spectrum majorizes the second, if the first Rayleigh quotient majorizes the second for all trial functions. The proof of this theorem depends on Courant's minimax principle⁹ (which can be found in Courant-Hilbert). Although simple, the proof goes further afield than the scope of this paper on practical applications; so I will use the theorem but not prove it here.

Example 3: Buckling of a tapered column

In the study of the buckling of a tapered column (see Fig. 20), the following eigenvalue problem arises

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Fig. 20. The buckling tapered column.

$$(1 + x) \frac{d^2 y}{dx^2} + \lambda y = 0$$

$$y(0) = 0, y'(1) = 0 .$$

Set

$$H = - \frac{d^2}{dx^2} \text{ and } G = (1 + x)^{-1} .$$

With the given boundary conditions, H is self-adjoint and positive-definite. Y must resemble Fig. 21. Because overall normalization does not matter in evaluating Rayleigh quotients, take

$$y_2 = x - \frac{x^3}{3}, \quad y_2(0) = 0, \quad y_2'(1) = 0 .$$

Then

$$y_1 = -(1 + x) \frac{d^2 y_2}{dx^2} = 2x(1 + x) .$$

Because G is simply a multiplicative factor, y_1 does not have to obey any of the boundary conditions for G to be self-adjoint and positive-definite. Then

$$S_3 = \int_0^1 [2x(1 + x)]^2 (1 + x)^{-1} dx = \frac{7}{3}$$

$$S_4 = \int_0^1 [2x(1 + x)] \left(x - \frac{x^3}{3} \right) (1 + x)^{-1} dx = \frac{8}{15}$$

$$S_5 = \int_0^1 \left(x - \frac{x^3}{3} \right)^2 (1 + x)^{-1} dx = 0.128439 \text{ (Simpson's rule, } h = 0.1)$$

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Fig. 21. Shape of the solution in the problem of the buckling tapered column.

$$\begin{array}{rcl}
 \mu_4 = \frac{S_3}{S_4} = 4.375 & & \\
 \mu_5 = \frac{S_4}{S_5} = 4.152413 & & \\
 & \searrow \quad \nearrow & \\
 & \mu_4 - \mu_5 = 0.222587 & .
 \end{array}$$

Now

$$R(u) = \frac{-\int_0^1 u(d^2u/dx^2)dx}{\int_0^1 u^2(1+x)^{-1}dx} > \frac{-\int_0^1 u(d^2u/dx^2)dx}{\int_0^1 u^2dx} = R'(u) .$$

Now $R'(u)$ is the Rayleigh quotient for the comparison problem

$$-\frac{d^2y}{dx^2} = \lambda y, \quad y(0) = 0, \quad y'(1) = 0,$$

for which $\lambda = \pi^2/4, 9\pi^2/4, 25\pi^2/4, \dots$. Thus $\ell_2 = 9\pi^2/4$ and $(\mu_4 - \mu_5)/(\ell_2/\mu_5 - 1) = 0.051194$. Then

$$4.152413 > \lambda_1 > 4.101219 .$$

(Fractional difference between these extremes is 1.25%.)

Summary and Conclusions

This section has been devoted to those applications of the variational method in which the functional being extremized is the quantity of interest. Not all problems can be handled this way, but many interesting ones can.

4. PERTURBATION METHODS

Perturbation methods are widely used to find the effect of small terms on the behavior of the solutions of differential equations. They have been used extensively in celestial mechanics, fluid dynamics, and quantum mechanics. There are certain difficulties which occur from time to time in the application of perturbation methods, and in this section I shall try to make them clear by means of example.

Singular Perturbations of the Boundary-Layer Type

The first characteristic difficulty with perturbation theory arises because the "small" term involves the highest derivative in the equation. Consider a model problem suggested by Friedrichs:

$$\epsilon \frac{d^2 y}{dx^2} + \frac{dy}{dx} = \frac{1}{2}, \quad y(0) = 0, \quad y(1) = 1, \quad \epsilon \ll 1. \quad (203)$$

If one tries to treat Eq. (203) by conventional perturbation methods, one sets

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots \quad (204)$$

Then

$$\text{order } 1: \quad y_0' = \frac{1}{2}$$

$$\text{order } \epsilon: \quad y_0'' + y_1' = 0$$

$$\text{order } \epsilon^2: \quad y_1'' + y_2' = 0$$

$$\dots$$

(205)

The classical procedure is then to solve Eq. (205) successively, requiring the boundary conditions to be satisfied in each order. But Eq. (205) consists of first-order equations so that one cannot simultaneously satisfy the two conditions $y(0) = 0$ and $y(1) = 1$. One cannot proceed any further.

The cause for this difficulty arises from the manner in which the true solution y varies as $\epsilon \rightarrow 0$. The model problem has been chosen because it is exactly soluble, enabling study of this limit.

$$y = \frac{x}{2} + \frac{1}{2} \frac{1 - e^{-(x/\epsilon)}}{1 - e^{-(1/\epsilon)}} \quad (206)$$

Shown in Fig. 22 is y vs x with ϵ as the parameter. The exact solution approaches the line $y = (1 + x)/2$ more and more closely as $\epsilon \rightarrow 0$ for all $x > 0$, but exactly at $x = 0$, y always equals 0.

Now $y = (1 + x)/2$ would be the solution if Eq. (205) had been solved with the exterior boundary condition $y(1) = 1$. The exterior perturbation solution can be augmented for small x (in the boundary layer) by introducing $x/\epsilon = \xi$ as a new variable and trying to solve the differential equation for $\xi \leq 1$ (interior solution). Then

$$\frac{d^2 y}{d\xi^2} + \frac{dy}{d\xi} = \frac{\epsilon}{2} \quad (207)$$

If a perturbation series such as Eq. (202) is introduced, then

$$y_0'' + y_0' = 0$$

$$y_1'' + y_1' = \frac{1}{2}$$

$$y_2'' + y_2' = 0$$

. . . .

(208)

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Fig. 22. Plot of the solution given in Eq. (206) for various ϵ .

Because this interior perturbation solution should only be good for small ϵ and therefore small x , drop the boundary condition $y(x = 1) = 1$. One therefore has some arbitrary constants in the interior solution which will be determined by suitably matching interior and exterior solutions. Then

$$y = A(1 - e^{-\xi}) + \epsilon \left[\frac{\xi}{2} + B(1 - e^{-\xi}) \right] + \epsilon^2 C(1 - e^{-\xi}) + \dots \quad (209)$$

When $\epsilon \rightarrow 0$, this solution should approach the exterior solution $(1 + x)/2$ for $x > 0$. As $\epsilon \rightarrow 0$, Eq. (209) becomes

$$y = A + \frac{x}{2} + O(\epsilon) \quad (210)$$

Therefore, $A = 1/2$ and the lowest-order interior perturbation solution becomes

$$y = \frac{1}{2} \left(1 - e^{-x/\epsilon} \right) + \frac{x}{2}, \quad (211)$$

which is a good representation of the exact solution of Eq. (206) for small ϵ . [As it turns out, if one chooses $B, C, \dots$ as appropriate functions of ϵ , one can satisfy the boundary condition $f(1) = 1$ for all ϵ , in which case the perturbation solution becomes identical to Eq. (206).] This method is called the method of matched inner and outer expansions.

Perturbation Theory near a Moving Singularity

The next example is the problem

$$(x + \epsilon y) \frac{dy}{dx} + y = 1 \quad (\epsilon \ll 1) \quad (212a)$$

$$y(1) = 2 \quad (212b)$$

$$y(0) = ? . \quad (212c)$$

Figure 23 shows the direction field in the x-y plane. On line 1 of Fig. 23 ($y = -x/\epsilon$), $dy/dx = \pm\infty$. On line 2 ($y = 1$), $dy/dx = 0$. The slope cannot change sign in the four quadrants around P; it can only change sign crossing lines 1 and 2. The integral curves must resemble the heavy curves. There are two separatrices, and using the group theoretic methods of Sect. 2, one can find them. The differential equation (212a) is invariant to the group

$$\begin{aligned} x' &= \lambda x - \epsilon(1 - \lambda) \\ y' &= \lambda y + (1 - \lambda) \end{aligned} \quad 0 < \lambda < \infty . \quad (213)$$

Then

$$\begin{aligned} \xi &= \left(\frac{\partial x'}{\partial \lambda} \right)_{\lambda=1} = x + \epsilon \\ \eta &= \left(\frac{\partial y'}{\partial \lambda} \right)_{\lambda=1} = y - 1 . \end{aligned}$$

Therefore on the separatrices

$$\frac{dy}{dx} = \frac{\eta}{\xi} = \frac{y - 1}{x + \epsilon} .$$

Thus the separatrices are given by

$$(x + \epsilon y) \frac{y - 1}{x + \epsilon} + y = 1 ,$$

which can be satisfied only if

$$y = 1 \quad \text{or} \quad 2x + \epsilon y + \epsilon = 0 \quad (\text{curve 3}) . \quad (214)$$

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Fig. 23. The direction field of Eq. (212a).

(Some care appears to be necessary in finding the separatrix 3. Both Ames¹⁰ and Jordan and Smith¹¹ erroneously show $x + \epsilon y$ as the separatrix!)

These constructions show that the separatrix 3 becomes steeper and moves closer to the y -axis as $\epsilon \rightarrow 0$. The value of $y(0)$ should then depend strongly on ϵ , increasing without limit as $\epsilon \rightarrow 0$. This may well cause trouble in the use of perturbation theory.

A straightforward perturbation expansion of the type in Eq. (204) yields the coupled equations

$$xy'_0 + y_0 = 1$$

$$xy'_1 + y_1 + y_0 y'_0 = 0$$

$$xy'_2 + y_2 + (y_1 y_0)' = 0$$

. . . .

When solved, these lead to the expansion

$$y = \frac{x+1}{x} + \epsilon \frac{(x-1)(3x+1)}{2x^3} + \epsilon^2 \frac{(1-x)(1+x)(3x+1)}{2x^5} + \dots, \quad (215)$$

which represents a curve such as 5 that is singular at $x = 0$. Thus the ordinary perturbation expansion moves the singularity from line 3 to the y -axis.

Matched Inner and Outer Expansions

Equation (215) is a suitable expansion away from the singularity of line 3 — it is thus an outer expansion. In order to calculate an inner expansion, suitable inner variables are needed.

The choice is not unique, though the accuracy of the resulting formula will depend on it. Van Dyke makes the choice

$$Y = \epsilon^{1/2} y$$

$$X = \frac{x}{\epsilon^{1/2}} \quad (216)$$

for the sake of illustrating the method. Then the differential equation becomes

$$(X + Y) \frac{dY}{dX} + Y = \sqrt{\epsilon} . \quad (217)$$

If

$$Y = Y_0 + \sqrt{\epsilon} Y_1 + \epsilon Y_2 + \dots ,$$

the following coupled equations result:

$$(X + Y_0) Y_0' + Y_0 = 0 \quad :O(1)$$

$$Y_1 Y_0' + Y_1' (X + Y_0) + Y_1 = 1 \quad :O(\sqrt{\epsilon})$$

$$Y_2 Y_0' + Y_1 Y_1' + Y_2' (X + Y_0) + Y_2 = 0 \quad :O(\epsilon)$$

The right-hand side of the first of these equations is a perfect differential; therefore,

$$X Y_0' + \frac{Y_0^2}{2} = A \quad (A = \text{constant of integration})$$

or

$$Y_0 = -X + \sqrt{X^2 + 2A} . \quad (218)$$

(I chose the upper sign because I want Y_0 to be positive.)

The integration constant A is found by matching the leading term of Eq. (218) with the leading term of Eq. (215). To do this, rewrite Eq. (218) in "outer" variables

$$\sqrt{\epsilon y_0} = -\frac{x}{\sqrt{\epsilon}} + \sqrt{\frac{x^2}{\epsilon} + 2A} . \quad (219)$$

If Eq. (219) is expanded in powers of ϵ , one finds to lowest order in ϵ that

$$y_0 = \frac{A}{x} .$$

Comparing this with the leading term of Eq. (215) for small x , one finds $A = 1$ and

$$Y_0 = -X + \sqrt{X^2 + 2} . \quad (220)$$

The equation for Y_1 can now be solved; it, too, is a perfect differential.

$$Y_1(X + Y_0) = X + B \quad (B = \text{constant of integration})$$

$$Y_1 = \frac{X + B}{X + Y_0} = \frac{X + B}{\sqrt{X^2 + 2}} . \quad (221)$$

Again the integration constant is determined by matching inner and outer solutions. In outer variables, Eq. (221) becomes

$$\sqrt{\epsilon} y_1 = \frac{x/\sqrt{\epsilon} + B}{\sqrt{(x^2/\epsilon) + 2}} . \quad (222)$$

When expanded in terms of ϵ , $y_0 + \sqrt{\epsilon} y_1$ becomes

$$y_0 + \sqrt{\epsilon} y_1 = \left(\frac{1}{x} - \frac{\epsilon}{2x^3} + \dots \right) + \left(1 + \frac{B\sqrt{\epsilon}}{x} - \frac{\epsilon}{x^2} + \dots \right) .$$

When compared with Eq. (215), $B = 0$ because no terms proportional to $\sqrt{\epsilon}$ occur in Eq. (215). Thus

$$Y_1 = \frac{X}{\sqrt{X^2 + 2}} . \quad (223)$$

The equation for Y_2 can now be solved.

$$Y_2 = \frac{C - Y_1^2}{2\sqrt{X^2 + 2}} = \frac{C + (C - 1)X^2/2}{(X^2 + 2)^{3/2}} \quad C = \text{integration constant} .$$

Then

$$\begin{aligned} y_0 + \sqrt{\epsilon} y_1 + \epsilon y_2 &= \left(\frac{1}{x} - \frac{\epsilon}{2x^3} + \dots \right) + \left(1 - \frac{\epsilon}{x^2} + \dots \right) + \epsilon \frac{C - 1}{2x} \\ &= \frac{1 + x}{x} + \epsilon \frac{(C - 1)x^2 - 2x - 1}{2x^3} + \dots \end{aligned}$$

Comparing with Eq. (215), one sees that $C = 4$. Thus

$$Y_2 = \frac{4 + (3/2)X^2}{(X^2 + 2)^{3/2}} . \quad (224)$$

When $x = 0$, $X = 0$, and $Y_0 = \sqrt{2}$, $Y_1 = 0$, $Y_2 = \sqrt{2}$. Thus

$$Y(0) = \sqrt{2} + \epsilon\sqrt{2} + \dots = \sqrt{2}(1 + \epsilon + \dots) \quad (225a)$$

$$y(0) = \sqrt{\frac{2}{\epsilon}} (1 + \epsilon + \dots) \quad (225b)$$

Because the differential equation is invariant to group (213), an integrating factor can be found. The exact solution is

$$y = \sqrt{\left(\frac{x}{\epsilon}\right)^2 + \frac{2}{\epsilon}(1+x) + 4} - \frac{x}{\epsilon} \quad (226)$$

and

$$y(0) = \sqrt{\frac{2}{\epsilon} + 4} = \sqrt{\frac{2}{\epsilon}} (1 + \epsilon + \dots) \quad (227)$$

Thus Eq. (225b) is correct to order ϵ , as expected.

Lighthill's Method

In Lighthill's method, also known as the method of strained coordinates, both the dependent and independent variables are expanded in powers of the smallness parameter ϵ . The expansion of the independent variable allows the singularity to shift towards its correct position. The method is best clarified by example. Turn again to differential equation (212).

$$y = y_0(t) + \epsilon y_1(t) + \dots \quad (228a)$$

$$x = t + \epsilon x_1(t) + \dots \quad (228b)$$

Here t is an auxiliary variable, which in lowest order is identical with x .

If Eqs. (228a) and (228b) are substituted into (212), then

$$ty_0' + y_0 = 1 \quad \text{order (1)} \quad (229a)$$

$$ty_1' + y_1 + y_0' (y_0 + x_1 - tx_1') = 0 \quad \text{order } (\epsilon) \quad (229b)$$

. . . . .

(The primes denote differentiation with respect to t .) The first of these equations is easy to solve:

$$y_0 = \frac{t + A}{t}, \quad A = \text{constant of integration}.$$

If $y_0(1) = 2$, $A = 1$. Thus the lowest-order solution is

$$y = y_0 = \frac{t + 1}{t}, \quad x = t, \quad (230)$$

which is the ordinary perturbation solution. The next higher order solution depends on the solution of Eq. (229b), which can now be written as

$$(ty_1)' = \frac{1 + t}{t^3} - \left(\frac{x_1}{t} \right)'. \quad (231)$$

Equation (231) can be solved easily, though I actually do not need to do so. It is already apparent that if the right-hand side of Eq. (231) is not zero, y_1 will be more singular at $t = 0$ than y_0 . In fact

$$ty_1 = -\frac{x_1}{t} - \frac{1}{2t^2} - \frac{1}{t} + B, \quad B = \text{constant of integration}. \quad (232)$$

Lighthill's principle is that higher approximations shall be no more singular than the first. This means that the first three terms on the right-hand side of Eq. (232) must sum to zero, i.e.,

$$x_1 = - \frac{2t + 1}{2t} . \quad (233)$$

Thus Lighthill's principle gives the lowest-order "strain" relation between x and t

$$x = t - \epsilon \frac{2t + 1}{2t} . \quad (234)$$

When $x = 0$,

$$t = \frac{1}{2} (\epsilon + \sqrt{2\epsilon + \epsilon^2}) = \sqrt{\frac{\epsilon}{2}} + . . . \quad (235)$$

so that to lowest order,

$$y(0) \sim y_0(0) = \sqrt{\frac{2}{\epsilon}} + ... \quad (236)$$

The computations involved in continuing this process are tedious because of the necessity to continue both series, and I shall stop here. Lighthill's method is somewhat mysterious and for that reason has been the subject of much research. My own preference is for the method of matched inner and outer expansions.

Method of Casting out Secular Terms (Poincaré-Linsted)

This method is used to study perturbed periodic motions, e.g., as in the following problem involving Duffing's equation:

$$\frac{d^2x}{dt^2} + x + \epsilon x^3 = 0 \quad (\epsilon \ll 1) \quad (237a)$$

$$x(0) = 1, \quad \left(\frac{dx}{dt}\right)_0 = 0. \quad (237b)$$

An ordinary perturbation expansion gives

$$x_0'' + x_0 = 0 \quad (238a)$$

$$x_1'' + x_1 = -x_0^3. \quad (238b)$$

The lowest-order solution is $x_0 = \cos t$. Now

$$x_0^3 = \cos^3 t = \frac{1}{4} \cos 3t + \frac{3}{4} \cos t$$

so that Eq. (238b) becomes

$$x_1'' + x_1 = -\frac{1}{4} \cos 3t - \frac{3}{4} \cos t. \quad (239)$$

The first term on the right-hand side leads to the special solution $1/32 \times \cos 3t$. The second term on the right-hand side is slightly more difficult, but a special solution can be found by the method of variation of parameters

$$-\frac{3}{4} \int \sin(t - t') \cos t' dt' = \underbrace{-\frac{3}{8} t \sin t - \frac{3}{16} \cos t}_{\text{special solution}},$$

Thus

$$x_1 = \frac{1}{32} \cos 3t - \frac{3}{8} t \sin t - \frac{1}{32} \cos t + \frac{3}{8} \sin t, \quad (240)$$

which satisfies the initial conditions $x_1(0) = x_1'(0) = 0$.

The difficulty with this procedure is the occurrence of the "secular" term $t \sin t$ in what is expected to be a purely periodic problem. According to Poincaré and Linsted, the difficulty arises because the perturbation ϵx^3 causes a shift in the fundamental frequency of the vibration. They proceed by setting $\tau = \omega t$ where

$$x = x_0(\tau) + \epsilon x_1(\tau) + \dots \quad (241a)$$

$$\omega = 1 + \epsilon \omega_1 + \dots \quad (241b)$$

Now Eq. (237a) becomes

$$\omega^2 \frac{d^2 x}{d\tau^2} + x + \epsilon x^3 = 0 ,$$

which gives

$$x_0'' + x_0 = 0 \quad x_0 = x_0(\tau) \quad (242a)$$

$$2\omega_1 x_0'' + x_1'' + x_1 + x_0^3 = 0 \quad x_1 = x_1(\tau) . \quad (242b)$$

Now

$$x_0(\tau) = \cos \tau$$

so that Eq. (242b) becomes

$$x_1'' + x_1 = -\cos^3 \tau + 2\omega_1 \cos \tau = -\frac{1}{4} \cos 3\tau - \frac{3}{4} \cos \tau + 2\omega_1 \cos \tau . \quad (243)$$

The secular terms will be absent if ω_1 is chosen equal to $3/8$. Then

$$w = 1 + \frac{3}{8} \epsilon + \dots, \quad \tau = wt \quad (244a)$$

$$x = \cos \tau + \frac{\epsilon}{32} (\cos 3\tau - \cos \tau) + \dots \quad (244b)$$

Summary and Conclusions

Perturbation theory is midway between art and science. When difficulties occur, one must rely on physical understanding of the situation to fashion tools to circumvent the difficulties. I hope the examples given here have shown this. Finally, perturbation theory is at its best when low-order (usually first-order) solutions are sought because the computations involved in higher order theory are often very time consuming.

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