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THE IMPLoding TO EQUILIBRIUM OF HELICALLY SYMMETRIC
THETA PINCHES

by

Nazih Nicholas Sharky

1978

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ABSTRACT

THE IMPLODING TO EQUILIBRIUM OF HELICALLY SYMMETRIC THETA PINCHES

Nazih Nicholas Sharky

The time-dependent, single-fluid, dissipative magnetohydrodynamic equations are solved in helical coordinates (r, ϕ) , where $\phi = \theta - kz$, $k = \frac{2\pi}{L}$ and L is the periodicity length in the z -direction. The two-dimensional numerical calculations simulate theta pinches which have an $\ell=1$ helical field added to them. Given the applied magnetic fields and the initial state of the plasma, we study the time evolution of the system. The plasma is found to experience two kinds of oscillations, occurring on different time scales. These are the radial compression oscillations, and the slower helical oscillations of the plasma column. The plasma motion is followed until these oscillations disappear and an equilibrium is nearly reached. Hence given the amplitude and the rise time of the applied magnetic fields, our calculations allow us to relate the initial state of a cold, homogeneous plasma to its final equilibrium state where it is heated and compressed.

Furthermore, using the parameters of various experiments, we evaluate the amount of plasma helical distortion. Analytical expressions for this quantity, in terms of physical parameters, are all obtained with the assumption that the helical distortion δ_1 (normalized to the plasma radius) is small compared to unity. This assumption being violated in most experiments, we consider the regime where $\delta_1 \sim 1$. This experimentally interesting case has to be solved numerically. Our numerical results are then

compared with the analytical predictions based on the sharp boundary model, which are usually used in designing the experiments. We find that for Scyllac parameters, the plasma helical distortion agrees within 10% with the value predicted by the sharp boundary theory (of Blank et al.), although the assumptions made in this theory are violated. The parameters which affect δ_1 are then varied, yielding values of δ_1 between 0.4 and 1.5. In all cases studied, good agreement is obtained with sharp boundary theory regarding the equilibrium value of δ_1 . However, we find some helical equilibria which violate the condition $\beta \delta_1 < \frac{\pi^2}{16}$ (where β is the ratio of plasma pressure to the external magnetic field pressure) predicted by Freidberg's analytical theory with finite δ_1 .

Besides these two-dimensional calculations, two one-dimensional codes are developed in order to check our two-dimensional code, and to study the effects of various physical and numerical factors on the results. Several boundary conditions are tried at the center. The method of averaging is found to be more numerically dissipative than the method of applying one-sided difference equations at the center. However, the smoothing effects of the averaging method on the two-dimensional results are usually small. Finally, the numerical dissipation of the schemes we used, is found to have a minor effect on the results during the initial phase of the motion. However, for long-time calculations the amount of numerical dissipation should be minimized, since its cumulative effects on the results may be important.

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Chapter I. INTRODUCTION

The linear theta pinch is an experimental configuration used for the production of plasmas of high density and temperature. Unlike the z-pinch it is not subject to dangerous instabilities during the lifetime of the plasma. However, the end losses in such an open system present a serious problem. To overcome this difficulty, the cylindrical configuration can be replaced by a toroidal one. But in this case the plasma experiences an outward toroidal force which cannot be opposed by a purely toroidal magnetic field. One way to cure this lack of equilibrium is by adding small amounts of helical fields to the main toroidal field.^{1,2} The resulting systems, called high-beta stellarators, are characterized by a plasma equilibrium with essentially no net toroidal current. This method has been used in the Scyllac³ and ISAR T1⁴ experiments.

In this chapter we review the equilibrium theory which formed the original theoretical basis for high-beta stellarators, and summarize the important experimental results related to these systems. We define the major points studied in this work, and briefly describe a typical time-dependent calculation performed with our numerical code.

The configuration of helical fields is first introduced. Consider ℓ pairs of wires wound on a cylinder of radius R_0 , having a constant pitch L , and carrying currents alternately $+I$ and $-I$. Fig. (1) shows the case of $\ell = 2$. Using cylindrical coordinates (r, θ, z) , it can be shown that the vacuum magnetic scalar potential in the region $r < R_0$, is given by^{1,5}:

$$\Phi = \frac{8k\ell R_0}{c} \sum_{j=0}^{\infty} K'_N(NkR_0) I_N(Nkr) \sin[N(\theta - kz)], \quad (1)$$

Fig. 1: $l = 2$ system of helical windings.

where $k = \frac{2\pi}{L}$, $N = (2j + 1)\ell$, c is the speed of light, and I_N and K_N are the modified Bessel functions. Besides these $\ell \neq 0$ vacuum helical fields, there is an $\ell = 0$ magnetic field, which is periodic in z , but has no θ dependence.

Consider the sharp boundary, constant pressure model for a plasma column of radius a . A magnetic field characterized by ℓ produces then a first order distortion in the plasma surface of the form:

$$r = a[1 + \delta_\ell \cos(\ell\theta - kz)]$$

It has been shown that the combination of fields characterized by ℓ and $\ell \pm 1$, results in a transverse force which can be made to counteract the outward toroidal force. The following expressions are derived for this transverse force per unit length²:

$$F_{\ell, \ell+1} = \frac{1}{8} \beta (2 - \beta) B_0^2 k^2 a^3 \delta_\ell \delta_{\ell+1}, \quad (2.a)$$

$$F_{0,1} = \frac{1}{8} \beta (3 - 2\beta) B_0^2 k^2 a^3 \delta_1 \delta_0, \quad (2.b)$$

where β is the ratio of plasma pressure to the external magnetic field pressure, and B_0 is the theta pinch field outside the plasma. Both equations above are valid if $ka \ll 1$, and Eq. (2.a) is for the case when both fields are characterized by $\ell \neq 0$. The toroidal equilibrium condition is then obtained by equating the interference force per unit length $F_{\ell, \ell+1}$ or $F_{0,1}$ to the outward toroidal force per unit length: $F_R = \frac{\beta B_0^2 a^2}{4R}$, R being the major radius. The equilibrium condition can be written in the following form:

$$\delta_\ell \delta_{\ell+1} = \left(\frac{2}{2 - \beta} \right) k^2 a R, \quad (3.a)$$

$$\delta_1 \delta_0 = \left(\frac{2}{3 - 2\beta} \right) k^2 a R, \quad (3.b)$$

corresponding to Eqs. (2.a) and (2.b) respectively.

As will be discussed in section 4.1, the equilibrium plasma distortion δ_ℓ can be expressed analytically¹ in terms of the parameters entering the problem, e.g., ka , β and the ratio of helical field to the main theta pinch field. The design values of these parameters are then chosen to satisfy the toroidal equilibrium condition Eq. (3.a) or Eq. (3.b). It should be noted, however, that these equations and the expression of δ_ℓ in terms of physical parameters, were derived with the sharp boundary model and with the assumption $\delta_\ell \ll 1$. This model and assumption are strongly violated in the Scyllac³ and ISAR T1⁴ experiments, thereby possibly affecting the plasma equilibrium in these systems. This point will be discussed further when we present the experimental results.

The $\ell = 1$ configuration is chosen in all high-beta stellarator experiments, because the theory predicts that it is less unstable to the dominant $m = 1$ mode^{6,7}, and it is favorable in view of local stability criteria⁸ as well. To this field and to the main theta pinch field, small amounts of $\ell = 0$ or $\ell = 2$ fields are added. A general review of high-beta stellarator equilibrium and stability theory can be found in Ref. (9).

Some experimental results related to these systems are now presented. The first step in the experimental investigations consisted of adding an $\ell = 1$ field to a straight theta pinch as in Scylla IV-3¹⁰ and ISAR I¹¹. It was found that in the presence of this helical field, a high density and high temperature plasma can be produced by shock heating. The $m = 1$ unstable mode was observed and its growth rate was in approximate agreement with theory⁶. The next set of experiments were performed on a toroidal sector, where the forces $F_{1,2}^{12}$ or $F_{0,1}^{13}$ were found to

counteract the toroidal drift force F_R and increase the plasma lifetime. The behavior of the plasma in a complete torus was then studied in both ISAR T1⁴ and Scyllac³ experiments. The results were generally similar to those found earlier in the sector experiments, the plasma nearly achieving a toroidal equilibrium and then moving to the wall with an $m = 1$ type motion. The plasma confinement time was later extended by feedback techniques¹⁶ and wall stabilization^{17, 18}.

As mentioned above, it is important for the design of various experiments, to test the predictions of sharp boundary theory regarding the toroidal equilibrium condition Eq. (3.a) or (3.b), as well as the equilibrium helical shift δ_ℓ . Both of these theoretical predictions were compared with experimental results under conditions which violate the assumptions made in the theory (namely the plasma profiles were diffuse and $\delta_1 \sim 1$ in the experiments). In some cases there was good agreement with sharp boundary theory, and in others there was some discrepancy. Regarding the toroidal equilibrium condition Eq. (3.a) or (3.b), good agreement was obtained in Scyllac toroidal sector¹³, but there was about 50% discrepancy with the theory in the ISAR T1 experiments⁴. Similarly, regarding the equilibrium helical shift δ_1 , the experimental value was about 30% lower than the sharp boundary prediction in the straight, $\ell = 1$ experiment Helix¹⁵, but the agreement with theory was very good in ISAR T1⁴.

Therefore, the sharp boundary analytical theories do not always correctly predict the plasma behavior in a high-beta stellarator. This is not surprising, since the assumptions and model used in these theories,

are violated in almost all experiments. Naturally, it is desirable to predict theoretically the equilibrium conditions with a model and assumptions as close as possible to the experimental conditions. However, the equilibrium problem of a diffuse plasma in a high-beta stellarator, with arbitrary helical distortions δ_ℓ , has not been solved analytically.

The major aim of our numerical calculations is then to find the normalized helical shift with a diffuse profile model, and with no implied restrictions on δ_ℓ . Our results can then be compared with sharp boundary theory (which is usually used in designing the experiments), to see how well it predicts the equilibrium helical shift when the assumptions made in that theory are violated. Comparison can also be made with experimental results.

Therefore, the purpose of our numerical computations is to help in understanding the experimental results of various helical theta pinches, of straight and toroidal type. Furthermore, they allow us to predict the plasma behavior when physical parameters are varied. We note in passing that although Scyllac's experiments were discontinued near the middle of 1977, our results (obtained before that date¹⁹) and code apply to other helical machines currently in operation^{28, 17} and being planned.¹⁴

Besides studying helical theta pinches, we analyze in this work the effect of various numerical factors on our results, such as numerical dissipation and the boundary condition at the center. This analysis applies additionally to other computations performed by the Columbia University group with a similar two-dimensional code^{20-22, 25}. Furthermore, two one-dimensional codes are developed and used as a validity check on our two-dimensional code.

In order to study the plasma dynamics in the high-beta stellarator configuration, the physical quantities must be allowed to depend on three spatial variables and time. Since a three-dimensional problem requires too much computer time, and since we are mainly interested in the equilibrium dynamics, we assume that the systems we study are helically symmetric, that is, the $\ell = 1$ winding alone is accounted for. This will reduce the spatial dimensionality of the problem to two, but at the same time it will limit the validity of the results to certain experiments, as will be discussed in section 2.1.

A few two-dimensional simulations of a plasma in a pinch configuration have been performed in straight and toroidal geometries^{20,21,23,24}. These calculations do not go beyond the first two or three compression oscillations of the plasma. In this work, however, and for other pinch configurations simulated recently,^{22,25-27} the motion of the plasma is followed until the oscillations are damped and a quasi-equilibrium is reached. The results of a typical calculation are now described briefly.

The plasma, which is initially cold and homogeneous, is heated and compressed by a rising magnetic field. After reaching its peak value, the applied field remains essentially constant; this corresponds to crow-barring (shorting) the capacitor banks in the experiment. In our long-time computations, which are carried out beyond the field rise time, the plasma response to the applied fields appears in the form of a radial and helical motion. After experiencing damped, radial compression oscillations, the plasma reaches a state where it is compressed, hot and in radial equilibrium with the main theta pinch field. At the same time, the plasma moves helically in response to the applied helical field. This results in damped, helical oscillations before a final helical

equilibrium is reached. These oscillations take place on a longer time scale than the radial compression oscillations. When a toroidal pinch is simulated, and the calculation is carried out for a sufficiently long time, toroidal oscillations (instead of helical ones) are then observed.^{22, 25}

A second and different approach has been used in this work, where the initial plasma is already compressed and in radial equilibrium, instead of being cold and homogeneous. When the helical field is applied, the plasma then seeks a new equilibrium by taking a helical shape. This method avoids the phase of fast radial compressions, where some complicated anomalous diffusion usually takes place. However, it describes well the phase of slower helical oscillations of the plasma column.

The long-time results of these two dynamic approaches, especially the plasma helical shift near equilibrium, are then compared with sharp boundary equilibrium theory¹ and with the results of various experiments. It should be kept in mind, however, that unstable modes whose periodicity length is different from the one assumed in our helically symmetric model, will not appear in the results. Hence our calculations can predict the behavior of the plasma up to the onset time of the fastest growing instabilities. In some experiments it is expected that finite Larmor radius effects²⁸ or wall stabilization^{17, 18} can delay, or even suppress, the major instabilities. In other experiments feedback stabilization¹⁶ is used. In these cases the long-time behavior of the plasma can be predicted reasonably well by our calculations. Finally, end loss effects in straight systems can be neglected up to a time²⁹

$$t \approx \frac{\ell}{v_s \sqrt{1 - \beta}}, \text{ where } \ell \text{ is the machine length and } v_s \text{ is the sound speed.}$$

Since our calculations do not exceed this time (which is typically 10 to 50 μsec in Scylla IV-3¹⁰ and ISAR I¹¹ experiments), it is reasonable to neglect end loss effects in our code.

In the next chapter, where we basically follow Weitzner's formulation³⁰, the plasma model and equations are introduced. The methods of numerically solving the equations are presented in the third chapter, and the results are shown and discussed in the fourth chapter.

Chapter II. PHYSICAL AND MATHEMATICAL FORMULATION OF THE PROBLEM

2.1 Helical Symmetry and Electromagnetic Quantities:

Throughout this work a straight cylindrical coordinate system is used. A physical or geometric quantity is said to be helically symmetric if it spatially depends only on the two variables r and $\phi = \theta - kz$, where $k = 2\pi/L$ and L is the periodicity length in the z -direction. Hence the assumption of helical symmetry reduces the number of spatial variables by one; naturally it limits the range of validity of the theory in that all perturbations studied have this same symmetry.

First we discuss this point in some detail. By Eq. (1) we expect our results to apply to a straight theta pinch with an additional helical field corresponding to one value of ℓ , if $\ell \neq 0$; e.g., the Scylla IV-3¹⁰, ISAR-1¹¹ and Helix¹⁷ experiments with their $\ell = 1$ field. Furthermore, our results approximate the motion in a toroidal theta pinch if the inverse aspect ratio is small, and if the effects of the second helical field - e.g., the $\ell = 0$ or $\ell = 2$ field, when the main helical field is of the $\ell = 1$ type - are of higher order. The first condition is satisfied in the Scyllac experiments where the aspect ratio is approximately equal to 400. It is found experimentally¹⁶ that without helical fields the plasma shift to the outer wall due to toroidal curvature, occurs in about 6 μsec . However, with the helical fields and feedback system, the lifetime of the plasma is extended to about 20 μsec ¹⁶. Our numerical calculations do not go beyond this time, and enough information about the plasma equilibrium is obtained from our results. The other condition is related to the

effects of the second helical field, such as helical oscillations and changes in the plasma shape. It can be shown² that even when the strengths of the $\ell = 0$ and $\ell = 1$ fields are comparable, the $\ell = 0$ plasma distortion is much smaller than that due to the $\ell = 1$ field (provided that $ka \ll 1$, where a is the plasma radius). Therefore, the plasma shape and its helical distortion obtained in our calculations, should approximate quite well the results of a toroidal experiment such as Scyllac. However, the effects of the interference force $F_{\ell, \ell+1}$, mentioned in the introduction, can be seen only with a three-dimensional calculation.

We now turn to some geometric considerations related to helical symmetry. With the assumption of this symmetry we can use the r - ϕ plane to represent various geometric curves and surfaces needed in the problem. The point (r_0, ϕ_0) in the r - ϕ plane corresponds to the helix

$$\vec{r} = r_0 \hat{r} (\phi_0 + kz) + z \hat{z}$$

where $\hat{r}(\theta)$ and $\hat{z}$ are the unit vectors respectively in the r and z directions. In other words, if a physical quantity is helically symmetric it is constant along a helix (r, ϕ) .

We introduce now two surfaces. The first one (S_1), shown in Fig. (2), is the image of the straight line in the r - ϕ plane:

$$0 \leq r \leq R, \quad \phi = \phi_0.$$

The equation of this surface in three-dimensional space is:

$$\vec{r} = r' \hat{r} (\phi_0 + kz) + z \hat{z}$$

Fig. 2: Helically symmetric surface (S_1).

with: $0 \leq r' \leq R$, and $z_1 \leq z \leq z_2$.

Two linearly independent tangent vectors are:

$$\vec{t}_1 = \frac{d\vec{r}}{dr'} = \hat{r} \quad (4-a)$$

$$\vec{t}_2 = \frac{d\vec{r}}{dz} = kr' \hat{\theta} + \hat{z} , \quad (4-b)$$

the normal vector to the surface is:

$$\vec{n} = \vec{t}_2 \times \vec{t}_1 = \hat{\theta} - kr' \hat{z} , \quad (5-a)$$

and the differential element of area is:

$$d\vec{a} = \vec{n} dr' dz . \quad (5-b)$$

The second surface (S_2) is the image of an arbitrary curve $r = r(\phi)$ in the r - ϕ plane, to be made specific later on to coincide with the compression coil surface. In cylindrical coordinates it is:

$$\vec{r} = r (\hat{\theta} - kz) \hat{r}(\theta) + z\hat{z} . \quad (6)$$

Here two tangent vectors are:

$$\vec{t}_1 = \frac{d\vec{r}}{d\theta} = \frac{dr}{d\phi} \hat{r} + r\hat{\theta}$$

$$\vec{t}_2 = \frac{d\vec{r}}{dz} = -k \frac{dr}{d\phi} \hat{r} + \hat{z} ,$$

an outward normal is:

$$\vec{n} = \vec{t}_1 \times \vec{t}_2 = r\hat{r} - \frac{dr}{d\phi} \hat{\theta} + kr \frac{dr}{d\phi} \hat{z} , \quad (7)$$

and the element of surface area is:

$$d\vec{a} = \vec{n} d\theta dz .$$

The assumption of helical symmetry allows us to introduce two flux functions. If the magnetic field $\vec{B}$ is helically symmetric the equation:

$$\vec{\nabla} \cdot \vec{B} = \frac{1}{r} \frac{\partial}{\partial r} (rB_r) + \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

becomes:

$$\frac{1}{r} \left(\frac{\partial}{\partial r} (rB_r) + \frac{\partial}{\partial \phi} (B_\theta - kr B_z) \right) = 0 .$$

This equation is automatically satisfied if we use a function $\psi(r, \phi, t)$ defined by:

$$B_r = - \frac{1}{r} \frac{\partial \psi}{\partial \phi} \quad (8)$$

$$B_\theta - kr B_z = \frac{\partial \psi}{\partial r} . \quad (9)$$

As will be discussed in Section (2.3), the displacement current is neglected in Ampere's law. Therefore, if the current density $\vec{J}$ is helically symmetric we get:

$$\vec{\nabla} \cdot \vec{J} = \frac{1}{r} \left(\frac{\partial}{\partial r} (rJ_r) + \frac{\partial}{\partial \phi} (J_\theta - kr J_z) \right) = 0 ,$$

which allows us to introduce a function $\chi(r, \phi, t)$ such that:

$$J_r = \frac{1}{r} \frac{\partial \chi}{\partial \phi} \quad (10)$$

$$J_\theta - kr J_z = - \frac{\partial \chi}{\partial r} . \quad (11)$$

The physical meaning of these two functions is seen if we calculate the flux of $\vec{B}$ through surface (S_1) defined above. Using

Eqs. (5-a), (5-b) and (9) we get:

$$\int_{S_1} \vec{B} \cdot d\vec{a} = \int_{z_1}^{z_2} \int_0^R (B_\theta - kr B_z) dr dz$$

or:
$$\int_{S_1} \vec{B} \cdot d\vec{a} = (z_2 - z_1) [\psi(R, \phi_0, t) - \psi(r=0, t)] \quad (12)$$

Hence up to an additive function of time, $\psi(r, \phi, t)$ is the magnetic flux per unit length through surface (S_1). Similarly $-\chi(r, \phi, t)$ is the current flux per unit length through surface (S_1), up to an additive function of time. It will be shown below that these functions of time are arbitrary; hence they can be set equal to zero if preferable.

In order to express the three components of $\vec{B}$ and $\vec{J}$ in terms of the fluxes, we relate these two vectors by:

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} \quad (13)$$

From the components of this equation we get:

$$\frac{4\pi}{c} r J_r = \frac{4\pi}{c} \frac{\partial \chi}{\partial \phi} = \frac{\partial}{\partial \phi} (B_z + kr B_\theta)$$

and:
$$\frac{4\pi}{c} (kr J_z - J_\theta) = \frac{4\pi}{c} \frac{\partial \chi}{\partial r} = \frac{\partial}{\partial r} (B_z + kr B_\theta) .$$

Hence:
$$\frac{4\pi}{c} \chi(r, \phi, t) = B_z + kr B_\theta + f(t) ,$$

where $f(t)$ is an arbitrary function of time. If we define the function

$\chi'(r, \phi, t) = \chi(r, \phi, t) - f(t) \frac{c}{4\pi}$, then χ' satisfies the same equations

(10) and (11) which define χ . Hence we can always use χ' instead of

χ without affecting the equations. From now on we will drop the prime

from χ' and use the equation:

$$\frac{4\pi}{c} \chi(r, \phi, t) = B_z + kr B_\theta \quad (14)$$

From Eqs. (9) and (14) we get:

$$B_\theta = \frac{\frac{4\pi}{c} kr \chi + \frac{\partial \psi}{\partial r}}{1 + k^2 r^2} \quad (15)$$

$$B_z = \frac{\frac{4\pi}{c} \chi - kr \frac{\partial \psi}{\partial r}}{1 + k^2 r^2} \quad (16)$$

Then using the z-component of Eq. (13), as well as Eqs. (8) and (15), it is easy to derive the following expression:

$$J_z = \frac{c}{\pi} \Delta^* \psi + \frac{kr \frac{\partial \chi}{\partial r}}{1 + k^2 r^2} + \frac{2k\chi}{(1 + k^2 r^2)^2} \quad (17)$$

where:

$$\Delta^* \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r \frac{\partial \psi}{\partial r}}{1 + k^2 r^2} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2}$$

Then from Eqs. (11) and (17) we get a corresponding expression for J_θ . All components of $\vec{B}$ and $\vec{J}$ are now expressible in terms of ψ, χ and their spatial derivatives. As usual, the assumption of symmetry (in this case, helical symmetry) allows us to decrease the number of dependent variables by one.

Finally Faraday's law in integral form:

$$\frac{1}{c} \frac{d}{dt} \oint \vec{B} \cdot d\vec{a} = - \oint \vec{E} \cdot d\vec{\ell} ,$$

is applied to a portion of surface (S_1) of length $L = \frac{2\pi}{k}$ in the z-direction. We assume that $\vec{B}$ is helically symmetric and that the electric field $\vec{E}$ is periodic in z of period L, as will be shown later.

Then using Eqs. (4-b) and (12) we get:

$$\frac{L}{c} \left(\frac{\partial \psi}{\partial t}(r, \phi, t) - \frac{\partial}{\partial t} \psi(r=0, t) \right) = - \left(\int_0^L E_z(r=0, t) dz - \int_0^L (E_z + kr E_\theta) dz \right)$$

where the contributions to the line integral from the two lines at $z = 0$ and $z = L$, cancel because of the periodicity of $\vec{E}$. The direction of integration along the contour follows from the direction of $\vec{n}$ as shown in Fig. (3).

Now if we choose the function of time in ψ such that:

$$\psi'(r, \phi, t) = \psi(r, \phi, t) + g(t)$$

where:
$$g(t) = \frac{c}{L} \int_0^t \int_0^L E_z(r=0, t') dz dt' - \psi(r=0, t),$$

we get the equation:

$$\frac{\partial \psi'}{\partial t} = \frac{c}{L} \int_0^L (E_z + kr E_\theta) dz \quad (18)$$

Since Eqs. (8), (15), (16) and (17), which define $\vec{B}$ and $\vec{J}$, depend only on the spatial derivatives of $\psi(r, \phi, t)$, they will have exactly the same form in terms of $\psi'(r, \phi, t)$. Therefore from now on, we will use ψ' instead of ψ , and drop the prime. All equations derived so far will then remain the same and Eq. (18) becomes:

$$\frac{\partial \psi}{\partial t}(r, \phi, t) = \frac{c}{L} \int_0^L (E_z + kr E_\theta) dz \quad (19)$$

2.2 Model and Electromagnetic Equations

In the experiments performed in both open ended and

Fig. 3: Contour used in Faraday's law leading to Eq. (18).

toroidal theta-pinchs, e.g., Scylla IV-3¹⁰, ISAR-1¹¹, Helix¹⁷, Scyllac¹⁴, ISART-1¹⁵, etc., the plasma is enclosed in an insulating discharge tube (vacuum vessel) which is either uniform or helically shaped. Outside the discharge tube there is a compression coil, and between them there is usually a vacuum region. The main magnetic field is produced by the θ -current applied to the compression coil. The helical field can be generated either by placing helical windings between the compression coil and the discharge tube (as shown in Fig. 4), or by shaping helically the inner surface of the compression coil.

Therefore we choose a model which consists of a plasma region bounded by an insulating discharge tube, and a vacuum region between the discharge tube and the compression coil. The system is assumed to be infinitely long in the z -direction (i.e., with infinite aspect ratio). The region inside the discharge tube will be called the plasma region, even though a vacuum or low density region will develop outside the main plasma column. Both the discharge tube and the compression coil are helically symmetric, except for the slot in the compression coil. This slot is usually straight. We can, therefore, assume that the magnetic field is helically symmetric in both plasma and vacuum regions. The electric field, on the other hand, is helically symmetric only inside the discharge tube; whereas in the vacuum region it is periodic in z , of period $L = 2\pi/k$. The helical symmetry of $\vec{E}$ is destroyed there because of the electromotive force which develops across the (non-helically-symmetric) slot. All other physical quantities are assumed to be helically symmetric

Fig. 4: Schematic view of a straight theta pinch experiment with helical windings for generating the $\ell = 1$ field.

in the plasma region.

Two ways of generating the helical fields are studied, and they are shown in Figs. (5.a) and (5.b). These two figures are in the r - ϕ plane; in other words, they are at constant z , and as we vary z they rotate about the center of the figures which is the axis of the discharge tube. In Fig. (5.a) both the discharge tube and the compression coil are straight cylinders, and between them there are two $\ell = 1$ helical windings, as is with ISAR-1. In Fig. (5.b) the discharge tube is a straight cylinder, but the compression coil has helical grooves in its inner surface. This approximates Scyllac's helically shaped compression coil.

Before investigating the mathematical consistency of the proposed model, we discuss the boundary conditions for ψ and χ . First it is easy to see that on the perfectly conducting compression coil ψ is a function of time alone. Using the condition that $\vec{n} \cdot \vec{B} = 0$ near a perfect conductor, along with Eq. (7) for a helically symmetric surface, and Eqs. (8), (15) and (16), we find that:

$$\vec{n} \cdot \vec{B} = -\frac{\partial \psi}{\partial \phi} - \frac{dr}{d\phi} \frac{\partial \psi}{\partial r} = 0$$

on the compression coil. This implies that ψ on the helically symmetric compression coil depends only on time.

In order to find the value of ψ on the compression coil, we can use Eq. (19) where the integral is evaluated over one period of a helix adjacent to the compression coil. From Eq. (4-b) we see that $(E_z + kr E_\theta)$ is the component of E tangential to the helically symmetric compression coil. Hence the integrand vanishes except near

Fig. 5: View in the $r - \phi$ plane of a model with (a) helical windings, (b) helical grooves in the compression coil.

the slot. Since the helical contour cuts the non-helically-symmetric slot once, the integral will be related to the electromotive force across the slot which we define by:

$$V(t) = - \int \vec{E} \cdot d\vec{\ell} ,$$

where the path of integration is along the $+\theta$ direction. Finally along a helix, the variable $\phi = \theta - kz$ remains constant; hence if z increases, θ increases, and we get the equation:

$$\left(\frac{\partial \psi}{\partial t} \right)_{c.c.} = -\frac{c}{L} V(t) \quad (20)$$

which can be used as a boundary condition for ψ on the compression coil.

Next we can easily prove that in the vacuum region, $\chi(r, \phi, t)$ is a function of time alone. In the case of a grooved compression coil with no helical windings, $\vec{J} = 0$ throughout the vacuum region; hence from Eqs. (10) and (11), the spatial derivatives of χ must vanish there. It remains to prove now that in the presence of helical windings, χ does not experience a jump across them.

Assume that there is a current carrying helical winding at $r = r_0$, $\phi = \phi_0$. Consider then a surface (S_3) defined by:

$$r_0 - \Delta \leq r \leq r_0 + \Delta \quad , \quad \phi = \phi_0 \quad \text{where} \quad \Delta \ll r_0 .$$

This surface has the same tangent and normal vectors as surface (S_1), given by Eqs. (4) and (5). Hence:

$$\begin{aligned}
\int_{S_3} \vec{J} \cdot d\vec{a} &= \int_{z_1}^{z_2} \int_{r_0-\Delta}^{r_0+\Delta} (J_\theta - kr J_z) dr dz \\
&= - \int_{z_1}^{z_2} \int_{r_0-\Delta}^{r_0+\Delta} \frac{\partial \chi}{\partial r} dr \\
&= - (z_2 - z_1) [\chi(r_0+\Delta, \phi_0, t) - \chi(r_0-\Delta, \phi_0, t)] , \quad (21)
\end{aligned}$$

where Eq. (11) has been used. Since the current in the helical winding is tangent to surface (S_3), the left-hand-side of Eq. (21) vanishes, and we find that $\chi(r, \phi, t)$ does not experience any jump across a current-carrying helical winding. Hence χ is a function of time alone in the vacuum region.

In order to determine $\chi(t)$ in this region, we have to relate it to the current in the compression coil. Let then $r(\phi)$ be the equation of the inner surface of the compression coil, and $\theta(z)$ the equation of a curve on this surface representing the slot. Then using Eq. (6) the equation of the slot can be written in the form:

$$\vec{r}(z) = r(\theta(z) - kz) \hat{r}(\theta(z)) + z\hat{z} . \quad (22)$$

Apply now Ampere's circuital law to a narrow contour (C) of length L in the z -direction, shown in Fig. (6). The contour consists of two curves parallel to one edge of the slot, one just above it (inside the conductor) and the other just below it (in the vacuum region).

We get then:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \int_0^L \vec{B} \cdot \frac{d\vec{r}}{dz} dz = \frac{4\pi}{c} I(t) , \quad (23)$$

where the second integral is evaluated over the part of curve (C) below the conductor, $I(t)$ is the current per helical period L across

Fig. 6: Closed contour (c) used in Eq. (23).

the edge of the slot, and $\frac{d\vec{r}}{dz}$ is a vector tangent to the slot which can be obtained by deriving Eq. (22) with respect to z . It is then easy to prove that Eq. (23) can be written in the following form:

$$I(t) = \chi(t) \int_0^L dz \frac{1+kr^2}{1+k^2r^2} \frac{d\theta}{dz} - \frac{c}{4\pi} \int_0^{2\pi} d\phi \left[\frac{r}{1+k^2r^2} \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{dr}{d\phi} \frac{\partial \psi}{\partial \phi} \right] \quad (24)$$

where the first integral is evaluated along the slot and the second over the inner surface of the compression coil. Eq. (24) can be used to determine $\chi(t)$ in the vacuum region if the current $I(t)$ per helical period across an edge of the slot is known.

Eq. (24) can be rewritten in another useful form if we make use of the following relation:

$$\frac{\vec{n} \cdot \vec{\nabla} \psi}{1+k^2r^2} = \frac{r}{1+k^2r^2} \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{dr}{d\phi} \frac{\partial \psi}{\partial \phi} \quad (25)$$

where the normal vector $\vec{n}$ is given by Eq. (7). Then using Gauss's theorem, the surface integral in Eq. (24) can be converted into a volume integral. After some manipulations and use of Eq. (17) we get:

$$I(t) = L \chi(t) - I_z(t) \quad , \quad (26)$$

where: $I_z(t) = \int r dr d\phi J_z(r, \phi)$

is the net z -current in the plasma region as well as in the vacuum region if there are helical windings in the latter. Eq. (26) can also be used to determine $\chi(t)$ if $I(t)$ and $I_z(t)$ are known.

Therefore the values of $\psi(t)$ on the compression coil and $\chi(t)$ in the vacuum region, can be found from Eqs. (20) and (24) if we know the two input functions of time $V(t)$ and $I(t)$ which are related by an external

circuit equation. Instead of specifying these two functions, we can prescribe the currents $I(t)$ and $I_z(t)$ as was mentioned above. In all helically symmetric theta pinches, it is found experimentally that the net plasma z-current (or toroidal current in a closed system), is of the order of a KAmper or less. On the other hand the current $I(t)$ per length L is usually of the order of a MAmper. Therefore we can safely neglect $I_z(t)$ in Eq. (26), even when helical windings are placed in the vacuum region, because these windings carry equal and opposite currents. We then get the following equation:

$$\chi(t) = \frac{I(t)}{L} \quad (27)$$

Now if we combine Eqs. (24) and (27) we get:

$$\int_0^{2\pi} d\phi \left[\frac{r}{1+k^2r^2} \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{dr}{d\phi} \frac{\partial \psi}{\partial \phi} \right] = - \frac{4\pi}{c} \frac{I(t)}{2\pi} \int_0^{2\pi} d\phi \frac{k^2r^2}{1+k^2r^2} \quad (28)$$

where the integrals are evaluated on the surface of the compression coil. Given the current $I(t)$, Eq. (28) can be considered as an implicit boundary condition for ψ on the compression coil. Therefore in order to calculate $\psi(t)$ and $\chi(t)$ we have the choice between Eqs. (20) and (24) or Eqs. (27) and (28). The latter have been generally used because they do not require any knowledge of the circuit impedance.

Then the boundary conditions at the insulating discharge tube can be easily obtained. Since no surface currents can flow on the latter, the tangential components of the magnetic field must be continuous across it. Hence all components of $\vec{B}$ are continuous, and from Eqs. (8), (9), and (14) we see that $\chi(r, \phi, t)$ and the spatial

derivatives of $\psi(r, \phi, t)$ must be continuous across the discharge tube. This is equivalent to the requirement that χ , ψ and the normal derivative of ψ must be continuous, as can be seen from Eq. (25) where $r(\phi)$ is the equation of the surface of the discharge tube.

After having discussed some of the boundary conditions, we derive now the equations for ψ and χ . In the vacuum region, since no currents can flow, $\chi(r, \phi, t) = \chi(t)$, and we get from Eq. (17):

$$\Delta^* \psi = - \frac{4\pi}{c} \frac{2k \chi(t)}{(1 + k^2 r^2)^2} . \quad (29)$$

In the plasma region we can use Eq. (19) along with the assumption of helical symmetry of $\vec{E}$ in that region. A helically symmetric quantity being constant over a helix, we get the following equation inside the discharge tube:

$$\frac{1}{c} \frac{\partial \psi(r, \phi, t)}{\partial t} = E_z(r, \phi, t) + kr E_\theta(r, \phi, t) . \quad (30)$$

The equation for χ in that region can be obtained by deriving Eq. (14) with respect to time and using Faraday's law. This results in:

$$\frac{\partial \chi(r, \phi, t)}{\partial t} = \frac{c^2}{4\pi} \left(\frac{1+k^2 r^2}{r} \frac{\partial E_r}{\partial \phi} - \frac{\partial}{\partial r} (E_\theta - kr E_z) - \frac{1}{r} (E_\theta + kr E_z) \right) . \quad (31)$$

In order to eliminate the electric field from the last two equations, we use the simple form of Ohm's law:

$$\vec{E} = \frac{1}{c} \vec{v} \times \vec{B} + \eta \vec{J} , \quad (32)$$

where $\vec{v}$ is the fluid velocity and η is the scalar electric resistivity.

Before writing the resulting equations for ψ and χ , it is convenient to normalize all variables. We choose the quantities: r_0 , B_0 , ρ_0 and T_0 , which are respectively a typical distance, magnetic field, mass density and temperature. We then define the following quantities:

$$\alpha = kr_0, \quad v_0 = \frac{B_0}{\sqrt{4\pi\rho_0}}, \quad t_0 = \frac{r_0}{v_0}, \quad \psi_0 = r_0 B_0, \\ \chi_0 = \frac{c}{4\pi} B_0, \quad J_0 = \frac{c}{4\pi} \frac{B_0}{r_0}, \quad E_0 = \frac{B_0 v_0}{c}, \quad I_0 = \chi_0 r_0.$$

Normalized variables are then given by:

$$\tilde{r} = \frac{r}{r_0}, \quad \tilde{t} = \frac{t}{t_0}, \quad \tilde{\vec{v}} = \frac{\vec{v}}{v_0}, \quad \tilde{\vec{B}} = \frac{\vec{B}}{B_0}, \dots \text{etc.}$$

If we rewrite all equations in terms of normalized variables and omit the wiggles, the equations defining $\vec{B}$ and $\vec{J}$ become:

$$B_r = -\frac{1}{r} \frac{\partial \psi}{\partial \phi}$$

$$B_\theta = \frac{1}{1 + \alpha^2 r^2} (\alpha r \chi + \frac{\partial \psi}{\partial r}) \quad (33)$$

$$B_z = \frac{1}{1 + \alpha^2 r^2} (\chi - \alpha r \frac{\partial \psi}{\partial r})$$

$$J_r = \frac{1}{r} \frac{\partial \chi}{\partial \phi}$$

$$J_z = \Delta^* \psi + \left(\frac{\alpha r}{1 + \alpha^2 r^2} \right) \frac{\partial \chi}{\partial r} + \frac{2\alpha \chi}{(1 + \alpha^2 r^2)^2}, \quad \text{where} \quad (34)$$

$$\Delta^* \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{1 + \alpha^2 r^2} \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2}$$

$$J_{\theta} = \alpha r J_z - \frac{\partial \chi}{\partial r}.$$

Then $\chi(t)$ in the vacuum region and $\psi(t)$ on the compression coil are given by:

$$\chi(t) = \frac{\alpha}{2\pi} I(t) \quad (35)$$

$$\int_0^{2\pi} d\phi \left(\frac{r}{1 + \alpha^2 r^2} \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{dr}{d\phi} \frac{\partial \psi}{\partial \phi} \right) = - \frac{I(t)}{2\pi} \int_0^{2\pi} d\phi \frac{\alpha^2 r^2}{1 + \alpha^2 r^2} \quad (36)$$

and the equations for $\psi(r, \phi, t)$ and $\chi(r, \phi, t)$ in the plasma region are obtained from Eqs. (30), (31) and (32), and they take the following form:

$$\frac{\partial \psi}{\partial t} = A_4(1 + \alpha^2 r^2) \eta \left(\Delta^* \psi + \frac{2\alpha\chi}{(1 + \alpha^2 r^2)^2} \right) - u \frac{\partial \psi}{\partial r} - \frac{v}{r} \frac{\partial \psi}{\partial \phi}; \quad (37)$$

$$\begin{aligned} \frac{\partial \chi}{\partial t} = & A_4(1 + \alpha^2 r^2) \left(\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r\eta}{1 + \alpha^2 r^2} \frac{\partial \chi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \phi} \left(\eta \frac{\partial \chi}{\partial \phi} \right) \right) \\ & - 2A_4\alpha\eta \left(\Delta^* \psi + \frac{2\alpha\chi}{(1 + \alpha^2 r^2)^2} \right) + \frac{2\alpha}{1 + \alpha^2 r^2} u \frac{\partial \psi}{\partial r} \\ & - \left(\frac{1 + \alpha^2 r^2}{r} \right) \frac{\partial}{\partial r} \left(\frac{r}{1 + \alpha^2 r^2} u\chi \right) - \frac{1}{r} \frac{\partial}{\partial \phi} (v\chi) \\ & + \frac{2\alpha}{r} \left(\frac{1}{1 + \alpha^2 r^2} \right) (v + \alpha r w) \frac{\partial \psi}{\partial \phi} - \frac{1}{r} \frac{\partial \psi}{\partial \phi} \frac{\partial w}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial r} \frac{\partial w}{\partial \phi}; \quad (38) \end{aligned}$$

where:

$$u = v_r$$

$$v = v_{\theta} - \alpha r v_z$$

$$w = v_z + \alpha r v_{\theta}$$

(39)

(v_r, v_θ, v_z) are the components of $\vec{v}$ in cylindrical coordinates,
 $A_4 = \frac{c^2}{4\pi} \frac{n_0 t_0}{r_0^2}$, and n_0 is a normalization constant for the electrical resistivity.

Hence $\chi(r, \phi, t)$ satisfies Eq. (38), a diffusion equation, in the plasma region; it is given by Eq. (35) in the vacuum region, and the boundary condition at the discharge tube is $\chi(r, \phi, t) = \chi(t)$. The function $\psi(r, \phi, t)$, on the other hand, satisfies an elliptic equation in the vacuum region and a diffusion equation in the plasma region, given respectively by Eqs. (29) and (37). The boundary conditions are Eq. (36) on the compression coil, and the continuity of ψ and its normal derivative across the discharge tube (this counts as one boundary condition since we do not specify these quantities). Therefore the model used is mathematically consistent, and it should allow us to determine $\vec{B}$ in both vacuum and plasma regions.

2.3 Fluid Equations

The plasma is described by a single-fluid model with mass density ρ , velocity $\vec{v}$ and temperature T . The conservation form is used for the momentum equations but not for the energy equation. In other words the variables to be solved for are (ρ, m, n, s, T) , where: $m = \rho u$, $n = \rho v$, $s = \rho w$, and (u, v, w) are given by Eqs. (39). We then have the following differential equations in normalized form:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} (rm) - \frac{1}{r} \frac{\partial n}{\partial \phi}, \quad (40)$$

$$\frac{\partial m}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{rm^2}{\rho} \right) - \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{mn}{\rho} \right) + \frac{1}{\rho r} \left(\frac{n + \alpha r s}{1 + \alpha^2 r^2} \right)^2$$

$$-\frac{1}{A_1} \frac{\partial}{\partial r} (\rho T) - \frac{\partial \psi}{\partial r} \left(\Delta^* \psi + \frac{2\alpha\chi}{(1+\alpha^2 r^2)^2} \right) - \frac{1}{\alpha^2 r^2 + 1} \chi \frac{\partial \chi}{\partial r}, \quad (41)$$

$$\begin{aligned} \frac{\partial n}{\partial t} = & -\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r m n}{\rho} \right) - \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{n^2}{\rho} \right) - \left(\frac{1}{1+\alpha^2 r^2} \right) \frac{m}{\rho r} [2\alpha r s + (1-\alpha^2 r^2)n] \\ & - \frac{1}{A_1} \left(\frac{1+\alpha^2 r^2}{r} \right) \frac{\partial}{\partial \phi} (\rho T) - \left(\frac{1+\alpha^2 r^2}{r} \right) \frac{\partial \psi}{\partial \phi} \left(\Delta^* \psi + \frac{2\alpha\chi}{(1+\alpha^2 r^2)^2} \right) - \frac{\chi}{r} \frac{\partial \chi}{\partial \phi}, \quad (42) \end{aligned}$$

$$\frac{\partial s}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r m s}{\rho} \right) - \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{n s}{\rho} \right) + \frac{1}{r} \left(\frac{\partial \chi}{\partial \phi} \frac{\partial \psi}{\partial r} - \frac{\partial \chi}{\partial r} \frac{\partial \psi}{\partial \phi} \right), \quad (43)$$

$$\begin{aligned} \frac{\partial T}{\partial t} = & -\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r m T}{\rho} \right) + \frac{1}{3} \frac{T}{r} \frac{\partial}{\partial r} \left(\frac{r m}{\rho} \right) - \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{n T}{\rho} \right) + \frac{1}{3} \frac{T}{r} \frac{\partial}{\partial \phi} \left(\frac{n}{\rho} \right) \\ & + \frac{A_2}{\rho} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r K \frac{\partial T}{\partial r} \right) + \left(\frac{1+\alpha^2 r^2}{r^2} \right) \frac{\partial}{\partial \phi} \left(K \frac{\partial T}{\partial \phi} \right) \right] \\ & + A_3 \frac{\eta}{\rho} \left[\frac{1}{1+\alpha^2 r^2} \left(\frac{\partial \chi}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \chi}{\partial \phi} \right)^2 + (1+\alpha^2 r^2) \left(\Delta^* \psi + \frac{2\alpha\chi}{(1+\alpha^2 r^2)^2} \right)^2 \right], \quad (44) \end{aligned}$$

where: $A_1 = \frac{v_0^2}{RT_0}$, $A_2 = \frac{2 K_0 t_0}{3R \rho_0 r_0^2}$,

$$A_3 = \frac{2A_1 A_4}{3} = \frac{c^2}{6\pi} \frac{\eta_0 v_0}{RT_0 r_0}$$

Here K_0 and η_0 are normalization constants for the scalar heat conductivity K and electrical resistivity η . R denotes the gas constant $R = \frac{2k}{m_i}$, where k is Boltzman constant and m_i the ion mass.

The ratio γ of specific heats is assumed to be equal to 5/3 for

three degrees of freedom. For the total pressure in the momentum equation the ideal gas law: $p = \rho RT$ is used. The factor 2, resulting from the assumption $p_e \approx p_i$, is absorbed in the constant R . The plasma is assumed to be fully ionized. This is a good approximation in our case, because the initial ionization level in Scylla IV-3¹⁰ and Scyllac's¹⁴ experiments is quite high (90% to 100%).

Classical resistivity in the direction parallel to the magnetic field is given by Spitzer's³¹ expression:

$$\eta_{||, \text{class}} = \frac{m_e v_{\text{class}}}{ne^2} \quad (45)$$

where m_e , n , e and v_{class} are the electron mass, density, charge and Coulomb collision frequency, respectively. Using a constant value for the Coulomb logarithm: $\ln \Lambda = 10$, and a scalar resistivity:

$$\eta_{\text{class}} = \eta_{\perp, \text{class}} = 2\eta_{||, \text{class}}$$

we get:

$$\eta_{\text{class}} = \eta_0 \tilde{\eta}_{\text{class}} = \frac{\eta_0}{\tilde{T}^{3/2}} \quad (46)$$

with: $\eta_0 = 2.3 \times 10^{-13}$ cgs. This constant is obtained here by choosing $kT_0 = 1$ eV.

In order to include the effects of microinstabilities on the resistivity, we replace v_{class} in Eq. (45) by:

$$v = \max(v_{\text{class}}, v_{\text{anom}}) \quad (47)$$

where v_{anom} is an anomalous effective collision frequency between the electrons and the turbulent waves set up by the instabilities. Chodura³²

introduced an empirical expression for this quantity thought to be caused by the ion-acoustic instability:

$$v_{anom} = c \omega_{pi} [1 - e^{-|v_d|/fv_{thi}}] \quad (48)$$

where c and f are adjustable parameters, ω_{pi} and v_{thi} are, respectively, the ion plasma frequency and thermal velocity, and: $v_d = J/ne$ is the electron-ion drift velocity. If departures from a one-dimensional theta pinch are small, it is reasonable to assume that $J_{II} \ll J_{\perp}$, and thereby to use $J_{\perp}$ to calculate v_d . Expression (48) was used in ZT-1 and Scylla 1-B simulations³³, and it was found that for $c = 1$ and $f = 3$, there is good agreement with these experiments. Since Scylla 1-B parameters are close to Scyllac's, the same values for c and f were used in our calculations.

For a more careful treatment of anomalous resistivity, the effects of turbulence must be calculated self-consistently³⁴. Several microinstabilities are usually included, and their relative importance depends on the parameter regime. It is shown³⁵ that for high density pinches (e.g. Scyllac), the lower-hybrid drift instability usually plays a dominant role in the anomalous transport. This approach, which has been used in several one-dimensional codes³⁶, is difficult to apply to our two-dimensional calculations. We chose then the empirical method described above.

For the coefficient of heat conductivity we assume that the helical distortion of the plasma is small, and thereby we use the coefficient $K_{\perp}$ as in the one-dimensional case. The following expression

is then obtained³⁷:

$$K = K_0 (\tilde{K}_e + \tilde{K}_i) = K_0 \tilde{K} \quad (49)$$

where: $K_0 = 8.4 \times 10^3$ cgs,

and:

$$\tilde{K} = \tilde{\eta}^{5/2} \left[\frac{4.66 X_e^2 + 11.9}{X_e^4 + 14.8 X_e^2 + 3.77} + \sqrt{\lambda} \frac{2 \lambda X_e^2 + 2.64}{\lambda^2 X_e^4 + 2.70 \lambda X_e^2 + 0.68} \right].$$

Here $\lambda = m_e/m_i$ is the electron to ion mass ratio, and $X_e = \omega_{ce}/\nu_{ce}$ is the ratio of electron cyclotron frequency to electron collision frequency. The values $kT_0 = 1$ eV and $\ln \Lambda = 10$ were used in the evaluation of K_0 . Finally, it should be noted that η and K in Eq. (44) are the dimensionless coefficients $\tilde{\eta}$ and $\tilde{K}$ defined above.

Viscosity can be safely neglected in the momentum equations because the conditions, under which this approximation is valid³⁸, are satisfied in our parameter regime. However, a numerical viscosity enters the calculations. This point is discussed in Section 3.3.

Chapter III. NUMERICAL METHODS

3.1 The Plasma Region

In this section we describe the numerical method used in solving the equations inside the vacuum vessel. If an initial value problem is solved with an explicit finite difference scheme, it is well-known that numerical stability causes usually severe limitations on the time step Δt . This difficulty can be alleviated by using an implicit difference scheme. However, for a two-dimensional problem, the number of simultaneous equations to be solved becomes too large. This can be remedied by using the alternating direction implicit (A.D.I.) method³⁹, or other fractional step methods⁴⁰. The former, which is chosen in our calculations, is described briefly in the next paragraph.

The set of equations to be solved, Eqs. (37), (38), (40-44), are written in difference form as a sequence of two steps. In the first step all terms involving r -derivatives (including the convection terms) are evaluated implicitly, and the other terms explicitly. In the second step the roles are reversed. It can be shown that when these two approximations are used on alternate time steps, the overall accuracy is second order in time. In our calculations, the transport coefficients and both their spatial derivatives were treated explicitly; hence the truncation error associated with these terms is first order in Δt . All spatial derivatives, however, were approximated by a centered difference, making the truncation error second order in the grid sizes Δr and $\Delta \phi$.

The differential equations are kept in Eulerian form, and the mesh used is not staggered in space or time. Nonlinear terms in the fluid equations, which are to be evaluated at the new time step, are linearized by use of a Taylor expansion in time, with accuracy of order $(\Delta t)^2$. Therefore the method is basically similar to that used by Lui and Chu²⁰ in their pinch simulations. However, the temperature equation is solved here simultaneously with the rest of the fluid equations, and the two electromagnetic equations are solved simultaneously as well. A predictor-corrector step is sometimes added as in Ref. (20), but it is not necessary to include. The splitting method is not used either in our calculations.

The set of difference equations can be written after linearization in block tridiagonal form. The blocks are 5 by 5 and 2 by 2 matrices, for the fluid and field equations, respectively. These systems of linear equations can be solved by one of the standard techniques, e.g. by an iterative method or by Gauss' elimination.⁴¹ We used a method named after Crout⁴², which is similar to Gauss' elimination but it requires fewer operations.

A small artificial dissipation term is usually added to the fluid equations in order to damp possible numerical oscillations. If Q is a fluid variable, then the following term is added to the equation for Q :

$$c \frac{(\Delta r)^2}{r} \frac{\partial}{\partial r} \left(r \left| \frac{\partial u}{\partial r} \right| \frac{\partial Q}{\partial r} \right) + c(\Delta \phi)^2 \frac{\partial}{\partial \phi} \left(\left| \frac{\partial v}{\partial \phi} \right| \frac{\partial Q}{\partial \phi} \right)$$

where c is a small constant (typically $c \approx 0.10$). This Lapidus⁴³ term is mildly dissipative because the numerical dissipation it introduces

is of second order, and it acts at the discontinuities without affecting much the remaining part of the profiles. Furthermore, it is needed mainly during the first implosion, and the constant c can be reduced subsequently in the calculation.

Finally, we discuss the region surrounding the dense plasma. In most pinch experiments, when the applied magnetic field sweeps the uncompressed plasma towards the axis of the machine, some low density plasma is left behind. In other words, diffusion causes the magnetic piston to be "leaky". If this remaining plasma has a very low density, the signal speed becomes very large in this region, causing severe restrictions on the time step. One way to overcome this difficulty is by keeping the density everywhere above some cut-off value (typically 10% of the initial plasma density). This low density plasma filling the region between the vacuum vessel and the main plasma column, corresponds physically to a plasma emitted from the wall, and it is observed in most pinch experiments.

Another way of overcoming the same problem consists of assuming the low density plasma to be a vacuum. This method was used by Brackbill⁴⁴ and Brackbill et al⁴⁵ in their two- and three-dimensional calculations, respectively. The advantage of this approach is that it requires less computer time than the previous one, and when it is used in conjunction with a Lagrangian grid, it provides a good resolution of the compressed plasma. However, this vacuum model may not simulate well enough the transient phase of some experiments, where currents are induced outside the main plasma column. Since these currents often have important effects on the results, we chose

the first method described above. This approach was used by Hain et al⁴⁶ in their one-dimensional calculations, and later by Lui and Chu²⁰ in their two-dimensional simulations. Improved versions of this method can be found in Refs. (47) and (48), and the effect of the low density cut-off on the results is discussed briefly in Section 4.4 .

3.2 The Vacuum Region

In the vacuum region between the vacuum vessel and the compression coil, the elliptic equation Eq. (29) must be solved at each time step, $\chi(t)$ being determined by Eq. (35). Using successive overrelaxation⁴⁹ method we get the equation:

$$\begin{aligned} \psi_{j,K}^{(r+1)} = \frac{\omega}{D} \left[\frac{1}{\Delta r^2} \left[\left(\frac{r_{j-\frac{1}{2}}}{1+\alpha^2 r_{j-\frac{1}{2}}^2} \right) \psi_{j-1,k}^{(r)} + \left(\frac{r_{j+\frac{1}{2}}}{1+\alpha^2 r_{j+\frac{1}{2}}^2} \right) \psi_{j+1,k}^{(r+1)} \right] \right. \\ \left. + \frac{1}{r_j \Delta \phi^2} \left[\psi_{j,K-1}^{(r+1)} + \psi_{j,K+1}^{(r)} \right] + \frac{2\alpha r_j \chi}{(1+\alpha^2 r_j^2)^2} \right] + (1-\omega) \psi_{j,K}^{(r)} \quad (50) \end{aligned}$$

where:

$$D = \frac{1}{\Delta r^2} \left[\frac{r_{j-\frac{1}{2}}}{1+\alpha^2 r_{j-\frac{1}{2}}^2} + \frac{r_{j+\frac{1}{2}}}{1+\alpha^2 r_{j+\frac{1}{2}}^2} \right] + \frac{2}{r_j \Delta \phi^2} ,$$

and the following notation is used: $f(r,\phi) = f(j\Delta r, k\Delta\phi) = f_{j,K}$.

The subscript (r) is the iteration index, and the value of the parameter ω is such as to maximize the speed of convergence. Eq. (50) which is valid for a uniform grid, is second order accurate because of centered space differencing. However, in the case of a

circularly shaped compression coil the mesh is nonuniform, and Eq. (50) is rewritten with an appropriate, second order differencing.

In order to solve Eq. (50) at a given time step, the solution of the previous time step is used as an initial guess. The stopping criterion is: $\max_{(j,K)} \left| \psi_{j,K}^{(r+1)} - \psi_{j,K}^{(r)} \right| < c\psi_{CC}$, where ψ_{CC} is the value of ψ on the compression coil, and c is a constant typically equal to 10^{-4} . When c is varied by an order of magnitude, the changes in the solutions are found to be very small. The number of iterations needed depends on c , Δt , Δr , $\Delta \phi$ and other factors. In our calculations it is typically equal to 20 when the magnetic field reaches half its maximum value.

The boundary conditions for ψ in the vacuum region are Eq. (36) on the compression coil, and the continuity of ψ and $\frac{\partial \psi}{\partial r}$ across the vacuum vessel. Once ψ is known in the vacuum region, one then solves for it in the plasma region as explained in Section 3.1. This method of solving for ψ separately in the two regions introduces some errors. However, for small time steps these errors are negligible. This has been verified by solving for ψ simultaneously in both regions. This approach is discussed briefly in the next paragraph.

The method was used earlier by Chu and Kranzer⁵⁰ to solve the problem of a plasma surrounded by a vacuum region, described by a parabolic equation in the plasma region and an elliptic equation in vacuum. In our case this method has been applied to the ψ equation which includes convection terms besides the diffusion terms. Eq. (37) is first rewritten in the following form:

$$c \frac{\partial \psi}{\partial t} = A_4(1+\alpha^2 r^2) \eta \left[\Delta^* \psi + \frac{2\alpha x}{(1+\alpha^2 r^2)^2} \right] - c \left[u \frac{\partial \psi}{\partial r} + \frac{v}{r} \frac{\partial \psi}{\partial \phi} \right] \quad (51)$$

where $c = 1$ in the plasma region, and $c = 0$ in the vacuum region. Eq. (51) is then solved throughout the two regions. A fully implicit scheme is used in this case, and the difference equations are solved by the successive overrelaxation method. The results agree well with those obtained when the ψ equations are solved separately in the two regions. The latter method has been used subsequently because it requires less computer time.

The boundary condition related to the strength of the helical field is discussed in the appendix.

3.3 Boundary Conditions and Numerical Dissipation

Periodic boundary conditions are usually used in the angular direction when the angle varies between zero and 2π . However, this range can be reduced if one makes use of the symmetry properties of the solutions. It can be easily shown that the MHD equations in helical coordinates Eqs. (37), (38), (40-44), are invariant under the transformation $\phi \rightarrow -\phi$ if ψ, χ, ρ, m, T are even functions of ϕ , and if n, s are odd functions of ϕ . Furthermore, in both cases of helical coils and grooved compression coil shown in Fig. (5), there is a symmetry with respect to the direction $\phi = 0, \phi = \pi$. Hence it is sufficient to solve the equations for $0 \leq \phi \leq \pi$, and use the symmetry or antisymmetry properties of the dependent variables. It should be noted, however, that unstable motions which violate this symmetry cannot be detected then.

Regarding the boundary conditions in the radial direction, the most important ones are those imposed on ψ and χ on the compression

coil (and helical coils if they are present), because the magnetic field which drives and confines the plasma is mainly determined by these conditions. We have discussed these at length in Section 2.2. In applying Eqs. (20) and (24), care must be used in determining $V(t)$ based on the external circuit, or else drastic effects may appear in the results. For instance, the helical shift of the plasma may even occur in the wrong direction. The boundary conditions at the center and at the vacuum vessel, which have much less effect on the motion of the plasma, are now discussed.

Contrary to differential equations, when difference equations are to be solved in cylindrical coordinates, the center must be considered as a boundary point. Furthermore, solving the MHD equations in difference form usually requires more boundary conditions than the corresponding differential equations. For instance, if a centered space difference is used in the r -direction, then ρ , m , n and s must be known at the vacuum vessel and at the center although the continuity and momentum equations are hyperbolic.

There are various ways of evaluating the dependent variables at the boundary. A few boundary conditions were studied by Chu and Sereny⁵¹ in their one-dimensional gas dynamic problem, and numerical results were compared with analytical ones. They found that the method of integration along characteristics on the boundary gives the best results. Among the other methods they tried, first-order extrapolation (or reflection) generally works better than higher order extrapolation or one-sided difference equations. Although this method is sluggish, it does not cause numerical oscillations near the boundary. Guided by these results, and since the characteristic method is too involved

in our two-dimensional problem, we chose first-order extrapolation to determine some of the variables on the boundary. Hence the following set of boundary conditions were applied to the fluid variables on the vacuum vessel:

$$v_{\text{normal}} = 0, \quad T = \text{constant}, \quad \frac{\partial \rho}{\partial n} = 0, \quad \frac{\partial v_{\text{tang}}}{\partial n} = 0;$$

where $\partial/\partial n$ denotes the normal difference quotient. The first two boundary conditions are physical, while the other two are extraneous. Furthermore, the boundary condition $T = \text{constant}$, can be replaced by $\frac{\partial T}{\partial n} = 0$ if the vacuum vessel acts as a heat insulator.

At the center, the extrapolation method is replaced by a simple averaging over the neighboring grid points, which are at a distance Δr from the center. This is straightforward for scalar quantities. However, in the averaging of vector quantities, the variation of the unit vectors with the angle ϕ must be taken into account. Once the averaged vector quantity is obtained at the center, appropriate components are then used when the equations are solved in a given ϕ -direction.

This boundary condition at the center has been used in most calculations. The reason it was selected is that it does not cause any numerical oscillations near the center. With other methods which have been tried (e.g., the difference equations applied to the center), there were such oscillations when the magnetic field was large. The effect on the results of the boundary condition at the center, is studied with a 1-D code in Section 4.4.

The question of dissipation is now discussed briefly.

It is well-known that in solving an initial-value problem by a finite difference method, numerical errors enter the solution in the form of nonphysical numerical dissipation and dispersion. These errors depend on the numerical scheme used, and their effect on the solution is similar to that of physical dissipation and dispersion. Naturally, for the numerical results to be trusted, the numerical dissipation coefficient should be small compared to the coefficient of physical dissipation (e.g., electrical resistivity). For a linear differential equation with periodic boundary conditions, one can use Fourier analysis or Taylor expansion⁵² to calculate the coefficient of numerical dissipation, and compare it with the physical one. However, this is very difficult to apply to the MHD equations which are coupled, nonlinear and with non-periodic boundary conditions. This problem will be studied in Section 4.4, where the amount of numerical dissipation and its effect on the results are analyzed with a 1-D code.

Finally we discuss the choice of the time step. Although implicit schemes are known to be unconditionally stable when they are applied to the diffusion equation, the time step cannot be made arbitrarily large in our calculations. This is mainly due to the presence of convection terms in the equations. Another possible reason is that the stability conditions are usually derived for simple, linearized equations, and it is not clear how the nonlinear terms and the boundary conditions affect the stability. Furthermore, even if the time step is not limited by stability conditions, it is not desirable to use too large a value for this quantity, because this will

destroy the accuracy of the results by increasing the truncation error.

For typical theta pinch parameters, if we fix Δt and increase the electrical resistivity and heat conductivity by one or two orders of magnitude, we find that the diffusion terms don't cause any instabilities when they are treated implicitly. Therefore in our parameter regime the time step is controlled mainly by the Courant-Friedricks-Lewy⁵³ stability factor, which has the following form in terms of the non-normalized quantities:

$$Q_1 = \frac{\Delta t}{\Delta r} \left[|v_r| + \left[\frac{\gamma k T}{m_i} + \frac{B_z^2 + B_\theta^2}{4\pi\rho} \right]^{\frac{1}{2}} \right]$$

At each time step the maximum value of Q_1 throughout the grid is evaluated, and Δt is chosen to make this value typically 3 or 4 (here Δt corresponds to one cycle of the A.D.I. method). This control on Δt saves some computer time because it allows us to use a large value for Δt in the beginning of the calculation when the magneto-acoustic speed is small.

3.4 One-Dimensional Codes

In Section 4.4 the two-dimensional A.D.I. code is converted to a one-dimensional (1-D) code by setting $\partial/\partial\phi = 0$ and $k = 0$; the corresponding results are then compared with those of two 1-D codes. The purpose of these 1-D calculations is to check the 2-D code, and to study the effects of some numerical factors on the results. In this section we describe briefly these two 1-D codes.

The first one is based on the fully implicit scheme, which

is first-order accurate in time, but has good stability properties, allowing the use of a large time step. The method of linearization and solving the resulting block tridiagonal system, is similar to the one described above for the A.D.I. code. The Lapidus artificial dissipation is added here too, and it is kept to a minimum level ($c \approx 0.05$).

The other 1-D code uses a second-order accurate explicit scheme, based on the flux-corrected transport (FCT) technique⁵⁴. The time step limitation, however, is rather strong in this code. Equations are written for the variables (ρ , u , T , x), and they take the same form as in Ref. (54), except for the diffusion terms which are not present in their equations. The splitting method discussed in Ref. (46), is then used. At each time step the diffusion term is first dropped, and the dependent variables are advanced in time with the convection term only (using the FCT algorithm). The resulting quantities are then acted on by the diffusion term. This splitting method is second-order accurate in time. An advantage of the FCT code is its low numerical dissipation, and this property is used in Section 4.4.

Chapter IV. RESULTS AND DISCUSSION

4.1 Review of Helical Equilibrium Theory

Because of the importance of the $\ell = 1$ systems, we will limit our discussion to this case only. The typical shape of the plasma in such a system is shown in Fig. (7). The first theoretical equilibrium calculation¹ is based on the sharp boundary model. The following analytical expression is obtained for the helical distortion of the plasma column:

$$\delta_1 = \frac{B_{hel}/B_o}{ka(1 - \beta/2)}, \quad (52)$$

where B_{hel} is the vacuum helical field on the axis of the discharge tube, δ_1 is normalized to a , and the other parameters have the same meaning as in Chapter I. Eq. (52) is valid for the experimentally

Fig. 7: Schematic view of a plasma column with normalized helical distortion $\delta_1 \approx 1$.

interesting case $ka \ll 1$. For arbitrary ka , correction terms involving modified Bessel functions must be included.²

Later a more realistic model of a plasma with a diffuse profile has been considered⁵⁵. For the "rigid-rotor" equilibrium⁵⁶ and with the assumption $ka \ll 1$, the following expression is obtained:⁵⁵

$$\delta_1 = \frac{B_{hel}^{pl}/B_0}{ka(1-\beta)^{\frac{1}{2}}}, \quad (53)$$

where both δ_1 and β correspond to the center of the plasma, and B_{hel}^{pl} is the helical field at $r = 0$ in the presence of the plasma. Expressions (52) and (53) differ by their β -dependence, the diffuse profile model predicting a higher shift. This difference is significant only at high β .

In both analytical derivations mentioned above, it is assumed that $\delta_1 \ll 1$. However, in most experiments δ_1 is of order unity. This interesting regime was considered by Weitzner⁵⁷ and by Freidberg⁵⁸ in their stability calculations, but no analytical expression was derived for δ_1 . In Ref. (58), Freidberg obtains the condition: $\beta\delta_1 < \frac{\pi^2}{16}$, for the existence of equilibrium in a straight theta pinch with an $\ell = 1$ field. The derivation is based on the sharp boundary model, and the plasma is assumed to have a circular cross-section. As shown in section 4.3, this condition is numerically tested with our 2-D code using a diffuse plasma model.

The case of a diffuse plasma with arbitrary δ_1 was finally studied by Marder⁵⁹. He solves the equilibrium problem numerically by using the Grad-Shafranov equation in helical coordinates. Surprisingly

he finds that for finite δ_1 , and high β (e.g. $\delta_1 = 1.2$ at the compression coil and $\beta = 0.8$), the β -dependence of the plasma shift agrees well with sharp boundary theory (Eq. 52).

In contrast to all analytical and numerical calculations mentioned above, our calculations are time-dependent. Hence they provide information on the dynamic phase before the plasma reaches equilibrium, and the attainability of the equilibrium. Except for the numerical computations done by H. Weitzner and T.A. Oliphant (unpublished), to the author's knowledge, the only two-dimensional, time-dependent numerical calculations for a helical theta pinch, are the ones presented in this work.

There is, however, a three-dimensional code which can simulate high beta stellarators by including two helical fields simultaneously.^{45,60} The major aim of those calculations is to test numerically the analytical toroidal equilibrium condition Eq. (3.a) or (3.b) derived with the sharp boundary model. They obtain good agreement with these analytical expressions when a sharp boundary model is used. However, a large deviation from the analytical predictions is found in the case of a diffuse profile plasma, especially for high values of β . For instance, a deviation of about 50% is found for typical Scyllac parameters.

We mention now a final point related to our calculations. It has been predicted⁵⁷, and experimentally verified later¹¹, that the plasma helical shift δ_1 oscillates before reaching its equilibrium value. With the sharp boundary model, it can be shown that the frequency of these helical oscillations is to leading order⁵⁷:

$$\omega_H = (2 - \beta)^{\frac{1}{2}} k v_A, \quad (54)$$

where v_A is the Alfvén speed calculated with the plasma density on axis and the magnetic field outside.

In the next two sections our numerical results are shown for two kinds of initial conditions. In the first case the calculation is started with a homogeneous plasma filling the discharge tube, and in the second case the plasma is initially in a theta pinch radial equilibrium.

4.2 Initially Uncompressed Homogeneous Plasma

In this section the plasma is initially uncompressed, at rest, and fully ionized with deuterium density $n = 10^{15} \text{ cm}^{-3}$ and temperature $T = 2 \text{ eV}$, both uniform in space. The main magnetic field which is zero initially (no bias field), rises to 42 KGauss or 17 KGauss as in Scylla IV-3¹⁰ and Scyllac experiments.³

Two kinds of oscillations, occurring on different time scales, are observed in our calculations. Damped, radial compression (or hydro-magnetic) oscillations are first set up, their period being approximately equal to the transit time of an Alfvén wave across the radius r_d of the discharge tube. When these oscillations disappear the plasma reaches a state where it is compressed, hot and in radial equilibrium with the magnetic field. The second kind of oscillations seen in our results, are the helical oscillations of the plasma shift $\delta_1 a$. Their period is approximately L/r_d times larger than the previous one, L being the periodicity length of the helical field in the z -direction.

As discussed in the previous section, it is important in our problem to follow the motion of the plasma until it nearly reaches an equilibrium state. This can easily be done with a one-dimensional code, and such 1-D results are shown in section 4.4. However,

two-dimensional codes may present certain difficulties, and it would be desirable to see first whether our 2-D code can connect an initially homogeneous state to a final state where the plasma is nearly in equilibrium.

For this purpose we use Scylla IV-3 parameters¹⁰ except for the magnetic field which we replace by a weaker field in order to avoid excessive computer time. A model with helical coils is used in these first calculations, and the corresponding results are shown in Figs. (8-10). The radial compression oscillations can be seen in Fig. (8), where the dashed curve is the magnetic field outside the plasma, and the solid curve is the field at the plasma center. When the magnetic field reaches its peak at 4 μsec , the capacitor bank (supplying current to the compression coil) is crowbarred, causing the applied field to be nearly constant subsequently. The plasma density and the magnetic field profiles at $t = 7.05 \mu\text{sec}$, in the $\phi = 0$ and $\phi = \pi$ directions, can be seen in Fig. (9). Constant flux contours in the $r - \phi$ plane are shown for the same instant of time in Fig. (10), where ψ_3 corresponds to a contour just outside the plasma column. The maximum plasma velocity is then two orders of magnitude lower than the radial speed during the first implosion. Therefore we can say that at $t \approx 7 \mu\text{sec}$, the plasma has practically reached an equilibrium in both the r and ϕ directions.

We now study the plasma motion in Scyllac for the case of an applied magnetic field which rises to 17 KGauss. A model with helically grooved compression coil is used, and the initial conditions are similar to those introduced above. Furthermore, the anomalous resistivity defined in section 2.3 is now used besides the classical one. The results are shown in Figs. (11-14). First the compression oscillations can be seen

Fig. 8: Time dependence of the magnetic field outside and at the center of the plasma (with Scylla IV-3 parameters except for $B_{z_{\max}}$ which is equal to 10.5 KGAuss).

Fig. 9: Plasma density and magnetic field profiles at $t = 7.05 \mu\text{sec}$ (same parameters as Fig. 8).

Fig. 10: Constant flux contours at $t = 7.05 \mu_{\text{sec}}$ (same parameters as Fig. 8).

in Fig. (11), where again the dashed and the solid curves represent, respectively, the B_z field outside and at the center of the plasma. Near $t = 4 \mu\text{sec}$, the compression oscillations are damped out and a radial equilibrium is reached with $\beta \approx .22$. However, the helical oscillations of the plasma center are still present at that time. This can be seen in Fig. (12) where the transverse shift $\delta_1 a$ of the plasma center, defined in section 4.1, is plotted versus time. In this figure the origin corresponds to the axis of the discharge tube, and $\delta_1 a$ is obtained from the position of maximum (or minimum) flux ψ . The period of these helical oscillations is an order of magnitude larger than the period of the compression oscillations. This agrees quite well with Eq. (54), as can be easily seen if we use Scyllac parameters ($k = 0.1 \text{ cm}^{-1}$ and radius of discharge tube = 5 cm). Then using these parameters, with $\beta = .22$, we find that sharp boundary theory (Eq. 52) predicts a value of 1.07 cm for the equilibrium shift of the plasma center. Fig. (12) shows that $\delta_1 a$ does oscillate about this value. In Fig. (13) the shift of the contour just outside the plasma (with $\beta \approx 0$) is plotted versus time, showing much weaker helical oscillations in this case. Finally, constant flux contours at $t = 5.50 \mu\text{sec}$, can be seen in Fig. (14), where ψ_3 corresponds to a contour just outside the plasma column. The fact that the contours of Fig. (10) are not circular as in Fig. (14), is due to the model with helical coils used in the previous calculation.

In order to see whether there is good agreement with sharp boundary theory for other values of k and B_{hel}/B_o , these two quantities were varied keeping all other parameters unchanged. The results are

Fig. 11: Time dependence of the magnetic field (for Scyllac parameters with $B_{z_{\max}} = 17$ KGAUSS).

Fig. 12: Time dependence of the transverse helical shift of the plasma center, i.e. for $\beta = \beta_{\text{max}}$ (same parameters as Fig. 11).

Fig. 13: Time dependence of the transverse helical shift for a contour just outside the plasma, i.e. for $\beta \approx 0$ (same parameters as Fig. 11).

$B_z \text{ max} = 17 \text{ KGAUSS}$

$k = 0.1 \text{ cm}^{-1}$

$\frac{B_{\text{hel}}}{B_0} = 9.5 \times 10^{-2}$

TIME = $5.50 \mu \text{ sec}$

Fig. 14: Constant flux contours at $t = 5.50 \mu \text{ sec}$ (same parameters as Fig. 11).

shown in Figs. (15) and (16). The agreement with sharp boundary theory is clear for the case of Fig. (15), where sharp boundary theory predicts an equilibrium value of 0.5 cm for $\beta = .22$. In Fig. (16) the helical shift oscillates about the value 2.0 cm obtained from Eq. (52).

Hence for these parameters, sharp boundary theory apparently predicts quite well the dependence of the shift on k and B_{hel}/B_o , although actual profiles are diffuse and $\delta_1 \sim 1$. It should be noted, however, that the value of β being low in this section, it is not possible to study the dependence of the shift on this quantity. This is done with the second method presented in the next section.

4.3 Plasma Initially in a Rigid Rotor Equilibrium

Our aim in this case is to find the same kind of helical equilibrium as above, but starting with different initial conditions. The plasma is now assumed to be already compressed, heated and in radial equilibrium in the presence of a B_z field. A "rigid-rotor" equilibrium⁵⁶ is chosen to represent the initial density and magnetic field profiles:

$$\begin{aligned}\rho(r) &= \rho_o \operatorname{sech}^2 \left[\left(\frac{r}{a_1} \right)^2 + \gamma \right] / \operatorname{sech}^2 \gamma \\ B_z(r) &= B_o \tanh \left[\left(\frac{r}{a_1} \right)^2 + \gamma \right]\end{aligned}$$

where ρ_o is the density on axis, $a_1 = a [1 + (1 - \beta)^{\frac{1}{2}}]^{\frac{1}{2}}$, $\gamma = \tanh^{-1} (1 - \beta)^{\frac{1}{2}}$, with a the plasma radius, and β is measured on axis. The temperature is chosen such that pressure balance is satisfied.

When the helical field is applied the plasma seeks a new equilibrium position by taking a helical shape. It should be noted that

Fig. 15: Time dependence of the helical distortion of the plasma center.

Fig. 16: Time dependence of the helical distortion of the plasma center.

the rise time of this field is not important in this case. Hence a small value can be chosen for this quantity in order to get a fast plasma response. In this method the fast compression oscillations are not excited, leaving only the helical oscillations.

The advantage of this approach is that β , the peak density and the plasma radius are now arbitrary. Hence by choosing for these quantities their experimental values in the post-implosion phase, we can better simulate the plasma motion in that phase. Furthermore, from the theoretical point of view, we can study now the dependence of the helical shift on β .

Two sets of parameters corresponding to Scyllac experiments, were used, and they are listed in table I. The results for the case of the strong magnetic field (42 KGauss), are shown in Figs. (17-19). In Fig. (17) the helical shift of the plasma $\delta_1 a$ is plotted as a function

TABLE I

$B_{z \max} = B_o$ (KGauss)	42	17
B_{hel}/B_o	6.83×10^{-2}	9.50×10^{-2}
k (cm^{-1})	0.15	0.10
β	0.80	0.65
$n_{\max}$ (10^{16} cm^{-3})	2.7	2.7
$T_{\max}$ (eV)	590	60
Rise time of B_{hel} (μsec)	0.25	0.50
Plasma radius a (cm)	1.7	1.7
Discharge tube radius (cm)	5	5
Mean radius of compression coil (cm)	10	10

of time. The upper curve, which corresponds to the center of the plasma (where $\beta = 0.80$), oscillates about the value predicted by sharp boundary theory (0.77 cm), and the period of these oscillations agrees well with the sharp boundary value (0.97 μ sec). The lower curve represents the shift of the contour just outside the plasma column where $\beta \approx 0$ locally. The value of $\delta_1 a$ at this contour is a little higher than the sharp boundary value of 0.46 cm. This is due to the fact that $kr \approx \frac{1}{2}$ on this contour, hence it is not small compared to unity. As was mentioned in the comment following Eq. (52), the expression for $\delta_1 a$ involves then modified Bessel functions, causing the shift to be larger by a factor: $2 I_1'(kr)$.

Fig. (18) shows the plasma density and the B_z field at $t = 1.68 \mu$ sec. The small oscillations which are superposed on the density profile are of numerical nature. They appear in this calculation because of the large values of B_z , T and β . High magnetic fields usually cause more numerical oscillations in the velocity and hence in the density. Furthermore, at high temperatures the classical resistivity is low, and the oscillations are less damped. Finally, a high β causes large gradients in the density and field profiles, which are more difficult to handle numerically. It should be noted that anomalous resistivity is not included in this method, because turbulent effects are not expected to appear for a rigid rotor equilibrium⁵⁶. Furthermore, β remains high throughout the calculation, implying that there is very little numerical and physical dissipation in this case. A two-dimensional view of the plasma at $t = 1.68 \mu$ sec is shown in Fig. (19), where the outermost constant flux contour ($\psi = \psi_4$) is just outside the plasma column.

The results for the case of the lower magnetic field (17 KGauss) are shown in Figs. (20-22). The plasma shift in Fig. (20) agrees quite well with the sharp boundary value of 1.41 cm for $\beta = 0.65$. The

Fig. 17: Time dependence of the helical shift for $\beta = \beta_{\text{max}}$ and $\beta \approx 0$ (Scyllac parameters with $B_{z \text{ max}} = 42 \text{ KGauss}$).

Fig. 18: Plasma density and magnetic field profiles at $t = 1.68 \mu\text{sec}$ (same parameters as Fig. 17).

$B_z \text{ max} = 42 \text{ KGAUSS}$

$k = 0.15 \text{ cm}^{-1}$

$\frac{B_{\text{hel}}}{B_0} = 6.8 \times 10^{-2}$

TIME = $1.68 \mu\text{sec}$

Fig. 19: Constant flux contours at $t = 1.68 \mu\text{sec}$ (same parameters as Fig. 17).

lower curve in this figure again corresponds to a contour just outside the plasma column ($\beta \approx 0$). Sharp boundary theory predicts a shift of 0.95 cm for this contour. In Fig. (21) the density and magnetic field profiles are plotted for $t = 2.67 \mu\text{sec}$. Finally, constant flux profiles are drawn in Fig. (22), where a slight distortion from a circular shape can be seen in the contour closest to the center ($\psi = \psi_2$). This is caused by the increase of the helical shift of a given contour with the local value of β .

All results shown in the last two sections (except the case of $B_{\text{max}} = 10 \text{ KGauss}$), were obtained with the grooved compression coil model (Fig. 5,b). In order to see whether the sharp edge of the helical grooves affects the results, some of these cases were tried again using a circularly shaped compression coil. The value of the helical shift $\delta_1 a(t)$ was practically the same as that obtained with the previous model (the maximum change was less than 2%).

Therefore both types of initial conditions lead us to conclude that Eq. (52) predicts quite well the dependence of the helical shift on various parameters, even when the assumptions made in that theory ($\delta_1 \ll 1$ and sharp boundary model) are violated. As mentioned in chapter I, a similar agreement with sharp boundary theory occurred in the ISAR T1 experiments⁴. It should be kept in mind, however, that the parameters used in all our calculations are close to those of Scyllac. Hence our conclusion, regarding the agreement with sharp boundary theory, does not necessarily apply to other machines whose parameters are very different from those of Scyllac.

Fig. 20: Time dependence of the helical shift for $\beta = \beta_{\text{max}}$ and $\beta \approx 0$ (Scyllac parameters with $B_{z \text{ max}} = 17 \text{ KGauss}$).

Fig. 21: Plasma density and magnetic field profiles at $t = 2.67 \mu\text{sec}$ (same parameters as Fig. 20).

$B_z \text{ max} = 17 \text{ KGAUSS}$

$k = 0.1 \text{ cm}^{-1}$

$\frac{B_{\text{hel}}}{B_0} = 9.5 \times 10^{-2}$

TIME = $3.03 \mu\text{sec}$

Fig. 22: Constant flux contours at $t = 3.03 \mu\text{sec}$ (same parameters as Fig. 20).

In order to test the sharp boundary helical equilibrium condition obtained by Freidberg⁵⁸ and discussed in section 4.1, Scyllac's parameters with $B_{z\max} = 17$ KGauss were used (see table I), except for the plasma radius which was lowered to the value of $a = 1$ cm. An equilibrium was then obtained with $\beta\delta_1 \approx 0.9$, which violates the condition derived in Ref. (58). Therefore $\ell = 1$ helical equilibria exist beyond the condition set by sharp boundary theory. This conclusion agrees with previous numerical⁵⁹ and experimental results.⁴ It should be noted also that the constant flux contours obtained in our calculations, were not circular as assumed in the sharp boundary derivation⁵⁸.

As discussed above, our numerical calculations, yielding the equilibrium helical shift for a diffuse plasma and arbitrary δ_1 , have important applications in regards to the experimental design of helical theta pinches. However, our 2-D code cannot be used for numerically testing. The sharp boundary toroidal equilibrium condition Eq. (3.a) or (3.b), since this would require the presence of two helical fields simultaneously. This was done in the 3-D numerical computations⁶⁰ mentioned in section 4.1, and in another analytical calculation with a diffuse plasma model but with the assumption $\delta_2 \ll 1$.⁶¹

4.4 Numerical Effects on the Results

In this section one-dimensional theta pinch results are shown. The purpose of these calculations is to check our 2-D code, and to better understand the effects of various physical and numerical factors on the results; e.g. the choice of the boundary conditions at the center, and the numerical dissipation in the 2-D code. This study

also applies to other calculations done in Columbia University with a similar 2-D code.^{20-22,25}

The 1-D equations can be obtained from the 2-D set (Eqs. 37, 38, 40-44) by letting $\frac{\partial}{\partial \phi} = 0$ and $k = 0$. This implies that:

$$\chi = B_z, \psi = 0, \quad u_\theta = u_z = 0, \quad B_\theta = B_r = 0.$$

Hence there are only three fluid equations and one field equation to be solved in this case. Classical Spitzer's resistivity (Eq. 46), and the coefficient of heat conductivity $K_\perp$ (Eq. 49) are used in these calculations. No anomalous resistivity is included.

TABLE II

B_{zmax}	10 KGauss
Field rise time	4 μ sec
Initial density	10^{15} cm^{-3}
Initial temperature	2 eV
Discharge tube radius	4 cm

First we compare the results of the 2-D code (where we set $k = 0$ and $\frac{\partial}{\partial \phi} = 0$) with the fully implicit code using the same initial and boundary conditions. The parameters used in the 1-D calculations are listed in table II. The density and temperature profiles at three instants of time during the first implosion are shown in Figs. (23-26). Comparison between Figs. (23) and (24), where $n(r)$ denotes the particle density, shows a very good agreement between the results of the A.D.I. and fully implicit codes. The temperature profiles of Figs. (25) and (26) agree very well too.

The long-time scale effects of the boundary conditions at the center are now studied with the fully implicit code. This will help us understand the same effects on the 2-D results. Two types of boundary conditions are tried. The first one consists of applying the difference equations to a cell next to the center. The resulting equations which are first order accurate in space in time, are then solved explicitly for the physical quantities at the center. The second method, which is a first order extrapolation, sets the value at the center equal to the neighboring value at each time step. This is equivalent to the averaging method mentioned in chapter III. Because of the symmetry of the 1-D problem, the velocity must vanish at the center. Hence the boundary conditions described above can be applied to the density, temperature and field equations.

Fig. (27), which shows the magnetic field outside the plasma and at the center as a function of time, is obtained when the three difference equations are applied to the center point. The particle density,

Fig. 23: Density profiles during the first implosion obtained with the A.D.I. (2-D) code.

Fig. 24: Density profiles during the first implosion obtained with the fully implicit (1-D) code.

Fig. 25: Temperature profiles from the A.D.I. code.

Fig. 26: Temperature profiles from the fully implicit code.

temperature and magnetic field profiles at $t = 7 \mu\text{sec}$ are shown in Figs. (28) and (29). The velocity is then three orders of magnitude lower than the initial implosion velocity. However, the compression oscillations have not yet completely disappeared, and they cause small variations, of a few percent, in the density and magnetic field at the center. Therefore the center values which are shown from now on, will be averaged over a period of compression oscillations. Furthermore, the plasma being nearly in equilibrium, β at the center can be easily evaluated using:

$$\beta = 1 - \left(\frac{B_{\text{center}}}{B_{\text{out}}} \right)^2. \quad \text{In Fig. (28) we find that } \beta = 0.36.$$

When the extrapolation method is applied to the three physical quantities at the center, there is more density and field diffusion. The density profile at $7 \mu\text{sec}$ is a little broader than the previous case with a peak value of 10.5, the temperature profile is unchanged, and the magnetic field at the center is 8.6 KGauss, corresponding to $\beta = 0.27$. Finally, when the difference equations are applied to the density and temperature at the center, but the field is obtained with the extrapolation method, the profiles at $7 \mu\text{sec}$ are very similar to those of Fig. (28), and again $\beta = 0.36$.

We conclude that the extrapolation method is more dissipative, as found by Chu and Sereny⁵¹, and its cumulative effect on the density profile can be quite severe. Furthermore, as the density profile is flattened, the magnetic field must increase at the center in order to satisfy pressure balance. Therefore, the magnetic field profile is affected as well by this boundary condition.

Fig. 27: Time dependence of the magnetic field (with the fully implicit code and the difference equations applied to the center).

Fig. 28: Plasma density and magnetic field profiles at $t = 7 \mu\text{sec}$ (same code and boundary conditions as Fig. 27).

Fig. 29: Temperature profile at $t = 7 \mu\text{sec}$ (same code and boundary condition as Fig. 27).

It should be noted, however, that the effect of the boundary conditions at the center is less pronounced in the 2-D code, because the maxima and minima (where dissipation is highest) are helically shifted away from the center. These effects are expected to be unimportant if the plasma peak density is more than two grid points away from the center. However, when the plasma shift is small, the extrapolation (or averaging) method should be avoided, especially in the density equation.

The question of numerical dissipation in the 2-D (A.D.I.) code is now studied, and its effects on the results are analyzed on both implosion and equilibrium time scales. First we consider the profiles during the initial implosion. The similarity between the results of the A.D.I. and fully implicit codes, shown earlier in Figs. (23-26), implies that their numerical dissipations are comparable. It will be seen later that this conclusion holds on the equilibrium time scale as well. Therefore in the remaining analysis we will use the fully implicit code since it requires less computer time.

When an initial-value problem is solved numerically, the numerical dissipation is proportional to $(\Delta r)^n$, where Δr is the grid size and n is an integer which increases with the scheme's order of accuracy in time⁵². The fully implicit scheme being first order accurate, its numerical dissipation is expected to be proportional to $(\Delta r)^2$, with a proportionality factor usually weakly dependent on Δr . Therefore by varying Δr we can change the amount of numerical dissipation, and see how sensitive the results are to these variations. In Figs. (24) and (26) there were 19 grid points in the radial direction ($N = 19$). These calculations are now repeated with $N = 37$, corresponding to Δr smaller by a factor of two, and the results are shown in Figs. (30) and (31).

Fig. 30: Density profiles during the first implosion (with the fully implicit code and double the number of grid points, i.e. $N = 37$).

Fig. 31: Temperature profiles during the first implosion (same code and number of grid points as Fig. 30).

Comparing these with Figs. (24) and (26), we find that with lower numerical dissipation the profiles are sharper as expected, and the motion is slightly faster. Besides that there is very little change in the results. The profiles obtained with the F.C.T. code are comparable to these as well. Therefore we can say that on the Alfvén or implosion time scale, the effects of numerical dissipation are quite small. We now study these effects on the equilibrium time scale.

The results of the A.D.I. and fully implicit codes are first compared using the extrapolation method at the center. The profiles obtained at 7 μsec are quite similar. With the A.D.I. code the peak density is 9.8, the temperature is 31 eV at the center, and the magnetic field inside is 8.7 KGauss. Therefore the numerical dissipation of the A.D.I. and fully implicit codes must be comparable.

We now study the effect of varying the grid size, and consequently the numerical dissipation, on a long time scale. The results of the fully implicit code at 7 μsec are shown in Fig. (32), where $N = 37$ and the difference equations were applied at the center (since this method is less dissipative). The temperature at the center in this case is 40 eV and the value of β is 0.50. Comparing Figs. (32) and (28), where the only different parameter is the grid size, we conclude that on the equilibrium time scale, the cumulative effect of numerical dissipation can be important.

The effect of decreasing the numerical dissipation is studied now with the F.C.T. code, which is second-order accurate in time, and consequently has less numerical dissipation than the fully

Fig. 32: Plasma density and magnetic field profiles at $t = 7 \mu\text{sec}$ (same code and number of grid points as Fig. 30).

implicit code. The corresponding results with $N = 19$ are shown in Figs. (33) and (34). The compression oscillations of Fig. (33) disappear after about 6 μsec , and the velocity is then controlled by diffusion. If we estimate the physical diffusion term near the center at 6 μsec (with a temperature of about 40 eV), and compare it with the slope of $B_z(t)$ at this time, we find that the increase in B_z at the center is caused by physical, and not by numerical, dissipation. The density and field profiles at 7 μsec are seen in Fig. (34) where $\beta = 0.60$ and the temperature at the center is 39 eV. Comparison between this figure and Fig. (28) shows again that numerical dissipation can have serious effects on the results on the equilibrium time scale.

We found that lower numerical dissipation leads to higher values of β . However, it should be noted that this was not caused purely by varying the numerical dissipation with its smoothing effect on various profiles. It was caused additionally by the lower value of physical resistivity in these cases. Decreasing the numerical dissipation usually causes the temperature to become higher. Because of the temperature dependence of physical resistivity (Eq. 46), this quantity is then lower thereby causing less field diffusion and a higher beta.

Finally, in order to have an idea of the relative amounts of physical and numerical dissipation in our calculations, we vary now the physical resistivity. Using the fully implicit code with the difference equations at the center, and dividing the physical resistivity by a factor 4, we obtain the results of Fig. (35) at 7 μsec . If we compare these results with Figs. (28) and (32), we find that decreasing the

Fig. 33: Time dependence of the magnetic field obtained with the F.C.T. code.

Fig. 34: Plasma density and magnetic field profiles at $t = 7 \mu\text{sec}$ from the F.C.T. code.

Fig. 35: Plasma density and magnetic field profiles at $t = 7 \mu\text{sec}$ (with the fully implicit code and the physical resistivity divided by 4).

physical resistivity by four and doubling the number of grid points, have nearly the same effect on the field diffusion on the equilibrium time scale. However, as mentioned in the previous paragraph, a decrease in numerical dissipation is usually accompanied by an increase in temperature and thereby a decrease in physical resistivity as well.

The calculation is then repeated with the physical resistivity divided by 10, and in another case multiplied by 10. The corresponding compression oscillations can be seen in Figs. (36) and (37) respectively. These results show the drastic effects of varying the physical resistivity on the final β . The density and temperature profiles produced in these two cases are obviously very different. For the lower physical resistivity the normalized peak density is 25, and the peak temperature is 29 eV at 7 μ sec. The corresponding values for the higher physical resistivity are 7, and 23 eV.

We conclude from this analysis that the numerical dissipation has only a small effect on our 2-D results during the initial plasma motion. On the other hand, if the purpose of the calculation is to find the equilibrium profiles accurately, then the cumulative effects of numerical dissipation could cause more difficulty. This can be remedied, however, by using a large number of grid points or a code with low numerical dissipation. Furthermore, even when a relatively small number of grid points (e.g., $N = 19$) are used in the A.D.I. code, the effect of numerical dissipation on the final profiles does not exceed that of physical dissipation. It should be noted that these conclusions are valid for parameters close to the ones used in these calculations. However, if the temperature is much higher or the density much lower, anomalous field diffusion becomes more important than classical and numerical diffusion.

Fig. 36: Time dependence of the magnetic field (with the fully implicit code and the physical resistivity divided by 10).

Fig. 37: Time dependence of the magnetic field (with the fully implicit code and the physical resistivity multiplied by 10).

Before ending this section we discuss briefly the effects of the low density region on the results. In all calculations shown above, the density outside the main plasma column was not allowed to decrease beyond the cut-off value of 10% of the initial density. For comparison a cut-off value of 1% was used later on (with the fully implicit code). At 7 μ sec, the peak density averaged over a period of compression oscillations, is 14.1 in the first case and 13.5 in the second case. The average magnetic field at the center is 8.00 KGauss in the first case and 7.90 KGauss in the second. Therefore we can say that the effects of this method on the final profiles are quite small.

APPENDIX I

BOUNDARY CONDITION DETERMINING THE SIZE OF THE HELICAL FIELD

In this appendix we consider an important boundary condition related to the strength of the helical field. As was discussed in section 4.1, the helical shift of the plasma depends on the ratio B_{hel}/B_o , where B_{hel} is the helical field on the axis of the vacuum vessel with no plasma present, and B_o is the vacuum axial theta pinch field. The quantity B_{hel}/B_o can be easily measured experimentally, and the boundary conditions which yield this prescribed value are now discussed.

We consider separately the two methods of generating the helical fields which are shown in Figs. 5.a and 5.b. First, for the case of helical coils one can use a boundary condition similar to Eq. (20) in the cgs system:

$$\left(\frac{\partial \psi}{\partial t}\right)_{hel \text{ coil}} = - \frac{c}{L} V_{hel}(t), \quad (55)$$

where $V_{hel}(t)$ is the electromotive force across the ends of a helical coil of period L in the z -direction. However, in experiments involving helical coils, the two $\ell = 1$ coils carry equal and opposite currents. Hence in order to make sure that this will be true in our calculations, we use a different boundary condition involving the current in the helical windings I_{hel} . It has been shown with Eq. (21) that X does not experience any jump

across a current carrying helical winding. However $\frac{\partial \psi}{\partial r}$ does, and the jump in this quantity can be related to I_{hel} if the thickness of the winding in the r -direction is negligible compared to its width in the θ or z -direction. Using Ampere's law we get the following condition in dimensionless form:

$$\left(\frac{\partial \psi}{\partial r}\right)_{r_h^+} - \left(\frac{\partial \psi}{\partial r}\right)_{r_h^-} = \frac{I_{hel}(t)}{\Delta l} (1 + \alpha^2 r_h^2)^{\frac{1}{2}}, \quad (56)$$

where r_h denotes the position of the helical coils in the radial direction, and Δl is the width of the coils in the ϕ -direction. If $I_{hel}(t)$ rises sinusoidally in time, we obtain the values of ψ at the two helical coils from the following equations:

$$\begin{aligned} \psi_{JHEL,2} &= \frac{1}{2}(\psi_{JHEL+1,2} + \psi_{JHEL-1,2}) + D \sin\left(\frac{\pi}{2} \frac{t}{\tau_{hel}}\right), \\ \psi_{JHEL,M-1} &= \frac{1}{2}(\psi_{JHEL+1,M-1} + \psi_{JHEL-1,M-1}) - D \sin\left(\frac{\pi}{2} \frac{t}{\tau_{hel}}\right), \end{aligned} \quad (57)$$

where $J = JHEL$ denotes the radial position of the helical coils, τ_{hel} is the rise time of the helical currents, and $K = 2, K = M-1$ correspond respectively to the direction $\phi = 0$ and $\phi = \pi$ of the helical coils.

Eqs. (57) which are valid for $t \leq \tau_{hel}$, are obtained by one-sided differencing. For $t > \tau_{hel}$ the terms $\pm D \sin\left(\frac{\pi}{2} \frac{t}{\tau_{hel}}\right)$ are replaced by $\pm D$; this corresponds to crowbarring the capacitor bank at $t = \tau_{hel}$. The constant D which is proportional to the maximum value of $I_{hel}(t)$, is determined in the following way. The vacuum field equation Eq. (29) is written in dimensionless form, and it is solved numerically as described in section 3.2. However, the domain of the solution is now the entire

region bounded by the compression coil (i.e., inside and outside the discharge tube). The electromagnetic boundary conditions are Eqs. (57) on the helical coils and Eq. (36) on the compression coil. The constant D is then chosen so that the ratio B_{hel}/B_o takes the desired experimental value. It should be noted here that the value of D depends on the choice of the grid size $\Delta\phi$ in the ϕ -direction, because we are effectively assuming that $\Delta l = r_h \Delta\phi$ is Eq. (56). Hence the same value of $\Delta\phi$ must be used later on, when Eq. (57) is applied in the presence of the plasma.

Now for the case of a helically grooved compression coil, we are usually given the experimental value of B_{hel}/B_o and the radius b of the compression coil. In our model: $b = \frac{1}{2}(b_1 + b_2)$, where b_1 and b_2 are the radii corresponding to the inner surface of the compression coil in the grooved and non-grooved regions, respectively (Fig. 5.b). In this case keeping b constant, we adjust b_1 and b_2 such that the desired value of B_{hel}/B_o is obtained. The method of solving for the vacuum field is the same as above, except that there is now one electromagnetic boundary condition only (Eq. 36). The values of b_1 and b_2 do not depend on $\Delta\phi$ in this case.


```

C *** NOTATION
C *** Y,X,R-----PSI,KI,MASS DENSITY RHO.
C *** PM,PN,PS-----COMPONENTS OF MOMENTUM DENSITY: M,N,S.
C *** T,HC-----TEMPERATURE,HEAT CONDUCTIVITY.
C *** A,S-----MATRIX ELEMENTS USED IN CROUT'S METHOD.
C *** YC,XX,RR,PPM,PPN,PPS,TT-----VALUES OF DEPENDENT VARIABLES AT THE
C -----NEW TIME STEP.
C *** R2,PM2,PN2,PS2-----PREDICTED VALUES OF R,PM,PN,PS.
C *** YII,YV-----OLD AND NEW VALUES OF PSI IN THE VACUUM REGION.
C *** J,K-----COORDINATES OF AN ARBITRARY GRID POINT.
C *** N,NIN,NOUT1,NOUT2-----VALUE OF J AT: THE DISCHARGE TUBE, THE
C -----HELICAL COILS, THE INNER,AND THE OUTER
C -----SURFACE OF THE COMPRESSION COIL.
C *** M-----NUMBER OF GRID POINTS IN THE PHI-DIRECTION.
C *** Z-----NORMALIZED HELICAL WAVE NUMBER (K*RZERO).
C *** DR,DPH,DT-----GRID SIZE IN R- AND PHI-DIRECTION, AND TIME STEP.
C *** ANMLS,CLASL-----ANOMALOUS AND CLASSICAL RESISTIVITY.
C *** QPSI,QKI-----ARTIFICIAL LAPIDUS DISSIPATION FOR PSI AND KI.
C *** QR,QM,QN,QS,QT----ARTIFICIAL LAPIDUS DISSIPATION FOR R,PM,PN,PS,T.
C *** YWALL,XWALL-----VALUES OF PSI AND KI AT THE COMPRESSION COIL
C
C
C DIMENSION A(85,15), Y(19,11), X(19,11), R(19,11), PM(19,11), PN(19,
1,11), PS(19,11), T(19,11), HC(19,11), ST(19,11), DEL(19,11), S(85)
2, YY(18,10), XX(18,10), RR(18,10), PPM(18,10), PPN(18,10), PPS(18,
310), TT(18,10), R2(19,11), PM2(19,11), PN2(19,11), PS2(19,11)
4, YU(36,11), YV(36,11)
5, ANMLS(19,11), CLASL(19,11), RSMAX(19,11)
WRITE (6, 11)
11 FORMAT (/////)
WRITE (6, 2)
2 FORMAT (17H 2-D THETA PINCH)
WRITE (6, 5)
5 FORMAT (18H -----)
WRITE (6, 7)
7 FORMAT (//)
WRITE (6, 3)
3 FORMAT(113H B0=42KGAUSS, DENSITY=10**15, T0=1EV, U0=2.05*(10**8)C
1M/SEC, (TIME)0=1.95*(10**-8)SEC, RISE TIME=4.0*(10**-6)SEC)
WRITE (6,1158)
1142 FORMAT (132H B.FLUX FCT J.FLUX FCT DENSITY U-RAD. U-THETA
1 U-Z TEMPERATURE B-Z B-THETA J-Z J-THETA
2 (J*B)-RAD.)
1141 FORMAT (131H -----)
1 -----)
2 -----)
1158 FORMAT (/////)
1168 FORMAT (/////////)
1146 FORMAT (///)
1122 FORMAT (7E14.4)

```

```

1123 FORMAT (9E14.4)
1133 FORMAT (10E13.4)
1131 FORMAT (8E12.3)
1134 FORMAT (12E11.3)
1124 FORMAT (7E13.4)
  FCURR = 1.5*(6.83/2.67)
  A1 = 8.81E+4
  A1ST = A1/1.67
  A2 = 2.48E-5
  A3 = 442
  A4 = 1.50E-2
  B10 = .15
  B12 = .10
  TWOTHR = 2./3.
  TIME0 = 1.95E-8
  RTIME1 = 4.0E-6/TIME0
  RTIME2 = RTIME1
  FINAL = 2.5E-6
  OMEGA = 1.0E-30
  EPSILO = 1.0E-16
  DELTA = 1.0E-30
  DFLTA1 = -1.0E-13
  N = 16
  FN = N
  NM1 = N-1
  NM2 = N-2
  NM3 = N-3
  NM4 = N-4
  NP1 = N+1
  NP2 = N+2
  NOUT1 = 26
  FNOUT1 = NOUT1
  NOUT1M = NOUT1-1
  NOUT1P = NOUT1+1
  NOUT1X = NOUT1+2
  NOUT1Q = NOUT1-2
  NOUT = NOUT1
  FNOUT = NOUT
  NOUTM = NOUTM
  NIN = 26
  NINP = NIN+1
  NINP2 = NIN+2
  NINM = NIN-1
  NINM2 = NIN-2
  NOUT2 = 34
  FNOUT2 = NOUT2
  NOUT2M = NOUT2-1
  NOUT2Q = NOUT2-2
  RZERO = 4.
  RVV = 5./RZERO
  RCC1 = 11.05/RZERO
  RCC2 = 8.95/RZERO
  DR = RVV/(FN-1.5)
  D=DR = (RCC1-RVV)/(FNOUT1-FN)
  D=DR = (RCC2-RCC1)/(FNOUT2-FNOUT1)
  D=INV = 1./DR+1./DDR
  D=DDR = DR/DDR
  DT = 0.25
  D=DT = DR/DT
  DTDR = DT/DR
  Z = 0.15*RZERO

```

```

DRZ = DR*Z
DDRZ = DDR*Z
ZMAX1 = Z*RCC1
ZMAX2 = Z*RCC2
ZMAX = ZMAX1
REFLAX = 1.375
GAMA = 1.0E-4*3.
BZERO = 42.
B7MAX = 17./BZERO
RMIN = 0.1
M = 11
MM1 = M-1
MM2 = M-2
MM3 = M-3
FMM2 = MM2
FMM3 = M-3
FNEW = 2.*FMM2-2.
MHALF = 2+MM2/2
MHALFP = MHALF+1
DPH = 3.1416/FMM3
KMID = (M+1)/2
KMIDP = KMID+1
KMIDM = KMID-1
FKMID = KMID
KMIDP2 = KMID+2
KMIDM2 = KMID-2
PI = 3.1416
QUANT1 = .5*(1./(1.+ZMAX1*ZMAX1)+1./(1.+ZMAX2*ZMAX2))
QUANT2 = RCC1/(1.+ZMAX1*ZMAX1)
QUANT3 = RCC2/(1.+ZMAX2*ZMAX2)
FKMID = 2*KMID-4
FQUANT = (RCC2-RCC1)/(PI*DPH)
FNEW1 = 2.*FKMID-3.
FNEW2 = FNEW1-2.
QUANT4 = .5*(QUANT2/DDR+QUANT3/DDDR)-FQUANT/(.5*(RCC1+RCC2))
RSMAX(1,5) = 1./(TWALL*SQRT(TWALL))
YWALL = 0.
KSHIFT = MM1
INITIAL CONDITIONS
DO 8 K = 1, M
DO 9 J = 1, N
Y(J,K) = 0.
X(J,K) = 0.
R(J,K) = 1.
PM(J,K) = 0.
PN(J,K) = 0.
PS(J,K) = 0.
T(J,K) = TWALL
9 CONTINUE
DO 6766 J = N, NOUT2
YU(J,K) = 0.
YV(J,K) = 0.
6766 CONTINUE
8 CONTINUE
DO 3965 J = 2, NM1
FFJ = J
FJ = FFJ-.5
JP1 = J+1
JM1 = J-1
RJ = (FJ-1.)*DR
BF = 1.+(RJ*Z)**2

```

```

DO 3965 K = 2, MM1
  KD1 = K+1
  KM1 = K-1
  DEL(J,K) = Z*(X(J,K)+X(J,K))/(BF*BF)+((FJ-.5)/(1.+(FJ-.5)*DRZ)**2
1)* (Y(JP1,K)-Y(J,K))-(FJ-1.5)/(1.+(FJ-1.5)*DRZ)**2*(Y(J,K)-Y(JM1,
2K)))/(RJ*DR)+(Y(J,KP1)-Y(J,K)-Y(J,K)+Y(J,KM1))/(RJ*DPH)**2
3965 CONTINUE
  TIME = 0.
  KD = 0
8000 CONTINUE
  DO 7000 NT = 1, 30
1523 KD = KD+1
  FNT = KD
  TIME = TIME+DT
  TIME2 = TIME*TIME0
C  DEFINITION OF SQRT OF T(J,K)
  DO 333 J = 1, N
  DO 332 K = 2, MM1
  IF (T(J,K).LE.0.) GO TO 1234
  ST(J,K) = SQRT(T(J,K))
332 CONTINUE
  ST(J,1) = ST(J,MM1)
  ST(J,M) = ST(J,2)
333 CONTINUE
C  DEFINITION OF HEAT CONDUCTIVITY
  DO 802 J = 1, N
  DO 803 K = 2, MM1
  HC(J,K) = 75.*A2/DR
  HC(J,K) = HC(J,K)/4.
803 CONTINUE
  HC(J,1) = HC(J,MM1)
  HC(J,M) = HC(J,2)
802 CONTINUE
C  ****FIRST TIME STEP STARTS HERE
C  DEFINITIONS OF PREDICTED VALUES
  GO TO 8500
8501 CONTINUE
C  SOME BOUNDARY CONDITIONS
  DO 4289 K = 2, MM1
  KPR = M-K+1
  Y(1,K) = Y(2,KPR)
  X(1,K) = X(2,KPR)
4289 CONTINUE
  ALPHA = 0.
C  SOLUTION OF ELLIPTIC EQ. IN VACUUM REGION
  GO TO 6700
6701 CONTINUE
C  DEFINITION OF RESISTIVITY
  DO 3870 J = 2, NM1
  FFJ = J
  FJ = FFJ-.5
  JP1 = J+1
  JM1 = J-1
  ZRJ = DRZ*(FJ-1.)
  BF = 1.+ZRJ*ZRJ
  DO 3860 K = 2, MM1
  KP1 = K+1
  KM1 = K-1
  FJTH = ZRJ*DEL(J,K)-(X(JP1,K)-X(JM1,K))/(2.*DR*BF)+2.*ZRJ*X(J,K)
1/(BF*BF)
  FJR = (X(J,KP1)-X(J,KM1))/(2.*DPH*(FJ-1.)*DR)

```

```

RATIO = 25.1*SQRT(FJTH*FJTH+FJR*FJR)/(R(J,K)*ST(J,K))
ANMLS(J,K) = (1.-EXP(-RATIO))/(1.32*SQRT(R(J,K)))
CLASL(J,K) = 1./(T(J,K)*ST(J,K))
RSMAX(J,K) = ANMLS(J,K)
IF (RSMAX(J,K).LT.CLASL(J,K)) RSMAX(J,K)=CLASL(J,K)
3860 CONTINUE
RSMAX(J,1) = RSMAX(J,3)
RSMAX(J,M) = RSMAX(J,MM2)
3870 CONTINUE
DO 3850 K = 2, MM1
KPR = M-K+1
RSMAX(1,K) = RSMAX(2,KPR)
RSMAX(N,K) = RSMAX(NM1,K)
3850 CONTINUE
C MATRIX ELEMENTS OF FIELD EQS. - FIRST TIME STEP
K = 2
565 KM1 = K-1
Kp1 = K+1
Rp1 = .5/(1.+(.5*DRZ)**2)
RJ = +RP1
RPH = 0.
U = PM2(1,K)/R2(1,K)
Up1 = PM2(2,K)/R2(2,K)
DO 70 J = 2, NM1
FFJ = J
FJ = FFJ-.5
Jp1 = J+1
Jm1 = J-1
RM1 = RJ
RJ = RP1
Rp1 = FJ/(1.+(FJ*DRZ)**2)
RMH = RPH
RPH = (FJ-.5)/(1.+(FJ-.5)*DRZ)**2)
EM = RSMAX(JM1,K)*A4/DR
EJ = RSMAX(J,K)*A4/DR
EP = RSMAX(JP1,K)*A4/DR
Um1 = U
U = UP1
Up1 = PM2(JP1,K)/R2(JP1,K)
BF = (FJ-1.)/RJ
RATIO = (PS2(J,KP1)/R2(J,KP1)-PS2(J,KM1)/R2(J,KM1))/(4.*DPH*(FJ-1.
1)*DR)
Zp = (FJ-.5)*ABS(PM(JP1,K)/R(JP1,K)-PM(J,K)/R(J,K))
Zm = (FJ-1.5)*ABS(PM(J,K)/R(J,K)-PM(JM1,K)/R(JM1,K))
Z1 = ABS(PN(J,KP1)/R(J,KP1)-PN(J,K)/R(J,K))
Z2 = ABS(PN(J,K)/R(J,K)-PN(J,KM1)/R(J,KM1))
QpSI = B12/(FJ-1.)*(ZP*(Y(JP1,K)-Y(J,K))-ZM*(Y(J,K)-Y(JM1,K))+(Z1*
1(Y(J,KP1)-Y(J,K))-Z2*(Y(J,K)-Y(J,KM1)))/DPH)
QkI = B12/(FJ-1.)*(ZP*(X(JP1,K)-X(J,K))-ZM*(X(J,K)-X(JM1,K))+(Z1*(
1X(J,KP1)-X(J,K))-Z2*(X(J,K)-X(J,KM1)))/DPH)
Jx = J+J-3
A(JX,1) = -(EJ*RMH/RJ+.5*U)
A(JX,2) = 0.
A(JX,3) = EJ*(RPH+RMH)/RJ+DRDT
A(JX,4) = -(EJ+EJ)*DR*DRZ/BF
A(JX,5) = -EJ*RPH/RJ+.5*U
A(JX,6) = 0.
S(JX) = DRDT*Y(J,K)+(EJ/(DPH*RJ)*(Y(J,KP1)-Y(J,K)-Y(J,K)+Y(J,KM1))
1-.5*PN(J,K)/R(J,K)*(Y(J,KP1)-Y(J,KM1)))/(FJ-1.)*DPH)+ALPHA
S(JX) = S(JX)+QPSI
4284 JX = J+J-2

```

```

A(JX,1) = Z*(U/BF+(EJ+EJ)*RMH/(FJ-1.))+RATIO
A(JX,2) = -((EJ+EM)*RMH+RM1*UM1)/(RJ+RJ)
A(JX,3) = -Z*(EJ+EJ)*(RPH+RMH)/(FJ-1.)
A(JX,4) = DRDT+(RPH*(EP+EJ)+RMH*(EJ+EM))/(RJ+RJ)+4.*EJ*(DRZ/BF)**2
A(JX,5) = Z*((EJ+EJ)*RPH/(FJ-1.))-U/BF)-RATIO
A(JX,6) = (-RPH*(EJ+EP)+RP1*UP1)/(RJ+RJ)
S(JX) = DRDT*X(J,K)+((((EJ+RSMAX(J,KP1)*A4/DR)*(X(J,KP1)-X
1(J,K))- (EJ+A4/DR*RSMAX(J,KM1))*(X(J,K)-X(J,KM1)))*BF-4.*EJ
2*7*(Y(J,KP1)-Y(J,K)-Y(J,K)+Y(J,KM1)))/(DPH*(FJ-1.))+PN(J,KM1)/R(J,
3KM1)*X(J,KM1)-PN(J,KP1)/R(J,KP1)*X(J,KP1)+(Y(J,KP1)-Y(J,KM1))*(2.*
4Z*(PN(J,K)+(FJ-1.)*DR7*PS(J,K))/(BF*R(J,K))-(PS(JP1,K)/R(JP1,K)-PS
5(JM1,K)/R(JM1,K))/(DR+DR)))/((DPH+DPH)*(FJ-1.))+QKI

```

70 CONTINUE

C MATRIX ELEMENTS NEAR THE BOUNDARY

```

S(1) = S(1)-Y(1,K)*A(1,1)
S(2) = S(2)-Y(1,K)*A(2,1)
S(2) = S(2)-X(1,K)*A(2,2)
J = NM1
JX = J+J-3
S(JX) = S(JX)-Y(N,K)*A(JX,5)
JX = J+J-2

```

```

S(JX) = S(JX)-Y(N,K)*A(JX,5)-X(N,K)*A(JX,6)

```

C CROUT,S INVERSION OF FIELD MATRIX - FIRST TIME STEP

```

L = NM2+NM2

```

```

LM1 = L-1

```

```

LM2 = L-2

```

```

LM3 = L-3

```

```

GO TO 4500

```

4501 CONTINUE

C END OF CROUT'S INVERSION

C VALUES AT THE NEW TIME STEP

```

X(NM1,K) = S(L)

```

```

Y(NM1,K) = S(LM1)-A(LM1,4)*XX(NM1,K)

```

```

IM = 1

```

```

DO 150 IJ = 3, LM1, 2

```

```

I = L-IJ

```

```

I+1 = I+1

```

```

IM = IM+1

```

```

NA = N-IM

```

```

NB = NA+1

```

```

X(NA,K) = S(IP1)-(A(IP1,5)*YY(NB,K)+A(IP1,6)*XX(NB,K))

```

```

Y(NA,K) = S(I)-(A(I,4)*XX(NB,K)+A(I,5)*YY(NB,K))

```

150 CONTINUE

```

K = K+1

```

```

IF (K.LE.MM1) GO TO 565

```

```

DO 1972 J = 2, NM1

```

```

DO 1972 K = 2, MM1

```

```

Y(J,K) = YY(J,K)

```

```

X(J,K) = XX(J,K)

```

1972 CONTINUE

```

DO 502 J = 1, N

```

```

Y(J,1) = Y(J,3)

```

```

X(J,1) = X(J,3)

```

```

Y(J,M) = Y(J,MM2)

```

```

X(J,M) = X(J,MM2)

```

502 CONTINUE

C SOME BOUNDARY CONDITIONS

```

DO 4277 K = 2, MM1

```

```

KPR = M-K+1

```

```

Y(1,K) = Y(2,KPR)

```

```

X(1,K) = X(2,KPR)

```

```

R(1,K) = R(2,KPR)
PM(1,K) = -PM(2,KPR)
PN(1,K) = PN(2,KPR)
PS(1,K) = -PS(2,KPR)
T(1,K) = T(2,KPR)
4277 CONTINUE
C   MATRIX ELEMENTS OF FLUID EQS. - FIRST TIME STEP
K = 2
575 KM1 = K-1
    KP1 = K+1
    U = PM2(1,K)/R2(1,K)
    UP1 = PM2(2,K)/R2(2,K)
    V = PN2(1,K)/R2(1,K)
    VP1 = PN2(2,K)/R2(2,K)
    W = PS2(1,K)/R2(1,K)
    WP1 = PS2(2,K)/R2(2,K)
    FP1 = .5*UP1
    GP1 = .5*VP1
    FM = .5*U
    GM = .5*V
    DO 60 J = 2, NM1
    FFJ = J
    FJ = FFJ-.5
    ALPHA2 = 1.
    RJ = DR*(FJ-1.)
    ZRJ = Z*RJ
    BF = 1.+ZRJ*ZRJ
    JP1 = J+1
    JM1 = J-1
    BPH = BF/(DPH*DPH*(FJ-1.))
    DRDJ = RJ/DT
    UM1 = U
    U = UP1
    UP1 = PM2(JP1,K)/R2(JP1,K)
    VM1 = V
    V = VP1
    VP1 = PN2(JP1,K)/R2(JP1,K)
    WM1 = W
    W = WP1
    WP1 = PS2(JP1,K)/R2(JP1,K)
    FM1 = FM
    FM = FP1
    FP1 = FJ*UP1
    GM1 = GM
    GM = GP1
    GP1 = FJ*VP1
    FM2 = FM1
    GM2 = GM1
    VKM1 = PN(J,KM1)/R(J,KM1)
    VKP1 = PN(J,KP1)/R(J,KP1)
    DEL(J,K) = Z*(X(J,K)+X(J,K))/(BF*BF)+((FJ-.5)/(1.+((FJ-.5)*DRZ)**2
1)*((Y(JP1,K)-Y(J,K))-(FJ-1.5)/(1.+((FJ-1.5)*DRZ)**2)*(Y(J,K)-Y(JM1
2K))))/(RJ*DR)+(Y(J,KP1)-Y(J,K)-Y(J,K)+Y(J,KM1))/(RJ*DPH)**2
    ALPHA = (BPH*((HC(J,KP1)+HC(J,K))*(T(J,KP1)-T(J,K))-(HC(J,K)+HC(J
1KM1))*(T(J,K)-T(J,KM1))))+.5*A3*RJ*RSMAX(J,K)*((X(JP1,K)-X(JM
21,K))**2/(DR*DR*BF)+((X(J,KP1)-X(J,KM1))/(DPH*RJ))**2+.4.*BF*DEL(J
3K)*DEL(J,K))*R(J,K)/R2(J,K))/R(J,K)
    ZP = (FJ-.5)*ABS(PM(JP1,K)/R(JP1,K)-PM(J,K)/R(J,K))
    ZM = (FJ-1.5)*ABS(PM(J,K)/R(J,K)-PM(JM1,K)/R(JM1,K))
    QP = B10*(ZP*(R(JP1,K)-R(J,K))-ZM*(R(J,K)-R(JM1,K)))+(ABS(VKP1-V)*
1(R(J,KP1)-R(J,K))-ABS(V-VKM1)*(R(J,K)-R(J,KM1)))/DPH)

```

```

QM = B10*(ZP*(PM(JP1,K)-PM(J,K))-ZM*(PM(J,K)-PM(JM1,K)))+(ABS(VKP1
1-V)*(PM(J,KP1)-PM(J,K))-ABS(V-VKM1)*(PM(J,K)-PM(J,KM1)))/DPH)
QN = B10*(ZP*(PN(JP1,K)-PN(J,K))-ZM*(PN(J,K)-PN(JM1,K)))+(ABS(VKP1
1-V)*(PN(J,KP1)-PN(J,K))-ABS(V-VKM1)*(PN(J,K)-PN(J,KM1)))/DPH)
QS = B10*(ZP*(PS(JP1,K)-PS(J,K))-ZM*(PS(J,K)-PS(JM1,K)))+(ABS(VKP1
1-V)*(PS(J,KP1)-PS(J,K))-ABS(V-VKM1)*(PS(J,K)-PS(J,KM1)))/DPH)
QT = B10*(ZP*(T(JP1,K)-T(J,K))-ZM*(T(J,K)-T(JM1,K)))+(ABS(VKP1-V)*
1*(T(J,KP1)-T(J,K))-ABS(V-VKM1)*(T(J,K)-T(J,KM1)))/DPH)

```

```
JX = 5*J-9
```

```
A(JX,1) = 0.
```

```
A(JX,2) = -(FJ-2.)
```

```
IF (J.EQ.2) A(JX,2)=-A(JX,2)
```

```
A(JX,3) = 0.
```

```
A(JX,4) = 0.
```

```
A(JX,5) = 0.
```

```
A(JX,6) = DRDJ*DRDJ
```

```
A(JX,7) = 0.
```

```
A(JX,8) = 0.
```

```
A(JX,9) = 0.
```

```
A(JX,10) = 0.
```

```
A(JX,11) = 0.
```

```
A(JX,12) = FJ
```

```
A(JX,13) = 0.
```

```
A(JX,14) = 0.
```

```
A(JX,15) = 0.
```

```
S(JX) = (DRDJ+DRDJ)*R(J,K)-(PN(J,KP1)-PN(J,KM1))/DPH+(QR+QR)
```

```
JX = 5*J-8
```

```
A(JX,1) = .5*FM2*UM1
```

```
A(JX,2) = -FM2
```

```
A(JX,3) = 0.
```

```
A(JX,4) = 0.
```

```
A(JX,5) = -(FJ-1.)*R2(JM1,K)/A1
```

```
A(JX,6) = ((V+ZRJ*W)/BF)**2
```

```
A(JX,7) = DRDJ
```

```
A(JX,8) = -2.*(V+ZRJ*W)/(BF*BF)
```

```
A(JX,9) = ZRJ*A(JX,8)
```

```
A(JX,10) = 0.
```

```
A(JX,11) = -.5*FP1*UP1*ALPHA2
```

```
A(JX,12) = FP1*ALPHA2
```

```
A(JX,13) = 0.
```

```
A(JX,14) = 0.
```

```
A(JX,15) = (FJ-1.)*R2(JP1,K)/A1
```

```
S(JX) = DRDJ*PM(J,K)-(PM(J,KP1)*VKP1-PM(J,KM1)*VKM1)/(DPH+DPH)+(FJ
1-1.)*(-(X(JP1,K)*X(JP1,K)-X(JM1,K)*X(JM1,K))/(4.*BF)-.5*(Y(JP1,K)-
2Y(JM1,K))*DEL(J,K))+QM
```

```
4281 JX = 5*J-7
```

```
A(JX,1) = FM2*VM1
```

```
A(JX,2) = -GM2
```

```
A(JX,3) = -FM2
```

```
A(JX,4) = 0.
```

```
A(JX,5) = 0.
```

```
A(JX,6) = -(U+U)/BF*(ZRJ*(W+W)+(2.-BF)*V)
```

```
A(JX,7) = 2./BF*(ZRJ*(W+W)+(2.-BF)*V)
```

```
A(JX,8) = 2.*(DRDJ+(2.-BF)/BF*U)
```

```
A(JX,9) = 4.*FM*DRZ/BF
```

```
A(JX,10) = 0.
```

```
A(JX,11) = -FP1*VP1*ALPHA2
```

```
A(JX,12) = GP1*ALPHA2
```

```
A(JX,13) = FP1*ALPHA2
```

```
A(JX,14) = 0.
```

```
A(JX,15) = 0.
```


S(JX) = (DRDJ+DRDJ)*PN(J,K)-(PN(J,KP1)*VKP1-PN(J,KM1)*VKM1+.5*(X(J
1,KP1)*X(J,KP1)-X(J,KM1)*X(J,KM1)))+8F*((Y(J,KP1)-Y(J,KM1))*DEL(J,K)
2+.5/A1*(R(J,KP1)*T(J,KP1)-R(J,KM1)*T(J,KM1)))/DPH+(QN+QN)

7125 JX = 5*J-6

A(JX,1) = FM2*WM1

A(JX,2) = -(FJ-2.)*WM1

IF (J.EQ.2) A(JX,2)=-A(JX,2)

A(JX,3) = 0.

A(JX,4) = -FM2

A(JX,5) = 0.

A(JX,6) = 0.

A(JX,7) = 0.

A(JX,8) = 0.

A(JX,9) = DRDJ+DRDJ

A(JX,10) = 0.

A(JX,11) = -FP1*WP1*ALPHA2

A(JX,12) = FJ*WP1*ALPHA2

A(JX,13) = 0.

A(JX,14) = FP1*ALPHA2

A(JX,15) = 0.

S(JX) = (DRDJ+DRDJ)*PS(J,K)+(VKM1*PS(J,KM1)-VKP1*PS(J,KP1)+((X(J,K
1P1)-X(J,KM1))*(Y(JP1,K)-Y(JM1,K))-(X(JP1,K)-X(JM1,K))*(Y(J,KP1)-Y(
2J,KM1)))/((NR+DR))/DPH+(QS+QS)

4282 JX = 5*J-5

A(JX,1) = 0.

A(JX,2) = 0.

A(JX,3) = 0.

A(JX,4) = 0.

A(JX,5) = -FM2-(HC(J,K)+HC(JM1,K))*(FJ-1.5)/R2(J,K)

A(JX,6) = 0.

A(JX,7) = 0.

A(JX,8) = 0.

A(JX,9) = 0.

A(JX,10) = DRDJ+DRDJ+((FJ-.5)*(HC(JP1,K)+HC(J,K))+(FJ
1-1.5)*(HC(J,K)+HC(JM1,K)))/R2(J,K)

A(JX,10) = A(JX,10)-(FP1-FM1)/3.

A(JX,11) = 0.

A(JX,12) = 0.

A(JX,13) = 0.

A(JX,14) = 0.

A(JX,15) = FP1*ALPHA2-(FJ-.5)/R2(J,K)*(HC(J,K)+HC(JP1,K))

S(JX) = ALPHA+(DRDJ+DRDJ)*T(J,K)+(VKM1*T(J,KM1)-VKP1*T(J,KP1)+T(J,
1K)/3.+(VKP1-VKM1))/DPH+(QT+QT).

60 CONTINUE

C MATRIX ELEMENTS NEAR THE BOUNDARY

S(2) = S(2)-T(1,K)*A(2,5)

S(1) = S(1)-PM(1,K)*A(1,2)

S(2) = S(2)-(R(1,K)*A(2,1)+PM(1,K)*A(2,2))

S(3) = S(3)-(R(1,K)*A(3,1)+PM(1,K)*A(3,2)+PN(1,K)*A(3,3))

S(4) = S(4)-(R(1,K)*A(4,1)+PM(1,K)*A(4,2)+PS(1,K)*A(4,4))

S(5) = S(5)-(PM(1,K)*A(5,2)+T(1,K)*A(5,5))

J = NM1

JX = 5*J-8

S(JX) = S(JX)-(R(N,K)*A(JX,11)+T(N,K)*A(JX,15))

JX = 5*J-5

S(JX) = S(JX)-T(N,K)*A(JX,15)

C CROUT,S INVERSION-FIRST DT-FLUID EQS.

L = 5*NM2

LM1 = L-1

LM2 = L-2

LM3 = L-3

```

LM4 = L-4
LM9 = L-9
GO TO 6500
6501 CONTINUE
C   END OF CROUT'S INVERSION
C   VALUES AT THE NEW TIME STEP
TT(NM1,K) = S(L)
PDS(NM1,K) = S(LM1) - A(LM1,10)*TT(NM1,K)
PDN(NM1,K) = S(LM2) - (A(LM2,9)*PPS(NM1,K) + A(LM2,10)*TT(NM1,K))
PDM(NM1,K) = S(LM3) - (A(LM3,8)*PPN(NM1,K) + A(LM3,9)*PPS(NM1,K) + A(LM3,
1,10)*TT(NM1,K))
RDN(NM1,K) = S(LM4) - (A(LM4,7)*PPM(NM1,K) + A(LM4,8)*PPN(NM1,K) + A(LM4,
19)*PPS(NM1,K) + A(LM4,10)*TT(NM1,K))
IM = 1
DO 50 IJ = 9, LM1, 5
I = L-IJ
Ip1 = I+1
Ip2 = I+2
Ip3 = I+3
Ip4 = I+4
IM = IM+1
NA = N-IM
NB = NA+1
TT(NA,K) = S(IP4) - (A(IP4,11)*RR(NB,K) + A(IP4,12)*PPM(NB,K) + A(IP4,13
1)*PPN(NB,K) + A(IP4,14)*PPS(NB,K) + A(IP4,15)*TT(NB,K))
IF (ABS(PPN(NB,K)).LT.OMEGA) PPN(NB,K)=0.
IF (ABS(PPS(NB,K)).LT.OMEGA) PPS(NB,K)=0.
PDS(NA,K) = S(IP3) - (A(IP3,10)*TT(NA,K) + A(IP3,11)*RR(NB,K) + A(IP3,12
1)*PPM(NB,K) + A(IP3,13)*PPN(NB,K) + A(IP3,14)*PPS(NB,K) + A(IP3,15)*TT(N
2B,K))
IF (ABS(PPS(NA,K)).LT.OMEGA) PPS(NA,K)=0.
PDN(NA,K) = S(IP2) - (A(IP2,9)*PPS(NA,K) + A(IP2,10)*TT(NA,K) + A(IP2,11)
1*RR(NB,K) + A(IP2,12)*PPM(NB,K) + A(IP2,13)*PPN(NB,K) + A(IP2,15)*TT(NB,
2K))
IF (ABS(PPN(NA,K)).LT.OMEGA) PPN(NA,K)=0.
PDM(NA,K) = S(IP1) - (A(IP1,8)*PPN(NA,K) + A(IP1,9)*PPS(NA,K) + A(IP1,10)
1*TT(NA,K) + A(IP1,11)*RR(NB,K) + A(IP1,12)*PPM(NB,K) + A(IP1,15)*TT(NR,K
2))
RDN(NA,K) = S(I) - (A(I,7)*PPM(NA,K) + A(I,8)*PPN(NA,K) + A(I,9)*PPS(NA,K
1) + A(I,10)*TT(NA,K) + A(I,12)*PPM(NB,K))
50 CONTINUE
K = K+1
IF (K.LE.MM1) GO TO 575
DO 1973 K = 2, MM1
DO 1973 J = 2, NM1
R(J,K) = RR(J,K)
IF (R(J,K).LT.RMIN) R(J,K)=RMIN
PM(J,K) = PPM(J,K)
PN(J,K) = PPN(J,K)
IF (ABS(PN(J,K)).LT.OMEGA) PN(J,K) = 0.
PS(J,K) = PPS(J,K)
IF (ABS(PS(J,K)).LT.OMEGA) PS(J,K)=0.
T(J,K) = TT(J,K)
1973 CONTINUE
C   SOME BOUNDARY CONDITIONS
DO 4262 K = 2, MM1
KPR = M-K+1
R(1,K) = R(2,KPR)
PM(1,K) = -PM(2,KPR)
PN(1,K) = PN(2,KPR)
PS(1,K) = -PS(2,KPR)

```

```

      T(1,K) = T(2,KPR)
4262 CONTINUE
      DO 503 K = 2, MM1
      R(N,K) = R(NM1,K)
      PM(N,K) = 0.
      PN(N,K) = PN(NM1,K)
      PS(N,K) = PS(NM1,K)
      T(N,K) = TWALL
503 CONTINUE
      DO 504 J = 1, N
      R(J,1) = R(J,3)
      PM(J,1) = PM(J,3)
      PN(J,1) = -PN(J,3)
      PS(J,1) = -PS(J,3)
      T(J,1) = T(J,3)
      R(J,M) = R(J,MM2)
      PM(J,M) = PM(J,MM2)
      PN(J,M) = -PN(J,MM2)
      PS(J,M) = -PS(J,MM2)
      T(J,M) = T(J,MM2)
504 CONTINUE
C   DEFINITION OF SQRT OF T(J,K)
      DO 336 J = 1, N
      DO 337 K = 2, MM1
      IF (T(J,K).LE.0.) GO TO 1234
      ST(J,K) = SQRT(T(J,K))
337 CONTINUE
      ST(J,1) = ST(J,MM1)
      ST(J,M) = ST(J,2)
336 CONTINUE
C   ****SECOND TIME STEP STARTS HERE
3210 KD = KD+1
      FNT = KD
      TIME = TIME+DT
      TIME2 = TIME*TIME0
C   DEFINITIONS OF PREDICTED VALUES
      GO TO 8500
8502 CONTINUE
      ALPHA = 0.
C   MATRIX ELEMENTS OF FIELD EQS. - SECOND TIME STEP
      RPH = 0.
      RP1 = .5/(1.+(.5*DRZ)**2)
      RJ = +RP1
      J = 2
95 FFJ = J
      FJ = FFJ-.5
      JM1 = J-1
      JP1 = J+1
      BF = 1.
      BF = BF+((FJ-1.)*DRZ)**2
      RDPH = (FJ-1.)*DPH
      RM1 = RJ
      RJ = RP1
      RP1 = FJ/(1.+(FJ*DRZ)**2)
      RMH = RPH
      RPH = (FJ-.5)/(1.+(FJ-.5)*DRZ)**2)
      V = PN2(J,1)/R2(J,1)
      VP1 = PN2(J,2)/R2(J,2)
      DO 90 K = 2, MM1
      KM1 = K-1
      KP1 = K+1

```

```

VM1 = V
V = VP1
VP1 = PN2(J,KP1)/R2(J,KP1)
EM = A4/DR*RSMAX(J,KM1)
EJ = A4/DR*RSMAX(J,K)
EP = A4/DR*RSMAX(J,KP1)
RATIO = Z/(BF*R2(J,K))*(PN2(J,K)+(FJ-1.)*DRZ*PS2(J,K))-(PS2(JP1,K)
1/R2(JP1,K)-PS2(JM1,K)/R2(JM1,K))/(4.*DR)
ZP = (FJ-.5)*ABS(PM(JP1,K)/R(JP1,K)-PM(J,K)/R(J,K))
ZM = (FJ-1.5)*ABS(PM(J,K)/R(J,K)-PM(JM1,K)/R(JM1,K))
Z1 = ABS(PN(J,KP1)/R(J,KP1)-PN(J,K)/R(J,K))
Z2 = ABS(PN(J,K)/R(J,K)-PN(J,KM1)/R(J,KM1))
QPSI = B12*(DPH*(ZP*(Y(JP1,K)-Y(J,K))-ZM*(Y(J,K)-Y(JM1,K)))+Z1*(Y(
1J,KP1)-Y(J,K))-Z2*(Y(J,K)-Y(J,KM1)))
QKI = B12*(DPH*(ZP*(X(JP1,K)-X(J,K))-ZM*(X(J,K)-X(JM1,K)))+Z1*(X(J
1,KP1)-X(J,K))-Z2*(X(J,K)-X(J,KM1)))
KX = K+K-3
A(KX,1) = -EJ/(DPH*RJ) -.5*V
A(KX,2) = 0.
A(KX,3) = RDPH*DRDT+(EJ+EJ)/(DPH*RJ)
A(KX,4) = 0.
A(KX,5) = -EJ/(DPH*RJ) +.5*V
A(KX,6) = 0.
S(KX) = RDPH*(DRDT*Y(J,K)+EJ/RJ*(RPH*(Y(JP1,K)-Y(J,K))-RMH*(Y(J,K)
1-Y(JM1,K)))-.5*PM(J,K)/R(J,K)*(Y(JP1,K)-Y(JM1,K))+ALPHA)
S(KX) = S(KX)+(EJ+EJ)*DR*DRZ*RJ*DPH*X(J,K)
S(KX) = S(KX)+QPSI
4294 KX = K+K-2
A(KX,1) = Z*(EJ+EJ)/RDPH+RATIO
A(KX,2) = -(EJ+EM)/(RDPH+RDPH)*BF-.5*VM1
A(KX,3) = -4.*EJ*Z/RDPH
A(KX,4) = RDPH*DRDT+BF*(EP+EJ+EJ+EM)/(RDPH+RDPH)
A(KX,5) = Z*(EJ+EJ)/RDPH-RATIO
A(KX,6) = -(EP+EJ)/(RDPH+RDPH)*BF+.5*VP1
S(KX) = (Y(JP1,K)-Y(JM1,K))*(PS(J,KP1)/R(J,KP1)-PS(J,KM1)/R(J,KM1)
1)/(4.*DR)+RDPH*(DRDT*X(J,K)+(RPH*(X(JP1,K)-X(J,K))*(EJ+A4/DR*
2RSMAX(JP1,K))-RMH*(X(J,K)-X(JM1,K))*(EJ+A4/DR*RSMAX(JM1,K)))
3/(RJ+RJ)-Z*(EJ+EJ)/(FJ-1.)*(RPH*(Y(JP1,K)-Y(J,K))-RMH*(Y(J,K)
4-Y(JM1,K)))+(Y(JP1,K)-Y(JM1,K))*Z*PM(J,K)/(R(J,K)*BF)-(RP1*PM(JP1.
5K)*X(JP1,K)/R(JP1,K)-RM1*PM(JM1,K)*X(JM1,K)/R(JM1,K))/(RJ+RJ))+QKI
S(KX) = S(KX)-4.*EJ*RDPH*(DRZ/BF)**2*X(J,K)
90 CONTINUE
C MATRIX ELEMENTS NEAR THE BOUNDARIES
A(1,5) = A(1,5)+A(1,1)
A(2,5) = A(2,5)+A(2,1)
A(2,6) = A(2,6)+A(2,2)
L = MM2+MM2
LM1 = L-1
A(LM1,1) = A(LM1,1)+A(LM1,5)
A(L,1) = A(L,1)+A(L,5)
A(L,2) = A(L,2)+A(L,6)
C CROUT'S INVERSION OF FIELD MATRIX - SECOND TIME STEP
1617 L = MM2+MM2
LM1 = L-1
LM2 = L-2
LM3 = L-3
GO TO 4500
4502 CONTINUE
C END OF CROUT'S INVERSION
C VALUES AT THE NEW TIME STEP
XV(J,MM1) = S(L)

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YV(J,MM1) = S(LM1)-A(LM1,4)*XX(J,MM1)
IM = 1
DO 200 IJ = 3 ,LM1, 2
I = L-IJ
ID1 = I+1
IM = IM+1
MA = M-IM
MR = MA+1
XY(J,MA) = S(IP1)-(A(IP1,5)*YY(J,MB)+A(IP1,6)*XX(J,MR))
YV(J,MA) = S(I)-(A(I,4)*XX(J,MA)+A(I,5)*YY(J,MB))
200 CONTINUE
J = J+1
IF (J.LE.NM1) GO TO 95
DO 1974 J = 2, NM1
DO 1974 K = 2, MM1
Y(J,K) = YY(J,K)
X(J,K) = XX(J,K)
1974 CONTINUE
C SOLUTION OF ELLIPTIC EQ. IN VACUUM REGION
GO TO 6700
6702 CONTINUE
C SOME BOUNDARY CONDITIONS
DO 505 J = 1, N
Y(J,1) = Y(J,3)
X(J,1) = X(J,3)
Y(J,M) = Y(J,MM2)
X(J,M) = X(J,MM2)
505 CONTINUE
C MATRIX ELEMENTS OF FLUID EQS. - SECOND TIME STEP
J = 2
960 FFJ = J
FJ = FFJ-.5
ALPHA2 = 1.
JM1 = J-1
JP1 = J+1
RJ = (FJ-1.)*DR
ZRJ = Z*RJ
DRDJ = RJ/DT
BF = 1.+ZRJ*ZRJ
BPH = BF/(DPH*(FJ-1.))
U = PM2(J,1)/R2(J,1)
UP1 = PM2(J,2)/R2(J,2)
V = PN2(J,1)/R2(J,1)
VP1 = PN2(J,2)/R2(J,2)
W = PS2(J,1)/R2(J,1)
WP1 = PS2(J,2)/R2(J,2)
DO 80 K = 2, MM1
KM1 = K-1
KP1 = K+1
UM1 = U
U = UP1
UP1 = PM2(J,KP1)/R2(J,KP1)
VM1 = V
V = VP1
VP1 = PN2(J,KP1)/R2(J,KP1)
WM1 = W
W = WP1
WP1 = PS2(J,KP1)/R2(J,KP1)
UJM1 = PM(JM1,K)/R(JM1,K)
UJP1 = PM(JP1,K)/R(JP1,K)
FM1 = (FJ-2.)*UJM1

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```

IF (J.EQ.2) FM1=-FM1
FM2 = FM1
FP1 = FJ*UJP1
DEL(J,K) = Z*(X(J,K)+X(J,K))/(RF*BF)+((FJ-.5)/(1.+(FJ-.5)*DRZ)**2
1)*(Y(JP1,K)-Y(J,K))-(FJ-1.5)/(1.+(FJ-1.5)*DRZ)**2*(Y(J,K)-Y(JM1,
2K)))/(RJ*DR)+(Y(J,KP1)-Y(J,K)-Y(J,K)+Y(J,KM1))/(RJ*DPH)**2
ALPHA = DPH/R(J,K)*((FJ-.5)*(HC(JP1,K)+HC(J,K))*(T(JP1,K)-T(J,K))-
1*(FJ-1.5)*(HC(J,K)+HC(JM1,K))*(T(J,K)-T(JM1,K))+.5*A3*RJ*RSMAX(J,K)
2*((X(JP1,K)-X(JM1,K))**2/(DR*DR*RF)+((X(J,KP1)-X(J,KM1))/(RJ
3*DPH))**2+.4*BF*DEL(J,K)*DEL(J,K)))
RATIO = (PN(J,K)+ZRJ*PS(J,K))/(R(J,K)*BF)
ZM = (FJ-1.5)*ABS(U-UJM1)
ZP = (FJ-.5)*ABS(UJP1-U)
QR = B10*(ZP*(R(JP1,K)-R(J,K))-ZM*(R(J,K)-R(JM1,K)))+(ABS(VP1-V)*(
1R(J,KP1)-R(J,K))-ABS(V-VM1)*(R(J,K)-R(J,KM1)))/DPH)
QM = B10*(ZP*(PM(JP1,K)-PM(J,K))-ZM*(PM(J,K)-PM(JM1,K)))+(ABS(VP1-
1V)*(PM(J,KP1)-PM(J,K))-ABS(V-VM1)*(PM(J,K)-PM(J,KM1)))/DPH)
QN = B10*(ZP*(PN(JP1,K)-PN(J,K))-ZM*(PN(J,K)-PN(JM1,K)))+(ABS(VP1-
1V)*(PN(J,KP1)-PN(J,K))-ABS(V-VM1)*(PN(J,K)-PN(J,KM1)))/DPH)
QS = B10*(ZP*(PS(JP1,K)-PS(J,K))-ZM*(PS(J,K)-PS(JM1,K)))+(ABS(VP1-
1V)*(PS(J,KP1)-PS(J,K))-ABS(V-VM1)*(PS(J,K)-PS(J,KM1)))/DPH)
QT = B10*(ZP*(T(JP1,K)-T(J,K))-ZM*(T(J,K)-T(JM1,K)))+(ABS(VP1-V)*(
1T(J,KP1)-T(J,K))-ABS(V-VM1)*(T(J,K)-T(J,KM1)))/DPH)
KX = 5*K-9
A(KX,1) = 0.
A(KX,2) = 0.
A(KX,3) = -1./DPH
A(KX,4) = 0.
A(KX,5) = 0.
A(KX,6) = DRDJ+DRDJ
A(KX,7) = 0.
A(KX,8) = 0.
A(KX,9) = 0.
A(KX,10) = 0.
A(KX,11) = 0.
A(KX,12) = 0.
A(KX,13) = 1./DPH
A(KX,14) = 0.
A(KX,15) = 0.
S(KX) = (DRDJ+DRDJ)*R(J,K)-FJ*PM(JP1,K)+(FJ-2.)*PM(JM1,K)+(QR+QR)
IF (J.EQ.2) S(KX)=S(KX)+PM(JM1,K)
KX = 5*K-8
A(KX,1) = UM1*VM1
A(KX,2) = -VM1
A(KX,3) = -UM1
A(KX,4) = 0.
A(KX,5) = 0.
A(KX,6) = 0.
A(KX,7) = (DRDJ+DRDJ)*DPH
A(KX,8) = 0.
A(KX,9) = 0.
A(KX,10) = 0.
A(KX,11) = -UP1*VP1
A(KX,12) = VP1
A(KX,13) = UP1
A(KX,14) = 0.
A(KX,15) = 0.
S(KX) = (DPH+DPH)*RJ*(PM(J,K)/DT-((R(JP1,K)*T(JP1,K)-R(JM1,K)*T(JM
11,K))/A1+.5*(Y(JP1,K)-Y(JM1,K))*DEL(J,K)+(X(JP1,K)*X(JP1,K)-X(JM1,
2K)*X(JM1,K))/(4.*BF))/DR)-DPH*(FP1*ALPHA2*PM(JP1,K)-FM2*PM(JM1,K)+
3 0.)*2.*DPH*QM

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S(KX) = S(KX) + (DPH+DPH)*R(J,K)*RATIO*RATIO
4291 KX = 5*K-7
A(KX,1) = .5*VM1*VM1
A(KX,2) = 0.
A(KX,3) = -VM1
A(KX,4) = 0.
A(KX,5) = -BF*R2(J,KM1)/A1
A(KX,6) = 0.
A(KX,7) = 0.
A(KX,8) = DRDJ*DPH
A(KX,9) = 0.
A(KX,10) = 0.
A(KX,11) = -.5*VP1*VP1
A(KX,12) = 0.
A(KX,13) = VP1
A(KX,14) = 0.
A(KX,15) = BF*R2(J,KP1)/A1
S(KX) = -.5*(DPH*(FP1*ALPHA2*PN(JP1,K)-FM2*PN(JM1,K))+BF*(Y(J,KP1)
1-Y(J,KM1))*DEL(J,K)+.5*(X(J,KP
21)*X(J,KP1)-X(J,KM1)*X(J,KM1)))+DPH*(DRDJ*PN(J,K)+QN)
3-DPH*PM(J,K)/(R(J,K)*BF)*((ZRJ+ZRJ)*PS(J,K)+(2.-BF)*PN(J,K))
KX = 5*K-6
A(KX,1) = VM1*WM1
A(KX,2) = 0.
A(KX,3) = -WM1
A(KX,4) = -VM1
A(KX,5) = 0.
A(KX,6) = 0.
A(KX,7) = 0.
A(KX,8) = 0.
A(KX,9) = (DPH+DPH)*DRDJ
A(KX,10) = 0.
A(KX,11) = -VP1*WP1
A(KX,12) = 0.
A(KX,13) = WP1
A(KX,14) = VP1
A(KX,15) = 0.
S(KX) = DPH*(2.*DRDJ*PS(J,K))+((X(J,KP1)-X(J,KM1))*(Y(JP1,K)-Y(JM1
1,K))-(X(JP1,K)-X(JM1,K))*(Y(J,KP1)-Y(J,KM1)))/(DR+DR)+DPH*(QS+QS)
S(KX) = S(KX)-DPH*(FP1*ALPHA2*PS(JP1,K)-FM2*PS(JM1,K))
4292 KX = 5*K-5
A(KX,1) = 0.
A(KX,2) = 0.
A(KX,3) = 0.
A(KX,4) = 0.
A(KX,5) = -VM1-(HC(J,K)+HC(J,KM1))*BPH/R2(J,K)
A(KX,6) = 0.
A(KX,7) = 0.
A(KX,8) = 0.
A(KX,9) = 0.
A(KX,10) = (DRDJ+DRDJ)*DPH+(VM1-VP1)/3.+BPH/R2(J,K)*
1(HC(J,KP1)+HC(J,K)+HC(J,K)+HC(J,KM1))
A(KX,11) = 0.
A(KX,12) = 0.
A(KX,13) = 0.
A(KX,14) = 0.
A(KX,15) = VP1-BPH*(HC(J,KP1)+HC(J,K))/R2(J,K)
S(KX) = ALPHA+DPH*((DRDJ+DRDJ)*T(J,K)+FM2*T(JM1,K)-FP1*ALPHA2*T(JP
11,K)*T(J,K)/3.+(FP1*ALPHA2-FM2)+2.*QT)
80 CONTINUE
C MATRIX ELEMENTS NEAR THE BOUNDARIES

```

```

DO 6508 KY = 1, 5
A(KY,11) = A(KY,11)+A(KY,1)
A(KY,12) = A(KY,12)+A(KY,2)
A(KY,13) = A(KY,13)-A(KY,3)
A(KY,14) = A(KY,14)-A(KY,4)
A(KY,15) = A(KY,15)+A(KY,5)

```

```

6508 CONTINUE

```

```

L = 5*MM2
LM4 = L-4

```

```

DO 6509 KY = LM4, L

```

```

A(KY,1) = A(KY,1)+A(KY,11)
A(KY,2) = A(KY,2)+A(KY,12)
A(KY,3) = A(KY,3)-A(KY,13)
A(KY,4) = A(KY,4)-A(KY,14)
A(KY,5) = A(KY,5)+A(KY,15)

```

```

6509 CONTINUE

```

```

C CROUT'S INVERSION OF FLUID MATRIX - SECOND TIME STEP

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```

1625 L = 5*MM2

```

```

LM1 = L-1

```

```

LM2 = L-2

```

```

LM3 = L-3

```

```

LM4 = L-4

```

```

LM9 = L-9

```

```

GO TO 6500

```

```

6502 CONTINUE

```

```

C END OF CROUT'S INVERSION

```

```

C VALUES AT THE NEW TIME STEP

```

```

TT(J,MM1) = S(L)

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```

PPN(J,MM1) = S(LM1)-A(LM1,10)*TT(J,MM1)

```

```

PPN(J,MM1) = S(LM2)-(A(LM2,9)*PPS(J,MM1)+A(LM2,10)*TT(J,MM1))

```

```

IF (ABS(PPN(J,MM1)).LT.OMEGA) PPN(J,MM1)=0.

```

```

IF (ABS(PPS(J,MM1)).LT.OMEGA) PPS(J,MM1)=0.

```

```

PPM(J,MM1) = S(LM3)-(A(LM3,8)*PPN(J,MM1)+A(LM3,9)*PPS(J,MM1)+A(LM3,10)*TT(J,MM1))

```

```

R0(J,MM1) = S(LM4)-(A(LM4,7)*PPM(J,MM1)+A(LM4,8)*PPN(J,MM1)+A(LM4,9)*PPS(J,MM1)+A(LM4,10)*TT(J,MM1))

```

```

IM = 1

```

```

DO 100 IJ = 9, LM1, 5

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```

I = L-IJ

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```

IP1 = I+1

```

```

IP2 = I+2

```

```

IP3 = I+3

```

```

IP4 = I+4

```

```

IM = IM+1

```

```

MA = M-IM

```

```

MB = MA+1

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```

TT(J,MA) = S(IP4)-(A(IP4,11)*RR(J,MB)+A(IP4,12)*PPM(J,MB)+A(IP4,13)*PPN(J,MB)+A(IP4,14)*PPS(J,MB)+A(IP4,15)*TT(J,MB))

```

```

IF (ABS(PPN(J,MB)).LT.OMEGA) PPN(J,MB)=0.

```

```

IF (ABS(PPS(J,MB)).LT.OMEGA) PPS(J,MB)=0.

```

```

IF (ABS(A(IP3,13)).LT.DELTA) A(IP3,13)=0.

```

```

IF (ABS(A(IP3,14)).LT.DELTA) A(IP3,14)=0.

```

```

PPS(J,MA) = S(IP3)-(A(IP3,10)*TT(J,MA)+A(IP3,11)*RR(J,MB)+A(IP3,12)*PPM(J,MB)+A(IP3,13)*PPN(J,MB)+A(IP3,14)*PPS(J,MB)+A(IP3,15)*TT(J,MA))

```

```

IF (ABS(PPS(J,MA)).LT.OMEGA) PPS(J,MA)=0.

```

```

PPN(J,MA) = S(IP2)-(A(IP2,9)*PPS(J,MA)+A(IP2,10)*TT(J,MA)+A(IP2,11)*RR(J,MB)+A(IP2,12)*PPM(J,MB)+A(IP2,13)*PPN(J,MB)+A(IP2,14)*PPS(J,MA)+A(IP2,15)*TT(J,MA))

```

```

PPM(J,MA) = S(IP1)-(A(IP1,8)*PPN(J,MA)+A(IP1,9)*PPS(J,MA)+A(IP1,10)*TT(J,MA)+A(IP1,11)*RR(J,MB)+A(IP1,12)*PPM(J,MB)+A(IP1,13)*PPN(J,MA))

```


2MR))

$$R_0(J,MA) = S(I) - (A(I,7)*PPM(J,MA) + A(I,8)*PPN(J,MA) + A(I,9)*PPS(J,MA) + A(I,10)*TT(J,MA) + A(I,13)*PPN(J,MB))$$

100 CONTINUE

J = J+1

IF (J.LE.NM1) GO TO 960

DO 1976 K = 2, MM1

DO 1976 J = 2, NM1

R(J,K) = RR(J,K)

IF (R(J,K).LT.RMIN) R(J,K)=RMIN

PM(J,K) = PPM(J,K)

PN(J,K) = PPN(J,K)

IF (ABS(PN(J,K)).LT.OMEGA) PN(J,K) = 0.

PS(J,K) = PPS(J,K)

IF (ABS(PS(J,K)).LT.OMEGA) PS(J,K)=0.

T(J,K) = TT(J,K)

1976 CONTINUE

C SOME BOUNDARY CONDITIONS

DO 506 K = 2, MM1

R(N,K) = R(NM1,K)

PM(N,K) = 0.

PN(N,K) = PN(NM1,K)

PS(N,K) = PS(NM1,K)

T(N,K) = TWALL

506 CONTINUE

DO 507 J = 1, N

R(J,1) = R(J,3)

PM(J,1) = PM(J,3)

PN(J,1) = -PN(J,3)

PS(J,1) = -PS(J,3)

T(J,1) = T(J,3)

R(J,M) = R(J,MM2)

PM(J,M) = PM(J,MM2)

PN(J,M) = -PN(J,MM2)

PS(J,M) = -PS(J,MM2)

T(J,M) = T(J,MM2)

507 CONTINUE

IF (KD.LT.20000) GO TO 2974

C
C
C
C

SOLUTION OF ELLIPTIC EQUATION IN THE VACUUM REGION

6700 CONTINUE

VCCOIL = Z*VCCMAX*COS(CONST3*TIME)

VHEL = Z*VHLMAX*COS(CONST4*(TIME-DELAY))

IF (TIME.LT.RTIME1) GO TO 1140

VCCOIL = 0.

1140 QUANT = TIME-DELAY

IF (QUANT.LT.RTIME2) GO TO 1150

VHEL = 0.

1150 CONTINUE

XWALL = BZMAX*SIN(CONST3*TIME)

IF (TIME.GT.RTIME1) XWALL=BZMAX

DO 2901 K = 1, M

X(N,K) = XWALL

2901 CONTINUE

DO 5722 K = 1, M

$$YV(N,K) = ((4.*Y(NM1,K) - 3.*Y(N,K) - Y(NM2,K)) / DRDDR + 4.*YU(NP1,K) - YU(INP2,K)) / 3.$$

YU(N,K) = YV(N,K)

5722 CONTINUE

```

LTEST = 0
6710 LTEST = LTEST+1
DO 6723 J = NP1, NOUT2M
  YV(J,1) = YU(J,3)
  YU(J,1) = YV(J,1)
  YU(J,M) = YU(J,MM2)
6723 CONTINUE
  FSUM1 = 0.
  FSUM2 = 0.
  FSUM3 = 0.
  DO 2902 K = 2, KMID
    FSUM1 = FSUM1+YU(NOUT1M,K)
2902 CONTINUE
  FSUM1 = (2.*FSUM1-YU(NOUT1M,2))/FNEW1
  DO 2903 K = KMIDP, MM1
    FSUM2 = FSUM2+YU(NOUT2M,K)
2903 CONTINUE
  FSUM2 = (2.*FSUM2-YU(NOUT2M,MM1))/FNEW2
  DO 2904 J = NOUT1, NOUT2M
    FJ = J
    FSUM3 = FSUM3+YU(J,KMIDP)/(RCC1+(FJ-FNOUT1)*DDDR)
2904 CONTINUE
  FSUM3 = FSUM3/(FNOUT2-FNOUT1)
  YWALL = (XWALL*(QUANT1-1.)/Z+.5*(QUANT2*FSUM1/DDR+QUANT3*FSUM2/DDD
1R)-FQUANT*FSUM3)/QUANT4
  DO 1960 K = 1, KMID
    YU(NOUT1,K) = YWALL
1960 CONTINUE
  DO 1961 K = KMID, M
    YU(NOUT2,K) = YWALL
1961 CONTINUE
  DO 1962 J = NOUT1, NOUT2
    YV(J,KMID) = YWALL
    YU(J,KMID) = YWALL
1962 CONTINUE
  RJP = RVV/DDR+.5
  RPH = RJP/(1.+(DDRZ*RJP)**2)
  DO 6725 J = NP1, NOUT1M
    FJ = J
    JP1 = J+1
    JM1 = J-1
    DRMIN = DDR
    DRPLUS = DDR
    IF (J.EQ.N) DRMIN=DR
    RJ = RVV*(FJ-FN)*DDR
    RJP = RJ/DRPLUS+.5
    RMH = RPH
    RPH = RJP/(1.+(Z*DRPLUS*RJP)**2)
    BF = 1.+(Z*RJ)**2
    PROD1 = 1./(RJ*DPH*DPH)
    PROD2 = 2./(DRPLUS+DRMIN)
    DO 6725 K = 2, MM1
      KP1 = K+1
      KM1 = K-1
      YV(J,K) = RELAX*(PROD2*(RPH*YU(JP1,K)+RMH*YV(JM1,K))+(YU(J,KP1)+YV
1(J,KM1))*PROD1+2.*Z*RJ*XWALL/(BF*BF))/(PROD2*(RPH+RMH)+2.*PROD1)+(
21.-RELAX)*YU(J,K)
6725 CONTINUE
  RJP = RCC1/DDR-.5
  RPH = RJP/(1.+(Z*DDR*RJP)**2)
  DO 2725 J = NOUT1, NOUT2M

```

```

FJ = J
JP1 = J+1
JM1 = J-1
DRMIN = DDDR
DRPLUS = DDDR
IF (J.EQ.NOUT1) DRMIN=DDR
RJ = RCC1+(FJ-FNOUT1)*DDDR
RJP = RJ/DRPLUS+.5
RMH = RPH
RPH = RJP/(1.+(Z*DRPLUS*RJP)**2)
BF = 1.+(Z*RJ)**2
PROD1 = 1./(RJ*DPH*DPH)
PROD2 = 2./(DRPLUS+DRMIN)
Dn 2735 K = KMIDP, MM1
Kp1 = K+1
Km1 = K-1
YV(J,K) = RELAX*(PROD2*(RPH*YU(JP1,K)+RMH*YV(JM1,K))+(YU(J,KP1)+YV
1(J,KM1))*PROD1+2.*Z*RJ*XWALL/(BF*BF))/(PROD2*(RPH+RMH)+2.*PROD1)+(
21.-RELAX)*YU(J,K)
2735 CONTINUE
Dn 2745 K = 2, KMID
YV(J,K) = YU(J,K)
2745 CONTINUE
2725 CONTINUE
YMAX = ABS(YV(NP1,2)-YU(NP1,2))
Dn 6726 J = NP1, NOUT2M
Dn 6726 K = 2, MM1
DIFF = ABS(YV(J,K)-YU(J,K))
IF (YMAX.LT.DIFF) YMAX=DIFF
YU(J,K) = YV(J,K)
6726 CONTINUE
Dn 6727 J = NOUT1, NOUT2M
Dn 6727 K = KMIDP, MM1
DIFF = ABS(YV(J,K)-YU(J,K))
IF (YMAX.LT.DIFF) YMAX=DIFF
YU(J,K) = YV(J,K)
6727 CONTINUE
SMALL = GAMA*ABS(XWALL)
IF (YMAX.LT.SMALL) GO TO 5700
GO TO 6710
5700 CONTINUE
6730 HALF1 = KD/2
Dn 6792 K = 1, M
Y(N,K) = (4.*Y(NM1,K)-Y(NM2,K)+DRDDR*(4.*YU(NP1,K)-YU(NP2,K)-3.*YU
1(N,K)))/3.
6792 CONTINUE
HALF2 = FNT/2.
IF (ABS(HALF2-HALF1).LT.0.25) GO TO 6702
GO TO 6701

C
C
C
CALCULATION OF PREDICTED VALUES
8500 Dn 1635 K = 2, MM1
Km1 = K-1
Kp1 = K+1
Up1 = .5*PM(2,K)/R(2,K)
U = .5*PM(1,K)/R(1,K)
Dn 1635 J = 2, NM1
JP1 = J+1
JM1 = J-1
FJ = J

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FJ = FFJ-.5
RJ = DR*(FJ-1.)
R7 = RJ*Z
BF = 1.+RZ*RZ
DRDJ = RJ/DT
UM1 = U
U = UP1
Up1 = FJ*PM(JP1,K)/R(JP1,K)
Up2 = UP1
Um2 = UM1
VKP1 = PN(J,KP1)/R(J,KP1)
VKM1 = PN(J,KM1)/R(J,KM1)
R2(J,K) = (R(JP1,K)+R(JM1,K)+R(J,KP1)+R(J,KM1))/4.-(FJ*PM(JP1,K)-(
1FJ-2.)*PM(JM1,K)+(PN(J,KP1)-PN(J,KM1))/DPH)/(DRDJ+DRDJ)
IF (J.EQ.2) R2(J,K)=R2(J,K)+PM(JM1,K)/(DPH*2.*DRDJ)
PM2(J,K) = (PM(JP1,K)+PM(JM1,K)+PM(J,KP1)+PM(J,KM1))/4.-(.5*(UP2*P
1M(JP1,K)-UM2*PM(JM1,K)+(VKP1*PM(J,KP1)-VKM1*PM(J,KM1))/(DPH+DPH)-
2*(PN(J,K)+RZ*PS(J,K))/BF)**2/R(J,K))/DRDJ-((R(JP1,K)*T(JP1,K)-R(JM
31,K)*T(JM1,K))/A1+.5*(Y(JP1,K)-Y(JM1,K))*DEL(J,K)+(X(JP1,K)*X(JP1.
4K)-X(JM1,K)*X(JM1,K))/(4.*BF))/DRDT
PN2(J,K) = (PN(JP1,K)+PN(JM1,K)+PN(J,KP1)+PN(J,KM1))/4.-(UP2*PN(JP
11,K)-UM2*PN(JM1,K)+2./BF*PM(J,K)/R(J,K)*((RZ+RZ)*PS(J,K)+(2.-BF)*P
1N(J,K)+(VKP
21*PN(J,KP1)-VKM1*PN(J,KM1)+(BF+BF)/A1*(R(J,KP1)*T(J,KP1)-R(J,KM1)*
3T(J,KM1))+BF*(Y(J,KP1)-Y(J,KM1))*DEL(J,K)+.5*(X(J,KP1)*X(J,KP1)-X(
4J,KM1)*X(J,KM1)))/DPH)/(DRDJ+DRDJ)
IF (ABS(PN2(J,K)).LT.OMEGA) PN2(J,K)=0.
PS2(J,K) = (PS(JP1,K)+PS(JM1,K)+PS(J,KP1)+PS(J,KM1))/4.-(UP2*PS(JP
11,K)-UM2*PS(JM1,K)+(VKP1*PS(J,KP1)-VKM1*PS(J,KM1))/DPH-((X(J,KP1)-
2X(J,KM1))*(Y(JP1,K)-Y(JM1,K))-(X(JP1,K)-X(JM1,K))*(Y(J,KP1)-Y(J,KM
31))))/(DPH*(DR+DR)))/(DRDJ+DRDJ)
IF (ABS(PS2(J,K)).LT.OMEGA) PS2(J,K)=0.
1635 CONTINUE
DO 1636 J = 2, NM1
DO 1637 K = 2, MM1
IF (R2(J,K).LT.0.1) R2(J,K)=0.1
1637 CONTINUE
R2(J,1) = R2(J,3)
R2(J,M) = R2(J,MM2)
PM2(J,1) = PM2(J,3)
PM2(J,M) = PM2(J,MM2)
PN2(J,1) = -PN2(J,3)
PN2(J,M) = -PN2(J,MM2)
PS2(J,1) = -PS2(J,3)
PS2(J,M) = -PS2(J,MM2)
1636 CONTINUE
DO 1638 K = 2, MM1
KPR = M-K+1
R2(1,K) = R2(2,KPR)
PM2(1,K) = -PM2(2,KPR)
PN2(1,K) = PN2(2,KPR)
PS2(1,K) = -PS2(2,KPR)
R2(N,K) = R(N,K)+PM2(NM1,K)/DRDT
IF (R2(N,K).LT.0.1) R2(N,K)=0.1
PM2(N,K) = 0.
PN2(N,K) = PN2(NM1,K)
PS2(N,K) = PS2(NM1,K)
1638 CONTINUE
HALF1 = KD/2
HALF2 = FNT/2.
IF (ABS(HALF2-HALF1).LT.0.25) GO TO 8502

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C
C CROUT INVERSION - FIELD EQS.

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4500 A(1,4) = A(1,4)/A(1,3)
      A(1,5) = A(1,5)/A(1,3)
      S(1) = S(1)/A(1,3)
      A(2,4) = A(2,4)-A(1,4)*A(2,3)
      A(3,2) = A(3,2)-A(1,4)*A(3,1)
      A(4,2) = A(4,2)-A(1,4)*A(4,1)
      D = D-A(1,4)*C
      A(2,5) = (A(2,5)-A(2,3)*A(1,5))/A(2,4)
      A(2,6) = A(2,6)/A(2,4)
      S(2) = (S(2)-S(1)*A(2,3))/A(2,4)
      DO 400 I = 3, LM3, 2
        IP1 = I+1
        IP2 = I+2
        IP3 = I+3
        IM1 = I-1
        IM2 = I-2
        A(I,3) = A(I,3)-(A(IM2,5)*A(I,1)+A(IM1,5)*A(I,2))
        A(IP1,3) = A(IP1,3)-(A(IM2,5)*A(IP1,1)+A(IM1,5)*A(IP1,2))
        A(I,4) = (A(I,4)-A(I,2)*A(IM1,6))/A(I,3)
        A(I,5) = A(I,5)/A(I,3)
        S(I) = (S(I)-(A(I,1)*S(IM2)+A(I,2)*S(IM1)))/A(I,3)
        A(IP1,4) = A(IP1,4)-(A(IP1,2)*A(IM1,6)+A(IP1,3)*A(I,4))
        A(IP2,2) = A(IP2,2)-A(I,4)*A(IP2,1)
        A(IP3,2) = A(IP3,2)-A(I,4)*A(IP3,1)
        A(IP1,5) = (A(IP1,5)-A(IP1,3)*A(I,5))/A(IP1,4)
        A(IP1,6) = A(IP1,6)/A(IP1,4)
        S(IP1) = (S(IP1)-(S(IM2)*A(IP1,1)+S(IM1)*A(IP1,2)+S(I)*A(IP1,3)))/
1A(IP1,4)
400 CONTINUE
      A(LM1,3) = A(LM1,3)-(A(LM3,5)*A(LM1,1)+A(LM2,5)*A(LM1,2))
      A(L,3) = A(L,3)-(A(LM3,5)*A(L,1)+A(LM2,5)*A(L,2))
      A(LM1,4) = (A(LM1,4)-A(LM1,2)*A(LM2,6))/A(LM1,3)
      S(LM1) = (S(LM1)-(C*S(1)+D*S(2)+A(LM1,1)*S(LM3)+A(LM1,2)*S(LM2)))/
1A(LM1,3)
      A(L,4) = A(L,4)-(A(L,2)*A(LM2,6)+A(L,3)*A(LM1,4))
      S(L) = (S(L)-(A(L,1)*S(LM3)+A(L,2)*S(LM2)+A(L,3)*S(LM1)))/A(L,4)
      HALF1 = KD/2
      HALF2 = FNT/2.
      IF (ABS(HALF2-HALF1).LT.0.25) GO TO 4502
      GO TO 4501

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C
C CROUT INVERSION - FLUID EQS.

```

6500 I = 1
      IP1 = I+1
      IP2 = I+2
      IP3 = I+3
      IP4 = I+4
      IP5 = I+5
      IP6 = I+6
      IP7 = I+7
      IP8 = I+8
      IP9 = I+9
      A(1,7) = A(1,7)/A(1,6)
      A(1,8) = A(1,8)/A(1,6)
      A(1,9) = A(1,9)/A(1,6)
      A(1,10) = A(1,10)/A(1,6)
      A(I,12) = A(I,12)/A(I,6)
      A(I,13) = A(I,13)/A(I,6)

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S(1) = S(1)/A(1,6)
A(2,7) = A(2,7)-A(2,6)*A(1,7)
A(3,7) = A(3,7)-A(3,6)*A(1,7)
A(4,7) = A(4,7)-A(4,6)*A(1,7)
A(5,7) = A(5,7)-A(5,6)*A(1,7)
A(IP6,2) = A(IP6,2)-A(I,7)*A(IP6,1)
A(IP7,2) = A(IP7,2)-A(I,7)*A(IP7,1)
A(IP8,2) = A(IP8,2)-A(I,7)*A(IP8,1)
A(IP9,2) = A(IP9,2)-A(I,7)*A(IP9,1)
A(2,8) = (A(2,8)-A(1,8)*A(2,6))/A(2,7)
A(2,9) = (A(2,9)-A(1,9)*A(2,6))/A(2,7)
A(2,10) = (A(2,10)-A(1,10)*A(2,6))/A(2,7)
A(IP1,11) = A(IP1,11)/A(IP1,7)
A(IP1,12) = (A(IP1,12)-A(I,12)*A(IP1,6))/A(IP1,7)
A(IP1,13) = (A(IP1,13)-A(IP1,6)*A(I,13))/A(IP1,7)
A(IP1,15) = A(IP1,15)/A(IP1,7)
S(2) = (S(2)-S(1)*A(2,6))/A(2,7)
A(3,8) = A(3,8)-(A(3,6)*A(1,8)+A(3,7)*A(2,8))
A(4,8) = A(4,8)-(A(4,6)*A(1,8)+A(4,7)*A(2,8))
A(5,8) = A(5,8)-(A(5,6)*A(1,8)+A(5,7)*A(2,8))
A(IP5,3) = A(IP5,3)-A(IP1,8)*A(IP5,2)
DO 124 IU = IP6, IP9
A(IU,3) = A(IU,3)-(A(I,8)*A(IU,1)+A(IP1,8)*A(IU,2))
134 CONTINUE
A(3,9) = (A(3,9)-(A(1,9)*A(3,6)+A(2,9)*A(3,7)))/A(3,8)
A(3,10) = (A(3,10)-(A(1,10)*A(3,6)+A(2,10)*A(3,7)))/A(3,8)
A(IP2,11) = (A(IP2,11)-(A(IP2,7)*A(IP1,11)))/A(IP2,8)
A(IP2,12) = (A(IP2,12)-(A(I,12)*A(IP2,6)+A(IP1,12)*A(IP2,7)))/A(IP
12,8)
A(IP2,13) = (A(IP2,13)-(A(IP2,6)*A(I,13)+A(IP2,7)*A(IP1,13)))/A(IP
12,8)
A(IP2,15) = (A(IP2,15)-A(IP1,15)*A(IP2,7))/A(IP2,8)
S(3) = (S(3)-(S(1)*A(3,6)+S(2)*A(3,7)))/A(3,8)
A(4,9) = A(4,9)-(A(4,6)*A(1,9)+A(4,7)*A(2,9)+A(4,8)*A(3,9))
A(5,9) = A(5,9)-(A(5,6)*A(1,9)+A(5,7)*A(2,9)+A(5,8)*A(3,9))
A(IP5,4) = A(IP5,4)-(A(IP1,9)*A(IP5,2)+A(IP2,9)*A(IP5,3))
DO 125 IU = IP6, IP9
A(IU,4) = A(IU,4)-(A(I,9)*A(IU,1)+A(IP1,9)*A(IU,2)+A(IP2,9)*A(IU,3
1))
135 CONTINUE
A(4,10) = (A(4,10)-(A(4,6)*A(1,10)+A(4,7)*A(2,10)+A(4,8)*A(3,10))
1)/A(4,9)
A(IP3,11) = (A(IP3,11)-(A(IP3,7)*A(IP1,11)+A(IP3,8)*A(IP2,11)))/A(
1IP3,9)
A(IP3,12) = (A(IP3,12)-(A(I,12)*A(IP3,6)+A(IP1,12)*A(IP3,7)+A(IP2,
112)*A(IP3,8)))/A(IP3,9)
A(IP3,13) = (A(IP3,13)-(A(IP3,6)*A(I,13)+A(IP3,7)*A(IP1,13)+A(IP3,
18)*A(IP2,13)))/A(IP3,9)
A(IP3,14) = A(IP3,14)/A(IP3,9)
A(IP3,15) = (A(IP3,15)-(A(IP1,15)*A(IP3,7)+A(IP2,15)*A(IP3,8)))/A(
1IP3,9)
S(4) = (S(4)-(S(1)*A(4,6)+S(2)*A(4,7)+S(3)*A(4,8)))/A(4,9)
A(5,10) = A(5,10)-(A(5,6)*A(1,10)+A(5,7)*A(2,10)+A(5,8)*A(3,10)+A(
15,9)*A(4,10))
A(IP5,5) = A(IP5,5)-(A(IP1,10)*A(IP5,2)+A(IP2,10)*A(IP5,3)+A(IP3,1
10)*A(IP5,4))
DO 136 IU = IP6, IP9
A(IU,5) = A(IU,5)-(A(I,10)*A(IU,1)+A(IP1,10)*A(IU,2)+A(IP2,10)*A(I
IU,3)+A(IP3,10)*A(IU,4))
136 CONTINUE
A(IP4,11) = (A(IP4,11)-(A(IP4,7)*A(IP1,11)+A(IP4,8)*A(IP2,11)+A(IP

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14,9)*A(IP3,11)))/A(IP4,10)
A(IP4,12) = (A(IP4,12)-(A(I,12)*A(IP4,6)+A(IP1,12)*A(IP4,7)+A(IP2,
11)*A(IP4,8)+A(IP3,12)*A(IP4,9)))/A(IP4,10)
A(IP4,13) = (A(IP4,13)-(A(IP4,6)*A(I,13)+A(IP4,7)*A(IP1,13)+A(IP4
1,8)*A(IP2,13)+A(IP4,9)*A(IP3,13)))/A(IP4,10)
A(IP4,14) = (A(IP4,14)-A(IP4,9)*A(IP3,14))/A(IP4,10)
A(IP4,15) = (A(IP4,15)-(A(IP1,15)*A(IP4,7)+A(IP2,15)*A(IP4,8)+A(IP
13,15)*A(IP4,9)))/A(IP4,10)
S(5) = (S(5)-(S(1)*A(5,6)+S(2)*A(5,7)+S(3)*A(5,8)+S(4)*A(5,9)))/A(
15,10)
D0 600 I = 6, LM9, 5
IP1 = I+1
IP2 = I+2
IP3 = I+3
IP4 = I+4
IP5 = I+5
IP6 = I+6
IP7 = I+7
IP8 = I+8
IP9 = I+9
IM1 = I-1
IM2 = I-2
IM3 = I-3
IM4 = I-4
IM5 = I-5
D0 121 IU = I, IP4
A(IU,6) = A(IU,6)-(A(IU,2)*A(IM4,11)+A(IU,3)*A(IM3,11)+A(IU,4)*A(I
IM2,11)+A(IU,5)*A(IM1,11))
121 CONTINUE
A(I,7) = (A(I,7)-(A(I,2)*A(IM4,12)+A(I,3)*A(IM3,12)+A(I,4)*A(IM2,1
12)+A(I,5)*A(IM1,12)))/A(I,6)
A(I,8) = (A(I,8)-(A(I,2)*A(IM4,13)+A(I,3)*A(IM3,13)+A(I,4)*A(IM2,1
13)+A(I,5)*A(IM1,13)))/A(I,6)
A(I,9) = (A(I,9)-(A(I,4)*A(IM2,14)+A(I,5)*A(IM1,14)))/A(I,6)
A(I,10) = (A(I,10)-(A(I,2)*A(IM4,15)+A(I,3)*A(IM3,15)+A(I,4)*A(IM2
1,15)+A(I,5)*A(IM1,15)))/A(I,6)
A(I,12) = A(I,12)/A(I,6)
A(I,13) = A(I,13)/A(I,6)
S(I) = (S(I)-(A(I,2)*S(IM4)+A(I,3)*S(IM3)+A(I,4)*S(IM2)+A(I,5)*S(I
IM1)))/A(I,6)
D0 122 IU = IP1, IP4
A(IU,7) = A(IU,7)-(A(IM5,12)*A(IU,1)+A(IM4,12)*A(IU,2)+A(IM3,12)*A
1(IU,3)+A(IM2,12)*A(IU,4)+A(IM1,12)*A(IU,5)+A(I,7)*A(IU,6))
122 CONTINUE
A(IP6,2) = A(IP6,2)-A(I,7)*A(IP6,1)
A(IP7,2) = A(IP7,2)-A(I,7)*A(IP7,1)
A(IP8,2) = A(IP8,2)-A(I,7)*A(IP8,1)
A(IP9,2) = A(IP9,2)-A(I,7)*A(IP9,1)
A(IP1,8) = (A(IP1,8)-(A(IP1,1)*A(IM5,13)+A(IP1,2)*A(IM4,13)+A(IP1,
13)*A(IM3,13)+A(IP1,4)*A(IM2,13)+A(IP1,5)*A(IM1,13)+A(IP1,6)*A(I,8
2))/A(IP1,7)
A(IP1,9) = (A(IP1,9)-(A(IP1,4)*A(IM2,14)+A(IP1,5)*A(IM1,14)+A(IP1,
16)*A(I,9)))/A(IP1,7)
A(IP1,10) = (A(IP1,10)-(A(IP1,2)*A(IM4,15)+A(IP1,3)*A(IM3,15)+A(IP
11,4)*A(IM2,15)+A(IP1,5)*A(IM1,15)+A(IP1,6)*A(I,10)))/A(IP1,7)
A(IP1,11) = A(IP1,11)/A(IP1,7)
A(IP1,12) = (A(IP1,12)-A(I,12)*A(IP1,6))/A(IP1,7)
A(IP1,13) = (A(IP1,13)-A(IP1,6)*A(I,13))/A(IP1,7)
A(IP1,15) = A(IP1,15)/A(IP1,7)
S(IP1) = (S(IP1)-(A(IP1,1)*S(IM5)+A(IP1,2)*S(IM4)+A(IP1,3)*S(IM3)+
1A(IP1,4)*S(IM2)+A(IP1,5)*S(IM1)+A(IP1,6)*S(I)))/A(IP1,7)

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Do 123 IU = IP2, IP4

A(IU,8) = A(IU,8) - (A(IM5,13)*A(IU,1) + A(IM4,13)*A(IU,2) + A(IM3,13)*A(IU,3) + A(IM2,13)*A(IU,4) + A(IM1,13)*A(IU,5) + A(I,8)*A(IU,6) + A(IP1,8)*2*A(IU,7))

123 CONTINUE

A(IP5,3) = A(IP5,3) - A(IP1,8)*A(IP5,2)

Do 124 IU = IP6, IP9

A(IU,3) = A(IU,3) - (A(I,8)*A(IU,1) + A(IP1,8)*A(IU,2))

124 CONTINUE

A(IP2,9) = (A(IP2,9) - (A(IP2,3)*A(IM3,14) + A(IP2,4)*A(IM2,14) + A(IP2,15)*A(IM1,14) + A(IP2,6)*A(I,9) + A(IP2,7)*A(IP1,9))) / A(IP2,8)

A(IP2,10) = (A(IP2,10) - (A(IP2,2)*A(IM4,15) + A(IP2,3)*A(IM3,15) + A(IP12,4)*A(IM2,15) + A(IP2,5)*A(IM1,15) + A(IP2,6)*A(I,10) + A(IP2,7)*A(IP1,210))) / A(IP2,8)

A(IP2,11) = (A(IP2,11) - (A(IP2,7)*A(IP1,11))) / A(IP2,8)

A(IP2,12) = (A(IP2,12) - (A(I,12)*A(IP2,6) + A(IP1,12)*A(IP2,7))) / A(IP12,8)

A(IP2,13) = (A(IP2,13) - (A(IP2,6)*A(I,13) + A(IP2,7)*A(IP1,13))) / A(IP12,8)

A(IP2,15) = (A(IP2,15) - A(IP1,15)*A(IP2,7)) / A(IP2,8)

S(IP2) = (S(IP2) - (A(IP2,1)*S(IM5) + A(IP2,2)*S(IM4) + A(IP2,3)*S(IM3) + A(IP2,4)*S(IM2) + A(IP2,5)*S(IM1) + A(IP2,6)*S(I) + A(IP2,7)*S(IP1))) / A(2IP2,8)

A(IP3,9) = A(IP3,9) - (A(IM2,14)*A(IP3,4) + A(IM1,14)*A(IP3,5) + A(I,9)*A(IP3,6) + A(IP1,9)*A(IP3,7) + A(IP2,9)*A(IP3,8))

A(IP4,9) = A(IP4,9) - (A(IM2,14)*A(IP4,4) + A(IM1,14)*A(IP4,5) + A(I,9)*A(IP4,6) + A(IP1,9)*A(IP4,7) + A(IP2,9)*A(IP4,8))

A(IP5,4) = A(IP5,4) - (A(IP1,9)*A(IP5,2) + A(IP2,9)*A(IP5,3))

Do 125 IU = IP6, IP9

A(IU,4) = A(IU,4) - (A(I,9)*A(IU,1) + A(IP1,9)*A(IU,2) + A(IP2,9)*A(IU,3))

125 CONTINUE

A(IP3,10) = (A(IP3,10) - (A(IP3,2)*A(IM4,15) + A(IP3,3)*A(IM3,15) + A(IP13,4)*A(IM2,15) + A(IP3,5)*A(IM1,15) + A(IP3,6)*A(I,10) + A(IP3,7)*A(IP1,210) + A(IP3,8)*A(IP2,10))) / A(IP3,9)

A(IP3,11) = (A(IP3,11) - (A(IP3,7)*A(IP1,11) + A(IP3,8)*A(IP2,11))) / A(IP3,9)

A(IP3,12) = (A(IP3,12) - (A(I,12)*A(IP3,6) + A(IP1,12)*A(IP3,7) + A(IP2,112)*A(IP3,8))) / A(IP3,9)

A(IP3,13) = (A(IP3,13) - (A(IP3,6)*A(I,13) + A(IP3,7)*A(IP1,13) + A(IP3,18)*A(IP2,13))) / A(IP3,9)

A(IP3,14) = A(IP3,14) / A(IP3,9)

A(IP3,15) = (A(IP3,15) - (A(IP1,15)*A(IP3,7) + A(IP2,15)*A(IP3,8))) / A(IP3,9)

S(IP3) = (S(IP3) - (A(IP3,1)*S(IM5) + A(IP3,2)*S(IM4) + A(IP3,3)*S(IM3) + A(IP3,4)*S(IM2) + A(IP3,5)*S(IM1) + A(IP3,6)*S(I) + A(IP3,7)*S(IP1) + A(IP23,8)*S(IP2))) / A(IP3,9)

A(IP4,10) = A(IP4,10) - (A(IP4,2)*A(IM4,15) + A(IP4,3)*A(IM3,15) + A(IP14,4)*A(IM2,15) + A(IP4,5)*A(IM1,15) + A(IP4,6)*A(I,10) + A(IP4,7)*A(IP1,210) + A(IP4,8)*A(IP2,10) + A(IP4,9)*A(IP3,10))

A(IP5,5) = A(IP5,5) - (A(IP1,10)*A(IP5,2) + A(IP2,10)*A(IP5,3) + A(IP3,110)*A(IP5,4))

Do 126 IU = IP6, IP9

A(IU,5) = A(IU,5) - (A(I,10)*A(IU,1) + A(IP1,10)*A(IU,2) + A(IP2,10)*A(IU,3) + A(IP3,10)*A(IU,4))

126 CONTINUE

A(IP4,11) = (A(IP4,11) - (A(IP4,7)*A(IP1,11) + A(IP4,8)*A(IP2,11) + A(IP14,9)*A(IP3,11))) / A(IP4,10)

A(IP4,12) = (A(IP4,12) - (A(I,12)*A(IP4,6) + A(IP1,12)*A(IP4,7) + A(IP2,112)*A(IP4,8) + A(IP3,12)*A(IP4,9))) / A(IP4,10)

A(IP4,13) = (A(IP4,13) - (A(IP4,6)*A(I,13) + A(IP4,7)*A(IP1,13) + A(IP4


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1,2)*A(IP2,13)+A(IP4,9)*A(IP3,13)))/A(IP4,10)
A(IP4,14) = (A(IP4,14)-A(IP4,9)*A(IP3,14))/A(IP4,10)
A(IP4,15) = (A(IP4,15)-(A(IP1,15)*A(IP4,7)+A(IP2,15)*A(IP4,8)+A(IP
13,15)*A(IP4,9)))/A(IP4,10)
S(IP4) = (S(IP4)-(A(IP4,1)*S(IM5)+A(IP4,2)*S(IM4)+A(IP4,3)*S(IM3)+
1A(IP4,4)*S(IM2)+A(IP4,5)*S(IM1)+A(IP4,6)*S(I)+A(IP4,7)*S(IP1)+A(IP
24,8)*S(IP2)+A(IP4,9)*S(IP3)))/A(IP4,10)
600 CONTINUE
I = LM4
IM1 = I-1
IM2 = I-2
IM3 = I-3
IM4 = I-4
IM5 = I-5
IP1 = I+1
IP2 = I+2
IP3 = I+3
IP4 = I+4
DO 131 IU = I, IP4
A(IU,6) = A(IU,6)-(A(IU,2)*A(IM4,11)+A(IU,3)*A(IM3,11)+A(IU,4)*A(I
IM2,11)+A(IU,5)*A(IM1,11))
131 CONTINUE
A(I,7) = (A(I,7)-(A(I,2)*A(IM4,12)+A(I,3)*A(IM3,12)+A(I,4)*A(IM2,1
12)+A(I,5)*A(IM1,12)))/A(I,6)
A(I,8) = (A(I,8)-(A(I,2)*A(IM4,13)+A(I,3)*A(IM3,13)+A(I,4)*A(IM2,1
13)+A(I,5)*A(IM1,13)))/A(I,6)
A(I,10) = (A(I,10)-(A(I,2)*A(IM4,15)+A(I,3)*A(IM3,15)+A(I,4)*A(IM2
1,15)+A(I,5)*A(IM1,15)))/A(I,6)
S(I) = (S(I)-(A(I,2)*S(IM4)+A(I,3)*S(IM3)+A(I,4)*S(IM2)+A(I,5)*S(I
IM1)))/A(I,6)
DO 132 IU = IP1, IP4
A(IU,7) = A(IU,7)-(A(IM5,12)*A(IU,1)+A(IM4,12)*A(IU,2)+A(IM3,12)*A
1(IU,3)+A(IM2,12)*A(IU,4)+A(IM1,12)*A(IU,5)+A(I,7)*A(IU,6))
132 CONTINUE
A(IP1,8) = (A(IP1,8)-(A(IP1,1)*A(IM5,13)+A(IP1,2)*A(IM4,13)+A(IP1,
13)*A(IM3,13)+A(IP1,4)*A(IM2,13)+A(IP1,5)*A(IM1,13)+A(IP1,6)*A(I,8)
2))/A(IP1,7)
A(IP1,9) = (A(IP1,9)-(A(IP1,4)*A(IM2,14)+A(IP1,5)*A(IM1,14)+A(IP1,
16)*A(I,9)))/A(IP1,7)
A(IP1,10) = (A(IP1,10)-(A(IP1,2)*A(IM4,15)+A(IP1,3)*A(IM3,15)+A(IP
11,4)*A(IM2,15)+A(IP1,5)*A(IM1,15)+A(IP1,6)*A(I,10)))/A(IP1,7)
S(IP1) = (S(IP1)-(A(IP1,1)*S(IM5)+A(IP1,2)*S(IM4)+A(IP1,3)*S(IM3)+
1A(IP1,4)*S(IM2)+A(IP1,5)*S(IM1)+A(IP1,6)*S(I)))/A(IP1,7)
DO 133 IU = IP2, IP4
A(IU,8) = A(IU,8)-(A(IM5,13)*A(IU,1)+A(IM4,13)*A(IU,2)+A(IM3,13)*A
1(IU,3)+A(IM2,13)*A(IU,4)+A(IM1,13)*A(IU,5)+A(I,8)*A(IU,6)+A(IP1,8)
2)*A(IU,7))
133 CONTINUE
A(IP2,9) = (A(IP2,9)-(A(IP2,3)*A(IM3,14)+A(IP2,4)*A(IM2,14)+A(IP2,
15)*A(IM1,14)+A(IP2,6)*A(I,9)+A(IP2,7)*A(IP1,9)))/A(IP2,8)
A(IP2,10) = (A(IP2,10)-(A(IP2,2)*A(IM4,15)+A(IP2,3)*A(IM3,15)+A(IP
12,4)*A(IM2,15)+A(IP2,5)*A(IM1,15)+A(IP2,6)*A(I,10)+A(IP2,7)*A(IP1,
210)))/A(IP2,8)
S(IP2) = (S(IP2)-(A(IP2,1)*S(IM5)+A(IP2,2)*S(IM4)+A(IP2,3)*S(IM3)+
1A(IP2,4)*S(IM2)+A(IP2,5)*S(IM1)+A(IP2,6)*S(I)+A(IP2,7)*S(IP1)))/A(
2IP2,8)
A(IP3,9) = A(IP3,9)-(A(IM2,14)*A(IP3,4)+A(IM1,14)*A(IP3,5)+A(I,9)*
1A(IP3,6)+A(IP1,9)*A(IP3,7)+A(IP2,9)*A(IP3,8))
A(IP4,9) = A(IP4,9)-(A(IM2,14)*A(IP4,4)+A(IM1,14)*A(IP4,5)+A(I,9)*
1A(IP4,6)+A(IP1,9)*A(IP4,7)+A(IP2,9)*A(IP4,8))
A(IP3,10) = (A(IP3,10)-(A(IP3,2)*A(IM4,15)+A(IP3,3)*A(IM3,15)+A(IP

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13.4)*A(IM2,15)+A(IP3,5)*A(IM1,15)+A(IP3,6)*A(I,10)+A(IP3,7)*A(IP1,210)+A(IP3,8)*A(IP2,10)))/A(IP3,9)

S(IP3) = (S(IP3)-(A(IP3,1)*S(IM5)+A(IP3,2)*S(IM4)+A(IP3,3)*S(IM3)+1A(IP3,4)*S(IM2)+A(IP3,5)*S(IM1)+A(IP3,6)*S(I)+A(IP3,7)*S(IP1)+A(IP23,8)*S(IP2)))/A(IP3,9)

A(IP4,10) = A(IP4,10)-(A(IP4,2)*A(IM4,15)+A(IP4,3)*A(IM3,15)+A(IP14,4)*A(IM2,15)+A(IP4,5)*A(IM1,15)+A(IP4,6)*A(I,10)+A(IP4,7)*A(IP1,210)+A(IP4,8)*A(IP2,10)+A(IP4,9)*A(IP3,10))

S(IP4) = (S(IP4)-(A(IP4,1)*S(IM5)+A(IP4,2)*S(IM4)+A(IP4,3)*S(IM3)+1A(IP4,4)*S(IM2)+A(IP4,5)*S(IM1)+A(IP4,6)*S(I)+A(IP4,7)*S(IP1)+A(IP24,8)*S(IP2)+A(IP4,9)*S(IP3)))/A(IP4,10)

HALF1 = KD/2

HALF2 = FNT/2.

IF (ABS(HALF2-HALF1).LT.0.25) GO TO 6502

GO TO 6501

C

C

C

EVALUATION OF THE C.F.L. STABILITY FACTOR AND CONTROL OF TIME STEP

2974 QMAX = 0.

DO 1607 J = 2, NM1

FFJ = J

FJ = FFJ-.5

Jp1 = J+1

Jm1 = J-1

BF = 1.+((FJ-1.)*DR)**2

DO 1607 K = 2, MM1

BSQ = (X(J,K)*X(J,K)+((Y(Jp1,K)-Y(Jm1,K))/(DR+DR))**2)/BF

VELSQ = T(J,K)/A1ST+BSQ/R(J,K)

IF (T(J,K).LT.0.0) GO TO 1234

VLCITY = ABS(PM(J,K)/R(J,K))+SQRT(VELSQ)

IF(QMAX.LT.VLCITY) QMAX=VLCITY

1607 CONTINUE

QMAX = QMAX/DRDT

DT = DT*(1.1/QMAX)

DRDT = DR/DT

DTDR = DT/DR

7000 CONTINUE

C

PRINTING THE RESULTS

WRITE (6,1168)

WRITE (6,5530) KD, DT, TIME2, QMAX

5530 FORMAT (I5, 2E20.4, F14.3)

WRITE (6, 1158)

WRITE (6,1142)

WRITE(6,1141)

WRITE (6,7)

DO 488 K = 2, MM1

Kp1 = K+1

Km1 = K-1

DO 8830 J = 2, NM1

FFJ = J

FJ = FFJ-.5

Jp1 = J+1

Jm1 = J-1

BF = 1.+((FJ-1.)*DR7)**2

U = PM(J,K)/R(J,K)

UTH = (PN(J,K)+(FJ-1.)*DRZ*PS(J,K))/(BF*R(J,K))

U7 = (PS(J,K)-(FJ-1.)*DRZ*PN(J,K))/(BF*R(J,K))

DERIV = Y(Jp1,K)-Y(Jm1,K)

R7 = (X(J,K)-.5*(FJ-1.)*Z*DERIV)/BF

BTH = ((FJ-1.)*DRZ*X(J,K)+DERIV/(DR+DR))/BF

FJZ = DEL(J,K)+(Z*(FJ-1.)*(X(Jp1,K)-X(Jm1,K))/2.)/BF

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FJTH = (FJ-1.)*DRZ*FJZ-(X(JP1,K)-X(JM1,K))/(DR+DR)
FORCE = FJTH*BZ-FJZ*BTH
BZ = BZ*BZERO
BTH = BTH*BZERO
WRITE (6,1134) Y(J,K), X(J,K), R(J,K), U, UTH, UZ, T(J,K), BZ, BTH
1, FJZ, FJTH, FORCE
8830 CONTINUE
FFJ = N
FJ = FFJ-.5
U = 0.
BF = 1.+((FJ-1.)*DRZ)**2
UTH = (PN(N,K)+(FJ-1.)*DRZ*PS(N,K))/(BF*R(N,K))
U7 = (PS(N,K)-(FJ-1.)*DRZ*PN(N,K))/(BF*R(N,K))
DERIV = (3.*Y(N,K)-4.*Y(NM1,K)+Y(NM2,K))/2.
BZ = (X(N,K)-(FJ-1.)*Z*DERIV)/BF*BZERO
BTH = ((FJ-1.)*DRZ*X(N,K)+DERIV/DR)/BF*BZERO
WRITE (6,7225) Y(N,K), X(N,K), R(N,K), U, UTH, UZ, T(N,K), BZ, BTH
7225 FORMAT (9E11.3)
WRITE (6,1146)
488 CONTINUE
WRITE (6,1168)
6000 CONTINUE
IF (TIME2.LT.FINAL) GO TO 8000
GO TO 1231
11234 WRITE (6,1146)
WRITE (6,1178) KD, J, K, T(J,K), QMAX
1178 FORMAT (3I5,E14.4,F14.2)
1231 STOP
END

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REFERENCES

1. A.A. Blank, H. Grad and H. Weitzner, Toroidal High- β Equilibria, in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 3rd Int. Conf. Novosibirsk, 1968) 2, IAEA, Vienna (1969) 607.
2. F.L. Ribe and M.N. Rosenbluth, Feedback Stabilization of a High- β , Sharp-Bounded Plasma Column with Helical Fields, Phys. Fluids 13, 2572 (1970).
3. E.L. Cantrell, et al., Plasma Experiments in the Scyllac Toroidal Theta Pinch, in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 5th Int. Conf. Tokyo, 1974) 3, IAEA, Vienna (1975) 13.
4. E. Fünfer, et al., High-Beta Stellarator Experiments on ISAR T1, Nucl. Fusion 15, 133 (1975).
5. A.I. Morozov and L.S. Solov'ev, in Reviews of Plasma Physics, edited by M.A. Leontovich (Consultants Bureau, New York, 1966), Vol. 2, p. 56.
6. H. Grad and H. Weitzner, Critical β from Stellarator and Scyllac Expansions, Phys. Fluids 12, 1725 (1969).
7. J.P. Freidberg, Survey of Scyllac Theory, in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 4th Int. Conf. Madison, 1971) 3, IAEA, Vienna (1971) 215.
8. D. Correa and D. Lortz, A Class of Helically Symmetric MHD-Equilibria, Nucl. Fusion 13, 127 (1973).
9. R.C. Davidson and J.P. Freidberg, Review of Toroidal θ -pinch Theory, in Plasma Physics Suppl. (3rd Topical Conf. on Pulsed High Beta Plasmas, Culham, 1975) p. 13 (1976).
10. S.C. Burnett, et al., Theta-Pinch Experiments with Helical Equilibrium

- Fields in a 5-Meter Toroidal Sector and in a 3-Meter Linear Device
in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 4th
Int. Conf. Madison, 1971) 3, IAEA, Vienna (1971) 201.
11. E. Fünfer, et al., High-Beta ($\beta = 1$) - Stellarator Experiments,
in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 4th
Int. Conf. Madison, 1971) 3, IAEA, Vienna (1971) 189.
 12. E. Fünfer, et al., Recent Results from the Garching High-Beta
Stellarator Torus ISAR T1, in Controlled Fusion and Plasma Physics,
(Proc. 6th Europ. Conf. Moscow, 1973) 1, (1973) 109.
 13. W.R. Ellis, et al., Plasma Equilibrium and Stability in the Scyllac
Toroidal Sector Experiments, Nucl. Fusion 14, 841 (1974).
 14. M. Kaufmann, J. Neuhauser and H. Rohr, Experimental Test of Finite
Larmor Radius Stabilization in Toroidal Theta-Pinches and Extrapolation
to Reactor Parameters, Bull. Am. Phys. Soc. 21, (October 1976)
Paper 4E-16.
 15. K. Höthker, Private Communication.
 16. R.R. Bartsch, et al., Plasma Experiments on the Staged Theta Pinch,
the Implosion Heating Experiments, and Scyllac Feedback-Sector Experi-
ment, in Plasma Physics and Controlled Nuclear Fusion Research (Proc.
6th Int. Conf. Berchtesgaden, 1976) IAEA, Vienna (1977).
 17. P. Bogen, et al., Helical Equilibrium and Stability Experiments in
a Theta Pinch at Low Compression Ratio, in Plasma Physics and Control-
led Nuclear Fusion Research (Proc. 6th Int. Conf. Berchtesgaden, 1976)
1, IAEA, Vienna (1977) 539.
 18. W. Braun et al., Garching Toroidal High-Beta Stellarator Experiments,
in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 5th
Int. Conf. Tokyo, 1974) 3, IAEA, Vienna (1975) 25.

19. N. Sharky, Initial Motion of a Helically Symmetric Theta Pinch, Annual Controlled Fusion Theory Meeting, Wisconsin (April 1976) Paper 2C-16.
N. Sharky, H.C. Lui and C.K. Chu, The Imploding to Equilibrium of Helical Pinches, Bull. Am. Phys. Soc. 21 (October 1976) Paper 1E-3.
N. Sharky, H.C. Lui and C.K. Chu, The Evolution to Equilibrium of Helically Symmetric Theta Pinches, Annual Controlled Fusion Theory Meeting, San Diego (May 1977) Paper A-18.
20. H.C. Lui and C.K. Chu, Two-Dimensional Magnetohydrodynamic Simulation of Toroidal Pinches, Phys. Fluids 18, 1277 (1975).
21. H.C. Lui and C.K. Chu, Two-Dimensional Magnetohydrodynamic Simulation of Toroidal Pinches II: Belt Pinches, Phys. Fluids 19, 1947 (1976).
22. W. Park and C.K. Chu, Numerical Simulation of Padua Eta-Beta Reverse-Field Pinch, Nucl. Fusion 17, 1100 (1977).
23. D. Düchs, Two-Dimensional Theta-Pinch Dynamics with Transverse Magnetic Fields, Phys. Fluids 11, 2010 (1968).
24. I. Lindemuth and J. Killeen, Alternating Direction Implicit Techniques for Two-Dimensional Magnetohydrodynamic Calculations, Journal of Comp. Phys. 13, 181 (1973).
25. C.K. Chu, The Implosion to Equilibrium of Toroidal Plasmas, Bull. Am. Phys. Soc. 21 (October 1976) Papers 4E-10, 11, 12.
26. W. Park, H.C. Lui, and C.K. Chu, The Imploding to Equilibrium of Toroidal Screw Pinches, Bull. Am. Phys. Soc. 21 (October 1976) Paper 4E-13.

27. A. Aydemir , H.C. Lui and C.K. Chu, The Imploding to Equilibrium of Belt Pinches and High-Beta Tokamaks, Bull. Am. Phys. Soc. 21, (October 1976) Paper 6P-3.
28. J. Neuhauser, et al., Finite-Larmor-Radius Stabilization of the $m = 2$ Instability in the ISAR Tl-B High-Beta Stellarator, Nucl. Fusion 17, 3 (1977).
29. J.A. Wesson, Plasma Flow in the Theta Pinch, in Plasma Physics and Controlled Nuclear Fusion Research (Proc. Int. Conf. Culham, 1965) 1, IAEA, Vienna, (1966) 223.
30. H. Weitzner, The Formulation of a Helically Symmetric Magnetohydrodynamic Theta Pinch Problem, Los Alamos Report LA-4599-MS (1971).
31. L. Spitzer, Jr., Physics of Fully Ionized Gases, (Interscience Publishers, N.Y. 1962), Chapter 5.
32. R. Chodura, A Hybrid fluid-Particle Model of Ion Heating in High Mach-Number Shock Waves, Nucl. Fusion 15, 55 (1975).
33. A.G. Sgro and C.W. Nielson, Hybrid Model Studies of Ion Dynamics and Magnetic Field Diffusion During Pinch Implosions, Phys. Fluids 19, 126 (1976).
34. P.C. Liewer and N.A. Krall, Self-Consistent Approach to Anomalous Resistivity Applied to Theta Pinch Experiments, Phys. Fluids 16, 1953 (1973).
35. R.C. Davidson and N.T. Gladd, Anomalous Transport Properties Associated with the Lower-Hybrid-Drift Instability, Phys. Fluids 18, 1327 (1975).
N.T. Gladd, The Lower Hybrid Drift Instability and the Modified Two Stream Instability in High Density Theta Pinch Environments, Plasma Physics 18, 27 (1976).

36. B.H. Hui, S. Hamasaki and R.C. Davidson, Fluid-Numerical Studies of High-Density Theta-Pinch Implosion Including Classical and Anomalous Transport Processes, Nucl. Fusion 16, 73 (1976).
P.C. Liewer, Numerical Studies of Ion Reflection in Collisionless Theta-Pinch Implosions Using a Hybrid Vlasov-Fluid Model, Nucl. Fusion 16, 817 (1976).
37. S.I. Braginskii, in Reviews of Plasma Physics, edited by M.A. Leontovich (Consultants Bureau, N.Y., 1965) Vol. 1, p. 205 .
38. N. A. Krall and A.W. Trivelpiece, Principles of Plasma Physics, McGraw-Hill, N.Y. 1973) p. 94 and 97.
39. D.W. Peaceman and H.H. Rachford, Jr., The Numerical Solution of Parabolic and Elliptic Differential Equations, J. Soc. Industrial Appl. Math 3, 28 (1955).
40. K.A. Bagrinovskii and S.K. Godunov, Difference Schemes for Multi-dimensional Problems, Dokl. Akad. Nauk. USSR, 115, 431 (1957).
41. R.D. Richtmyer and K.W. Morton, Difference Methods for Initial-Value Problems, (Interscience Publishers, N.Y. 1967) 2nd Ed., p. 198.
42. F.B. Hildebrand, Methods of Applied Mathematics, (Prentice-Hall, N.J. 1963) p. 503.
43. A. Lapidus, A Detached Shock Calculation by Second-Order Finite Differences, J. Comput. Phys. 2, 154 (1967).
44. J.U. Brackbill, Numerical Magnetohydrodynamics for High-Beta Plasmas, in Methods in Computational Physics (Academic Press, N.Y. 1976) Vol. 16, p. 1.
45. J.V. Brackbill and D.C. Barnes, Time-Dependent Calculations of Equilibrium and Stability Problems in Scyllac and Tokamak Geometries, in Plasma Physics Suppl. (3rd Topical Conf. on Pulsed High Beta Plasmas, Culham, 1975) p. 111 (1976).

46. K. Hain, et al., Fully Ionized Pinch Collapse, Z. Naturforsch. 15a, 1039 (1960).
K. Hain and A.C. Kolb, Fast Theta-Pinch, Nucl. Fusion Suppl. Part. 2, 561 (1962).
47. I.R. Lindemuth, A Boundary Condition for Computational Magnetohydrodynamics, submitted to J. Comput. Phys.
48. W. Park, Ph.D. thesis, Plasma Physics Laboratory, Columbia University.
49. D.M. Young, in A Survey of Numerical Analysis, (J. Todd, ed.) Chapter 11, McGraw-Hill, New York (1962).
50. C.K. Chu and H. Kranzer, (unpublished).
51. C.K. Chu and A. Sereny, Boundary Conditions in Finite Difference Fluid Dynamic Codes, J. Comp. Phys. 15, 476 (1974).
52. K.W. Morton, Stability and Convergence in Fluid Flow Problems, Proc. Roy. Soc. Lond. A. 323, 237 (1971).
C.K. Chu, Physical and Numerical Dissipation and Dispersion in Viscous Compressible Flow Computation, Symposium on Unsteady Aerodynamics, University of Arizona (March 1975), Invited Paper.
53. R.D. Richtmyer and K.W. Morton, Difference Methods for Initial-Value Problems, (Interscience Publishers, N.Y. 1967), 2nd ed., p. 345.
54. J.P. Boris and D.L. Book, Flux-Corrected Transport. I. Shasta, A Transport Algorithm that Works, J. Comput. Phys. 11, 38 (1973).
55. G. Berge and J.P. Freidberg, Stability of a Diffuse, High β , $l=1$ System, Phys. Fluids 18, 1362 (1975).
56. R.L. Morse and J.P. Freidberg, Rigid Drift Model of High-Temperature Plasma Containment, Phys. Fluids 13, 531 (1970).
57. H. Weitzner, Free Boundary Long Helical Wavelength Equilibria, Phys. Fluids, 14, 658 (1971).

58. J.P. Freidberg, Stability of the Straight $\ell = 1$ Scyllac Configuration, Phys. Fluids 14, 2454 (1971).
59. B.M. Marder, Straight $\ell = 1$ Equilibrium and Stability, Phys. Fluids 19, 1395 (1976).
60. D.C. Barnes, J.U. Brackbill and W. Schneider, Advances in Scyllac Equilibrium Theory, Annual Controlled Fusion Theory Meeting, San Diego (May 1977) Paper PA10.
J.U. Brackbill and D.C. Barnes, Numerical Computations of Scyllac Equilibrium and Stability, Bull. Am. Phys. Soc. 21, (October 1976) Paper 3E-9.
61. D.C. Barnes, Toroidal Force Balance in the Scyllac High Beta Stellarator, Bull. Am. Phys. Soc. 21 (October 1976) Paper 3E-10.
D.C. Barnes, An Approximate Model for Toroidal Force Balance in the High-Beta Stellarator, submitted to Phys. Fluids (1977).

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