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AN ANALYTIC METHOD FOR SPIN TRANSFER MATRIX IN PRESENCE OF SNAKES

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INTRODUCTION

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Large accelerators can be made spin transparent using Siberian snakes. However, the number of snakes required is yet to be determined.

An algorithm for finding the spin transfer matrix analytically is developed. This is applied to find the $\cos \pi \nu_p$ for the case involving 6 snakes in two different configurations. This is in contrast to R. Ruth's¹ approach who found that the number of snakes is proportional to $|\epsilon|^2$, where ϵ is the depolarizing resonance strength.

Half the trace of the spin precession matrix with 6 equally spaced snakes is found analytically for two configurations. The first configuration involves alternating snakes with precession axes of $+45^\circ$ and -45° while the second configuration are alternating $+75^\circ$ and -75° as proposed by K. Steffen.² Then the largest resonance strength $|\epsilon|$ such that $|\cos \pi \nu_p| < \Delta < 1$ is determined. Finally, a comparison with tracking studies is made.

Transfer Matrix

We begin development of the spin transfer matrix by introducing the following matrices

$$C(x) = 1 \cos x + i\sigma_3 \sin x \quad (1)$$

$$S(x) = i\sigma_1 \cos x + i\sigma_2 \sin x,$$

where 1 is unit matrix and σ_1 , σ_2 , and σ_3 are the Pauli spin matrices. The first matrix represents a precession of x radians about the vertical axis. The second matrix represents a generalized snake where $\phi = \pi/2 - x$ is the angle between the precession axis and the longitudinal axis. It is useful to list some rules for manipulating these matrices,

$$S(x) = i\sigma_1 C(x)$$

$$-C(y-x) = S(x)S(y)$$

$$S(y+x) = S(x)C(y)$$

$$S(y-x) = C(x)S(y)$$

$$C(y+x) = C(x)C(y)$$

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$$2S\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = S(x) + S(y) \quad (2)$$

$$2C\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = C(x) + C(y)$$

Any precession can be represented as a combination of the S and C matrices defined by eq. (1). In Appendix A (eq. A.6) the spin precession in the arcs is found, for a storage ring with a single resonance, to be

$$T(\theta_f, \theta_i) = aC(\alpha - \kappa/2 (\theta_f - \theta_i)) + bS(\beta + \kappa/2 (\theta_f + \theta_i))$$

where all the variables are defined in Appendix A. Then, if the storage ring contains $2p$ snakes at azimuths θ_k the spin precession matrix for single a turn becomes:

$$M_p = \prod_{k=1}^{2p} S(\pi/2 - \phi_k) T(\theta_k, \theta_{k-1}) \quad (3)$$

with the convention that elements with higher k are to the left of elements of lower k and ϕ_k are the precession axes of the snakes. Choosing equally spaced points simplifies the T matrix since a , b , and α depend only on $\theta_f - \theta_i$. As an example, consider two snakes. The matrix becomes:

$$M_1 = -a^2 C(\phi_2 - \phi_1) + b^2 C(\phi_1 + \phi_2 + 2\beta + (2\kappa - 1)\pi) \\ - ab [S(\phi_1 - \phi_2 + \alpha + \beta) + S(\pi - \phi_1 - \phi_2 - \beta + \alpha - 2\kappa\pi)].$$

The trace for two snakes is now easily found to give

$$\cos \pi \nu_p = -a^2 \cos(\phi_2 - \phi_1) + b^2 \cos(\phi_1 + \phi_2 + 2\beta + (2\kappa - 1)\pi). \quad (4)$$

From this, we find the usual conclusion that $\phi_2 - \phi_1 = \pi/2$ etc.³

Six Snakes

For more than two snakes, equation (3) becomes unwieldy. Expressions for $\cos \pi \nu_p$ for 6 snakes in two configurations were worked out using MACSYMA.⁴ They can be expressed in the following form:

$$\cos \pi \nu_p = a_1(\kappa, \beta) b^2 + a_2(\kappa, \beta) b^4 + a_3(\kappa, \beta) b^6 \quad (5)$$

where $b = \sin \pi |\epsilon|/6$ when one is on resonance and the coefficients a_1 , a_2 and a_3 are given in Appendix B. Larger configurations (i.e., 10 snakes) couldn't be handled with the tools that were available. Note, J. Buon has found a recursion relation to evaluate eq. (3).⁸

Once equation (5) has been found, the rest is dealt with numerically. If κ is fixed and we set $\cos \pi \nu_p = \pm \Delta$ for fixed Δ , $0 < \Delta \leq 1$,

then eq. (5) becomes a function of only b and β . Now find β such that the smallest b satisfies this equation (denote this b as $b_{\min}$). This gives the largest $|\epsilon|$ for which $|\cos\pi v_p| \leq \Delta$.

Figures 1 and 2 show plots of $b_{\min}$ vs K for various choices of Δ . Note that the result is periodic in K with a period of 3. For $\Delta = 1$ (i.e. curve (1) in Fig. 1 and 2) $b_{\min} = 1$ except at $K = 3\ell$ where ℓ is an integer. The reason for this discontinuity is not clear, however, for $\Delta < 1$ there is no discontinuity.

The question we now ask is whether $|\cos\pi v_p| \leq \Delta$ is a good criterion for finding the strongest resonance that could be corrected by snakes i.e., such that the spin is preserved as one traverses the resonant energy. J. Buon has found a criteria $|\epsilon| < .1$ for two snakes using a spin tracking program.⁵ From eq. (4) this implies that $|\cos\pi v_p| < .025$. The corresponding $b_{\min}$ vs K are given in curves (4) of Fig. 1 and Fig. 2 for the two configurations considered. One observation is that the 6 snakes with alternating $\pm 45^\circ$ snakes gives $|\epsilon|$ values higher than the alternating $\pm 75^\circ$ snakes. (Note, this is overall; for a particular phase β the converse could be true.) Thus, one should expect the former to do better in tracking. This did not happen. Clearly this is oversimplified criterion.

CONCLUSION

Continued study involving tracking and eq. (5) is greatly needed. Preliminary results seem to show a discrepancy between the tracking and $\cos\pi v_p$ prescription. This suggests that finding the number of snakes needed to make an acceleration spin transparent cannot be found from the $\cos\pi v_p$ alone.

The above study is just the beginning. A more complete study involving acceleration, many resonances, finding the spin matrix over many revolutions etc. will be necessary to gain better understanding and come up with a better criterion.

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APPENDIX A

The equation for spin precession in the arcs of an accelerator with a single resonance and without snakes can be expressed as^{6,7}

$$\frac{d}{d\theta} \Psi = \frac{i}{2} \begin{pmatrix} -\gamma G & \epsilon e^{-i\kappa\theta} \\ \epsilon^* e^{i\kappa\theta} & \gamma G \end{pmatrix} \Psi \quad (A.1)$$

where Ψ is a two component spinor, θ is the particle's azimuth, ϵ is the resonance strength, κ the resonance position (i.e. $\kappa = kP \pm \nu, k$ etc.), γ is the Lorentz factor and $G = (g-2)/2$ where g is the anomalous magnet moment.

Transforming equation (A.1) to the rotating frame

$$\phi(\theta) = e^{i/2\kappa\theta\sigma_3} \Psi(\theta) \quad (A.2)$$

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$$\frac{d}{d\theta} \phi = \frac{i}{2} \begin{pmatrix} \delta & \epsilon \\ \epsilon^* & -\delta \end{pmatrix} \phi \quad (A.3)$$

where $\delta = \kappa - \gamma G$.

Equation (A.3) can be solved for ϕ as (with no acceleration)

$$\phi(\theta_f) = \exp \{ i/2 (\delta\sigma_3 + \epsilon_R\sigma_1 - \epsilon_I\sigma_2) (\theta_f - \theta_i) \} \phi(\theta_i) \quad (A.4)$$

where $\epsilon = \epsilon_R + i\epsilon_I$.

Expressing equation (A.4) in terms of the S and C matrices (see text) gives

$$\phi(\theta_f) = [aC(\alpha) + bS(\beta)]\phi(\theta_i) \quad (A.5)$$

where

$$\begin{aligned} a &= \sqrt{\cos^2 \frac{\lambda(\theta_f - \theta_i)}{2} + \frac{\delta^2}{\lambda^2} \sin^2 \frac{\lambda(\theta_f - \theta_i)}{2}} \\ \alpha &= \arg \left[\cos \frac{\lambda(\theta_f - \theta_i)}{2} + i \frac{\delta}{\lambda} \sin \frac{\lambda(\theta_f - \theta_i)}{2} \right] \\ b &= \frac{|\epsilon|}{\lambda} \sin \frac{\lambda(\theta_f - \theta_i)}{2} = \sqrt{1 - a^2} \\ \beta &= \arg \epsilon^* \text{ and } \lambda = \sqrt{\delta^2 + |\epsilon|^2} \end{aligned}$$

For equally spaced snakes, a , b and α are the same for each arc between snakes. Since the rotation in (A.2) can be represented as $C(-1/2\kappa\theta)$ the spin precession matrix in the lab frame is

$$T(\theta_f, \theta_i) = aC(\alpha - \kappa/2(\theta_f - \theta_i)) + bS(\beta + \kappa/2(\theta_f + \theta_i)) \quad (A.6)$$

for which $\Psi(\theta_f) = T(\theta_f, \theta_i) \Psi(\theta_i)$.

APPENDIX B

The following are half the trace of the spin transfer matrix for two configurations involving 6 snakes. Definitions for the variables can be found in Appendix A. The algorithm used is explained in the text. The form for half the trace is given as:

$$\cos \pi \nu_p = a_1(\kappa, \beta) b^2 + a_2(\kappa, \beta) b^4 + a_3(\kappa, \beta) b^6$$

where the coefficients are given below.

- I. Six snakes with alternating precession axes of $+45^\circ$ and -45° .

$$a_1(\kappa, \beta) = \cos(2\pi\kappa + 2\beta) \left(2\cos\left(\frac{4\pi\kappa}{3}\right) + 1 \right)$$

$$a_2(\kappa, \beta) = 2 \left[2\sin(4\pi\kappa + 4\beta) \sin(\pi\kappa/3) + \sin(3\pi\kappa + 2\beta) \right] \sin\pi\kappa \\ + 2\sin\left(\frac{5\pi\kappa}{3} + 2\beta\right) \sin\frac{\pi\kappa}{3} \left(2\cos\frac{4\pi\kappa}{3} + 1 \right)$$

$$a_3(\kappa, \beta) = -\cos(6\pi\kappa + 6\beta) + 2 \left[2\sin(4\pi\kappa + 4\beta) \sin(\pi\kappa/3) \right. \\ \left. + \sin(3\pi\kappa + 2\beta) \right] \sin\pi\kappa \\ - \cos\left(\frac{4\pi\kappa}{3} + 2\beta\right) \left(2\cos\frac{4\pi\kappa}{3} + 1 \right)$$

- II. Six snakes with alternating precession axes of $+75^\circ$ and -75° .

$$a_1(\kappa, \beta) = -2\sqrt{3} \sin(2\pi\kappa + 2\beta) \sin\frac{\pi\kappa}{3} \sin\pi\kappa \\ - \cos(2\pi\kappa + 2\beta) \left(\cos\frac{4\pi\kappa}{3} + 3\cos\frac{2\pi\kappa}{3} + 2 \right) \\ - \sqrt{3} \left(\cos\frac{4\pi\kappa}{3} + 2\cos\frac{2\pi\kappa}{3} \right)$$

$$a_2(\kappa, \beta) = -2(\sin(4\pi\kappa + 4\beta) + 2\sqrt{3} \sin(2\pi\kappa + 2\beta))$$

$$\sin\frac{\pi\kappa}{3}\sin\pi\kappa + (\sqrt{3} \cos(4\pi\kappa + 4\beta)$$

$$+ 4\cos(2\pi\kappa + 2\beta)) \left(\cos\frac{4\pi\kappa}{3} \right.$$

$$+ \cos\frac{2\pi\kappa}{3} + 1) + 2\cos(2\pi\kappa + 2\beta)\cos\frac{2\pi\kappa}{3}$$

$$\left(2\cos\frac{4\pi\kappa}{3} + 1 \right) + \sqrt{3}(\cos 2\pi\kappa$$

$$+ 2\cos\frac{4\pi\kappa}{3} + 4\cos\frac{2\pi\kappa}{3} + \frac{1}{2})$$

$$a_3(\kappa, \beta) = -\cos(6\pi\kappa + 6\beta) + 2(\sin(4\pi\kappa + 4\beta)$$

$$+ \sqrt{3} \sin(2\pi\kappa + 2\beta))\sin(\frac{\pi\kappa}{3})\sin(\pi\kappa)$$

$$- (\sqrt{3} \cos(4\pi\kappa + 4\beta) + \cos(2\pi\kappa + 2\beta))(\cos(4\pi\kappa/3)$$

$$+ \cos(\frac{2\pi\kappa}{3}) + 1) - \cos(2\pi\kappa + 2\beta)$$

$$(2\cos(\frac{4\pi\kappa}{3}) + 1)(2\cos(\frac{2\pi\kappa}{3}) + 1)$$

$$- \sqrt{3}(\cos 2\pi\kappa + \cos(4\pi\kappa/3) + 2\cos(2\pi\kappa/3) + 1/2)$$

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Figure 1. Plots $b_{\min}$ vs κ for four choices of Δ (i.e., 1, .9, .5 and .025) where $b = \sin \pi |\epsilon|/6$ for the 6 snakes with alternating $+45^\circ$ and -45° precession axis.

Figure 2. Same as Fig. 1, except for 6 snakes with alternating $+75^\circ$ and -75° precession axis.

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