October 1991

Weight diagram construction
of Lax operators*

DOE/ER/40272--140

DE92 008781

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Abstract: We review and expand methods introduced in our previous paper. It is proved that cyclic weight diagrams corresponding to representations of affine Lie algebras allow one to construct the associated Lax operator. The resultant Lax operator is in the Miura-like form and generates the modified KdV equations. The algorithm is extended to the super-symmetric case.

* Supported in part by U.S. Department of Energy under contract No. DE-FG05-86ER-40272, and a summer grant from the Institute for Fundamental Theory.

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I. INTRODUCTION

Several approaches to constructing the Lax operators have been developed. The matrix procedure discussed by Drinfeld and Sokolov¹ defines first a matrix eigenvalue equation. The system incorporates knowledge of the Cartan subalgebra and root system of some embedding affine Lie algebra $\hat{\mathcal{G}}$. Starting with an affine Lie algebra facilitates the construction of an integrable system from the resulting Lax pair operators. To fix the gauge invariance in the matrix system, the gradation conventions of Drinfeld and Sokolov require that one of the simple roots, say the m^{th} , must be removed from the affine system. The resulting system is denoted by $(\hat{\mathcal{G}}, c_m)$. This is equivalent to deleting the m^{th} Dynkin vertex. For the most part, Drinfeld and Sokolov choose the "canonical" gauge in which to express the coordinate dependent terms. In this gauge, Lax operators generate the regular KdV hierarchy equations.

If an endpoint vertex is deleted, the remaining matrix system generates a single Lax operator. For example, Drinfeld and Sokolov find that the canonical representation of $(A_n^{(1)}, c_0)$ creates a Lax operator related to the fundamental representation of A_n . On the other hand, if the eliminated root splits the Dynkin diagram, a coupled matrix system results. For example, splitting $A_{2n}^{(2)}$ at the m^{th} vertex gives Lax operators in the B_{n-m} and C_m series. Furthermore, the (pseudo)differential operator associated with the D_n series is derived using the embedding algebra $D_n^{(1)}$.

The modified KdV (mKdV) equations can be generated by expressing the coordinate term $q(x)$ in the "diagonal" gauge. The canonical Lax operators can then be recovered using the well-known Miura transformations. The diagonal gauge is technically simpler than the canonical gauge. Furthermore, the final Lax operator is in a factorized form which has been used to quantize the theory².

In this paper our focus will be on the explicit construction of Lax operators in the diagonal gauge as introduced in our previous paper³ where we have presented a simple diagrammatic technique for constructing these (pseudo)differential operators. In most cases this technique arrives at these operators much quicker than a direct application of the scheme of Drinfeld and Sokolov. Furthermore, the scheme also applies to higher representations of the embedding affine Lie algebra. In section II we briefly review affine and non-affine Lie algebras and construction of weight diagrams. Section III reviews the matrix method of Drinfeld and Sokolov for building Lax operators. Next is a presentation of our method, which replaces the matrix procedure with a scheme utilizing cyclic weight diagrams of representations of affine Lie algebras. Section IV presents a proof that the diagrammatic algorithm produces the correct Lax operator. Finally, section V discusses the generalization to Lax operators based on supersymmetric affine Lie algebras.

II. REVIEW OF LIE ALGEBRAS AND WEIGHT DIAGRAMS

As noted in the introduction, each Lax operator can be associated with a representation of some affine Lie algebra. As a result, in this section we give a brief review of relevant aspects from the theory of Lie algebras. In particular we wish to set the notation and discuss the construction of weight diagrams.

2.1 Basic Lie algebras

Recall⁴, a basis of generators for a basic semi-simple Lie algebra $\mathcal{G}$ is given by a system

of roots E_α , and the generators H_i of its Cartan subalgebra $\mathcal{H}$. These generators satisfy the commutation relations

$$\begin{aligned} [H_i, H_j] &= 0, \\ [H_i, E_\alpha] &= \alpha_i E_\alpha, \quad 1 \leq i \leq r, \end{aligned} \tag{2.1.1}$$

where r is the rank of $\mathcal{G}$. Each root can be written in the form $(a_1, \dots)$. A positive(negative) root is one whose first non-zero element has a positive(negative) value. There are an equal number of positive and negative roots. A simple positive(negative) root is one which can not be written as a linear combination of other positive(negative) roots with all positive coefficients. We denote the negative simple roots by E_i , $1 \leq i \leq r$, which we label with latin indices to distinguish them from the full root system E_α .

The simple roots, when expressed in the Cartan-Chevalley basis, satisfy the commutation relations

$$\begin{aligned} [E_i, F_j] &= \delta_{ij} H_i, \\ [H_i, E_j] &= A_{ij} E_j, \\ [H_i, F_j] &= -A_{ij} F_j, \end{aligned} \tag{2.1.2}$$

where A_{ij} , for $1 \leq i, j \leq r$, is the Cartan matrix of the Lie algebra.

Below, we will need explicit matrix representations. A straightforward procedure for computing the entries of the Cartan matrix originates with the Dynkin diagram. Recall, each of the r vertices of a Dynkin diagram corresponds to a positive simple root vector. Furthermore, the angle between the vectors is indicated by the number of lines connecting their respective Dynkin vertices. The Cartan matrix is then computed by

$$A_{ij} = \frac{2(F_i, F_j)}{(F_j, F_j)}. \tag{2.1.3}$$

Clearly, the diagonal elements will all equal two.

Now, one can associate uniquely to every irreducible representation of a basic Lie algebra a highest weight vector, that is a certain linear combination of positive simple roots

which has the property of being annihilated by some E_α for appropriate α . Conversely, to every linear combination of positive simple roots there corresponds a highest weight vector of some irreducible representation. Furthermore, for each highest weight one can construct a weight diagram which encodes all relevant information concerning the particular representation studied, *e.g.*, from it one can build explicit matrix representations of the generators of the Cartan subalgebra as well as the various raising and lowering operators. The level of a weight is the number of lowering operators applied to the highest weight which produces that weight. Finally, the height γ of the weight diagram is the level of the lowest weight.

Weight diagrams are generated by subtracting rows of the Cartan matrix initially from the highest weight vector. Rules of construction can be summed up as follows:

1. Subtract the i^{th} row of the Cartan matrix n times from a weight vector whose i^{th} component has a positive value n .
2. When weight vectors have more than one positive component, subtract all possible permutations of the appropriate Cartan rows.

A theorem due to Dynkin⁵ states that the final weight diagram is always "spindle shaped". In other words: i) the number of weight vectors at the level k is equal to the number at level $\gamma - k$, ii) the number of weights at level $k + 1$ is greater than or equal to the number at level k for $k < \frac{\gamma}{2}$.

For an explicit example consider the algebra A_2 . Though this is almost a trivial case, the results will be useful for the next section. The Dynkin diagram is given by

where, recall, the single bar represents 120° . The Cartan matrix is then easily found to be

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \quad (2.1.4)$$

The highest weight vector of the fundamental representation is $(10)^6$. Since a positive one appears in the first place, we subtract the first row of the Cartan matrix one time. This gives the weight (-11) . Now, due to the one in the second place, we subtract the second row of the Cartan matrix once to get $(0-1)$. This completes the process since no positive components remain. The result is the height two weight diagram

$$\begin{array}{c} (10)_1 \\ \downarrow 1 \\ (-11)_2 \\ \downarrow 1 \\ (0-1)_3 \end{array} \quad (2.1.5)$$

where the subscripts on the weight vectors indicate a counting of the vectors. The ones adjacent to the arrows represent the normalization factors of the corresponding negative simple roots. These values are fixed by the commutation relations (2.1.2). To simplify our diagrams, we will not display values of unity. Later, we will see that the procedure for building weight diagrams is slightly modified in the supersymmetric case.

The matrix element $(H_i)_{jj}$ is extracted from the i^{th} element of the j^{th} weight vector. Setting the off-diagonal elements to zero gives

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.1.6)$$

and

$$H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (2.1.7)$$

The matrix entry of the negative simple root $(E_i)_{jk}$ is assigned its normalization factor if the k^{th} weight vector branches into the j^{th} weight vector as a result of subtracting the i^{th} row of the Cartan matrix in the process. Other entries are by default zero giving the matrices

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.1.8)$$

and

$$E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2.1.9)$$

2.2 Affine Lie algebras

An affine Lie algebra is defined to be a tensor product of the basic Lie algebra with a Laurent polynomial algebra $C((\lambda, \lambda^{-1})^r)$. The simple root system of an affine Lie algebra consists of the simple roots of the corresponding Lie algebra, and an affine root which projects onto the basic Lie algebra as the corresponding lowest root. In the Cartan-Chevalley basis the simple roots of an affine Lie algebra obey the commutation relations

$$\begin{aligned} [H_i, H_j] &= 0, & [E_i, F_j] &= \delta_{ij} H_i, \\ [H_i, E_j] &= A_{ij} E_j, & [H_i, F_j] &= -A_{ij} F_j, \end{aligned} \quad (2.2.1)$$

where now $0 \leq i, j \leq r$. Unlike simple root systems of non-affine Lie algebras, there exists a linear relation among the positive(negative) simple roots in affine algebras. Consequently, affine Lie algebras are infinite dimensional.

Affine Lie algebras are invariant under an outer automorphism τ . The order k of the automorphism τ is the smallest integer such that $\tau^k = 1$. A twisted affine algebra is one such that $k \geq 2$. Let us denote a general affine Lie algebra $\hat{\mathcal{G}}^{(k)}$. Note, there may exist more than one affine extension of a particular basic Lie algebra, e.g., A_n extends to both the affine algebra $A_n^{(1)}$ and the twisted affine algebra $A_n^{(2)}$.

In general, due to the linear combination among simple roots, weight diagrams of affine representations have infinite extent. However, some affine representations give cyclic weight diagrams of finite extent. In fact, it is these cyclic cases that are crucial to the scheme below. To produce a cyclic weight diagram, the affine component which is appended to the highest weight vector of the underlying non-affine Lie algebra, unlike the non-affine weight components, may have to be assigned a negative value.

As an explicit example, consider the non-twisted affine algebra $A_2^{(1)}$. To generate the cyclic weight diagram corresponding to the canonical representation, start with the

weight vector (-110) , where -1 corresponds to the affine root. Figure 1 gives the Dynkin diagram and subsequent Cartan matrix which then generates the displayed resultant weight diagram.

This particular cyclic weight diagram can further be thought of as the affine extension of the highest weight diagram based on the fundamental representation of the basic Lie algebra A_2 . This is easy to see by removing everywhere the component due to the affine root. However, this is not always the case. For example, Figure 2 displays the cyclic weight diagram constructed with the weight vector (-211) , where now the affine component is -2 . Although (11) generates the highest weight diagram of the adjoint representation of A_2 , we see by comparing with Figure 3 that the affine extension contains an extra zero weight (000) .

In general, the affine component in the affinely extended vector, associated with the highest weight vector of a basic Lie algebra, will always be negative. However, we will give an example below showing that some supersymmetric cases require positive affine components.

III. LAX OPERATOR CONSTRUCTION

3.1 Standard construction of Lax operators

The Lax operator⁸ $L(x, t)$ is defined to be linear and Hermitian. Furthermore, it is required to have the property that its eigenvalues are constant under nonlinear evolution. In other words, for constant eigenvalues μ the eigenfunctions of the scalar characteristic

equation

$$L(x, t)\phi(x, t) = \mu\phi(x, t) \quad (3.1.1)$$

will evolve nonlinearly. The eigenfunctions have imposed upon them the property they are unitary with respect to the origin, *i.e.*,

$$\phi(x, t) = U(t)\phi(x, 0). \quad (3.1.2)$$

Then, this nonlinear behavior will be governed by some operator $A(t)$, which may be nonlinear, *via* the equation

$$\frac{\partial\phi(x, t)}{\partial t} = \frac{\partial U(t)}{\partial t}\phi(x, 0) = A(t)\phi(x, t), \quad (3.1.3)$$

where we have defined

$$\frac{\partial U(t)}{\partial t} \equiv A(t)U(t). \quad (3.1.4)$$

The pair of operators $L(x, t)$ and $A(t)$ is called the Lax pair.

Because $L(x, t)$ is a Hermitian operator its dynamics is generated by the unitary operator, *i.e.*,

$$U^\dagger(t)L(x, t)U(t) = L(x, 0). \quad (3.1.5)$$

Operating on both sides with $\frac{\partial}{\partial t}$, enforcing the unitary condition $U^\dagger(t)U(t) = 1$, and using (3.1.4) gives

$$U^\dagger(t)\left(\frac{\partial L(x, t)}{\partial t} - [A(t), L(x, t)]\right)U(t) = 0. \quad (3.1.6)$$

The final result is the differential scalar Lax equation

$$\frac{\partial L(x, t)}{\partial t} = [A(t), L(x, t)], \quad (3.1.7)$$

which generates the integrable KdV equations. Physically, the KdV equations describe the governing motion of solitons. Further, the spectral parameter μ is proportional to the speed of propagation of the soliton wave. This makes sense since solitons are non-dispersive and μ is required to be independent of time.

The matrix construction of Lax operators utilizing generators of some embedding affine Lie algebra $\hat{\mathcal{G}}^{(k)}$, reviewed by Drinfeld and Sokolov¹, begins with a matrix operator of the form

$$\mathcal{L} = I \frac{\partial}{\partial x} + \Lambda + q(x), \quad (3.1.8)$$

where I denotes the $N \times N$ dimensional unit matrix, and to simplify notation we have suppressed the argument t . The third term is discussed below. The second term is generated by the negative simple roots of the embedding affine Lie algebra. In the gradation conventions of Drinfeld and Sokolov, we have the circulant matrix

$$\Lambda = \sum_{i=0}^r c_i E_i. \quad (3.1.9)$$

The procedure is then to reduce the system of linear equations given by the kernel matrix equation

$$\mathcal{L}\vec{\psi}(x) = 0, \quad (3.1.10)$$

where $\vec{\psi} = (\psi_1, \psi_2, \dots, \psi_N)$, to the linear differential eigenvalue equation (3.1.1), where the vacuum solution ϕ is a function of the components of the eigenfunction $\vec{\psi}$. Drinfeld and Sokolov show that such reduction is possible if one removes a simple root, say the m^{th} , from the affine root system. They denote this situation $(\hat{\mathcal{G}}^{(k)}, c_m)$, which is in the homogeneous or standard gradation⁹. The coefficients in (3.1.9) are then assigned the values $c_{i \neq m} = 1$ and $c_m = \lambda$, where λ is a constant function of the spectral parameter μ .

Removing an element from the simple root system is equivalent to deleting the corresponding vertex from the Dynkin diagram. Thus, when an extremal vertex is deleted, the system $(\hat{\mathcal{G}}^{(k)}, c_m)$ represents a single residual basic Lie algebra. Removing the affine vertex obviously gives $\mathcal{G}$. On the other hand, deleting internal vertices splits the Dynkin diagram into two sections, corresponding to a pair of basic Lie algebras. In both situations, a pair of (pseudo)differential operators is found whose product gives the Lax operator L of Eq.(3.1.7).

To determine the exact form of the vacuum solution ϕ recall that negative simple roots are lowering operators on system eigenstates. Further, removal of a root in the affine system produces the simple root system of a non-affine Lie algebra. Thus, due to the linear combination among the roots of the affine system, one root must be singled out to act as a conventional state raising operator. This role is given to the removed root. Thus, the vacuum eigenstate will be annihilated by a vacuum projection operator Δ^- defined by

$$\Lambda = \sum_{i=0}^r E_{i \neq m} + \lambda E_m \equiv \Delta^- + \lambda E_m. \quad (3.1.11)$$

This requirement fixes the scalar vacuum solution ϕ by setting it equal to a linear combination of the components of $\vec{\psi}$ such that

$$\Delta^- \vec{\psi} = 0, \quad (3.1.12)$$

is satisfied. A direct relation between the scalar operator L and the matrix operator $\mathcal{L}$ will be given in the next section.

For the kernel equation (3.1.10) to produce a unique solution, we require that the number of independent degrees of freedom equal the rank of the embedding affine Lie algebra $\hat{\mathcal{G}}^{(k)}$, or equivalently the residual system $(\hat{\mathcal{G}}^{(k)}, c_m)$. The extra degrees of freedom generate gauge invariance. To fix the gauge invariance, one must find a matrix operator $S(x)$ that enforces the gauge transformation

$$\mathcal{L}_0 = e^{ad S} \mathcal{L}, \quad (3.1.13)$$

where ad denotes the adjoint mapping. The gauge freedom present in Eq.(3.1.10) allows one freedom in determining the form of the coordinate dependent term $q_0(x)$, i.e.,

$$\mathcal{L}_0 = I \frac{\partial}{\partial x} + \Lambda + q_0(x). \quad (3.1.14)$$

Drinfeld and Sokolov find the sufficient condition that $S \in C^\infty(\mathbb{R}, \eta)$, where η is generated by the positive simple roots $F_i, i \neq m$.

Many authors, including Drinfeld and Sokolov, work most frequently in the “canonical” gauge. However, in this paper we choose to work in their “diagonal” gauge which has the form

$$q^{diag}(x) = \begin{pmatrix} q_1 & 0 & \dots & 0 & 0 \\ 0 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & q_{N-1} & 0 \\ 0 & 0 & \dots & 0 & q_N \end{pmatrix}. \quad (3.1.15)$$

This gauge leads to the convenient form

$$q^{diag} = \sum_{i=0}^r v_i(q_1, q_2, \dots, q_N) H_i, \quad (3.1.16)$$

which is in the canonical or principal gradation⁹. Here, the functions v_i are linear combinations of the elements q_i . In this gauge, the gauge term q^{diag} associated with $(\hat{\mathcal{G}}^{(k)}, c_m)$ is the special case where the sum excludes $i = m$. The Lax operator L^{diag} generates the mKdV equations, and is related to L^{can} via the well-known Miura transformations.

3.2 Diagrammatic construction of Lax operators

To exploit gauge invariance of the Lax operators, one should choose a $q(x)$ gauge most suited to one's needs. Here, we are interested in developing a diagrammatic scheme for constructing L . In this regard, the diagonal gauge proves more useful than the other choices. In this section, we will demonstrate how the diagonal gauge allows one to build Lax operators directly from cyclic weight diagrams of representations of affine Lie algebras.

To motivate the algorithm, we first review the construction of L by solving the matrix system $\mathcal{L}\vec{\psi} = 0$. For the present discussion, it will be sufficient to consider embedding algebras of the form $(\hat{\mathcal{G}}^{(k)}, c_0)$ where the affine vertex is deleted. Thus, the diagonal gauge

simply reduces to the form

$$q^{diag((x))} = \sum_{i=1}^r v_i(x) H_i, \quad (3.2.1)$$

where we have excluded H_0 from the sum.

Consider again the canonical representation of the embedding affine Lie algebra $(A_2^{(1)}, c_0)$ presented in Figure 1. Matrix representations of the Cartan matrix can be read-off from the cyclic weight diagram, producing the previous results given in Eqs. (2.1.6) and (2.1.7). Similarly, the matrix representations of the simple roots are easily found to give

$$\Lambda = \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3.2.2)$$

where the effect of the affine root, indicated in Figure 1 by the dashed arrow line, is assigned the value λ . Plugging these values into the kernel equation (3.1.10) produces the system of equations

$$\begin{aligned} [\partial + v_1]\psi_1 &= -\lambda\psi_3, \\ [\partial - v_1 + v_2]\psi_2 &= -\psi_1, \\ [\partial - v_2]\psi_3 &= -\psi_2. \end{aligned} \quad (3.2.3)$$

Here, on the right-hand side we have placed the terms due to the matrix Λ .

The vacuum condition (3.1.12) determines the scalar function to be $\phi = \psi_3$. Thus, we must solve by starting with the last equation. First, we multiply this equation through by $[\partial - v_1 + v_2]$, and then eliminate ψ_2 using the second equation. This gives

$$[\partial - v_1 + v_2][\partial - v_2]\psi_3 = \psi_1. \quad (3.2.4)$$

Next, multiply through by $[\partial + v_1]$ and use the top equation in (3.2.3). The result is the scalar Lax eigenvalue equation

$$\begin{aligned} L^{(A_2^{(1)}, c_0)}\psi_3 &= [\partial + v_1][\partial - v_1 + v_2][\partial - v_2]\psi_3 \\ &= -\lambda\psi_3, \end{aligned} \quad (3.2.5)$$

where the spectral parameter is given by $\mu = -\lambda$. Imposing the field redefinitions

$$q_1 = v_1, \quad q_2 = v_2 - v_1, \quad (3.2.6)$$

gives the standard form

$$L^{(A_2^{(1)}, a_0)} = [\partial + q_1][\partial + q_2][\partial - q_1 - q_2]. \quad (3.2.7)$$

This example exhibits a common feature relevant for our scheme below. When the vacuum condition (3.1.12) requires the scalar eigenfunction to be given by a single component of the eigenfunction, say $\phi = \psi_i$, then the resulting characteristic equation satisfies

$$L\psi_i = \mu\psi_i. \quad (3.2.8)$$

Consequently, the system reduction must start with the i^{th} equation in the matrix system, and proceed upward till the top equation is reached. If $i \leq N$ the process continues with the bottom equation and moves upward until the i^{th} equation is reached again. We shall refer to this case as trivial since the corresponding cyclic affine weight diagram is linear, containing no branch points. A second feature brought out in this example, is that the number of factors in the resultant Lax operator (3.2.7) is equal to the number of weights in the weight diagram. Unfortunately, this is valid only for trivial cases. Nevertheless, this last observation is key to our scheme.

To highlight one more property of the general procedure we turn to a non-trivial example. For this, we require a representation of an affine Lie algebra whose cyclic weight diagram has at least one branching point. Thus, consider the canonical representation of the affine algebra $(D_4^{(1)}, c_0)$. Figure 4 presents the Dynkin diagram, Cartan matrix and corresponding cyclic weight diagram which has two branch points. Reading off from the weight diagram gives

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (3.2.9)$$

The branch points have manifested themselves by placing more than one non-zero entry in the second and sixth rows. Now, further reading off the elements of the Cartan matrices gives the system of equations

$$\begin{aligned}
[\partial + v_1]\psi_1 &= -\lambda\psi_7, \\
[\partial - v_1 + v_2]\psi_2 &= -\psi_1 - \lambda\psi_8, \\
[\partial - v_2 + v_3 + v_4]\psi_3 &= -\psi_2, \\
[\partial - v_3 + v_4]\psi_4 &= -\psi_3, \\
[\partial + v_3 - v_4]\psi_5 &= -\psi_3, \\
[\partial + v_2 - v_3 - v_4]\psi_6 &= -\psi_4 - \psi_5, \\
[\partial + v_1 - v_2]\psi_7 &= -\psi_6, \\
[\partial - v_1]\psi_8 &= -\psi_7.
\end{aligned} \tag{3.2.10}$$

The vacuum condition (3.1.12) produces two distinct solutions, ψ_8 and the linear combination $\psi_4 - \psi_5$. Here, we consider the first case. Proceeding as before, we eliminate ψ_6 and ψ_7 in the last two equations to get

$$[\partial + v_2 - v_3 - v_4][\partial + v_1 - v_2][\partial - v_1]\psi_8 = -\psi_4 - \psi_5. \tag{3.2.11}$$

Now, we encounter a well-known technical problem not found in the trivial case. The components ψ_4 and ψ_5 can not both be simultaneously eliminated since the expressions $[\partial - v_3 + v_4]$ and $[\partial + v_3 - v_4]$ do not commute. This dilemma is directly linked to the fact the corresponding cyclic weight diagram has a branch point connecting the fourth and fifth weights to a single weight located below them.

To overcome this obstacle the pseudo-differential operator ∂_x^{-1} must be introduced. Its operation on any function $f(x)$ is given by the expansion

$$\partial_x^{-1} f(x) = \sum_{i=0}^{\infty} (-1)^i f^{(i)}(x) \partial_x^{-1-i}. \tag{3.2.12}$$

Utilizing the pseudo-differential operator, we rewrite the fourth equation in (3.2.10) as

$$\psi_4 = -[\partial - v_3 + v_4]^{-1}\psi_3. \quad (3.2.13)$$

Thus, the combined effect of the bottom five equations is

$$\begin{aligned} \psi_3 = & \{[\partial - v_3 + v_4]^{-1} + [\partial + v_3 - v_4]^{-1}\}^{-1}[\partial + v_2 - v_3 - v_4] \\ & \times [\partial + v_1 - v_2][\partial - v_1]\psi_8. \end{aligned} \quad (3.2.14)$$

A helpful identity we use repeatedly is

$$\{A^{-1} + B^{-1}\}^{-1} = \{A^{-1}[A + B]B^{-1}\}^{-1} = B[A + B]^{-1}A. \quad (3.2.15)$$

When applied to Eq.(3.2.14), a cancellation occurs among the v_i 's appearing in the curly brackets. This simplifies the expression to

$$\psi_3 = \frac{1}{2}[\partial - v_3 + v_4]\partial^{-1}[\partial + v_3 - v_4][\partial + v_2 - v_3 - v_4][\partial + v_1 - v_2][\partial - v_1]\psi_8. \quad (3.2.16)$$

Continuing, incorporating the next two equations in (3.2.10) requires a second application of the relation (3.2.15). Finally, the Lax operator based on $(D_4^{(1)}, c_0)$ with vacuum ψ_8 is

$$\begin{aligned} L = & \frac{1}{4}\partial^{-1}[\partial + v_1][\partial - v_1 + v_2][\partial - v_2 + v_3 + v_4][\partial - v_3 + v_4] \\ & \times \partial^{-1}[\partial + v_3 - v_4][\partial + v_2 - v_3 - v_4][\partial + v_1 - v_2][\partial - v_1]. \end{aligned} \quad (3.2.17)$$

Using the field redefinitions

$$\begin{aligned} q_1 &= v_1, & q_2 &= v_2 - v_1, \\ q_3 &= -v_2 + v_3 + v_4, & q_4 &= -v_3 + v_4, \end{aligned} \quad (3.2.18)$$

we get

$$L = \frac{1}{4}\partial^{-1}[\partial + q_1][\partial + q_2][\partial + q_3][\partial + q_4]\partial^{-1}[\partial - q_4][\partial - q_3][\partial - q_2][\partial - q_1]. \quad (3.2.19)$$

which is proportional to the standard result.

We have chosen these two examples because they introduce the techniques needed to generate Lax operators associated with even the most complicated algebraic systems.

Furthermore, they show how closely the structure of cyclic affine weight diagrams is linked with the construction of general Lax operators. As a result, we propose a set of four steps which allows one to construct Lax operators associated with cyclic representations of affine Lie algebras.

First, we propose that to every weight vector of an affine cyclic weight diagram one can associate an operator as follows:

$$\text{Step 1 : } (a_0 a_1 a_2 \dots) \rightarrow [\partial_x + a_1 v_1(x) + a_2 v_2(x) + \dots]. \quad (3.2.20)$$

The coefficient a_0 does not appear on the right-hand side as it corresponds to the deleted vertex. Now, even before we specify how to build L from these factors, a significant conclusion can be inferred from this rule. Namely, in the context of scale invariant theories, two Lax operators L and L' will be equivalent if their respective cyclic weight diagrams have the same shape, normalization factors and number of weight vectors, and if there exists an isomorphism between the coefficients of the weight factors. In fact, some cases relax the condition that the number of weights in the two systems be equal.

Second, we introduce a step which is designed to facilitate the construction of Lax operators when branch points exist in the corresponding weight diagram. Essentially, this step reduces more complicated non-trivial cases to a sum of manageable trivial cases by reducing the branched weight diagram to a sum of linear subdiagrams.

Step 2: Replace branching weight diagrams by the sum

$$\text{of linear subdiagrams, each representing a vertical route} \quad (3.2.21)$$

beginning with, and ending on, the vacuum weight(s).

For example, Figure 5 presents the four linear subgraphs associated with the canonical representation of $D_4^{(1)}$.

In drawing cyclic weight diagrams, it is important that the arrows generated by the deleted vertex are distinguished from the others. Our convention is to use dashed lines.

Furthermore, the direction of the arrows must also be noted. The Lax operators associated with each subdiagram are then constructed as follows:

Step 3: Circulate around the loop beginning with the vacuum solution,

such that the flow is opposite most of the

arrows. If a weight vector is approached by an arrow's

a) tail, append its weight factor to the operator's left side, (3.2.22)

b) head, append the weight factor's inverse to the operator's left side

For weights at the tail end of both connecting arrows, do nothing.

Multiply by the product of the corresponding normalization factors.

The loop is to be circulated in a direction opposite most of the arrows so that the leading term of the Lax operator $L = \partial^n + \dots$ has positive exponent, i.e., $n > 0$.

For trivial cases, this completes the computation of L . However, for non-trivial cases with branching weight diagrams we can not naively build the final Lax operator from a sum of its constituent linear subgraphs. Instead, as we shall prove in the next section, they are added together analogously to how one computes total resistance of resistors in parallel.

Step 4: The Lax operator is given by the inverse of the sum of (3.2.23)
the inverses, of the operators resulting from step three.

For example, in the non-trivial case $(D_4^{(1)}, c_0)$, with $\phi = \psi_3$ as before, we build four operators corresponding to the linear subdiagrams in Figure 5,

$$L_1 = [\partial - v_1]^{-1} [\partial + v_1] [\partial - v_1 + v_2] [\partial - v_2 + v_3 + v_4] [\partial - v_3 + v_4] \times [\partial + v_2 - v_3 - v_4] [\partial + v_1 - v_2] [\partial - v_1], \quad (3.2.24)$$

$$L_2 = [\partial - v_1]^{-1} [\partial + v_1] [\partial - v_1 + v_2] [\partial - v_2 + v_3 + v_4] [\partial + v_3 - v_4] \times [\partial + v_2 - v_3 - v_4] [\partial + v_1 - v_2] [\partial - v_1], \quad (3.2.25)$$

$$L_3 = [\partial - v_1 + v_2] [\partial - v_2 + v_3 + v_4] [\partial - v_3 + v_4] [\partial + v_2 - v_3 - v_4] \times [\partial + v_1 - v_2] [\partial - v_1], \quad (3.2.26)$$

and

$$L_4 = [\partial - v_1 + v_3][\partial - v_2 + v_3 + v_4][\partial + v_3 - v_4][\partial + v_2 - v_3 - v_4] \\ \times [\partial + v_1 - v_2][\partial - v_1]. \quad (3.2.27)$$

Factoring out common terms, we find

$$L^{-1} = \sum_{i=1}^4 (L_i)^{-1} \\ = [\partial - v_1]^{-1} [\partial + v_1 - v_2]^{-1} [\partial + v_2 - v_3 - v_4]^{-1} \\ \times \{ [\partial - v_3 + v_4]^{-1} + [\partial + v_3 - v_4]^{-1} \} [\partial - v_2 + v_3 + v_4]^{-1} \\ \times [\partial - v_1 + v_2]^{-1} [\partial + v_1]^{-1} \{ 1 + [\partial - v_1]^{-1} [\partial + v_1] \}. \quad (3.2.28)$$

By taking the reciprocal, and simplifying, we reproduce the previous result Eq.(3.2.17).

To end this section we consider the alternative vacuum choice $\psi_3 - \psi_4$. It should be obvious that since we are dealing with cyclic weight diagrams, Lax operators associated with other vacuum states can be achieved by cyclically permuting factors in the primary Lax operator. Thus, this second vacuum choice immediately gives the Lax operator

$$L = \frac{1}{4} [\partial - q_3][\partial - q_2][\partial - q_1]\partial^{-1}[\partial + q_1][\partial + q_2][\partial + q_3][\partial + q_4]\partial^{-1}[\partial - q_4]. \quad (3.2.29)$$

IV. PROOF OF DIAGRAMMATIC SCHEME.

To prove the equivalence between the matrix system $\mathcal{L}\vec{\psi} = 0$ and the diagrammatic algorithm, we begin by rewriting the former as

$$\mathcal{D}\vec{\psi}(x) = -\Lambda\vec{\psi}(x), \quad (4.1)$$

where, to simplify notation, we have defined

$$\mathcal{D} \equiv \mathcal{L} - \Lambda = I \frac{\partial}{\partial x} + q(x). \quad (4.2)$$

The structure of the associated cyclic weight diagram is encoded entirely in the matrix Λ . Specifically, recall that the general matrix element Λ_{ij} is proportional to λ if the difference between the i^{th} weight and the connecting j^{th} weight equals the eliminated root of the embedding affine simple root system. All other connecting weights Λ_{ij} are proportional to 1. Otherwise, the matrix element is assigned the value 0. In all cases the proportionality constant is the normalization factor of the connecting root.

We construct the proof in stages. For the first stage we consider the trivial case, *i.e.*, a single entry in each row and column of Λ . Removing a root produces only a single vacuum state. This stage corresponds to weight diagrams with no branch points and only one arrow associated with the eliminated root. Clearly, we can rearrange the matrix equations in $\mathcal{L}\vec{\psi} = 0$ such that the vacuum state equation appears last. Furthermore, it can be arranged such that Λ is lower triangular with ones located along a diagonal once removed from the main diagonal, except for the eliminated root whose coefficient λ appears in the upper right-hand corner, *i.e.*, Λ is a circulant matrix. Thus, the Lax eigenvalue equation becomes

$$L\psi_N(x) = \mu\psi_N(x). \quad (4.3)$$

Since $\mathcal{D}$ is diagonal, the k^{th} equation in (4.1) can be written

$$\mathcal{D}_k\psi_k = - \sum_{i=1}^N \Lambda_{ki}\psi_i, \quad k = 2, 3, \dots, N. \quad (4.4)$$

Clearly, since Λ is a circulant matrix as specified above, the inequality $i < k$ holds for $k \neq 1$. Next, by repeatedly replacing the function ψ_i , appearing on the right-hand side,

with the i^{th} matrix equation we eventually reach the expression $\Lambda_{k1}\psi_1$, i.e.,

$$\begin{aligned}
\mathcal{D}_N \psi_N &= - \sum_{i=1}^N \Lambda_{Ni} \psi_i \\
&= \sum_{i=1}^N \sum_{j=1}^N \Lambda_{Ni} \mathcal{D}_i^{-1} \Lambda_{ij} \psi_j \\
&\vdots \\
&= (-)^{\gamma} \sum_{i=1}^N \dots \sum_{k=1}^N \Lambda_{Ni} \mathcal{D}_i^{-1} \Lambda_{ij} \dots \mathcal{D}_k^{-1} \Lambda_{k1} \psi_1
\end{aligned} \tag{4.5}$$

where γ is the height of the cyclic weight diagram. Due to the successive applications of the state lowering operators Λ_{ij} with $i > j$, this equation is interpreted as taking the highest state ψ_1 and lowering it to the vacuum state ψ_N .

Replacing ψ_1 through

$$\mathcal{D}_1 \psi_1 = -\mathcal{N}_0 \lambda \psi_N, \tag{4.6}$$

where we have used $\Lambda_{1N} = \mathcal{N}_0 \lambda$ which excites the level of the state since $1 < N$, gives

$$\mathcal{D}_N \psi_N = -(-)^{\gamma} \mathcal{N}_0 \lambda \sum_{i=1}^N \dots \sum_{k=1}^N \Lambda_{Ni} \mathcal{D}_i^{-1} \Lambda_{ij} \dots \mathcal{D}_k^{-1} \Lambda_{k1} \mathcal{D}_1^{-1} \psi_N. \tag{4.7}$$

where $\mathcal{N}_0$ is the normalization factor of the affine root. Finally, moving terms to the left-hand side we recover (4.3) where

$$L = \{ \mathcal{N}_0 \sum_{i=1}^N \dots \sum_{k=1}^N \mathcal{D}_N^{-1} \Lambda_{Ni} \mathcal{D}_i^{-1} \dots \Lambda_{k1} \mathcal{D}_k^{-1} \}^{-1}, \tag{4.8}$$

and the spectral parameter is given by

$$\mu = -(-)^{\gamma} \lambda. \tag{4.9}$$

Since each row and column of Λ contain only one entry the sum will generate a single term, i.e.,

$$L^{-1} = \{ \mathcal{N} \mathcal{D}_N^{-1} \mathcal{D}_{N-1}^{-1} \dots \mathcal{D}_1^{-1} \}^{-1}, \tag{4.10}$$

where $\mathcal{N}$ is the product of the normalization factors. Now, each $\mathcal{D}$ is a weight factor as defined in step 1. Thus, there is a direct mapping between the order of the weight factors and their location in the corresponding weight diagram.

Now, suppose we permit multiple row entries in Λ , i.e., branch points in the weight diagram. First consider the case where such multiple entries occur above the N^{th} row. As before, there is a single vacuum state, and the constant λ is located in the upper right-hand corner of Λ . Therefore, the constraint $i < k$, for $k \neq 1$, remains in effect for eq.(4.4). Hence, the derivation leading to (4.8) follows through unchanged. Now, each new entry in Λ causes an additional final term in (4.8). Clearly, *per* step 4 of the diagrammatic algorithm, the final Lax operator is obtained by taking the reciprocal of the sum of terms generated by (4.8).

Next, suppose the multiple row entries in Λ , due to the branch point, occur in the N^{th} row. The vacuum condition (3.1.12) shows that this is equivalent to a degenerate vacuum state with, say, degeneracy d . Subsequently, this row will be associated with the eliminated root, and the d integers will be assigned the value λ . In fact, λ appears only in this row. Clearly, in the weight diagram the d weights share the same level.

Let us first discuss the case where the coefficients λ occur in the first row. Thus, Eq.(4.4) remains valid, keeping intact the constraint $i < k$, for $k \neq 1$. Furthermore, the scalar eigenfunction $\phi(x)$ is now a linear combination of the components $\psi_N, \psi_{N-1}, \dots, \psi_{N-d+1}$, and the equation for ψ_1 becomes

$$\mathcal{D}_1 \psi_1 = - \sum_{s=N-d+1}^N \Lambda_{1s} \psi_s. \quad (4.11)$$

Consequently, eq.(4.7) is modified to

$$\begin{aligned} \psi_s = & -(-)^{\beta} \sum_{i=1}^N \dots \sum_{k=1}^N \sum_{j=N-d+1}^N \mathcal{D}_s^{-1} \Lambda_{si} \mathcal{D}_i^{-1} \dots \\ & \times \mathcal{D}_k^{-1} \Lambda_{k1} \mathcal{D}_1^{-1} \Lambda_{1j} \psi_j, \end{aligned} \quad (4.12)$$

where β is the number of field replacements performed. The characteristic equation is obtained by multiplying both sides by Λ_{1s} and then summing over s , i.e.,

$$\sum_{s=N-d+1}^N \Lambda_{1s} \psi_s = -(-)^{\beta} \sum_{s=1}^N \dots \sum_{k=1}^N \sum_{j=N-d+1}^N \Lambda_{1s} \mathcal{D}_s^{-1} \Lambda_{si} \dots \times \mathcal{D}_k^{-1} \Lambda_{ki} \mathcal{D}_i^{-1} \Lambda_{ij} \psi_j. \quad (4.13)$$

Note, the sum over s on the right-hand side has been extended to the entire range for convenience.

Each term in Λ_{1s} contains the factor λ , which can then be factored out. As a result, the scalar eigenfunction is found to be

$$\phi = \sum_{s=N-d+1}^N \Lambda_{1s} \psi_s, \quad (4.14)$$

and the Lax operator,

$$L = \lambda \left\{ \sum_{s=1}^N \dots \sum_{k=1}^N \Lambda_{1s} \mathcal{D}_s^{-1} \Lambda_{si} \dots \Lambda_{ki} \mathcal{D}_i^{-1} \right\}^{-1}. \quad (4.15)$$

Clearly, this has the same interpretation as the non-degenerate branching case.

For the last stage of the proof, we relax the condition that multiple occurrences of λ must all be in the first row of Λ . In the weight diagram this means not all the arrows associated with the eliminated root point to the bottom level. Recall from the discussion surrounding eq.(3.1.11), the eliminated root with coefficient λ acts as a state raising operator. Thus, every occurrence of λ will appear in the upper triangular portion of Λ , and the unit coefficients of the state lowering roots are in the lower triangular portion.

For λ in the k^{th} row of Λ , $k \neq 1$, Eq.(4.4) is modified to

$$\mathcal{D}_j \psi_j = -\mathcal{N}_0 \lambda \psi_k - \sum_{n \neq k}^N \Delta_{jn}^{-1} \psi_n, \quad (4.16)$$

where, since λ corresponds to the state raising operator, $k > j$. Consider the case where j is the largest such index to satisfy this equation. Then, allowing degenerate vacuum

states, we have

$$\begin{aligned}
\psi_s &= -(-)^{\beta} \sum_{i=1}^N \dots \sum_{j=1}^N \mathcal{D}_s^{-1} \Lambda_{si} \mathcal{D}_i^{-1} \dots \Lambda_{ij} \psi_j \\
&= (-)^{\beta} \sum_{i=1}^N \dots \sum_{j=1}^N \mathcal{D}_s^{-1} \Lambda_{si} \mathcal{D}_i^{-1} \dots \Lambda_{ij} \mathcal{D}_j^{-1} [\mathcal{N}_{0\lambda} \psi_k + \sum_{n \neq k}^N \Delta_{jn}^{-1} \psi_n],
\end{aligned} \tag{4.17}$$

where, again, β is the number of field replacements performed. The effect of the factor in front of ψ_k is to first, due to λ , raise this state to ψ_j and then to lower it till the vacuum state ψ_s is reached. Now, since the corresponding weight diagram is cyclic, there must exist some factor that will circulate ψ_s back to ψ_k .

First, as was the case with $(D_4^{(1)}, c_0)$, consider the situation where ψ_k is an intermediate state in (4.17), i.e.,

$$\psi_s = (-)^{\alpha} \sum_{i=1}^N \dots \sum_{k=1}^N \mathcal{D}_s^{-1} \Lambda_{si} \mathcal{D}_i^{-1} \dots \Lambda_{jk} \psi_k. \tag{4.18}$$

This gives

$$\psi_k = (-)^{\alpha} \left\{ \sum_{i=1}^N \dots \sum_{k=1}^N \mathcal{D}_s^{-1} \Lambda_{si} \mathcal{D}_i^{-1} \dots \Lambda_{jk} \right\}^{-1} \psi_s. \tag{4.19}$$

Thus, *per* step 3 of the diagrammatic algorithm, the factor $\mathcal{D}_k$, associated with the weight vector at the tail end of both connecting arrows does not appear. Further, proceeding from higher weights to lower weights in the weight diagram contributes factors of $\mathcal{D}_i^{-1}$ in the operator defined in step 2 for the linear subdiagram.

Finally, if ψ_k does not appear as an intermediate state of the vacuum state ψ_s in (4.17), then it must occur as an intermediate state for one of the other vacuum states. Again, since the weight diagram is cyclic, there is some closed path going from ψ_s to each of these other vacua. However, to write down a final expression is too unwielding. Nevertheless, it should be clear that the general rules of the diagrammatic algorithm are complete and provide an accurate mapping between weight diagrams and the scalar Lax equation.

V. SUPERSYMMETRIC LAX OPERATORS

In this section we generalize the diagrammatic scheme to supersymmetric affine Lie algebras. The classification of all possible supersymmetric extensions of the basic Lie algebras has been given by Kac¹⁰. The classical supersymmetric Lie algebras are separated into several series. The unitary series is denoted $A(m, n)$ or $SL(m+1 \parallel n+1)$. Next, the orthosymplectic series $OSp(m \parallel 2n)$ includes $B(m, n)$ or $OSp(2m+1 \parallel 2n)$, $C(n+1)$ or $OSp(2 \parallel 2n)$, and $D(m, n)$ or $OSp(2m \parallel 2n)$. These basic series are also called contragradient which means they have the same number of positive and negative roots. The exceptional contragradient extensions are given by $F(4)$, $G(3)$, and $D(2 \parallel 1; \alpha)$ which is a deformation of $D(2, 1)$ and is the only Lie algebra dependent on a continuous parameter. Finally, there are the non-contragradient or strange cases $P(n)$ and $Q(n)$.

In addition to the bosonic simple roots of the basic Lie algebra, the simple root system of the supersymmetric algebras contains two distinct kinds of fermionic roots. The Dynkin symbol of the first type is sometimes given by a shaded vertex representing a non-zero norm. The second fermionic root type has zero norm whose Dynkin symbol is given correspondingly by a crossed out vertex. As always, the bosonic root is denoted by a white vertex.

A new feature occurring in the supersymmetric Lie algebras is that they may have several non-equivalent simple root systems, corresponding to different Dynkin diagrams and Cartan matrices. In other words, the different root systems can not be transformed into each other through standard Weyl rotations. Instead, they are obtained by performing the "Weyl" transformation with respect to the nilpotent fermionic root. For more details, see Frappat *et al.*¹¹ which also presents a large collection of Dynkin diagrams associated with all of the classical contragradient supersymmetric cases, those of the affine and twisted

affine supersymmetric algebras.

Non-equivalent simple root systems which represent the same supersymmetric Lie algebra differ in the distribution of bosonic and fermionic roots. However, here we are interested in considering a natural extension of the Drinfeld-Sokolov procedure to the supersymmetric case. This restricts our choice of the simple root system for building the supersymmetric Lax operators¹². Recall, in the bosonic case the mKdV Lax operator constructed with the gradation choice of Drinfeld and Sokolov generates Toda lattice models^{1,13,14}. For supersymmetric algebras it has been shown¹⁵ that Toda lattices are possible only for simple root systems composed purely of fermionic roots. This choice leads to commutation relations

$$\begin{aligned} [H_i, H_j] &= 0, & \{E_i, F_j\} &= \delta_{ij} H_i, \\ [H_i, E_j] &= A_{ij} E_j, & [H_i, F_j] &= -A_{ij} F_j, \end{aligned} \quad (5.1)$$

where $0 \leq i, j \leq r$. The Cartan matrices can be obtained from the Dynkin diagram as in the non-supersymmetric case, although due to the zero-norm fermionic roots we now write

$$A_{ij} = (F_i, F_j). \quad (5.2)$$

Supersymmetric Lie algebras with purely fermionic root systems have been given by Leites *et al.*¹⁶,

$$SL(n+1 \parallel n), OSp(m \parallel 2n) \ (m = 2n, 2n+2, 2n \pm 1), D(2 \parallel 1; \alpha). \quad (5.3)$$

Further, the infinite-dimensional affine supersymmetric Lie algebras with purely fermionic simple root systems are

$$SL(n \parallel n)^{(1)}, OSp(2n+2 \parallel 2n)^{(1)}, D(2 \parallel 1; \alpha)^{(1)}, \quad (5.4)$$

while the infinite-dimensional twisted affine cases are

$$SQ(2n+1)^{(2)}, SL(n \parallel n)^{(2)}, OSp(2n \parallel 2n)^{(2)}. \quad (5.5)$$

The supersymmetric extension of the KdV equations was first discussed in Manin and Radul¹⁷. They suggested replacing the bosonic derivative ∂_x by its supersymmetric analog, *i.e.*,

$$\partial_x \rightarrow D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial w}. \quad (5.6)$$

Note that $D^2 = \frac{\partial}{\partial w}$. The system of matrix equations of Drinfeld and Sokolov can then be generalized to¹²

$$\mathcal{L}\vec{\psi}(x, \theta) \equiv [D + Q(x, \theta) + \Lambda]\vec{\psi}(x, \theta) = 0, \quad (5.7)$$

where Λ is generated by the purely negative fermionic roots, and $Q(x, \theta)$ is a Grassmann odd fermionic superfield which can be expanded as

$$Q(x, \theta) = \sum_{i=1}^r H_i \Psi_i(x, \theta), \quad (5.8)$$

where now H_i are elements of the Cartan-Kac subalgebra. The vacuum condition is as before,

$$\Delta^- \vec{\psi}(x, \theta) = 0. \quad (5.9)$$

Since the second type of fermionic root is nilpotent, they deserve special treatment when constructing cyclic weight diagrams. To illustrate how this comes about, consider the fundamental representation of the supersymmetric algebra $OSp(2 | 2)$. The Dynkin diagram of the purely fermionic root system is given by

where both fermionic roots are denoted as having zero norm, and where we have indicated the choice (11) for a highest weight vector. The Cartan matrix is then easily found to be

$$A = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}. \quad (5.10)$$

To construct the highest weight diagram we proceed as before. Since a positive one appears both in the first and second places we have two permutations of subtraction to perform.

In particular, we can start by subtracting the first row of the Cartan matrix giving (13), and then subtracting the second row resulting in (33). However, unlike the bosonic case, we may not subtract the first row of the Cartan another time from the weight (13). This is because here the fermionic weight vectors are nilpotent and subtracting any Cartan row twice gives a decoupled state. Similarly, we can start by subtracting the second Cartan row once (and only once) and then the first row giving (33). Thus, we find the weight diagram with height two:

$$\begin{array}{ccc}
 & (11)_1 & \\
 \swarrow & & \searrow \\
 (13)_2 & & (31)_3 \\
 \searrow & & \swarrow \\
 & (33)_4 &
 \end{array} \tag{5.11}$$

The decoupling which occurs when constructing a cyclic weight diagram for an affine supersymmetric algebra is almost as straightforward. For example, in figure 6 we display the partially decoupled weight diagram of $(SL(2 | 2)^{(1)}, c_0)$ where states were decoupled as we went from top to bottom. There are several ways to decouple the remaining weights since the lowering operators b_0, b_2 and b_3 still appear more than once. The only way for a cyclic weight diagram to emerge is by decoupling the weights outside the box. To see that this is also consistent, note that all paths leading from weight 5 to weights 2 or 4 require two applications of b_0 .

To construct a super-Lax operator let us take the vacuum solution $\phi = \Psi_4$. We easily find the super-Lax operator to be

$$L = [D + \Psi_3][D + \Psi_2 + \Psi_3][D + \Psi_1 + \Psi_2][D + \Psi_1]. \tag{5.12}$$

As in the non-supersymmetric case, Lax operators corresponding to the other three vacua are obtained through cyclic permutations of the above operator.

VI. DISCUSSIONS

In this paper we have shown how one can read off from cyclic weight diagrams, associated with representations of affine Lie algebras, Lax operators in the diagonal gauge. This method is most useful when tables of matrix representations are not at hand and must be generated by weight diagrams anyway. Furthermore, this procedure can easily be implemented on computer by virtue of the fact that computer generated algorithms currently exist for building highest weight diagrams¹⁸. With minor modifications, these programs can be adapted for cyclic weight diagrams.

It remains to be seen whether higher representations lead to any new physics. If so then a program of categorizing these results might be pursued to identify redundant solutions. This might be easier to answer for supersymmetric algebras since nilpotency projects out decoupled weight vectors. What is clear though, at least for the non-supersymmetric cases is that these higher representations lead to integrable systems. Recall, to prove the integrability of KdV systems Drinfeld and Sokolov found the necessary infinite set of conserved currents to be given by the coefficients of the Laurent expansion of $\mathcal{L}$ in the affine parameter λ . Our conclusion follows from the fact that every representation of a basic Lie algebra has an affine extension, and that defining properties of affine Lie algebras are representation independent.

Finally, it would be interesting to see if our procedure could be modified to directly generate Lax operators in other gauges. Furthermore, in light of recent work⁹ on generalizations of the Drinfeld and Sokolov scheme, one may also consider different gradations of the affine Lie algebra from which to obtain the matrix Λ and the form of $q(x)$.

ACKNOWLEDGEMENTS

We wish to acknowledge Zongan Qiu and Moustafa Awada for discussions, and Pierre Ramond for reading over the manuscript.

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FIGURE CAPTIONS

Figure 1. Dynkin diagram, Cartan matrix and cyclic weight diagram of the (-110) representation of $A_2^{(1)}$. The dashed arrow is the deleted root.

Figure 2. Cyclic weight diagram of $A_2^{(1)}$ from the weight (-211) . The dashed arrows are the deleted root.

Figure 3. Highest weight diagram of the adjoint representation of A_2 , (11) .

Figure 4. Dynkin diagram, Cartan matrix and cyclic weight diagram of the (-11000) representation of $D_4^{(1)}$. The dashed arrows are the deleted root.

Figure 5. Subdiagrams of the cyclic weight diagram of $D_4^{(1)}$. The dashed arrows are the deleted root.

Figure 6. Dynkin diagram, Cartan matrix and complete weight diagram of the (1001) representation of $SL(2 \parallel 2)^{(1)}$. The dashed arrows are the deleted root. The box encloses the cyclic weight diagram, excluding the decoupled weights outside.

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Figure 1

Figure 2

Figure 3

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

Figure 4

Figure 5

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

Figure 6

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