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SEPTEMBER 1979

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PPPL-1587

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WKB Theory for High- n Modes in Axisymmetric Toroidal Plasmas

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ABSTRACT

We demonstrate that the low-frequency, $k_{\parallel}/k_{\perp} \approx 0$ normal modes of an axisymmetric plasma, at large but finite toroidal mode number n , can be obtained by solving a novel WKB problem involving an infinite number of branches. Formulae for the frequencies of periodic normal modes are derived. The analysis is performed in the context of an ideal MHD model, and comparison is made with numerical ballooning mode results.

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Ballooning modes¹ and drift modes in toroidal plasmas share the properties of having short wavelength transverse to the sheared magnetic field $\hat{B}$, but only equilibrium-scale phase variation parallel to $\hat{B}$. However, if we try to construct a formal WKB-type expansion, the constraint $k_{||}/k_{\perp} = 0$ at lowest order leads to the phenomenon of a multivalued (aperiodic) lowest order eigenfunction, since the field lines do not in general close on themselves. Glasser² has proposed a way around this difficulty by representing the solution as an infinite sum of aperiodic functions, which add up to give a periodic solution. Connor et al.,³ have shown that this "ballooning representation" can be arrived at by a process of Fourier analysis, interpolation, and inverse Fourier transformation. Lee and Van Dam⁴ also arrived at the ballooning representation by Fourier methods. None of these authors has faced the problem of lack of uniqueness in the ballooning representation, although Ref. 3 has made the crucial observation that the aperiodic components can be assumed to obey the original, untransformed equation.

Rather than adopt the representational approach with its associated uniqueness questions, we here take the point of view that the infinite sum represents a linear superposition of waves propagating in a fictitious, multisheeted covering space, identical to physical space in all but its topology. We then find that a WKB theory can be constructed in a natural way, with the local dispersion relation having novel periodicity properties. We are led to equations for the normal mode frequencies which include the "1/n correction"^{3,5} as a limiting case, but give better agreement

with the ideal MHD toroidal eigenmode equation PEST⁶ at quite small n .

We shall use a (ψ, θ, ζ) coordinate system, where $2\pi\psi$ is the poloidal flux, θ is an arbitrary poloidal angle, and ζ is a toroidal angle chosen so as the magnetic field lines are straight in (ζ, θ) space. If ζ is the usual toroidal angle ϕ , then our coordinate system is essentially that of the PEST stability code,⁶ with the Jacobian $J \equiv (\vec{\nabla}_\psi \cdot \vec{\nabla}_\theta \times \vec{\nabla}_\zeta)^{-1} = X^2/F(\psi)$ where X is the distance from the axis of symmetry. If we instead require $J = \text{const.}$, then we have a Hamada coordinate system. In these coordinates we have $\vec{B} = \vec{\nabla}_\zeta + \vec{\nabla}_\psi + q(\psi)\vec{\nabla}_\theta \times \vec{\nabla}_\psi$. The "safety factor" $q(\psi)$ will be assumed to be a monotonic function of ψ .

As a simplifying device (and to facilitate comparison with PEST-II⁷) we adopt a tensor mass density $\rho = \rho^* \vec{\nabla}_\psi \vec{\nabla}_\psi$.

This model has the same marginal stability points as the physical scalar density model, but gives different growth rates. Its utility lies in the fact that we can eliminate two components of the fluid displacement $\vec{\xi}$ analytically,⁶ resulting in an equation of motion involving only the single scalar $\xi \equiv \vec{\nabla}_\psi \cdot \vec{\xi}$,

$$(\rho\omega^2 + F)\xi = 0, \quad (1)$$

where ξ has been assumed to depend on t and ζ only through the factor $\exp(-in\zeta - i\omega t)$, and F is an operator, $F(\psi, \theta; \partial_\theta - inq, -in^{-1}\partial_\theta, -in^{-1}\partial_\psi)$.

We now define our covering space by cutting the torus in the toroidal direction, along $\theta = \pi$ say, and continuing $\hat{\psi}$ onto an infinity of "Riemann sheets." Thus, $\hat{\psi}$ must obey Eq. (1), but need not be periodic in θ . The operator F is self-adjoint in the Hilbert space of functions $\psi(\psi, \theta)$ defined on our covering space. We can now trivially satisfy $k_{||} = 0$ to lowest order in n^{-1} by requiring that the only rapid θ dependence must be through the factor $\exp(in\theta)$, which insures that $\hat{\psi} \cdot \nabla_{\theta} \hat{\psi} = O(1)$. This leaves the fast ψ dependence unspecified, so we introduce an eikonal $S(\psi)$ and obtain the WKB representation of $\hat{\psi}$

$$\hat{\psi} = \hat{\psi}(\psi, \theta) \exp[-i n \theta - i n (a + S)] , \quad (2)$$

where $a = \psi - q\theta$ is constant on a field line, and the θ and ψ derivatives of $\hat{\psi}$ are of order unity.

The amplitude $\hat{\psi}$ obeys Eq. (1), with F replaced by the operator

$$\hat{F} = F[\psi, \theta; \partial_{\theta}, q - i n^{-1} \partial_{\theta}, q'(\theta - k) - i n^{-1} \partial_{\psi}] . \quad (3)$$

Here $k(\psi)$ is the radial wavenumber defined by $S = kq'$. Since the n dependence of $\hat{F}$ is explicit we can readily expand in inverse powers of n : $\hat{F} = \hat{F}^{(0)} + n^{-1} \hat{F}^{(1)} + \dots$, where the lowest order term $\hat{F}^{(0)} \equiv F[\psi, \theta; \partial_{\theta}, q, q'(\theta - k)]$ is an operator only in θ space. It is self-adjoint with respect to both the Hermitian and real inner products (assuming k is real).

Expanding ψ in a similar series to F , and demanding that Eq. (1) be satisfied order by order gives, at zeroth order $[\lambda^2 + F(0)] \psi(0) = 0$. This is an ordinary differential equation. We require $\psi(0)$ to be square integrable³ to keep the kinetic energy of the wave finite, which provides a sufficient boundary condition on the lowest order equation to define an eigenvalue problem. That is, solving under the boundary condition that $\psi(0)$ decays faster than $|r|^{-1/2}$ as $|r| \rightarrow \infty$ gives the local dispersion relation $\lambda^2 = \lambda(q, k)$, where we have here used q as radial variable rather than r , since q and k make a natural conjugate pair (assuming $q' \neq 0$). Self adjointness implies λ is real if k is real.

Contours of $\lambda(<0)$ in the $q - k$ plane for an unstable, low-shear, high- β equilibrium are shown in Fig. 1. Several features deserve comment, the most remarkable feature being that λ is a periodic function of k . This is a consequence of invariance of the eigenvalue equation under the symmetry operation $k \rightarrow k + 2\pi$, $\hat{\psi}(\theta) \rightarrow \hat{\psi}(\theta - 2\pi)$. There is also a less fundamental symmetry of λ about the lines $k = 0$ and $k = \pi$ due to the use of an up-down symmetric equilibrium. Thus, the symmetry properties of the $q - k$ phase space are the same as the $q - \theta$ physical space of the equilibrium, and in this sense k is an angle-like variable.

We also observe that the minima of λ (the most unstable points) occur on the lines $k = 2\pi\ell$; $\ell = 0, \pm 1, \pm 2, \dots$. This is because these points give eigenfunctions concentrated on the outside of the torus, near $\theta = 0$, where the destabilizing ballooning forces are strongest. No cases where this is not true

have so far been found. The occurrence of unstable ($\epsilon < 0$) saddle points of ϵ at $k = (2\ell + 1)$ is a feature of the particular equilibrium used, and is not universal. When it occurs, the unstable region of phase space divides into topologically distinct "trapping" and "passing" regions. These must be treated differently in deriving the quantization condition for J^2 .

In order to find $S = \int k dq$ we can invert the dispersion relation to find $k(q, J^2)$. Clearly, there is an infinite number of branches of this function, in contrast to the usual two. A preferred procedure is to parametrize the $\epsilon = J^2$ contour with a time-like parameter s , say, and to integrate the ray equations $\dot{q} = \partial\epsilon/\partial k$, $\dot{k} = -\partial\epsilon/\partial q$. We can then define the multivalued function $S = \int k \dot{q} ds$.

To find how the amplitude propagates between turning points, we use the solubility condition on the first order equation

$$\int_{-\infty}^{\infty} dq J^{\hat{F}}(0) F^{\hat{F}}(1) \hat{F}^{\hat{F}}(0) = 0. \quad (3)$$

Since we have used the real, rather than the Hermitian, inner product to annihilate $J^{\hat{F}}(0)$, $J^{\hat{F}}(1)$ is antiseif adjoint, except for the terms involving ∂_{ψ} . Only these terms survive in Eq. (4), and they combine to make a conservation equation for a complex wave action $\partial \bar{L}^{(0)}/\partial k$,⁸

$$(d/dq) \partial \bar{L}^{(0)}/\partial k = 0, \quad (5)$$

where $\bar{L}^{(0)}$ is a complex averaged Lagrangian, defined by

$$\bar{L}^{(0)} \dot{\bar{L}}^{(0)} = \frac{1}{2} \int_{-\infty}^{\infty} d\omega (J/q') \dot{\bar{L}}^{(0)} \omega^2 + F^{(0)} \dot{\bar{L}}^{(0)} \quad (6)$$

Defining the complex wave amplitude $a(q)$ by $\bar{L}^{(0)} = a(q) \dot{\bar{L}}^{(0)}$, where $\dot{\bar{L}}^{(0)}$ is a real eigenfunction of the zeroth order equation, normalized so that $\partial \bar{L}^{(0)} / \partial \omega^2 = 1$, and using the fact that $\bar{L}^{(0)} = 0$ when $\omega^2 = \omega^2(q, k)$, we can write Eq. (5) as $(\dot{q}a^2)^{\cdot} = 0$. Writing $a = |a| \exp(i \arg a)$, we see that Eq. (5) expresses both conservation of the action current $|\dot{q}a^2|$ and invariance of the phase $\arg a$ between turning points (i.e., points where $\dot{q} = 0 / \partial k = 0$). Near a turning point, however, a varies rapidly ($\rightarrow \infty$), and the expansion in n^{-1} breaks down.

In the neighborhood of a turning point we expand in powers of $n^{-1/2}$, thus deriving an Airy equation for a complex wave amplitude which matches onto the incoming and outgoing waves away from the turning point. Assuming the turning point to be well removed from the magnetic axis, the plasma boundary, and other turning points, we find a jump condition for $\dot{q}a^2$ as the ray equations propagate q and k through a turning point. In amplitude and phase representation, the jumps are

$$[|\dot{q}a^2|] = 0, \quad [\arg a] = (1/2)\pi \mu \operatorname{sgn}(n), \quad (7)$$

where we define the Maslov index⁹ to be $+1(-1)$ when the ray

trajectory traverses the turning point in a locally clockwise (anticlockwise) sense, and $\text{sgn}(n) = n/|n|$. These are the same results as are encountered in the WKB theory of waves obeying much simpler wave equations.⁹ We have also verified that the results hold true if the physical mass density $\hat{\rho} = \rho I$ is used. As Whitham⁸ has pointed out, the action conservation Eq. (5) is a result of phase invariance in conservative systems, and we conjecture that Eq. (7) is a similarly general result.

For trapped modes, whose ray trajectories in the $k - q$ phase space (Fig. 1) are topologically circular, $\oint \exp(-inS)$ is double valued. Then the quantization condition is the usual one for libratory phase space trajectories⁹

$$n \oint k \, dq = 2\pi(N + \frac{1}{2}) , \quad (8)$$

where $N = 0, 1, 2, \dots$. Although N should strictly be $O(n)$ for the WKB theory to apply, it is well known that Eq. (8) works for a harmonic oscillator potential even for $N = 0$, and it is therefore not surprising that using a Taylor expansion of γ about the most unstable point and Eq. (8) yields an $O(n^{-1})$ correction to the growth rate in agreement with that obtained from a rigorous localized analysis.^{3,5} We also have evidence that Eq. (8) is a reasonable approximation down to $n \sim 3$, in that (with $N = 0$) it predicts a critical n for stability between 3 and 4, in agreement with PEST⁶ applied to the same equilibrium (not the same one as in Fig. 1). In contrast, the localized "1/n correction" formula

gives a critical $n \sim 8$. The discrepancy arises from the highly nonelliptical character of the $\lambda = 0$ contour.

The case of passing modes, whose ray trajectories lie outside the separatrix in Fig. 1, is rather more novel, since $\hat{\xi} \exp(-inS)$ is an infinitely many valued function. Application of the jump conditions, Eq. (7), and equilibrium periodicity, allows us to write down the solution in terms of the two branches $\hat{\xi}_{\pm}$ in the "unit cell" - $\pi < k < \pi$:

$$\xi = \exp(-i\omega t - in\alpha) \sum_{\pm} \int_{-\infty}^{\infty} \exp[-in(\ell\Delta S + S_{\pm})] \hat{\xi}_{\pm}(q, \theta - 2\pi\ell\Delta k) \quad (9)$$

where $\Delta S(q)$ is the increment in S as s increases through one cycle; i.e., as $(q, k_{\pm})$ goes to $(q, k_{\pm} + \Delta k)$, where $\Delta k = 2\pi \operatorname{sgn}(\dot{k})$. The Maslov indices sum to zero, as is typical of rotating motions,⁹ again showing that k is topologically an angle.

The solutions (9) on the covering space form a continuum, but imposing periodicity yields the quantization condition

$$n[2\pi q - \operatorname{sgn}(\dot{k})\Delta S] = 2\pi N, \quad (10)$$

where N is an integer. The left hand side can be visualized geometrically as nA , where A is the area of the region in the $q - k$ plane bounded by the $\lambda = \omega^2$ contours, the horizontal lines $k = \pm\pi$,

and the vertical line $q = 0$ (which, except in reversed field geometries, is not a physical value of q).

For moderate values of n , contours which come close to the X point can couple by tunneling, leading to a more complicated quantization condition which reduces to Eqs. (8) and (10) in the appropriate limits.

The WKB approach can readily be generalized to nonaxisymmetric equilibria, the two-dimensional $q - k$ phase space being replaced by a four-dimensional $(\iota, q) - (k_\parallel, k_q)$ phase space. As long as the ray trajectories are not stochastic, Einstein-Brillouin-Keller quantization⁹ can then be applied. The approach has also proved fruitful in drift wave theory.^{10,11} Since the WKB method appears to give reasonably accurate results at quite modest n , it is expected that comparison with PEST-II⁷ results will play an important part in code validation. The ballooning analysis also provides insight into numerical questions, such as the optimal G coordinate to use. Further details and extensions of the theory will be reported elsewhere.

ACKNOWLEDGMENTS

This work was supported by United States Department of Energy Contract No. EY-76-C-02-3073.

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Fig. 1. Contours of constant λ .