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Fourier-Motzkin Elimination for Mixed Systems

G. E. Liepins

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FOURIER-MOTZKIN ELIMINATION FOR MIXED SYSTEMS

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CONTENTS

ABSTRACT

A simple extension of Fourier-Motzkin elimination is made to mixed systems of equations, that is, systems consisting of equalities in conjunction with inequalities and strict inequalities. The principal observation is that inequalities combined with strict inequalities result in strict inequalities.

Two applications are made to automatic data editing. First, a constructive method is provided to test for the existence of a linear objective function for the minimum weighted fields to impute (MWFI) problem with side constraints. If the linear objective function exists, it is determined; if it does not exist, the extension to a quadratic objective function is given. Next, for any fixed linear objective function, a solution algorithm based on extended Fourier-Motzkin elimination is given for the resultant MWFI and is illustrated with an example.

It is believed that the applications are significant in their own right: they provide solution techniques to difficult problems in the field of automatic data editing.

INTRODUCTION

Mixed systems of equalities in conjunction with inequalities and strict inequalities are not the systems traditionally encountered in mathematical programming: they have not been systematically investigated, and they possess few practical solution procedures (Stoer and Witzgall, 1970). One setting where such systems do arise naturally, however, is in the selection of an objective function to a mathematical programming problem. This type of problem is expected frequently in the error localization stage of automatic data editing (Liepins, Garfinkel, and Kunnathur, 1981). Formally, given an acceptance region (often determined by a system of linear inequalities) and a data record $x = (x_1, \ldots, x_n)$ which lies outside the region, find a linear objective function C( ) such that any solution to the subsequent minimum weighted fields to impute (MWFI) problem (1)-(3) is consistent with side constraints. (By abuse of language any such function will be called consistent with the side constraints.)

I. MWFI with objective function C( ): Let M be an m x n matrix and assume that for the vector $y^{(0)}$, $My^{(0)} \not\leq b$. (In automatic data editing, such a vector $y^{(0)}$ is said to be inconsistent with the constraints $My \leq b$.) Find the index set s which minimizes

$$C(s) = \sum_{i \in s} c_i \tag{1}$$

subject to

$$M[y^{(0)} + \varepsilon] \leq b \ , \tag{2}$$

$$\varepsilon_i \neq 0 \text{ if and only if } i \in s \ . \tag{3}$$

II. Consistency with side constraints: Let a partial preference ordering on the collection S of all 2^n subsets of the indices 1, ..., n be given in terms of indifference I, preference P, and strict preference SP. Then an objective function C( ) is consistent with side contraints if and only if whenever s_j and s_k are feasible solutions to (2)-(3), then

$$\text{if } s_j \text{ I } s_k \text{ then } C(s_j) = C(s_k) \ , \tag{4}$$

$$\text{if } s_j \text{ P } s_k \text{ then } C(s_j) \leq C(s_k) \ , \tag{5}$$

$$\text{if } s_j \text{ SP } s_k \text{ then } C(S_j) < C(s_k) \ . \tag{6}$$

[Clearly, no linear objective function C( ) with c_i nonnegative can be consistent with side constraints if there is a pair of index sets s_j and s_k satisfying $s_k \subset s_j$ and simultaneously s_j SP s_k.]

In the setting of automatic data editing, the side constraints are derived from partial information about the error process. For example, given that a data record $y^{(o)}$ is inconsistent, that is, $My^{(o)} \nleq b$, it might be known that, on the average, the third component is more likely to be wrong than the first two jointly. In terms of MWFI (1)-(3), this would require that the objective function $C(\ ) = (c_1, \ldots, c_n)$ be chosen so that whenever both $M[y^{(o)} + \varepsilon^{(1)}] \leq b$ and $M[y^{(o)} + \varepsilon^{(2)}] \leq b$ (where $\varepsilon^{(1)}_i \neq 0$ if and only if i = 1 or 2, and $\varepsilon^{(2)}_i \neq 0$ if and only if i = 3), then $c_3 < c_1 + c_2$.

Rather than solve the problem of determining the most general objective function for a MWFI consistent with side constraints, consider a simpler variant. If it is required that the implications (4)-(6) hold regardless of feasibility of solution, then the side constraints impose mixed partial ordering on the solutions. Thus, the actual problem undertaken is to find all linear objective functions which induce the specified mixed partial ordering.

The solution presented here is a variant of Fourier-Motzkin elimination and is a simple extension of results published by Duffin (1974) and others. The prescribed technique not only addresses the question of consistency but in the affirmative case allows all solutions to be determined and in the negative case allows higher order polynomial objective functions to be investigated. Moreover, what should be observed is that Fourier-Motzkin elimination is a completely general technique applicable to any system regardless of its origin.

EQUIVALENCE OF MIXED PARTIAL ORDERING AND MIXED SYSTEMS

Any mixed partial ordering is equivalent to a mixed system determined by differences of successive terms. For example, the mixed partial ordering (7) is equivalent to the system (8)-(11) formed by successive first differences. (Note, for example, that if $s_j = \{1\}$ and $s_k = \{1, 2\}$ then s_j SP s_k may be written as $c_1 < c_1 + c_2$ and similarly for other relations.)

Mixed partial ordering:

$$c_1 < c_1 + c_2 \leq c_3 = c_1 + c_3 < c_2 + c_3 \ . \tag{7}$$

Mixed system:

$$-c_2 < 0 \ , \tag{8}$$

$$c_1 + c_2 - c_3 \leq 0 \ , \tag{9}$$

$$-c_1 = 0 \ , \tag{10}$$

$$c_1 - c_2 < 0 \ . \tag{11}$$

It is easy to characterize the matrix of coefficients of a mixed system derived from a mixed partial ordering. This is done in the Appendix.

FOURIER-MOTZKIN ELIMINATION

Fourier-Motzkin elimination is described in various sources, for example Kohler (1967), so only the briefest details are presented here. Succinctly, for any system of inequalities,

$$Ax + By \geq d \tag{12}$$

with solution set $\{(x, y)\}$, Fourier-Motzkin elimination allows the determination of a set Y such that if (x, y) is a solution of equation (12), then $y \in Y$ and conversely, if $y \in Y$, there exists an element x such that the pair (x, y) satisfies (12). From the dual perspective, the solution involves finding all the extremal rays of the convex cone

$$wA = 0, \ w \geq 0 \ . \tag{13}$$

[In fact, the dual to the Fourier-Motzkin problem is the Chernikova problem. See either Abadie (1964) or Duffin (1974).]

Given a system of inequalities

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{m1} & \cdots & & a_{mn} \end{vmatrix} \begin{vmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{vmatrix} \geq \begin{vmatrix} d_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_m \end{vmatrix} , \tag{14}$$

the variable x_1 is eliminated as follows: Partition the set of row indices

$I^+ = \{i \mid a_{i1} > 0\}$,

$I^- = \{i \mid a_{i1} < 0\}$,

$I^0 = \{i \mid a_{i1} = 0\}$.

Case A. I^+ or I^- or both are empty. Drop the rows not indexed by I^0 and continue with x_2.

Case B. For every pair (s, t), such that $s \in I^+$ and $t \in I^-$, append the row $|a_{t1}| \cdot row_s + |a_{s1}| \cdot row_t$. When all such pairs of rows have been combined, drop the rows indexed by I^+ and I^- and continue with x_2.

Several points are worthy of observation:

(a) The processing of the inequalities at stage i can be represented as left multiplication by a matrix M_i with nonnegative entries. The cumulative processing can be represented as left multiplication by the product of the respective matrices, $M = M_i M_{i-1} \ldots M_1$.

(b) At any stage, if a value for y can be determined, a corresponding value for x can be determined by stepwise back substitution.

(c) If at any stage the matrix M of step (a) has two rows r and t satisfying $m_{r,j} = 0 \Rightarrow m_{t,j} = 0$, then row r can be dropped from M.

(d) If at any stage a row of the matrix M of (a) has more positive entries than the numbers of variables actively eliminated plus one, then the row can be dropped from M.

The rules regarding the omission of columns of M are more fully developed in Duffin (1974) as well as in expositions of the Chernikova algorithm. See, for example, Rubin (1977) or Liepins (1983).

MODIFICATIONS TO FOURIER-MOTZKIN ELIMINATION

Fourier-Motzkin elimination can be modified to solve mixed systems with or without nonnegativity constraints. The solution proceeds by eliminating one variable at a time until only one variable is left, in which case feasibility can be determined by inspection. If feasible, solutions can be constructed by backward substitution into the previous mixed systems.

The problem addressed is to determine the feasibility and solve a mixed system (15), with perhaps the additional constraints of non-negativity (16).

$$\begin{vmatrix} A_1 \\ A_2 \\ A_3 \end{vmatrix} x \begin{matrix} < \\ \leq \\ = \end{matrix} \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix} , \qquad (15)$$

$$x \geq 0 . \qquad (16)$$

Any such inhomogeneous problem can be transformed into a homogeneous problem by the enlargement of the original matrix by inclusion of the negative of the constant vector as the last column, as in (17)

$$\begin{vmatrix} A_1 & - b_1 \\ A_2 & - b_2 \\ A_3 & - b_3 \end{vmatrix} \begin{vmatrix} x \\ \xi \end{vmatrix} \begin{matrix} < \\ \leq \\ = \end{matrix} \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} , \ \xi \geq 0 , \ x \geq 0. \qquad (17)$$

Any solution of (17) with $\xi = 1$ is a solution of (15). Hence, only the homogeneous case will be dealt with and will be called the restricted (unrestricted) problem if nonnegativity constraints are imposed (variables unconstrained). The initial step of the homogeneous restricted problem requires that the negative identity be appended to the mixed system, to result in the mixed system (18).

$$\begin{vmatrix} A_1 \\ A_2 \\ A_3 \\ -I \end{vmatrix} x \begin{matrix} < \\ \leq \\ = \\ \leq \end{matrix} \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} . \tag{18}$$

Step 1. Solve the system $A_3x = 0$ by Gaussian elimination. If no solution exists, the original system is infeasible. Otherwise substitute into the remaining equations.

Step 2. Save the Gaussian reduced form of A_3, and remove it from the mixed system.

Step 3. Continue processing the remaining columns according to Fourier-Motzkin elimination, with the modification that any inequality combined with a strict inequality yields a strict inequality.

Proof: Straightforward.

Example 1

Mixed partial ordering

$$c_1 < c_2 \leq c_3 < c_2 + c_4 \leq c_3 + c_5 = c_1 + c_2, \; c_i \geq 0 . \tag{19}$$

Mixed system

$$c_1 - c_2 < 0 ,$$

$$c_2 - c_3 \leq 0 ,$$

$$-c_2 + c_3 - c_4 < 0 ,$$

$$c_2 - c_3 + c_4 - c_5 \le 0 \ ,$$

$$-c_1 - c_2 + c_3 + c_5 = 0 \ , \quad c_i \ge 0 \ .$$

Gaussian elimination: $c_1 = -c_2 + c_3 + c_5$.

row	A_1						A_2					A_3				A_4*	
1	-2	1	0	1	$<$	(1,2)	-1	0	1	$<$	(3)	-1	0	$<$	(2)	-1	$<$
2	1	-1	0	0	$\le$	(1,4)	-1	2	-1	$<$	(4)	0	-1	$<$	(4)	-1	$\le$**
3	-1	1	-1	0	$<$	(2,3)	0	-1	0	$<$	(8)	-1	0	$\le$**			
4	1	-1	1	-1	$\le$	(3,4)	0	0	-1	$<$	(9)	0	-1	$\le$**			
5	-1	0	0	0	$\le$	(2,5)	-1	0	0	$\le$**							
6	0	-1	0	0	$\le$	(4,5)	-1	1	-1	$\le$							
7	0	0	-1	0	$\le$	(6)	-1	0	0	$\le$							
8	0	0	0	-1	$\le$	(7)	0	-1	0	$\le$**							
9						(8)	0	0	-1	$\le$**							

Backward substitution: Use A_4 and choose c_5, say $c_5 = 1$. Use A_3 and choose c_4, say $c_4 = 1$. Use rows 1, 2, 6, and 7 of A_2 with $c_4 = c_5 = 1$ to choose c_3:

$$\left.\begin{array}{l} -c_3 < -1 \\ -c_3 < -2 + 1 \\ -c_3 \le -1 + 1 \\ -c_3 \le 0 \end{array}\right\} \text{Say } c_3 = 3 \ .$$

*The inequalities and strict inequalities indicate which of the relations is to hold for the row in question. The numbers in parentheses preceding rows of the respective matrices indicate which rows of the previous matrix were combined.

**Redundant constraints. May be dropped from further processing.

Use rows 1-5 of A_1 with $c_3 = 3$, $c_4 = c_5 = 1$ to choose c_2:

$$\left.\begin{array}{r} -2c_2 < -3 - 1 \\ c_2 \leq 3 \\ -c_2 < -2 \\ c_2 \leq 3 \\ -c_2 \leq 0 \end{array}\right\} \text{Say } c_2 = 3 \text{ .}$$

Set $c_1 = -3 + 3 + 1 = 1$.

Check: $1 < 3 \leq 3 < 4 \leq 4 = 4$.

If the original mixed partial ordering had $c_1 + c_2 < c_4 + c_5$ (20)
appended to it, the matrices A_1-A_4 would have the following additional rows $\tilde{A}_1$-$\tilde{A}_4$ (respectively) appended to them.

$\tilde{A}_1$					$\tilde{A}_2$				$\tilde{A}_3$			$\tilde{A}_4$	
0	1	-1	0	< (9)	1	-1	0	< (1,10)	-1	1	< (1,6)	-1	<
								(2,10)	1	-1	< (3,6)	-1	<
								(5,10)	-1	0	< (5,6)	0	<
								(6,10)	0	-1	< (7,6)	-1	<
								(7,10)	-1	0	< (9,6)	-1	<
											(8)	-1	<

It is already clear from the first two rows of $\tilde{A}_3$ that no solution exists for this enlarged system.

The process can be summarized by

$$M_{123} \begin{vmatrix} A_1 \\ \\ \tilde{A}_1 \end{vmatrix} = \begin{vmatrix} 0 & A_4 \\ & \\ 0 & \tilde{A}_4 \end{vmatrix} ,$$

where M_{123} is the 8 x 9 matrix below in (21), for which missing entries are all "0."

M_{123} = (21)

1	2	3	4	5	6	7	8	9	row
		1	1						1
							1		2
1	1	1	2					1	3
1			2			1		1	4
2	2		2					2	5
1	1		2	1				2	6
1			2		1			2	7
			1	1				1	8

It should be noted that only the extremal rows of the matrix M are necessary to determine the feasibility of the mixed system. Non-extremal rows can be identified by (c) and (d) parts of Case B following the Fourier-Motzkin algorithm (see p. 5). Hence, rows 3 and 6 can be dropped from the matrix M_{123} without affecting the solution or feasibility of the system.

EXTENSION TO QUADRATIC PARTIAL ORDERING

In this section, x will be an m-dimensional binary vector, that is, a vector such that $x^t = (x_1, \ldots, x_m)$ with $x_i = 0$ or 1 for each i, i = 1, ..., m, and R will be a preference ordering, either "<", "$\leq$", or "=". Specifically, given a sequence of binary vectors $x^{(i)}$, i = 1, ..., n; and preference orderings R_i, i = 1, ..., n - 1; the linear partial ordering problem can be viewed as finding a fixed (nonnegative) m-dimensional vector c which satisfies $c^t x^{(i)}\ R_i\ c^t x^{(i+1)}$ for i = 1, ..., n - 1.

A quadratic partial ordering problem can be defined similarly: Find a symmetric matrix C such that $x^{(i)t} C x^{(i)}\ R_i\ x^{(i+1)t} C x^{i+1}$. Now let $s_i = \{j | x^{(i)}_j = 1\}$. In the linear case, necessary and sufficient conditions for $s_i \subset s_j \Rightarrow c^t x^{(i)} \leq c^t x^{(j)}$ are that the vector c be nonnegative. For the quadratic case, the necessary and sufficient conditions for $s_i \subset s_j \Rightarrow x^{(i)t} C x^{(i)} \leq x^{(j)t} C x^{(j)}$ are somewhat more elaborate and are stated in (22) below: For s any subset of the indices 1, ..., m, and $i \in s$,

$$c_{ii} + \sum_{\substack{j \in s \\ j \neq i}} (c_{ij} + c_{ji}) \geq 0 . \qquad (22)$$

Clearly, sufficient conditions are that all the $c_{ij} \geq 0$.

Example 2

Although conditions (22) could be incorporated into a Fourier-Motzkin tableau, for purposes of this example, the partial quadratic ordering problem suggested by the partial linear ordering (19) and (20) will be considered with the additional constraints that $c_{ij} \geq 0$. Note that the required partial quadratic ordering can be represented as (23).

$$\begin{aligned} & c_1 < c_2 \leq c_3 < c_2 + c_{24} + c_4 \leq c_3 + c_{35} + c_5 = \\ & \quad c_1 + c_{12} + c_2 < c_4 + c_{45} - c_5 , \\ & c_{ij} \geq 0,\ c_i \geq 0 . \end{aligned} \qquad (23)$$

Moreover, to illustrate how the solution procedure can be relatively easily updated when no equalities are present and the degree of the partial ordering is increased (in this case, from linear to quadratic), consider the additional constaints $c_{35} = c_{12} = 0$. The addition of these variables and constraints leads to the initial system represented by the matrix $\tilde{\tilde{A}}_1$, given in (24).

row		c_2	c_3	c_4	c_5	c_{24}	c_{45}		
1		-2	1		1			<	
2		1	-1					≤	
3		-1	1	-1		-1		<	
4		1	-1	1	-1	1		≤	(24)
5		-1						≤	
6	$\tilde{\tilde{A}}_1 =$		-1					≤	
7				-1				≤	
8					-1			≤	
9			1	-1			-1	<	
10						-1		≤	Nonnegativity of
11							-1	≤	c_{24} and c_{25}

Rather than process $\tilde{\tilde{A}}_1$ anew, the matrix M_{123} given in (21) can be used. Drop nonextremal rows 3 and 6 and augment the matrix with two columns of zeros (columns 10 and 11) and two rows which are zero except in columns 10 and 11 where they form the 2 x 2 identity matrix (rows 9 and 10). Call this matrix $\tilde{\tilde{M}}_{123}$. Then $\tilde{\tilde{A}}_4 = \tilde{\tilde{M}}_{123}\tilde{\tilde{A}}_1$ represents the new system with the first three columns processed. This processing can be more easily affected. Rewrite $\begin{vmatrix} A_4 \\ \tilde{A}_4 \\ 0 \\ 0 \end{vmatrix}$ with the third and sixth entries omitted as the first column of $\tilde{\tilde{A}}_4$ and append the product of $\tilde{\tilde{M}}_{123}$ with the last two columns of $\tilde{\tilde{A}}_1$. The result $\tilde{\tilde{A}}_4$ for this example is given in (25).

$$\tilde{\tilde{A}}_4 = \begin{vmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 2 & -2 \\ -1 & 2 & -2 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \begin{matrix} < \\ \leq \\ < \\ < \\ < \\ < \\ \leq \\ \leq \end{matrix} \qquad (25)$$

A solution to the quadratic partial ordering can be seen to exist: say $c_{24} = 0$ and $c_{45} = 3$, $c_5 = 1$, $c_4 = 1$, $c_3 = 3$, $c_2 = 3$, and $c_1 = 1$. Check: $1 < 3 \leq 3 < 4 \leq 4 = 4 < 5$. (To formally backsubstitute to a solution, the tableaux $\tilde{\tilde{A}}_1$-$\tilde{\tilde{A}}_6$ are required. However, $\tilde{\tilde{A}}_2$ and $\tilde{\tilde{A}}_3$ can be generated with use of M_{12} much as A_4 was generated.) In terms of the quadratic partial ordering, a required symmetric matrix becomes (26), for which missing entires are all "0."

$$C = \begin{vmatrix} 1 & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 1 & 3/2 \\ & & & 3/2 & 1 \end{vmatrix} \qquad (26)$$

CONNECTIONS BETWEEN QUADRATIC PARTIAL ORDERING AND MWFI

It is clear that the determination of a symmetric matrix consistent with side constraints to a quadratic MWFI problem (27)-(30) leads to a quadratic partial ordering problem.

Let M be an m x n matrix and assume that for the vector $y^{(o)}$, $My^{(o)} \nleq b$.

Find the index set s which minimizes

$$\delta(\varepsilon)^t C\delta(\varepsilon) \tag{27}$$

subject to

$$M[y^{(o)} + \varepsilon] \leq b\ , \tag{28}$$

$$\delta(\varepsilon)^t = [\delta(\varepsilon_i),\ \ldots,\ \delta(\varepsilon_n)]\ , \tag{29a}$$

$$\delta(\varepsilon_i) = 1 \iff \varepsilon_i \neq 0\ , \tag{29b}$$

$$\delta(\varepsilon_i) = 0 \iff \varepsilon_i = 0\ . \tag{29c}$$

$$\varepsilon_i \neq 0 \iff i \in s. \tag{30}$$

FOURIER-MOTZKIN ELIMINATION AS A SOLUTION TECHNIQUE FOR MWFI

Given a MWFI with fixed objective function C( ), Fourier-Motzkin elimination can be substantially used to determine the solution. Moreover, it is expected that this approach is highly competitive with other solution algorithms.*

---

*The original version of this algorithm uses linear programming to test for feasibility of solutions. At present, it is unknown how the two variants compare, although there is a potential for generation of excessively many columns in Fourier-Motzkin elimination [see Kohler (1967) and Matheiss and Rubin (1980)].

Recall the MWFI problem: Let M be an m x n matrix, b a fixed n-dimensional vector, and $y^{(o)}$ a vector satisfying $My^{(o)} \not\leq b$. Find the index set s which minimizes

$$\sum_{i \in s} c_i \qquad (1)$$

subject to

$$M[y^{(o)} + \varepsilon] \leq b , \qquad (2)$$

$$\varepsilon_i \neq 0 \text{ if and only if } i \in s . \qquad (3)$$

FOURIER-MOTZKIN SOLUTION TO MWFI*

1. Let I index the equations for which $[My^{(o)}]_i > b_i$ for $i \in I$. Let the cardinality $|I|$ of I equal k. Construct the k x n binary failed edit matrix A according to $a_{ij} = 1 \iff m_{ij} \neq 0$, $a_{ij} = 0 \iff m_{ij} = 0$ (for $i \in I$).
2. Find an optimal solution w^* to the set covering problem that minimizes cw, subject to $Aw \geq 1$, w binary. Let s be the set of indices $\{i \mid w_i^* \neq 0\}$. Denote the cardinality of s by ℓ .
3. Let M_s be the m x ℓ submatrix of M determined by the columns of M indexed by s in step 2 above. Solve

$$x^t M_s = 0 , \qquad (31)$$

$$x^t[b-My^{(o)}] < 0 , \qquad (32)$$

$$x \geq 0 . \qquad (33)$$

---

*This algorithm is substantially motivated by an unpublished algorithm due to A. S. Kunnathur. A formal proof of the algorithms along with test results will appear in another paper. It should be noted that in this use, Fourier-Motzkin elimination involves rows rather than columns.

4. If no solution to (31)-(33) exists, then the set s identified in step 2 is the required solution to MWFI (1)-(3). Otherwise, let x^* be a solution and go to step 5.
5. Form $x^{*t}M$ and $x^{*t}b$ and append these as the last row and component to the matrix M and vector b, respectively. Return to step 1.

Example 3

Let

$$y^{(o)t} = (1, 2, 1, 0, 1),\ b^t = (5, 4, 0),\ c^t = (1, 1, 1, 1, 1)\ , \tag{34}$$

and let

$$M = \begin{pmatrix} 1 & 3 & -3 & 1 & 2 \\ -2 & 1 & 1 & 0 & 3 \\ 1 & -2 & 1 & 2 & 0 \end{pmatrix} .$$

Thus

$$[b - My^{(o)}]^t = (-1, 0, 2)\ . \tag{35}$$

1. y^o fails the first constraint: $1\cdot1 + 3\cdot2 - 3\cdot1 + 1\cdot0 + 1\cdot1 = 6 > 5$.
2. The failed edit matrix A is (1 1 1 1 1), and a prime cover to $Aw \geq 1$ is $w_1^* = 1$, $w_i^* = 0$ for $i \neq 1$.
3. $M_s = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and the system to be solved becomes (36)

$$(x_1, x_2, x_3) \begin{pmatrix} 1 & -1 & -1 & 0 & 0 \\ -2 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ \Vert & \wedge & \wedge\!| & \wedge\!| & \wedge\!| \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{37}$$

Gaussian elimination can be done within the matrix format* to result in the matrix

$$\left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ -2 & -2 & -2 & -1 & 0 \\ 1 & 3 & 1 & 0 & -1 \\ = & > & \geq & \geq & \geq \end{array}\right). \tag{38}$$

Fourier-Motzkin elimination of the variable x_2 yields

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ \geq \end{pmatrix}. \tag{39}$$

4-5. A solution x^{*t} is (2, 1, 0); consequently, the row (0 7 -5 2 7) is appended to M, the constant 14 is appended to b, and the corresponding entry of the extended vector $b - My^{(o)}$ is $(2, 1, 0) \cdot (-1, 0, 2) = -2$.

2. The failed edit matrix A is $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$ and a prime cover is $w_3^* = 1$, $w_i^* = 0$ for $i \neq 3$.

3. $M_s = \begin{pmatrix} -3 \\ 1 \\ 1 \\ -5 \end{pmatrix}$ and the system to be solved becomes (40)

$$(x_1, x_2, x_3, x_4)\begin{pmatrix} -3 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ -5 & -2 & 0 & 0 & 0 & -1 \\ = & > & \geq & \geq & \geq & \geq \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{41}$$

---

*Without a loss of generality, assume that the first column of the matrix M corresponds to an equality, and that $m_{k1} \neq 0$. Then Gaussian elimination of x_k is accomplished by the replacement of m_{ij} by $m_{ij} - \frac{m_{i1}}{m_{k1}} m_{kj}$ for $j > 1$.

Gaussian elimination of the variable x_2, and subsequent elimination of the variable x_1, results in the system

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} \end{pmatrix}. \qquad (42)$$

4-5. A solution x^{*t} is (1, 3, 0, 0) and consequently the row (-5 6 0 1 11) is appended to M and the constant 17 is appended to b.

2. A prime cover to the failed edit matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$ is $w_2^* = 1$, $w_i^* = 0$ for $i \neq 2$.

3. $M_s = \begin{pmatrix} 3 \\ 1 \\ -2 \\ 7 \\ 6 \end{pmatrix}$ and the system to be solved becomes (43)

$$(x_1, x_2, x_3, x_4, x_5) \begin{pmatrix} 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 7 & -2 & 0 & 0 & 0 & -1 & 0 \\ 6 & -1 & 0 & 0 & 0 & 0 & -1 \\ \rotatebox{90}{$=$} & \rotatebox{90}{$>$} & \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (44)$$

Gaussian elimination of the variable x_2 results in the system

$$\left(\begin{array}{c|cccccc} 3 & -1 & -1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & -2 & -1 & 0 & 0 \\ 7 & -2 & 0 & 7 & 0 & -1 & 0 \\ 6 & -1 & 0 & 6 & 0 & 0 & -1 \\ \rotatebox{90}{$=$} & \rotatebox{90}{$>$} & \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} & \rotatebox{90}{$\geq$} \end{array}\right). \qquad (45)$$

Elimination of the variable x_1 results in the system

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & -2 & -1 & 0 & 0 \\ 1 & 7 & 0 & -1 & 0 \\ 3 & 6 & 0 & 0 & -1 \\ \wedge & \wedge & \wedge| & \wedge| & \wedge| \end{pmatrix}. \qquad (46)$$

Since the first column is nonnegative, no solution to the system (47) exists, and hence a solution to the MWFI (35)-(36) is $s = \{2\}$.

CONCLUSIONS

The relationship between the MWFI problem with side constraints and the mixed partial ordering problem has been shown. The coefficient matrix of a mixed system derived from a mixed partial ordering has been characterized. It has been shown how Fourier-Motzkin elimination can be modified to solve mixed systems. Application has been made to the determination of an objective function for MWFI with side constraints. For a fixed linear objective function, a solution algorithm for the resultant MWFI has been presented and illustrated.

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APPENDIX

For $i = 1, \ldots, k - 1$, let each R_i be one of the relations I (indifference or equality), P (preference or inequality), or SP (strong preference or strict inequality). Let $\{s_i\}$ be a distinct collection of the $2^n - 1$ nonempty subsets of $\{1, \ldots, n\}$. Represent the relations I, P, and SP as follows:

$$s_j \; I \; s_k \iff \sum_{i \in s_j} c_i = \sum_{i \in s_k} c_i ,$$

$$s_j \; P \; s_k \iff \sum_{i \in s_j} c_i \leq \sum_{i \in s_k} c_i ,$$

$$s_j \; SP \; s_j \iff \sum_{i \in s_j} c_i < \sum_{i \in s_k} c_i .$$

Consider a mixed partial ordering $s_1 R_1 s_2 \ldots R_{k-1} s_k$ satisfying $s_{j+h} \not\subset s_j$. Define the $(k - 1) \times n$ matrix of the mixed system derived from the partial ordering by

$$a_{j\ell} = 1 \iff \ell \in s_j \text{ and } \ell \notin s_{j+1} ,$$

$$a_{j\ell} = -1 \iff \ell \notin s_j \text{ and } \ell \in s_{j+1} ,$$

$$a_{j\ell} = 0 \iff \text{either } (\ell \in s_j \text{ and } \ell \in s_{j+1}) \text{ or } (\ell \notin s_j \text{ and } \ell \notin s_{j+1}) .$$

(The matrix is determined as the matrix of indicators of first differences of the direct sums

$$\bigoplus \sum_{i \in s_j} c_i - \bigoplus \sum_{i \in s_{j+1}} c_i .)$$

Then an arbitrary (k - 1) x n matrix A is the matrix of coefficients for a mixed system $s_1R_1s_2 \ldots R_{k-1}s_k$ if and only if

i. All matrix elements are either -1, 0, or 1 .

ii. For any column j and any sum of consecutive rows, the sum

$$\sum_{i=k}^{k+\ell} a_{ij} = -1, 0, \text{ or } 1 .$$

iii. For any consecutive sequence of rows k, ..., k + ℓ , at least one column j exists such that

$$\sum_{i=k}^{k+\ell} a_{ij} = -1 .$$

iv. For any consecutive sequence of rows k, ..., k + ℓ at least one column j exists such that

$$\sum_{i=k}^{k+\ell} a_{ij} \neq -1 .$$

<u>Proof</u>: The "only if" portion is straightforward. For the "if" portion, set $V^{(i)} = [\delta_1(s_i), \ldots, \delta_n(s_i)]$ where $\delta_j(s_i) = 1 \Leftrightarrow j \in s_i$.

Set $\Delta^{(i)} = V^{(i)} - V^{(i+1)}$. It is clear that $V^{(j)} = -\sum_{i=1}^{j-1} \Delta^{(i)} + V^{(1)}$ and that the mixed partial ordering can be reconstructed from the mixed system so long as $V^{(1)}$ can be reconstructed. Moreover, $\Delta^{(1)} = [\delta_1(s_1) - \delta_1(s_2), \ldots, \delta_n(s_1) - \delta_n(s_2)]$. Hence $\Delta_i^{(1)} = 1 \Leftrightarrow \delta_i(s_1) = 1$ and $\delta_i(s_2) = 0$, $\Delta_i^{(1)} = -1 \Leftrightarrow \delta_i(s_1) = 0$ and $\delta_i(s_2) = 1$, $\Delta_i^{(1)} = 0 \Leftrightarrow [\delta_i(s_1) = 1$ and $\delta_i(s_2) = 1]$ or $[\delta_i(s_1) = 0$ and $\delta_i(s_2) = 0]$. So long as there exists some j such that $\Delta_i^{(j)} \neq 0$, the latter two cases are distinguishable. In the case that $\Delta_i^{(j)} = 0$ for j = 1, ..., k - 1, set $\delta_i(s_j) = 1$ for j = 1, ..., k - 1.

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