DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Magnetic fluctuations CAN contribute to plasma transport, "self-consistency constraints" notwithstanding

JOHN A. KROMMES

PPPL--2473

CHANG-BAE KIM

DE88 003458

Princeton University, Plasma Physics Laboratory

P.O. Box 451, Princeton, NJ 08544-0451

ABSTRACT

The recent conclusion of Terry, Diamond, and Hahm (TDH) [Phys. Rev. Lett. **57**, 1899 (1986)] that in a turbulent, collisionless plasma "magnetic transport including quasilinear magnetic flutter transport ... does not contribute to the relaxation of $\langle f \rangle$, and thus is not responsible for electron energy or momentum transport" is shown to be incorrect for a variety of situations of physical interest, including saturation by quasilinear plateau formation, induced scattering, and, most importantly, conventional mode coupling. The well-established theory of the mean infinitesimal response function and the spectral balance equation provides a unifying framework for understanding the work of TDH. In particular, the cancellations which lead to their conclusion are special cases of well-known relationships between the response function, particle propagator, and dielectric function. A more general, concise, and manifestly gauge-invariant algebraic derivation of the cancellations is given. Though the cancellations occur in a certain limit, the conclusions of TDH do not follow in general: Their picture of steady-state turbulence as consisting of small-scale "incoherent" ballistic "clumps" shielded by long-wavelength "coherent" dielectric response is physically misleading and mathematically incomplete, as it ignores or mistreats the often dominant process of renormalized n -wave coupling. Thus, when ion nonlinearities are considered, formulas for the magnetic contribution to transport emerge which are quite similar to the quasilinear one. Furthermore, limits are possible in which all or part of the noise can be negligible, yet in which the total fluctuation spectrum remains finite.

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

1. INTRODUCTION

Recently¹ Terry, Diamond, and Hahm (TDH) generalized earlier work² on electrostatic "clump"-driven turbulence in fluids and plasmas to include low frequency electromagnetic effects. They concluded that in fully developed, collisionless, low frequency, magnetized plasma turbulence "magnetic transport including quasilinear magnetic flutter transport ... does not contribute to the relaxation of $\langle f \rangle$, and thus is not responsible for electron energy or momentum transport!" This conclusion is remarkable: If true, it requires profound revisions of conventional theory and re-interpretations of experimental data, and the derivation substantially vitiates mixing-length estimates of the form³ $D \sim \bar{\gamma}/k_{\perp}^2$. Our goals in this paper are to provide additional insights to the calculations of TDH and to understand the relationships of their work to previously published research on renormalized turbulence theory. We shall argue that their result is far from a general conclusion: Because they have misinterpreted the role of "incoherent noise," they appear to have overlooked important or dominant terms in the theory for many, if not all, situations of practical interest. Our analysis upholds the general philosophy of the usual random walk estimate for thermal transport, at least in simple cases, although it is certainly not true in general that the electron transport is determined literally from the usual quasilinear expression. That is, the scaling of the usual expression may be correct, although the numerical coefficient may need to be computed from a fully nonlinear theory. We will explain this remark more completely as we proceed.

One feature of the result of TDH is particularly noteworthy: it applies to the relaxation of the entire distribution function, not just to a particular moment such as particle or energy transport. This raises issues which are more primitive and more general than the question of magnetic transport. Namely, in a wide class of (gyrokinetic) models appropriate for the low frequency, magnetized limit in question, particle transport is intrinsically ambipolar. However, it seems very unlikely that this important property is maintained by the approximations of TDH. Those authors, attempting to compute the electron flux and thus working in the limit $\omega \ll k_{\parallel} v_{te}$, studied certain nonlinear contributions due to resonant electrons and argued that certain important cancellations occur. Had they instead studied the ions—which would have been, by ambipolarity, equally correct as far as the particle transport is concerned—such electron effects would not have been in evidence and it seems inevitable that, because of the ion limit $\omega \gg k_{\parallel} v_{ti}$, they would have had to study instead conventional mode-coupling effects and would have arrived, we believe, at a quite different conclusion. Such an argument suggests that TDH overlooked important terms in their electron calculation. As we proceed, we shall explain where and how this occurred.

The fundamental physical point which we wish to make in this paper is that a careful analysis of the theory of plasma turbulence at the quite general level at which TDH apply it does not lead one to their conclusion that the self-consistent magnetic contribution to energy transport vanishes or is necessarily very small. This leaves open the question of what the true level actually is. We do not attempt a quantitative calculation here, for several reasons. First, TDH did not, and we wish to focus the discussion on the specific issues which

those authors raised. Second, it seems useful to separate the logical foundations of the theory from their practical implementation; this paper is concerned only with clarifying the foundations. Third, while the conceptual aspect of the theory is quite generally applicable, detailed results for transport coefficients are problem-specific. Thus, it is entirely possible that in some particular applications the magnetic contribution to thermal transport may be small. All we say here is that the logic of TDH does not support such a conclusion; if it holds, a much more detailed and quantitative analysis must be performed to verify it. Very little such work has been done, even within the context of low-order Eulerian closures such as the direct-interaction approximation; the numerical issues are not trivial. Furthermore, such closures may not be adequate, as difficult questions remain unresolved—for example, the role and mathematical description of coherent structures. TDH refer¹ to structures as motivation for their work and have interpreted *their* result in terms of them.⁴ In fact, whether or not coherent structures play a dominant role in the macroscopic transport properties of a turbulent plasma is problematical. It is certainly a challenging question deserving further study, and we do not claim to contribute to this subject in the present work. However, it is important to understand that this issue is also largely irrelevant as far as the quite general conclusions of TDH are concerned. To a great extent, their results are based on formal manipulations of equations so general that they can be considered to be valid through all orders of renormalization and to embrace all possible physical processes. Thus, in order to understand the work of TDH, it is not necessary to understand the precise role of subtle statistical correlations or to be able to actually compute the value of a transport coefficient; the logical discussion can be much less detailed.

1.1 The spectral balance equation, renormalized mode coupling, and the clump interpretation of plasma turbulence

It is important, however, to understand clearly the physical *content* of each term in the general equations. Here, we use the word “content” rather than “interpretation” quite deliberately. Physical “content” corresponds precisely to the rigorous mathematics; “interpretation” need not—it connotes a looser, more picturesque description. TDH rely heavily on the “clump” interpretation of plasma turbulence. (See Ref. 2 and references therein.) Briefly, this interpretation, as used by TDH, may be summarized by the assertion that the turbulence consists of a mixture of (1) long wavelength, “coherent” wavelike fluctuations, described by the nonlinear dielectric function $\mathcal{D}$, and (2) localized “incoherent” fluctuations, sometimes called “clumps” or “blobs.” This picture has considerable intuitive appeal, particularly in regimes of fully developed turbulence. It is motivated by the recognition⁵ that the description of steady-state turbulence which most immediately generalizes linear theory—namely, that some nonlinear dielectric function⁶ vanishes—is significantly incomplete, because such a theory predicts line spectra $[\mathcal{E}_{\mathbf{k},\omega} \propto \mathcal{E}_{\mathbf{k}}\delta(\omega-\omega_k)]$, where $\mathcal{E}$ is some spectral function], in contradiction to experimental observations. The observed spectral broadening ($\Delta\omega \sim \omega$) is interpreted to be the result of the incoherent noise. This much is reasonable. However, while this argument points to the necessity for terms (“incoherent noise”) additional to the nonlinear dielectric function, it does not follow

that the incoherent noise consists *solely* of localized fluctuations [in either v (resonant or "ballistic" approximation) and/or x space]. Indeed, such a description would not reduce correctly to weak turbulence theory, which in part describes effects due to π -wave coupling, the matrix elements for which are built primarily on *non*-resonant effects, nonlocal in velocity. It is known⁷ that significant parts of these effects are *not* contained in the dielectric function; thus, they must be in the incoherent noise. These effects, which persist beyond the domain of weak turbulence theory, must be accounted for (and are, in the usual mixing length formalism). We believe that their consideration significantly, if not totally, changes the conclusion of TDH. One of our goals in the present paper is to explain this point in more detail, in part by referring to a large body of previous work which TDH appear to have overlooked.

The ambiguity with the representation in the theory of renormalized mode coupling can be traced back to Dupree's original work^{8,5} on clumps as the stochastic remnant of self-consistent trapping. In one of his early papers,⁵ he wrote

$$\delta f = \delta f^{(c)} + \delta f^{(m)} + \delta \tilde{f},$$

where "c" denoted coherent (in a Fourier description, $\delta f_k^{(c)} \propto \delta \varphi_k$; see Sec. 2), "m" denoted mode coupling, and the tilde denoted the clumps, which were referred to as incoherent noise. He then neglected the mode-coupling effects in order to discuss the small-scale "clump" contributions, related to $\delta \tilde{f}$. This is a reasonable and useful procedure. Unfortunately, the nomenclature is not entirely consistent. Because of the conventional English usage of "coherent" and "incoherent" as antonyms, one may find a strong inclination to believe that if "coherent" refers to particle fluctuations in phase with the potential, then "incoherent" refers to *everything else*. Indeed, it turns out that this interpretation is much more amenable to systematic mathematical study. Thus, commonly,⁹ as in the work of TDH, one writes as an *identity*

$$\delta f = \delta f^{(c)} + \delta \tilde{f},$$

and it now becomes crucial to ask exactly where the mode coupling is, since this process is certainly not always negligible, and may often dominate. The answer is that the mode coupling is everywhere—part is in $\delta f^{(c)}$ (specifically, in the nonlinear dielectric $\mathcal{D}$), but an equally important part is in $\delta \tilde{f}$. Thus, it may indeed be useful to describe turbulence as a mixture of wavelike fluctuations and localized fluctuations. However, if one adopts as fundamental the definition of the "coherent" response $\delta f^{(c)}$ as that part of the distribution function which is in one-to-one correspondence with the nonlinear dielectric [essentially, the velocity integral of $\delta f^{(c)}$ is the susceptibility times the potential; see Eqs. (2.13) and (2.17)], then *he may not call the wavelike fluctuations "coherent,"* since they are not described solely by the dielectric. Equivalently, *"incoherent" fluctuations need not be localized or be related to coherent structures.* These observations are crucial for understanding the work of TDH, and substantially vitiate their conclusions.

In part, it was questions such as this which motivated the development of systematically renormalized plasma turbulence theory. This subject was developed in the 1970's by DuBois¹⁰ and Krommes^{11,9} following the seminal work of Martin, Siggia, and Rose¹² and earlier work by DuBois.¹³ Of course, much of this work was inspired by the pioneering papers of Kraichnan (cited in Ref. 9). TDH reference none of this research, relying instead on the more intuitive and informal interpretations of Dupree.⁵ However, it is important to be aware of the systematic results, because they resolve the ambiguity about mode-coupling effects explicitly and in detail, from a number of different points of view. Thus, we completely understand the reduction of the general renormalizations to weak turbulence theory,^{13,9} and have learned just how the familiar processes described by perturbation theory (and the renormalized versions thereof) are associated with both the incoherent noise and several different pieces of the nonlinear dielectric function.⁹ We understand how to fully renormalize the dielectric.^{14,9} We understand the conceptual and mathematical simplicity given to the theory by use of the mean infinitesimal response function,^{15,12} and understand the relation of that function to other quantities such as the particle propagator and dielectric function.⁹ (As we will see later, it is the latter relationship which forms the basis for the cancellations which TDH discuss.) In our discussion of the work of TDH we shall appeal to all of these results, which are particularly relevant because of the general nature of the proof which TDH try to give.

Therefore, before we review the arguments of TDH and explain in detail how they can fail, it is useful to recall various well-established results from the general theory of the statistical description of turbulence.^{12,9} As we will review in more detail in Sec. 2, it is possible to write a quite general "balance equation" relating the total fluctuation spectrum of the scalar potential φ and the parallel component $A_{\parallel}$ of the vector potential to a nonlinear mode-coupling term called the incoherent noise and to the nonlinear dielectric tensor $\mathbf{D}$. (We write $\mathbf{D}$ when we wish to emphasize the tensor nature of the dielectric; otherwise, we write $\mathcal{D}$.) Let "t" denote the transpose of a vector or matrix; when there are explicit wave number and frequency arguments, it will denote the Hermitean conjugate. (The more conventional "+" will be used for another purpose later.) Then, if we introduce the column vector $\mathbf{A} \doteq (\varphi, A_{\parallel})^t$, the steady-state balance equation for the Fourier transform of the covariance matrix has the general form

$$\langle \delta \mathbf{A} \delta \mathbf{A}^t \rangle_{\mathbf{k}, \omega} = \mathbf{D}^{-1}(\mathbf{k}, \omega) \cdot \langle \delta \tilde{\mathbf{A}} \delta \tilde{\mathbf{A}}^t \rangle_{\mathbf{k}, \omega} \cdot [\mathbf{D}^{-1}(\mathbf{k}, \omega)]^t, \quad (1.1)$$

where the tildes denote the incoherent noise and we assume homogeneity in space. A more detailed version of this equation can be written at the level of the particle distribution functions, and will be discussed in Sec. 2. [See Eqs. (2.15)–(2.21).] Equation (1.1) is formally exact. The most general way of constructing it has been given by Martin, Siggia, and Rose¹² and has been reviewed by Krommes.⁹ Difficulty arises, however, in interpreting it. Certainly, its [numerator/denominator] form is suggestive, and TDH follow Dupree⁵ and their own earlier work² in attempting to interpret $\langle \delta \mathbf{A} \delta \mathbf{A}^t \rangle$ as a ballistic or granular driving term, analogous to particle discreteness, which is shielded by the dielectric polarizability of the plasma to give the total observed response. However, while there may be such ballistic effects, one of us has previously pointed out¹⁶ that the clump interpretation is incomplete,

as it overlooks important mode-coupling effects. The difficulty with, and substantial ambiguity in, this clump picture of steady-state turbulence is apparent from the choice of language in Ref. 1: TDH state that they are considering "localized, phase-space 'blobs,' analogous to fluid eddies." Now in fluid turbulence the concept of "eddy" is, without further qualification, scale-neutral: eddies of all scales can be, and are, excited in steady-state turbulence. A complete theory of eddies is equivalent to a complete solution of the turbulence problem. Of course, when wavelike oscillations are an important constituent of the turbulence, it becomes less physically appealing (although still mathematically correct) to use the mathematically defined eddy^{17,18} as a basis¹⁷ (just as a plane wave can be constructed from a linear superposition of delta functions). We believe that TDH use "eddy" as approximately synonymous to "nonwavelike." However, since waves can be expanded in terms of eddies and *vice versa*, and since all scales are coupled nonlinearly, there is no sharp distinction between wavelike and nonwavelike scales. A proper mathematics should embrace both smoothly. Therefore, in attempting to understand just what approximation TDH make, it is clear that their use of the word "localized" is crucial. That TDH are in fact thinking of fluctuations of scale small compared to the energy-containing or production scales is supported by the repeated discussion and calculation, in the references they cite (see, e.g., Ref. 19), of the "clump lifetime," which includes the well-known logarithmic factor⁵ which arises from the exponentially rapid stretching of phase space elements in the limit where those elements are very small. However, as was emphasized in Ref. 16, such fluctuations are unlike particles in a very important way: they have no independent existence, but rather are *driven by* the shearing action of larger scales. In general, one expects the contributions of these small-scale eddies to macroscopic transport to be small. The situation is somewhat less clear when the possibility of coherent structures,²⁰ not well described by low order statistics, is admitted. However, although aspects of coherent structures may persist well into "fully" developed regimes of turbulence, and the destruction, re-formation, and plastic deformation of such objects may partially determine subtle statistical properties of the plasma, there is reason to believe that in many cases of fully developed turbulence a "reasonable" description of macroscopic transport properties is obtained at the level of low-order statistical closures such as the direct-interaction approximation, which does not recognize (individual) coherent structures at all. (It does capture much of the low-order statistics of a random collection of such structures.) Except for the smallest scales, where the notions of relative diffusion may have some validity, most workers (see, e.g., Ref. 21) have found it adequate to treat a mixture of waves and eddies through a more conventional description involving renormalized mode coupling. We shall say more about this below.

In any event, we want to stress that *all* of the physics of *every* scale is described by the formally exact balance equation (1.1) and its counterpart (2.15) for the distribution, which are definitively *self-consistent relationships* between the nonlinear noise, the nonlinear dielectric, and the spectrum. The structure of equations such as (1.1) can be deceptive. Superficially, it might appear that the spectral intensity is proportional to the intensity of the incoherent noise. This might imply, in turn, that the noise *drives* the turbulence and should be accorded a central role in the theory, and that is indeed what TDH do. However,

a better way of writing Eq. (1.1), which admits a sensible and immediate reduction to weak turbulence theory and the wave kinetic equation, is

$$\mathbf{D} \cdot \langle \delta \mathbf{A} \delta \mathbf{A}^t \rangle - \langle \delta \tilde{\mathbf{A}} \delta \tilde{\mathbf{A}}^t \rangle \cdot (\mathbf{D}^{-1})^t = 0. \quad (1.2)$$

It turns out that the noise term is nominally quadratic in the fluctuation intensity. Then, from the form (1.2), it is clear that the nonlinear contributions to $\mathbf{D}$ can be (and are; see, for example, Refs. 9, 16, and 22), when multiplied by $\langle \delta \mathbf{A} \delta \mathbf{A}^t \rangle$, of the same order as the noise term. That is, if in a strict perturbation expansion in the intensity we write

$$\mathbf{D} = \mathbf{D}^{(0)} + \mathbf{D}^{(1)} + \dots,$$

then Eq. (1.2) is through lowest nontrivial order

$$\mathbf{D}^{(0)} \cdot \langle \delta \mathbf{A} \delta \mathbf{A}^t \rangle + \{ \mathbf{D}^{(1)} \cdot \langle \delta \mathbf{A} \delta \mathbf{A}^t \rangle - \langle \delta \tilde{\mathbf{A}} \delta \tilde{\mathbf{A}}^t \rangle \cdot (\mathbf{D}^{(0)})^{-1} \}^t + \dots = 0,$$

and the terms in braces combine to give the total nonlinear effect. Of course, such perturbation expansion is not valid for strong turbulence. However, it remains true that it is a combination of nonlinear terms, some from the noise, some from the dielectric, which describes the rate of energy input to, or transfer through, the various excited scales. In particular, inertial ranges are possible in which the scales smaller than the production scales are driven by the larger production scales. ("Production" refers to the interaction between the fluctuations and the gradients of mean quantities,^{18,23} and is described at the level of the potential by contributions to $\mathbf{D}$.) One of us has emphasized these and related points in Ref. 16.

Balance equations of the type (1.1) have long been studied in fluid dynamics.²⁴ In most such discussions the notion of the nonlinear dielectric function is never introduced; the concept seems intuitively more useful when there is a hierarchy of spaces—e.g., v space and x space—and a self-consistent potential. However, Kraichnan has discussed²⁵ the interpretation of the equations of the direct-interaction approximation¹⁵ for steady-state, isotropic turbulence in terms of impedance concepts, and most of that discussion generalizes readily to more complicated situations. His discussion was centered on the infinitesimal response function R , which is one of the natural objects in the direct-interaction approximation and other renormalizations.¹² (The others are the mean field, and the covariance function C .) However, Krommes and Kleva^{14,6} have shown how to define the nonlinear dielectric function in terms of R , and Krommes and Similon²⁶ have explicitly demonstrated the connection between the nonlinear dielectric function and more familiar concepts of renormalized turbulence theory for the special, though quite pedagogical, case of the two-dimensional guiding center plasma. It is clear from these and many other discussions (see, for example, Ref. 22) that the form of Eq. (1.1), though formally correct, masks important aspects of the balance between (using Kraichnan's terminology²⁵) dynamical forcing (the incoherent noise) and dynamical resistance (described by the dielectric), and must be used with great care. Thus, it must again be emphasized that the form and content of the incoherent noise is not arbitrary: The noise describes certain contributions to renormalized

mode coupling due to fluctuations of all scales, not just the small ones. For example, the noise term describes the renormalized three-wave decay $k_1 + k_2 \rightarrow k$. The inverse process, $k \rightarrow k_1 + k_2$, is described by nonlinear contributions to the dielectric. That pieces of essentially one dynamical process are hidden in rather different parts of the balance equation suggests that the form (1.1) is, in general, a rather formal and possibly misleading representation of the physics of the dynamically important interactions. In any event, one must not forget that some conventional mode-coupling effects are contained in the incoherent noise.

So far, the discussion has been at the level of the potential. The particle and multi-species nature of plasma complicates the issue. Because of the conventional ordering $\omega \ll k_{\parallel} v_{te}$, ballistic effects (involving streaming of particles or particlelike fluctuations) are expected to be important only for electrons, and it is these which TDH study. Ignoring without discussion conventional mode-coupling effects due to either species, TDH discuss the contributions to electron transport due to a certain "ballistic" component of the incoherent electron noise. As we shall explain in more detail later, we are not convinced that their approximation treats the electron nonlinearities correctly. More importantly for many physics applications, however, we are unable to understand how their neglect of ion noise can be justified. The opinion of TDH on this point is expressed in Ref. 1 as follows (for consistency, we change the notation very slightly): "The coherent ion response is hydrodynamic, $\omega > k_{\parallel} v_{ti}$, so that $n_{i,k,\omega} = R_{k,\omega}^{\text{ion}} \delta\phi_{k,\omega}$, in accord with the usual drift-Alfvén model." [Unfortunately, they have chosen a nonstandard notation: Their R is not the conventional infinitesimal response function, but is rather a (linear) susceptibility.] Although they do not specifically state why they ignore the incoherent ion noise, one gains the impression that they feel that hydrodynamic or fluid ion response is adequately described by the coherent response $\delta f^{(c)}$, and this would be in accord with the words of the standard clump interpretation. However, as we have argued, this interpretation fails: The nonlinear incoherent response of the ions is just as important as their nonlinear coherent response; both contain important mode-coupling effects. Thus, by considering the ion nonlinearities, we will be led to a conclusion entirely different from that of TDH.

1.2 "Self-consistency constraints"

Nevertheless, we find it useful to consider the specific calculations of TDH, which in their method are somewhat difficult to follow algebraically. They describe a certain class of cancellations, and are led to conclude that because of these the magnetic contribution to transport cancels out. Although their study of these cancellations for the specific problem of drift-Alfvén turbulence is new, the general theory of these cancellations is actually well known, as they arise from a fundamental (dielectric shielding) property of the infinitesimal response function. We shall review the relevant mathematics in Sec. 2. However, it is useful here to give a few words of introduction, in order to convince the reader that the response function is not a strange, but rather, in essence, a very familiar object.

Consider first the familiar linear theory of the one-dimensional, unmagnetized Vlasov equation for a homogeneous background electron distribution $\langle f \rangle$ with neutralizing ion background:

$$-i(\omega - kv)\delta f_{k,\omega} + (-ik\delta\varphi_{k,\omega})\partial\langle f \rangle = \delta f_k(t=0),$$

where for conciseness we write

$$\partial\langle f \rangle \doteq \frac{q}{m} \frac{\partial\langle f \rangle}{\partial v} \quad (1.3)$$

($q = -e$). The potential is determined from Poisson's equation

$$k^2 \delta\varphi_{k,\omega} = 4\pi nq \int dv \delta f_{k,\omega}.$$

If one could ignore the potential (self-consistent response), the solution of the initial value problem would involve the (unperturbed) particle propagator $g^{(0)}$:

$$\delta f_{k,\omega} = \int d\bar{v} g_{k,\omega}^{(0)}(v; \bar{v}) \delta f_k(\bar{v}, t=0),$$

where

$$g^{(0)}(v; v') \doteq g^{(0)}(v) \delta(v-v')$$

and

$$g^{(0)}(v) \doteq \frac{1}{-i(\omega - kv + i\epsilon)}.$$

However, the true solution of Eq. (1.3), including the self-consistent potential, involves a Green's function $R^{(0)}$ which differs from $g^{(0)}$, and introduces the (linear) dielectric $\mathcal{D}^{(0)}$:

$$\delta f_{k,\omega}(v) = \int d\bar{v} R_{k,\omega}^{(0)}(v; \bar{v}) \delta f_k(\bar{v}, t=0), \quad (1.4)$$

where

$$R_{k,\omega}^{(0)}(v; \bar{v}) \doteq g_{k,\omega}^{(0)}(v; \bar{v}) + g_{k,\omega}^{(0)}(v) (\partial\langle f \rangle) [\mathcal{D}^{(0)}]^{-1} ik \hat{\varphi}_k g_{k,\omega}^{(0)}(\bar{v}), \quad (1.5)$$

$$\hat{\varphi}_k \doteq 4\pi(nq)/k^2,$$

and

$$\mathcal{D}^{(0)}(k, \omega) \doteq 1 - ik \int d\bar{v} \hat{\varphi}_k g_{k,\omega}^{(0)}(\bar{v}) \partial\langle f \rangle(\bar{v}).$$

(These formulas generalize readily to the case of several species and dimensions.) The difference between $R^{(0)}$ and $g^{(0)}$ [the second term in Eq. (1.5)] describes the dielectric response of the plasma to a perturbation. Furthermore, $R^{(0)}$, $g^{(0)}$, and $\mathcal{D}^{(0)}$ are not all independent, since we have

$$\int d\bar{v} \hat{\varphi}_k R_{k,\omega}^{(0)}(\bar{v}; v') = [\mathcal{D}^{(0)}]^{-1} \int d\bar{v} \hat{\varphi}_k g_{k,\omega}^{(0)}(\bar{v}; v'). \quad (1.6)$$

This important result states that the total potential due to a test particle is the bare potential shielded by the dielectric. It is the basis of the famous Test Particle Superposition Principle^{27,28} of plasma kinetic theory.²⁹

We reiterate these well-known results in order to emphasize that it is the response function $R^{(0)}$, rather than the bare propagator $g^{(0)}$, which is fundamental. Use of $R^{(0)}$ often leads to substantial algebraic and conceptual simplifications. [Note the concise form of Eq. (1.4).] Even more importantly, when one turns to a nonlinear, statistically renormalized description, it is $R^{(0)}$ which most immediately generalizes,¹² to a quantity R called the mean infinitesimal response function. If one desires, however, one can also introduce natural generalizations $g^{(0)} \rightarrow g$ and $\mathcal{D}^{(0)} \rightarrow \mathcal{D}$, and one can again show that the quantities R , g , and $\mathcal{D}$ are intimately related, through the renormalized generalization [Eq. (2.12)] of the shielding relation (1.6). As we will show, it is this relation which leads to the cancellations of TDH.

One tends to think of the dielectric as describing the effects of self-consistency—hence the title of Ref. 1. Certainly this is true in linear theory, where in the absence of the self-consistent potential the dielectric would have merely its vacuum value. However, it is interesting to note that fundamentally the relation between R , g , and $\mathcal{D}$ *has nothing to do with self-consistency!* In the conventional usage of the term, “self-consistent” refers to situations in which the potentials are determined from the distribution function, and is to be distinguished from “passive,” which describes problems in which the potentials are statistically specified and are thus not determined by the distribution. It is true, of course, that the gyrokinetic equations are self-consistent; however, the general relationship (2.12) holds as well for passive problems. At first glance, one might guess that the passive relations should be trivial, with $\mathcal{D} = 1$ and $R = g$. However, this is not correct: the dielectric function for a stochastically nonlinear problem of passive advection differs from unity! This arises because the particle propagator is a derivative holding constant the *mean*, rather than the total fluctuating, potential.¹⁰ Thus, because $\mathcal{D}$, R , and g are all statistically averaged quantities, second-order beats of random variables can contribute to $\mathcal{D}$, R , and g in nontrivial ways such that $\mathcal{D} \neq 1$, $R \neq g$. How this works for the dielectric can be seen in the work of Krommes and Kleva,¹⁴ for example.

Nevertheless, there is a profound difference between self-consistent and passive problems. This shows up in the self-consistent “polarization effects,”¹⁴ absent in passive problems, which affect the *form* of the nonlinear terms Σ and ∂f which appear in R , g , and $\mathcal{D}$. The existence of these effects can be understood very simply in terms of the response of the nonlinear term to a fluctuation (perturbation). Thus, a small perturbation of the nonlinearity Ef , where E is either specified passively or related self-consistently to f , gives rise to

$$\Delta(Ef) = \begin{cases} E\Delta f & (\text{passive}), \\ E\Delta f + f\Delta E & (\text{self-consistent}). \end{cases}$$

The extra term in the self-consistent perturbation can be loosely identified with the effect of dynamical friction; it gives rise to a variety of new terms statistically. These terms,

which are necessary for energy and momentum conservation,³⁰ may cancel in certain long wavelength limits against the (generalized diffusion) term already present in passive problems. Several authors have given physical interpretations of these terms^{31,32} (there are useful analogies to the linearized Balescu-Lenard operator) and Krommes and Similon have discussed²⁸ such cancellations explicitly for a particular strong turbulence problem. Note that these remarks do not affect the physical results regarding transport, which are our principle concern. We raise the issue in order to clarify just what TDH have done. Namely, the cancellations of TDH arise from the relation (1.6) between R , g , and $\mathcal{D}$; they are not related to the long wavelength cancellations just mentioned.

1.3 Failure of the arguments of Terry, Diamond, and Hahm

Given these remarks, we can now describe in words how the arguments of TDH can fail. There are several distinct problems.

First, certain saturation mechanisms are entirely ruled out by their theory, which is built in a fundamental way, both intuitively and formally, on the incoherent noise. As we have discussed, such a theory, especially represented by the form (1.1), may lead one to believe that in steady state the fluctuations vanish if the noise does. However, this precludes saturated steady states due to, for example, either quasilinear plateau formation or induced scattering, both of which ignore the noise entirely but are believed to be the dominant processes in certain physical situations.³³ The resolution of this paradox is that a limit is possible in which in steady state both the incoherent noise and the dielectric function vanish simultaneously (for real frequency), but in which the fluctuation spectrum remains finite. The value of the limit cannot be determined from the steady-state balance equation. Rather, one must resolve the indeterminacy by relaxing the assumption of steady state. That is, one must derive the wave kinetic equation—schematically,

$$\frac{d}{dT}\mathcal{E} = 2\gamma\{\langle f \rangle, \mathcal{E}\}\mathcal{E} + (\text{incoherent noise}), \quad (1.7)$$

where T is the mean time and γ is the nonlinear growth rate, a nonlinear functional of the mean distribution function and the spectrum. From Eq. (1.7), the possibility of saturated steady states with negligible incoherent noise ($\gamma\{\langle f \rangle, \mathcal{E}\} = 0$) is apparent: Quasilinear plateau formation is described by the approximation $\gamma \approx \gamma^{(1)}\{\langle f \rangle\} = 0$, where $\gamma^{(1)}$ is the linear growth rate; saturation by induced scattering emerges from $\gamma \approx \gamma^{(1)} + \gamma^{(n)} = 0$, where $\gamma^{(n)}$ is the first-order term of the expansion in $\mathcal{E}$ of the renormalized growth rate. Clearly, for such situations, which include the famous paradigm of saturation of the one-dimensional bump-on-tail instability, the conclusions of TDH cannot be correct.

We think the limiting process just described is subtle and interesting. However, we shall not dwell on it, because, although it is not necessarily obvious from Ref. 1, it is our understanding³⁴ that TDH did not intend to consider situations in which the growth rate was allowed to vanish nonlinearly. (They evidently exclude such possibilities with the

phrase "fully developed.") The desire to keep the linear growth rate finite is reasonable, motivated by the experimental situation in confinement devices, where mean gradients persist in the steady state. The assumption about the nonlinear growth rate is more difficult to understand in general, as it excludes a viable saturation mechanism. For example, one way that certain drift wave problems achieve saturation is by turning off the energy flow from the electrons to the waves, not by quasilinearly flattening the mean gradients but rather by generating nonlinear terms which dynamically change the sign of the effective growth rate,³⁵ so that the time average of the nonlinear growth vanishes. Nevertheless, in the present work we shall follow TDH and ignore the possibility of such an effect. However, the notion that limits are possible in which all or part of the incoherent noise is negligible although the fluctuation spectrum remains finite is also relevant to the case TDH treat of steady-state drift-Alfvén turbulence saturated by mode-coupling effects. Here, there are several other difficulties with their arguments. The most physical problem has to do with the neglect of ion nonlinearities. For simplicity, let us consider one popular limit, namely drift waves of small but finite β , saturated by primarily electrostatic ion nonlinearities.³ TDH would conclude that in steady state there would be no contribution to the radial flux proportional to $|v_{\parallel}| \langle (\delta B_r/B)^2 \rangle$ because of certain cancellations involving the electron incoherent noise. We agree that these cancellations occur in the limit that TDH consider—indeed, we shall show later that the cancellations are a special case of the very general and well-known relationships, between the renormalized particle propagator and the mean infinitesimal response function, to which we have alluded previously. Even so, their conclusion is not inevitable. They argue³⁴ that because magnetic fluctuations are related *via* Ampere's law to the fluctuating currents, which are primarily carried by the electrons, only the electron noise needs to be considered. However, this does not follow. It is true that the ion ballistic noise should not contribute substantially to the parallel current because of the very large ion inertia. However, we have already argued that ion noise is not primarily ballistic, and the real question is how the total fields are created. Given the fields, the (primarily electron) current can be driven by the fields through primarily linear susceptibility ($\delta E_{\parallel} \xrightarrow{\text{lin}} \delta j_{\parallel}$), so situations are possible in which the electron nonlinearities can be neglected entirely, yet in which the magnetic fluctuations are quite finite. For example, for the case of low β drift waves, the fluctuating vector potential is determined primarily from the electrostatic potential (including both coherent and incoherent ion contributions), *not* from the incoherent vector potential, and we have the scenario

$$\delta\varphi \xrightarrow{\text{lin}} \delta E_{\parallel} \xrightarrow{\text{lin}} \delta j_{\parallel} \xrightarrow{\text{lin}} \delta A_{\parallel} \xrightarrow{\text{lin}} \delta B_z,$$

in which nonlinearity is involved only in determining $\delta\varphi$ in the saturated steady state. That is, if we write the dielectric tensor in the form

$$\mathbf{D} = \begin{pmatrix} \mathcal{D}_{\varphi\varphi} & \mathcal{D}_{\varphi A} \\ \mathcal{D}_{A\varphi} & \mathcal{D}_{AA} \end{pmatrix},$$

then we may write the general relationship between the total potentials $\mathcal{A}$ and the incoherent potentials $\tilde{\mathcal{A}}$ as

$$\delta\mathcal{A} = \mathbf{D}^{-1} \cdot \delta\tilde{\mathcal{A}} \quad (1.8)$$

If we assume only that $\mathcal{D}_{\varphi A}$ is negligible (true for sufficiently low β), then the explicit form of Eq. (1.8) is

$$\delta\varphi \approx \frac{\delta\tilde{\varphi}}{\mathcal{D}_{\varphi\varphi}}, \quad (1.9)$$

$$\delta A_{\parallel} = -\left(\frac{\mathcal{D}_{A\varphi}}{\mathcal{D}_{AA}}\right)\delta\varphi + \frac{\delta\tilde{A}_{\parallel}}{\mathcal{D}_{AA}}. \quad (1.10)$$

[Equation (1.10) follows from Eq. (1.8) without approximation.] In general, the statistical dynamics of Eqs. (1.9) and (1.10) are far from trivial, since both electron and ion nonlinearities must be considered. However, even if we ignore either the electron nonlinearities or $\delta\tilde{A}_{\parallel}$ *entirely*—which is far more restrictive than to consider, as do TDH, cancellations which occur only for a restricted (resonant) subset of the particles—we still have a well-defined theory. It is important to understand that Eq. (1.9), when expressed in the statistical form

$$\langle |\delta\varphi|^2 \rangle = \frac{\langle |\delta\tilde{\varphi}|^2 \rangle}{|\mathcal{D}_{\varphi\varphi}|^2}, \quad (1.11)$$

is a formal expression (see our previous discussion of the balance equation) of the *full* (all terms included, in principle) renormalization of the nonlinear dynamical equation for the electrostatic potential. That equation might be a Hasegawa-Mima-like equation,³⁶ for example, which includes the important polarization drift nonlinearity of the ions. Thus, even though ion *ballistic* contributions to $\delta\tilde{\varphi}$ are negligible because of the large ion inertia, it is not true that $\delta\tilde{\varphi}$ is negligible. Rather, Eq. (1.11) describes the familiar mode-coupling effects which give rise to direct and inverse cascades and, ultimately, determine the fluctuation level and spectrum at saturation. Then, given $\langle |\delta\varphi|^2 \rangle$, the magnetic spectrum follows parasitically from

$$\langle |\delta A_{\parallel}|^2 \rangle = \left| \frac{\mathcal{D}_{A\varphi}}{\mathcal{D}_{AA}} \right|^2 \langle |\delta\varphi|^2 \rangle,$$

in which $\mathcal{D}_{A\varphi}$ and $\mathcal{D}_{AA}$ may be approximated by their linear values. Furthermore, since $\delta\tilde{A}_{\parallel}$ is primarily produced by electron incoherent noise, to the extent that that noise can be considered to be negligible, one can show that the usual quasilinear expression [Eq. (3.1b)] holds for the electron flux, contrary to the conclusion of TDH. [Here, and in the remainder of the paper, the term “quasilinear” does not refer to plateau formation, but merely to the presence of the factor $\delta(\omega - k_{\parallel}v_{\parallel})$ in expression (3.1b). See Sec. 3.1.] In general, both electron and ion noise must be considered, and in this case the result for the electron flux is not literally quasilinear. However, the complete result does share many features with the quasilinear expression, and does not appear to be small, because while TDH assert that everything is driven by the electron incoherent noise, in fact the ion coherent and incoherent noise can make comparable or larger contributions.

At this point a question of semantics arises. Another general conclusion of Ref. 1 is that “*transport and relaxation in drift-Alfvén turbulence are regulated by electrostatic fluctuations*”—that is, $\delta\varphi$. First of all, such a conclusion cannot be literally true, as it is

not properly gauge-invariant. [The mathematical result of TDH seems to be true only in the Coulomb gauge; our own more general calculations (see Sec. 3) are gauge-invariant.] However, if we assume that their words are somewhat picturesque but maintain some intuitive validity, then it might appear that our discussion in the last paragraph, which also emphasized the importance of $\delta\varphi$, is in agreement with this conclusion. However, this is not true; the two points are quite distinct. In fact, taken out of context the italicized conclusion of TDH is meaningless. In self-consistent problems, the potentials $\delta\varphi$ and $\delta A_{\parallel}$ are not independent; either one can be expressed in terms of the other. Thus, the form, although not the value, of the flux and relaxation of $\langle f \rangle$ is substantially ambiguous, and the question of what is regulated by what would appear to depend on the specific physics of particular problems. The more fundamental conclusion is the one quoted earlier, namely that "magnetic transport ... [$\sim v_{\parallel}^2 \langle (\delta B_r/B_0)^2 \rangle_k \delta(\omega - k_{\parallel} v_{\parallel})$] does not contribute to the relaxation of $\langle f \rangle$..." The velocity dependence of this result identifies the physical process of interest essentially unambiguously as the one which has been discussed by a number of previous authors³⁷ for the case of statistically specified magnetic fluctuations. Thus, to state the issue in the baldest terms, TDH conclude that this mechanism does not contribute to transport in self-consistent situations, whereas we conclude, on the basis of the arguments in the last paragraph, that it *can*.

We hasten to stress, though, that it is *not* true in general that the contribution of stochastic magnetic fields to transport can be assessed *rigorously* and *quantitatively* from the quasilinear formula quoted in the last paragraph. When both the electron and the ion nonlinearities are considered, there is no escaping a detailed nonlinear calculation. Of course, such a calculation is required even when only the ions are nonlinear, in order to determine the saturation level. More than that, though, inasmuch as electron incoherent noise of any form (including conventional mode-coupling effects) is important, the electron contribution will not be literally quasilinear, so that nonlinear calculations are required to determine the precise phase shifts and numerical coefficients. This does not mean, however, that the scaling of the result cannot be quasilinear, and it certainly does not mean that the result must be small.

We now come to the second difficulty. Situations exist in which only the electron nonlinearities operate or are important,³⁸ and it would seem that it is these to which the arguments of TDH should apply most directly. (Even if there were no physical applications, one could invent such situations as *gedanken* experiments.) We have already argued that one must exclude from consideration saturation by either quasilinear plateau formation or induced scattering. In addition, we now point out that use of the ballistic approximation for the electron noise precludes the contributions from various n -wave coupling processes—either renormalized or not—since for wave coupling the nonresonant contributions to the balance equation are crucial. That such processes can be important can be seen by considering the extreme, yet possible limit of a *one-component* plasma—say, a drift-kinetic or gyrokinetic electron plasma. In the presence of density or temperature gradients, there is no reason to believe that the transport in such a system should vanish,³⁹ yet that would be the prediction of TDH, whose result is proportional to the imaginary

part of the ion susceptibility, which in the assumed limit does not even exist and can be taken to vanish identically. Of course, we would not disagree that ion dissipation can often be important in applications; classical ion dissipation generally provides the sink necessary for achieving energetically steady states, and ion nonlinearities give rise to spectral broadening. However, this seems to be a point separate from that of TDH. It is certainly possible to invent one-component (i.e., one dependent variable) nonlinear systems which can achieve steady states. For example, one can follow Terry and Horton⁴⁰ and consider a fluid description with wave-number-dependent growth rates chosen to have negative sum: $\sum_k \gamma_k < 0$. Even more simply, the charge density of a thermal equilibrium guiding center plasma is such a system. The existence of transport in the latter system is well established both theoretically⁴¹ and numerically.⁴²

The authors of Ref. 1 have emphasized to us³⁴ that they did not have one-component situations in mind, and they view the quasineutrality condition, which relates electron fluctuations to ion fluctuations, as an important ingredient in their calculations. However, in our derivation in Sec. 3 of the cancellations which lead TDH to their conclusion, we never need to make use of the specific form of the Maxwell equations (including the quasineutrality condition), and we see nothing in their arguments which precludes the application of their nonlinear calculations to a one-component situation. Thus, we interpret the fact that their result vanishes identically for a one-component situation as evidence that an important class of terms, important not only for one-component but also for two-component situations, has been overlooked.

In the remainder of the paper we shall attempt to make these ideas more concrete. We report two specific results: (1) we give a more general and concise derivation of the cancellations of TDH, which utilizes the notion of a tensor "dual" to the dielectric; (2) we show how the inclusion of ion nonlinearities can substantially change their conclusions. First, in order to add some mathematical depth to the general discussion, we describe in Sec. 2 the systematic renormalization of the plasma equations of motion (in the gyrokinetic limit). In Sec. 3 we derive and interpret the cancellations discussed by TDH. We describe the ion contribution to electron flux in Sec. 4. Finally, we conclude by discussing and summarizing our conclusions in Sec. 5.

2. SELF-CONSISTENT RENORMALIZATION OF THE GYRO-KINETIC EQUATION

Here we briefly discuss the structure of the renormalized equations appropriate for low frequency drift-Alfvén turbulence. The general procedure has been reviewed by Krommes⁹ on the basis of the original work by Martin, Siggia, and Rose,¹² Krommes,¹¹ DuBois and Espedal,¹⁰ and others cited in Ref. 9. Many of the results specific to magnetic fluctuations are contained in the paper of TDH, although we believe that the present method, which recognizes the mean infinitesimal response function^{12,9} as a fundamental object, is substantially more transparent and compact than that of TDH. Use of the response function will permit major economies of calculation in our discussion in Sec. 3 of cancellations.

2.1 The gyrokinetic system

For definiteness and simplicity of notation, we shall write our equations in the limit of strongly magnetized electrons ($k_\perp \rho_e \rightarrow 0$) and cold ions [$k_\perp \rho_i \rightarrow 0$, $k_\perp \rho_s = \mathcal{O}(1)$], and in the absence of magnetic drifts. We are aware that such equations do not contain various modes of physical interest. However, the theory is quite readily generalized to include the effects of finite electron and ion gyroradii and of general magnetic geometry, and neither the conclusions of TDH nor our own discussion rely on the details of a particular mode. We believe the present system illustrates the key difficulties with the minimum of complication.

We let q_s be the signed charge of species s , and let T_s be the temperature of a Maxwellian distribution; we shall drop species labels unless there is the possibility of confusion. Then, for either species we have the gyrokinetic system⁴³

$$\left(\frac{\partial}{\partial t} + v_\parallel \hat{b} \cdot \nabla + \mathbf{V}_E \cdot \nabla \right) f + \left(\frac{q}{m} \right) E_\parallel \frac{\partial f}{\partial v_\parallel} = 0, \quad (2.1)$$

where

$$\begin{aligned} \mathbf{V}_E &\doteq \left(\frac{c}{B} \right) \mathbf{E} \times \hat{b}, \\ \hat{b} &\doteq \mathbf{B} / |\mathbf{B}|, \\ \mathbf{B} &\doteq \nabla \times \mathbf{A}, \\ \mathbf{A} &\approx \hat{b} A_\parallel, \\ \mathbf{E} &= -\nabla \varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \end{aligned}$$

and

$$\mathcal{D}_\perp \nabla_\perp^2 \varphi = -4\pi \rho, \quad (2.2a)$$

$$\nabla_\perp^2 A_\parallel = -\frac{4\pi}{c} j_\parallel, \quad (2.2b)$$

where

$$\left(\frac{\rho}{j_\parallel / c} \right) = \sum_s (n e)_s \int dv_\parallel \left(\frac{1}{v_\parallel / c} \right) f_s(v_\parallel). \quad (2.3)$$

Here $\mathcal{D}_\perp$ is the perpendicular dielectric constant: $\mathcal{D}_\perp \doteq 1 + \omega_{pi}^2 / \omega_{ci}^2$. For physical reasons we are particularly interested in the gyrokinetic limit⁴⁴ $\omega_{pi}^2 / \omega_{ci}^2 \gg 1$. Note that Eq. (2.2a) expresses quasineutrality in the laboratory frame; its left-hand side is the ion polarization charge density, and its right-hand side is the gyrocenter charge density.

Because $\mathbf{E}$ involves a time derivative, Eq. (2.1) is not in standard form¹² for renormalization. However, since we shall follow TDH and ignore the parallel nonlinearity, and also assume that the equilibrium distribution is Maxwellian ($\langle f \rangle = f_M$), it is easy to obtain a

standard form by extracting the magnetic part of the adiabatic response and by working with the subsidiary distribution⁴⁵

$$f^\dagger \doteq f + \left(\frac{q}{T}\right) \left(\frac{v_{\parallel}}{c}\right) A_{\parallel} f_M.$$

Then, correct through second order in the gyrokinetic expansion,⁴³ we obtain

$$\left(\frac{\partial}{\partial t} + v_{\parallel} \hat{b}_0 \cdot \nabla\right) f^\dagger + \partial(f) \cdot \mathbf{E}^\dagger + \mathbf{V}_E^\dagger \cdot \nabla \delta f^\dagger = 0. \quad (2.4)$$

Here $\hat{b}_0$ is the unit vector in the direction of the equilibrium magnetic field, the generalized derivative of the equilibrium distribution function is

$$\partial(f) \doteq -\left(\frac{q}{T}\right) (v_{\parallel} \hat{b}_0 - \mathbf{V}_d) f_M,$$

the diamagnetic velocity is conventionally defined as

$$\mathbf{V}_d \doteq \left(\frac{cT}{qB}\right) \left(\frac{\partial}{\partial x} \ln(f)\right) \hat{b}_0 \times \hat{x},$$

the effective electric field and $\mathbf{E} \times \mathbf{B}$ velocity are

$$\begin{aligned} \mathbf{E}^\dagger &\doteq -\nabla \psi, \\ \mathbf{V}_E^\dagger &\doteq \left(\frac{c}{B}\right) \mathbf{E}^\dagger \times \hat{b}_0, \end{aligned}$$

the effective potential is

$$\psi \doteq \varphi - \left(\frac{v_{\parallel}}{c}\right) A_{\parallel},$$

and the individual potentials are related to $f^\dagger$ through

$$\begin{aligned} \mathcal{D}_{\perp} \nabla_{\perp}^2 \varphi &= -4\pi \rho^\dagger, \\ (\nabla_{\perp}^2 - k_s^2) A_{\parallel} &= -\frac{4\pi}{c} j_{\parallel}^\dagger, \end{aligned}$$

where $k_s \doteq \delta_s^{-1}$, $\delta_s \doteq c/\omega_{pe}$ is the skin depth, and $\rho^\dagger$ and $j_{\parallel}^\dagger$ are defined as in Eq. (2.3) but with $f^\dagger$ replacing f .

It is significant that the evolution of $f^\dagger$ is determined from the potentials through only the combination ψ . This reflects an important covariance property of the gyrokinetic system which will be of use to us later. (Littlejohn⁴⁶ has given a much deeper discussion of the covariance properties of such systems.) Namely, let us introduce a two-vector notation,

where Greek subscripts and superscripts assume the values 0 or 1, corresponding to time and the unperturbed parallel direction. Define the coordinate vector

$$z^\alpha \doteq \begin{pmatrix} t \\ \ell/c \end{pmatrix},$$

(where ℓ is the arc length along the unperturbed field lines), the two-velocity

$$v^\alpha \doteq \begin{pmatrix} 1 \\ v_{\parallel}/c \end{pmatrix} = \frac{d}{dt} z^\alpha$$

[note that we distinguish the symbol v (upsilon) from v], the two-current

$$J^\alpha \doteq \begin{pmatrix} \rho^\dagger \\ j_{\parallel}^\dagger/c \end{pmatrix} = \sum_i J_i^\alpha,$$

where

$$J_s^\alpha \doteq (nq)_s \int dv_{\parallel} v^\alpha f_s^\dagger,$$

the two-potential

$$A^\alpha \doteq \begin{pmatrix} \varphi \\ A_{\parallel} \end{pmatrix},$$

the Maxwell operator

$$\widehat{M}^\alpha{}_\beta \doteq \begin{pmatrix} \mathcal{D}_\perp \nabla_\perp^2 & 0 \\ 0 & (\nabla_\perp^2 - k_s^2) \end{pmatrix},$$

and the metric tensor

$$g_{\alpha\beta} = g^{\alpha\beta} \doteq \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, if we lower or raise indices in the standard way⁴⁷—*e.g.*, $A_\alpha = g_{\alpha\beta} A^\beta$ — we have the covariant expression

$$\psi = -v^\alpha A_\alpha = -v_\alpha A^\alpha.$$

The streaming operator is

$$\left(\frac{\partial}{\partial t} + v_{\parallel} \hat{b}_0 \cdot \nabla \right) = v^\alpha \frac{\partial}{\partial z^\alpha},$$

and Maxwell's equations are

$$\widehat{M}^\alpha{}_\beta A^\beta = -4\pi J^\alpha.$$

Thus, the gyrokinetic system is covariant (under nonrelativistic transformations mixing t and z) if we interpret $\partial(f)$ as a covariant vector. This covariance property ensures that the gyrokinetic system is gauge-invariant, and this must be reflected in the results of any proper statistical theory. As it happens, the result of TDH is not manifestly gauge-invariant, and this seems to affect their interpretation of the result. (See Sec. 3.3.)

Because we assume statistical homogeneity in the y and z directions, and also because $\langle f^\dagger \rangle = \langle f \rangle$ since we assume $\langle A_\parallel \rangle = \langle \varphi \rangle = 0$, the average of Eq. (2.4) can be written in the form

$$\frac{\partial}{\partial t} \langle f \rangle = - \frac{\partial}{\partial x} \Gamma,$$

where the “radial” flux is

$$\Gamma(v_\parallel) \doteq - \left(\frac{c}{B} \right) \left\langle \left(\frac{\partial}{\partial y} \delta\psi \right) \delta f^\dagger \right\rangle. \quad (2.5)$$

If we introduce the “non-adiabatic” distribution

$$\delta h \doteq \varepsilon f + \left(\frac{q}{T} \right) \delta \varphi f_M = \delta f^\dagger + \left(\frac{q}{T} \right) \delta \psi f_M, \quad (2.6)$$

statistical homogeneity in y ensures that the last term of Eq. (2.6) does not contribute to Eq. (2.5), so we have

$$\Gamma = - \left(\frac{c}{B} \right) \left\langle \left(\frac{\partial}{\partial y} \delta\psi \right) \delta h \right\rangle,$$

which agrees with Eq. (5) of Ref. 1. (Equation 4 of Ref. 1 does not seem to be correct.) Because $\delta\psi$ is a linear functional of $\delta f^\dagger$, Γ can be determined from the fundamental covariance

$$C^\dagger(1, 1') \doteq \langle \delta f^\dagger(1) \delta f^\dagger(1') \rangle$$

—specifically, we have in steady state

$$\Gamma(v_\parallel) = \left(\frac{c}{B} \right) \int \frac{d\omega}{2\pi} \sum_k \int d\bar{v}_\parallel i k_y \hat{\psi}_k(v_\parallel; \bar{v}_\parallel) C_{k, \omega}^\dagger(v_\parallel; \bar{v}_\parallel), \quad (2.7)$$

where $\hat{\psi}$ is the linear operator which constructs $\delta\psi$ from $\delta f^\dagger$:

$$\hat{\psi}_s^\dagger(v_\parallel; \bar{v}_\parallel) = \hat{\psi}^s(v_\parallel; \bar{v}_\parallel) = -4\pi v_\alpha (\hat{M}^{-1})^\alpha_\beta \bar{v}^\beta \langle n e \rangle_s. \quad (2.8)$$

An important property of this and similar gyrokinetic systems is that the particle transport is intrinsically ambipolar. That is, the mean density evolves according to

$$\frac{\partial}{\partial t} \langle n_s \rangle = - \frac{\partial}{\partial x} \Gamma_s^{(n)},$$

where

$$\Gamma_s^{(n)} \doteq n_s \int dv_\parallel \Gamma_s(v_\parallel); \quad (2.9)$$

intrinsic ambipolarity states that

$$\Gamma_i^{(n)} = \Gamma_e^{(n)}$$

just because of the internal dynamics, not because of any mean fields. This is proven in Appendix A.

2.2 Renormalization of the gyrokinetic-Darwin system

We now proceed to describe the general equations for $C^\dagger$. Renormalization of Eq. (2.4) proceeds by standard techniques (reviewed in Ref. 9). The reader may find these methods somewhat formal, and may question to what extent formalism alone can be useful for practical problems. In fact, when we ourselves are faced with the practical evaluation of a transport coefficient, we do not employ most of the formalism in the form presented here, as it is not convenient for numerical evaluation. However, the methods do have the virtue of being rigorous, and the following discussion is the version of the formalism which most closely resembles the approach of TDH; thus, their approximations can be readily understood in the context of a complete theory.

In addition to $C^\dagger$, it is useful^{12,48} to introduce the mean infinitesimal response function R :

$$R(1; 1') \doteq \left\langle \frac{\delta f^\dagger(1)}{\delta \hat{\eta}(1')} \right\rangle \Big|_{\hat{\eta}=0},$$

where $\hat{\eta}$ is a source added to the right-hand side of Eq. (2.4). If we let $\hat{E}$ be the operator which constructs $E^\dagger$ from $f^\dagger$, viz., $E^\dagger = \hat{E}f^\dagger$ or, more explicitly, $E^\dagger(1) = \int d\bar{1} \hat{E}(1, \bar{1}) f^\dagger(\bar{1})$, then R can be shown^{12,9} to obey

$$g^{-1}R + (\partial \bar{f}) \cdot \hat{E}R = 1, \quad (2.10)$$

where the particle "propagator" obeys

$$g^{-1} \doteq \frac{\partial}{\partial t} + v_{||} \hat{b}_0 \cdot \nabla + \Sigma$$

and the operators Σ and $\partial \bar{f}$ are nonlinear functionals of $C^\dagger$ and R whose explicit forms we shall not need. (The overbar notation in $\bar{f}$ follows that of Ref. 9 and is intended to remind the reader that complicated nonlinear terms are included in this object; to zeroth order, $\partial \bar{f} \rightarrow \partial \langle f \rangle$.) These coupled equations for R and C are causal, and can in principle be integrated in time directly (probably numerically). However, for comparison with the work of TDH, it is useful to express¹¹ R in terms of the renormalized particle propagator g and the renormalized dielectric tensor D . This can be done by a procedure^{11,10,9} which exactly parallels the solution of the linearized Vlasov equation (where one first solves for the electric field, then determines the particle distribution from the field). It is convenient to write

$$\hat{E} = \nabla v^\dagger \cdot \hat{A},$$

where we have introduced the (covariant) vector

$$v \equiv v_\alpha \doteq \begin{pmatrix} -1 \\ v_{||}/c \end{pmatrix}$$

and the (contravariant) vector potential operator

$$\hat{\mathcal{A}} \equiv \hat{A}^\alpha \doteq \begin{pmatrix} \hat{\varphi} \\ \hat{A} \end{pmatrix}.$$

Then

$$R = g - g(\partial \bar{f}) \cdot \nabla v^t \cdot \hat{\mathcal{A}} R. \quad (2.11)$$

Equation (2.11) is an integral equation for R , given g . (Of course, g is self-consistently related to everything else.) It can be solved by applying the $\hat{\mathcal{A}}$ operator to Eq. (2.11) and solving for $\hat{\mathcal{A}} R$; we find

$$\hat{\mathcal{A}} R = \mathbf{D}^{-1} \cdot \hat{\mathcal{A}} g, \quad (2.12)$$

where the dielectric tensor has emerged:

$$\mathbf{D} \doteq \mathbf{I} + \mathbf{X}.$$

where

$$\mathbf{X} \doteq \hat{\mathcal{A}} g(\partial \bar{f}) \cdot \nabla v^t. \quad (2.13)$$

Thus

$$R = g - g(\partial \bar{f}) \cdot \nabla v^t \cdot \mathbf{D}^{-1} \cdot \hat{\mathcal{A}} g. \quad (2.14)$$

This result should be compared to Eq. (1.5), to which it reduces in the electrostatic and linear limit.

The significance of Eq. (2.12) cannot be overemphasized. This law of dielectric shielding is the renormalized generalization of the familiar result (1.6) from linear theory, which forms the basis of the Test Particle Superposition Principle, that the total plasma potential due to a test particle is the bare potential shielded by the dielectric polarization. It describes an intimate relationship between the renormalized infinitesimal response function, the renormalized particle propagator, and the renormalized dielectric function. It is this relationship which leads to the cancellations described by TDH, and discussed again by us in the next section.

Another result of the renormalization, of great importance to our discussions in the next section, is that the covariance of the distribution function can be written in the form

$$C^\dagger = R \tilde{F} R^\dagger, \quad (2.15)$$

where $\tilde{F}$, the "incoherent noise," is another nonlinear functional of $C^\dagger$ and R (nominally quadratic in $C^\dagger$). This is the most general form of the balance equation. It can be interpreted in terms of, or used to formalize, the notions of coherent and incoherent response by following the procedure described in Ref. 9. (See also Ref. 11.) That is, formally decompose the fluctuations $\delta f^\dagger$ into a coherent part $\delta f^{(c)}$ and an incoherent part $\delta \tilde{f}$:

$$\delta f^\dagger = \delta f^{(c)} + \delta \tilde{f}. \quad (2.16)$$

Define the coherent part by

$$\delta f^{(c)} = -g(\partial \tilde{f}) \cdot \delta E^t. \quad (2.17)$$

This definition is chosen so that the solution of Eq. (2.16) for $\delta \mathcal{A}$ introduces the same (infinitesimal response) dielectric defined previously:

$$\delta \mathcal{A} = \mathbf{D}^{-1} \cdot \delta \tilde{\mathcal{A}} \quad (2.18)$$

Furthermore, the solution of Eq. (2.16) is then

$$\delta f^t = R(g^{-1} \delta \tilde{f}), \quad (2.19)$$

from which

$$C^t = R[g^{-1} \tilde{C}(g^{-1})^t] R^t; \quad (2.20)$$

upon comparing this result to Eq. (2.15), we find

$$\tilde{C} = g \tilde{F} g^t. \quad (2.21)$$

Also, upon squaring and averaging Eq. (2.18), we obtain the balance equation for the spectrum:

$$\langle \delta \mathcal{A} \delta \mathcal{A}^t \rangle = \mathbf{D}^{-1} \cdot \langle \delta \tilde{\mathcal{A}} \delta \tilde{\mathcal{A}}^t \rangle \cdot (\mathbf{D}^{-1})^t,$$

where the interpretation

$$\langle \delta \tilde{\mathcal{A}} \delta \tilde{\mathcal{A}}^t \rangle \doteq \tilde{\mathcal{A}} \tilde{C} \tilde{\mathcal{A}}^t$$

can be seen, using Eq. (2.21), to be compatible with the result which follows from Eq. (2.15) with the aid of Eq. (2.12). All of these results are quite general, being formally valid through all orders of vertex renormalization.¹² Detailed expressions for Σ , $\partial \tilde{f}$, and $\tilde{F}$ can be written in the direct-interaction approximation⁹ or similar closures, if desired, but we shall not need them.

Interpretations involving the word “ballistic” stem from Eqs. (2.19)–(2.21). Thus, if one says that $\delta \tilde{f}$ is “ballistically” propagated from its initial condition, $\delta \tilde{f}(t) = g(t) \delta \tilde{f}(t=0)$, then Eq. (2.19) reads $\delta f^t(t) = R(t) \delta \tilde{f}(t=0)$ and says that the incoherent fluctuation is propagated by the total response function, giving rise to the total fluctuation. This again emphasizes the fundamental nature of the response function R . However, the word “ballistic” is quite misleading, which is why we have placed it in quotation marks. Namely, “ballistic” should refer to (the renormalized generalization of) $\delta(\omega - k_{\parallel} v_{\parallel})$, which is the real part of $g_{k,\omega}$; however, one must not overlook the nonresonant part $\text{Im } g_{k,\omega}$, which forms the heart of the conventional matrix elements of weak turbulence theory.

3. CANCELLATIONS

In this section we discuss the cancellations which form the core of the arguments of TDH. We show that these cancellations are a special case of the general result (2.14), which contains the important law of dielectric shielding (2.12).

3.1 Renormalized quasilinear theory

The arguments begin with the formal construction of $\langle \delta f^\dagger \delta \psi \rangle$, from which the flux Γ can be simply obtained according to Eq. (2.7). In general, we have

$$\langle \delta f^\dagger \delta \psi \rangle = \langle \delta f^{(c)} \delta \psi \rangle + \langle \delta \tilde{f} \delta \psi \rangle.$$

However, before we discuss this full result, it is useful to recover the familiar quasilinear formula, with which TDH take issue. This is obtained by neglecting the incoherent fluctuations:

$$\langle \delta f^\dagger \delta \psi \rangle \rightarrow \langle \delta f^{(c)} \delta \psi \rangle.$$

Now

$$\langle \delta f^{(c)} \delta \psi \rangle = \langle [g(\partial \bar{f}) \cdot \nabla \delta \psi] \delta \psi \rangle.$$

If we approximate $\bar{f} \approx \langle f \rangle$, then this result is proportional to the spectrum $\langle |\delta \psi|^2 \rangle$ and gives rise to the renormalized quasilinear contribution to the electron flux:

$$\begin{aligned} \Gamma_e^{(q)}(v_{\parallel}) &= \left(\frac{cT_e}{eB} \right) \int \frac{d\omega}{2\pi} \sum_{\mathbf{k}} (\omega_{*e} - k_{\parallel} v_{\parallel}) k_y \\ &\quad \times \int d\bar{v}_{\parallel} g_{\mathbf{k},\omega}(v_{\parallel}; \bar{v}_{\parallel}) \langle \delta \Psi(v_{\parallel}) \delta \Psi(\bar{v}_{\parallel}) \rangle_{\mathbf{k},\omega} f_M(\bar{v}_{\parallel}) \end{aligned} \quad (3.1a)$$

$$\begin{aligned} &\approx \pi \left(\frac{cT_e}{eB} \right) \int \frac{d\omega}{2\pi} \sum_{\mathbf{k}} k_y (\omega_{*e} - \omega) \delta(\omega - k_{\parallel} v_{\parallel}) \\ &\quad \times \langle \delta \Psi(v_{\parallel}) \delta \Psi(v_{\parallel}) \rangle_{\mathbf{k},\omega} f_M(v_{\parallel}), \end{aligned} \quad (3.1b)$$

where

$$\Psi(v_{\parallel}) \doteq e\psi(v_{\parallel})/T_e.$$

In deriving Eq. (3.1b) from Eq. (3.1a), we assumed that the spectral broadening was characteristic of drift wave turbulence: $\Delta\omega \lesssim \omega \ll k_{\parallel} v_{\parallel}$. Because of the presence of the delta function, we follow TDH and others in calling this formula “quasilinear.” Here, this word does not imply saturation by plateau formation. Note that we do not assume that the spectrum is linelike, as would be the case if both species were quasilinear (that is, if both the electrons and the ions followed linear trajectories, so that the spectral properties would be obtained from the zeros of the dielectric). Even when the electron nonlinearities are entirely neglected, ion nonlinearities can provide a substantial spectral broadening. Also, observe that because of the resonance condition $\omega = k_{\parallel} v_{\parallel}$, one has $\delta\psi = \delta\varphi - (\omega/k_{\parallel}c)\delta A_{\parallel} = \delta E_{\parallel}/(-ik_{\parallel})$; thus, we have the well-known (gauge-invariant) result that the quasilinear flux is proportional to $\langle |\delta E_{\parallel}|^2 \rangle$.

When the fluctuation spectrum is static ($\langle \delta \Psi \delta \Psi \rangle_{\mathbf{k},\omega} = 2\pi \delta(\omega) \langle \delta \Psi \delta \Psi \rangle_{\mathbf{k}}$) and purely magnetic ($\langle \delta \Psi \delta \Psi \rangle_{\mathbf{k}} = (v_{\parallel}/c)^2 \langle |e \delta A_{\parallel}/T_e|^2 \rangle_{\mathbf{k}}$), then the particle flux derived from Eq. (3.1b) reduces to the well-known result^{49,50,37}

$$\Gamma_e^{(n)} = -D \left(\frac{\langle n \rangle}{L_n} \right),$$

where

$$D \doteq \int dv_{\parallel} f_M(v_{\parallel}) D(v_{\parallel}) \quad (3.2a)$$

and

$$D(v_{\parallel}) \doteq \pi |v_{\parallel}| \sum_{\mathbf{k}} \delta(k_{\parallel}) \langle |\delta B_z / B|^2 \rangle_{\mathbf{k}}. \quad (3.2b)$$

(The proper interpretation of the wave number sum is discussed in Ref. 37.)

3.2 The dielectric relationship between response function and propagator

The issue which TDH raise concerns the importance of the incoherent electron fluctuations. TDH assert that the $\langle |\delta A_{\parallel}|^2 \rangle$ part of the quasilinear result is cancelled by the incoherent contribution $\langle \delta \tilde{f} \delta \psi \rangle$. We shall now review their arguments, and point out several difficulties. In particular, we disagree that the cancellations are complete, even in the “quasi-neutral” approximation and when only electron nonlinearities are included. More importantly, for some important practical applications the ion noise is important. TDH neglect this, implicitly arguing³⁴ that since magnetic fluctuations are due primarily to electron currents, ion noise should not contribute to $\delta \mathbf{B}$. We disagree with this latter conclusion, and can show precisely how the logic fails. In the process, we are able to give a very concise demonstration of the algebra of TDH.

The procedure of TDH is to write out the terms in $\langle \delta f^\dagger \delta \psi \rangle$ by using Eqs. (2.15) or (2.20) together with Eqs. (2.12) and (2.14) [or, equivalently, Eqs. (2.17) and (2.18)]:

$$\langle \delta f^{(c)} \delta \psi \rangle + \langle \delta \tilde{f} \delta \psi \rangle = [-g(\partial \tilde{f}) \cdot \nabla v^\dagger \cdot \mathbf{D}^{-1} \cdot \hat{\mathcal{A}} + 1] \tilde{C}(-v^\dagger \cdot \mathbf{D}^{-1} \cdot \hat{\mathcal{A}})^\dagger. \quad (3.3)$$

They retain only the electron noise $\tilde{C}_{ee}$, evidently arguing (incorrectly, in general) that this will dominate the contributions to $\delta \mathbf{B}$. However, before we consider this approximation, we wish to give a general discussion of the properties of the bracketed term in Eq. (3.3), which by Eq. (2.19) is equivalent to the operator $[Rg^{-1}]$. Our goal is to find the velocity- and species- space generalization of the shielding law (2.12). We shall use the index “s” to stand for both species *and* velocity. We assume homogeneous turbulence, so that all quantities can be Fourier-analyzed in space; for conciseness, we do not write the wave number index. Thus, let us define

$$w_s \doteq [g(\partial \tilde{f}) \cdot \nabla v]_s \\ \equiv \begin{pmatrix} p_s \\ a_s \end{pmatrix},$$

and, for some function C ,

$$\hat{\mathcal{A}}C \doteq \sum_i \int d\bar{v} \hat{\mathcal{A}}^i(\bar{v}) C_i(\bar{v}) \\ \doteq \hat{\mathcal{A}}^i C_i.$$

(We employ the summation convention over repeated species/velocity indices. When our intent is to sum only over velocity indices, for fixed species, we shall underline the index, as in $\chi_{\varphi\varphi}^{(i)} \doteq \hat{\varphi}^i p_i$.) In this notation, the dielectric tensor is compactly represented as the dyadic

$$\mathbf{D} = \mathbf{I} + \hat{\mathcal{A}}^i w_i^t, \quad (3.4)$$

or

$$\begin{pmatrix} \mathcal{D}_{\varphi\varphi} & \mathcal{D}_{\varphi A} \\ \mathcal{D}_{A\varphi} & \mathcal{D}_{AA} \end{pmatrix} \doteq \begin{pmatrix} 1 + \chi_{\varphi\varphi} & \chi_{\varphi A} \\ \chi_{A\varphi} & 1 + \chi_{AA} \end{pmatrix} = \begin{pmatrix} 1 + \hat{\varphi}^i p_i & \hat{\varphi}^i a_i \\ \hat{\mathcal{A}}^i p_i & 1 + \hat{\mathcal{A}}^i a_i \end{pmatrix}. \quad (3.5)$$

Our goal is to discuss

$$[Rg^{-1}]_s^t = -w_s^t \cdot \mathbf{D}^{-1} \cdot \hat{\mathcal{A}}^t + \delta_s^t, \quad (3.6)$$

in terms of which Eq. (3.3) can be written

$$\langle \delta f_s^\dagger \delta \psi \rangle = [Rg^{-1}]_s^t \tilde{C}_{\mp s}(-\mathbf{v}^t \cdot \mathbf{D}^{-1} \cdot \hat{\mathcal{A}}^t)^t.$$

The inverse of $\mathbf{D}$ can be found directly, and is

$$\mathbf{D}^{-1} = \frac{1}{\|\mathcal{D}\|} \begin{pmatrix} \mathcal{D}_{AA} & -\mathcal{D}_{\varphi A} \\ -\mathcal{D}_{A\varphi} & \mathcal{D}_{\varphi\varphi} \end{pmatrix}, \quad (3.7)$$

where $\|\mathcal{D}\|$ stands for the determinant:

$$\|\mathcal{D}\| \doteq 1 + \chi_{\varphi\varphi} + \chi_{AA} + \chi_{\varphi\varphi}\chi_{AA} - \chi_{\varphi A}\chi_{A\varphi}. \quad (3.8)$$

Ultimately, it will be this expression which we use to exhibit the cancellations explicitly. However, first it is useful to express $\mathbf{D}^{-1}$ in terms of its component vectors $\hat{\mathcal{A}}$ and w . This can be done using standard arguments of tensor invariance,⁵¹ and is

$$\mathbf{D}^{-1} = \mathbf{I} - \hat{\mathcal{A}}^i (\Delta^{-1})_i^j w_j^t, \quad (3.9)$$

where

$$\Delta_i^j \doteq \delta_i^j + w_i^t \cdot \hat{\mathcal{A}}^t \quad (3.10)$$

can be considered to be a tensor "complementary" or "dual" to $\mathbf{D}$. [We define the word "dual" by this discussion; we do not imply any other mathematical usage of the term. Also, note that while both $\mathbf{D}$ and Δ are second-rank tensors, $\mathbf{D}$ is 2×2 , while Δ is $\text{many} \times \text{many}$, where "many" means the (infinite) number of velocities times the number of species.] We have the interesting property

$$\mathbf{D}^{-1} \cdot \hat{\mathcal{A}}^t = \hat{\mathcal{A}}^t (\Delta^{-1})_i^t, \quad (3.11)$$

which is an alternative expression of dielectric shielding. Upon using this property to simplify Eq. (3.6), and then noting Eq. (3.10), we obtain

$$[Rg^{-1}]_s^{s'} = -w_s^i \cdot \hat{\mathcal{A}}^i (\Delta^{-1})_s^{s'} + \delta_s^{s'} \quad (3.12a)$$

$$= -(\Delta_s^i - \delta_s^i) (\Delta^{-1})_s^{s'} + \delta_s^{s'} \quad (3.12b)$$

$$= (\Delta^{-1})_s^{s'}. \quad (3.12c)$$

This generalizes to velocity and species space the shielding law (2.12). The appearance of the inverse of the dual tensor shows again that the response function R is more fundamental than either its bare or collective parts separately. The cancellations which occur in going from Eq. (3.12b) to Eq. (3.12c)—namely, all of the bare term cancels against part of the collective term—are the ones which TDH discuss for their special application.

To see that our calculations are consistent, we may obtain the inverse of Δ . If we compare the form (3.10) with Eq. (3.4), we see that we may obtain Δ^{-1} from $\mathbf{D}^{-1}$ by interchanging species and vector indices, as well as $\mathbf{w}$ and $\hat{\mathcal{A}}$. We obtain

$$(\Delta^{-1})_s^{s'} = \delta_s^{s'} - w_s^i \cdot \mathbf{D}^{-1} \cdot \hat{\mathcal{A}}^i,$$

which is, of course, the same as Eq. (3.6).

It is straightforward to obtain the actual value of Δ^{-1} by using the explicit form (3.7) with the expressions for the susceptibilities written in Eq. (3.5). Before proceeding with that full calculation, however, it is useful to gain some insight into the behavior by temporarily replacing all velocity-dependent quantities by constant vectors. That is, we treat in Eq. (3.10) the “ s ” labels as species labels only. (This approximation is closely related to the resonance approximation for the electron response discussed in the next subsection. However, here we assert it for the ions as well. This has no physical basis; it is for solely pedagogical purposes.) Then the 2×2 matrix Δ can be written down immediately from Eq. (3.10) (ordering the electron rows and columns first):

$$\Delta^{-1} = \frac{1}{\|\Delta\|} \begin{pmatrix} 1 + \hat{\mathcal{A}}^i \cdot w_i & -w_e^i \cdot \hat{\mathcal{A}}^i \\ -w_i^i \cdot \hat{\mathcal{A}}^e & 1 + \hat{\mathcal{A}}^e \cdot w_e \end{pmatrix}.$$

We see that the e - e component, which describes the contribution of electron noise to electron transport, involves only the ion response, and should be something like [note Eq. (3.4)]

$$(\Delta^{-1})_e^e \sim \frac{1}{\|\mathcal{D}\|} (1 + \chi_{\varphi\varphi}^{(i)} + \chi_{AA}^{(i)}).$$

[The true result, Eq. (3.16b) below, is slightly more complicated because the resonance approximation is not valid for the ions.] On the other hand, the coupling of ion noise to electron transport is essentially quasilinear:

$$\begin{aligned}(\Delta^{-1})_e^i &= -\frac{1}{\|\mathcal{D}\|} w_e^i \cdot \hat{A}^i \\ &= -g_e(\theta \bar{f}_e) \cdot \nabla \left(\frac{v_e^i \cdot \hat{A}^i}{\|\mathcal{D}\|} \right),\end{aligned}$$

or

$$(\Delta^{-1})_e^i \delta \tilde{f}_i = g_e(\theta \bar{f}_e) \cdot \nabla \psi^{(i)}(v_{\parallel e}).$$

Now, we turn to the full result. Upon multiplying the term $\delta_s^{s'}$ in Eq. (3.6) by $1 = \|\mathcal{D}\|/\|\mathcal{D}\|$, we obtain after rearranging some terms

$$\begin{aligned}[Rg^{-1}]_s^{s'} &= \|\mathcal{D}\|^{-1} \{ \delta_s^{s'} + [(\hat{\varphi}^s p_s) \delta_s^{s'} - (p_s \hat{\varphi}^{s'})] + [(\hat{A}^s a_s) \delta_s^{s'} - (a_s \hat{A}^{s'})] \\ &\quad + (\hat{\varphi}^s p_s)[(\hat{A}^s a_s) \delta_s^{s'} - a_s \hat{A}^{s'}] - (\hat{A}^s p_s)[(\hat{\varphi}^s a_s) \delta_s^{s'} - a_s \hat{\varphi}^{s'}] \\ &\quad + p_s(\hat{\varphi}^s \hat{A}^{s'} - \hat{A}^s \hat{\varphi}^{s'}) a_s \}. \quad (3.13)\end{aligned}$$

This is as far as we can proceed in general.

3.3 The ballistic approximation

At this point TDH make what they call the “ballistic” approximation for the electron noise $\bar{C}_{ee}$. They work in the reasonable limit $\Delta\omega \lesssim \omega$ (where $\Delta\omega$ is the spectral broadening due to the nonlinear interactions), which enables them to neglect the renormalization in the electron particle propagators:

$$g_e^{s'}(v_{\parallel}; v_{\parallel}') \approx \frac{1}{-i(\omega - k_{\parallel} v_{\parallel} + i\epsilon)} \delta(v_{\parallel} - v_{\parallel}') \delta_e^{s'}.$$

Furthermore, recalling formula (2.21), they note the identity

$$\begin{aligned}\tilde{G} &\doteq \left(\frac{\delta_e^s}{\omega - k_{\parallel} v_{\parallel} + i\epsilon} \right) \left(\frac{\delta_e^{s'}}{\omega - k_{\parallel} v_{\parallel}' - i\epsilon} \right) \\ &= \left(\frac{1}{\omega - k_{\parallel} v_{\parallel} + i\epsilon} - \frac{1}{\omega - k_{\parallel} v_{\parallel}' - i\epsilon} \right) \left(\frac{\delta_e^s \delta_e^{s'}}{k_{\parallel}(v_{\parallel} - v_{\parallel}') - 2i\epsilon} \right).\end{aligned}$$

They then retain only the singular part of the last factor,

$$\frac{1}{k_{\parallel}(v_{\parallel} - v_{\parallel}') - 2i\epsilon} \rightarrow \frac{\pi}{|k_{\parallel}|} \delta(v_{\parallel} - v_{\parallel}'),$$

whereupon the principal value parts of the first term cancel, so that

$$\tilde{G} \rightarrow [2\pi\delta(\omega - k_{\parallel}v_{\parallel})\delta_{\varepsilon}^{\varepsilon'}] \left[\frac{\pi}{|k_{\parallel}|} \delta(v_{\parallel} - v'_{\parallel})\delta_{\varepsilon}^{\varepsilon'} \right]. \quad (3.14)$$

Upon using this result in Eq. (2.21), one then finds

$$\tilde{C}_{k,\omega}^{\varepsilon\varepsilon}(v_{\parallel}, v'_{\parallel}) = 2\pi\delta(\omega - k_{\parallel}v_{\parallel})\tilde{C}_k^{\varepsilon\varepsilon}(v_{\parallel}, v'_{\parallel}),$$

where

$$\begin{aligned} \tilde{C}_k^{\varepsilon\varepsilon}(v_{\parallel}, v'_{\parallel}) &= \int \frac{d\omega}{2\pi} \tilde{C}_{k,\omega}^{\varepsilon\varepsilon}(v_{\parallel}, v'_{\parallel}) \\ &= \frac{\pi}{|k_{\parallel}|} \tilde{F}_{k,k_{\parallel}v_{\parallel}}^{\varepsilon\varepsilon}(v_{\parallel}, v_{\parallel})\delta(v_{\parallel} - v'_{\parallel}). \end{aligned} \quad (3.15)$$

This ballistic approximation is fundamental to their arguments. They call this form "standard," evidently referring to their intuition that the noise is dominated by fluctuations localized not only in $v_{\parallel}$ space but also in $\mathbf{x}$ space [because the clump lifetime approaches infinity as the relative coordinates $(x_{\perp}, v_{\perp}) \rightarrow 0$]. We have described the fallacy in this argument in Sec. 1 and elsewhere¹⁶; the approximation neglects conventional electron mode coupling. We shall return to this point later. For the moment, let us proceed with the approximation.

In addition to the ballistic approximation for the noise, TDH also make the reasonable and consistent approximation of retaining only the resonant parts of the nonadiabatic contributions to the electron susceptibilities. Furthermore, although they do not comment, they assume that the perpendicular electron temperature vanishes. (Simion has pointed out that this is not a trivial assumption; if $T_{e\perp} \neq 0$, an additional, possibly important contribution to electron energy, though not particle, transport is expected.⁵²) Then, all the velocity integrals over electron distributions can be performed, which amounts in our notation to treating all electron quantities relating to nonadiabatic response as simple numbers. It can then be seen that all of the resonant electron contributions cancel in Eq. (3.13). That is, writing out Eq. (3.13) explicitly and ignoring certain adiabatic parts, which will not contribute when the imaginary part is taken, we get

$$\begin{aligned} [Rg^{-1}]_{\varepsilon}^{\varepsilon} &= \|\mathcal{D}\|^{-1} \{ 1 + \chi_{\varphi\varphi}^{(i)} + \chi_{AA}^{(i)} \\ &\quad + [\chi_{\varphi\varphi}^{(i)} + (\hat{\varphi}^{\varepsilon} p_{\varepsilon})]\chi_{AA}^{(i)} - [\chi_{AA}^{(i)} + (\hat{A}^{\varepsilon} p_{\varepsilon})]\chi_{\varphi\varphi}^{(i)} \\ &\quad - \chi_{\varphi A}^{(i)}(p_{\varepsilon}\hat{A}^{\varepsilon}) - \chi_{AA}^{(i)}(p_{\varepsilon}\hat{\varphi}^{\varepsilon}) \} \end{aligned} \quad (3.16a)$$

$$= \|\mathcal{D}\|^{-1} (1 + \chi_{\varphi\varphi}^{(i)} + \chi_{AA}^{(i)} + \chi_{\varphi\varphi}^{(i)}\chi_{AA}^{(i)} - \chi_{\varphi A}^{(i)}\chi_{A\varphi}^{(i)}) \quad (3.16b)$$

$$\approx \|\mathcal{D}\|^{-1} (1 + \chi_{\varphi\varphi}^{(i)}), \quad (3.16c)$$

where the last line follows when ion current is neglected. The form (3.16b), symmetric in φ and A , should be compared to Eq. (3.8).

As a check on this result, consider the limit of a pure electron plasma, in which all ion quantities vanish; then

$$[Rg^{-1}] \rightarrow \|\mathcal{D}\|^{-1}.$$

This is the same form of the result which was obtained in the discussion of the dielectric for guiding center plasma.²⁶ It shows that one must be very cautious in neglecting "vacuum" contributions. If we neglected the "1" in Eq. (3.16c), the spectrum itself would vanish in the one-component limit.

We can now insert the result (3.16c) into Eq. (3.6) to determine the contribution to the electron flux due to the ballistic component of electron noise. We obtain

$$\Gamma_e^{(b)}(v_{\parallel}) = -\left(\frac{c}{B}\right) \int d\omega \sum_k k_y \delta(\omega - k_{\parallel} v_{\parallel}) \operatorname{Im} \left[(1 + \chi_{\varphi\varphi}^{(i)}) \frac{\langle \delta \tilde{f}_e \delta \psi^{(e)}(v_{\parallel}) \rangle_k}{\|\mathcal{D}\|(k, \omega)} \right]. \quad (3.17)$$

We note that this result, involving $\delta\psi$, is properly covariant.

In order to reduce Eq. (3.17) to the form given by TDH, we may use the relation Eq. (2.18) to relate $\delta\psi$ to $\delta\tilde{f}$. It is convenient to use Eq. (3.11) to write

$$\begin{aligned} \delta\psi &= -v^{\dagger} \cdot \mathbf{D}^{-1} \cdot \hat{\mathcal{A}}^{\dagger} \delta\tilde{f}_r \\ &= -v^{\dagger} \cdot \hat{\mathcal{A}}^{\dagger} (\Delta^{-1})_i^{\dagger} \delta\tilde{f}_r \\ &= \hat{\psi}^{\dagger} (\Delta^{-1})_i^{\dagger} \delta\tilde{f}_r. \end{aligned}$$

Thus,

$$\begin{aligned} \delta\psi^{(e)} &= \hat{\psi}^{\dagger} (\Delta^{-1})_i^{\dagger} \delta\tilde{f}_e \\ &= \|\mathcal{D}\|^{-1} [\hat{\psi}^{\dagger} (1 + \chi_{\varphi\varphi}^{(i)}) - \hat{\psi}^{\dagger} (p_i \hat{\varphi}^e + a_i \hat{A}^e)] \delta\tilde{f}_e, \end{aligned}$$

where the first term in parentheses is the approximation (3.16c) to $(\Delta^{-1})_i^{\dagger}$ and where the second term in parentheses is the result for $(\Delta^{-1})_i^{\dagger}$ and was obtained from Eq. (3.13). Since we have already neglected ion current in Eq. (3.16c), for consistency we should neglect the term in a_i as well as the $\hat{A}^i$ part of $\hat{\psi}^{\dagger}$. Thus, upon also recognizing that $\hat{\varphi}^{\dagger} p_i = \chi_{\varphi\varphi}^{(i)}$, we find

$$\delta\psi^{(e)} = \delta\varphi^{(e)} + v_{\parallel} \delta A_{\parallel}^{(e)} / c,$$

where

$$\delta\varphi^{(e)} = \frac{\hat{\varphi}^e \delta\tilde{f}_e}{\|\mathcal{D}\|} \quad (3.18a)$$

and

$$\delta A_{\parallel}^{(e)} = [1 + \chi_{\varphi\varphi}^{(i)}] \left(\frac{\hat{A}^e \delta\tilde{f}_e}{\|\mathcal{D}\|} \right). \quad (3.18b)$$

Upon inserting this result into Eq. (3.17), we note that the $\delta A_{\parallel}^{(e)}$ term vanishes completely by reality, and we finally find

$$\Gamma_e^{(b)} = -\left(\frac{cT_e}{eB}\right) \int d\omega \sum_{\mathbf{k}} k_y \delta(\omega - k_{\parallel} v_{\parallel}) [\text{Im} \chi_{\varphi\varphi}^{(i)}] \left(\frac{\langle [\delta \tilde{f}_e(v_{\parallel})/f_M] \delta \tilde{\Phi}^{(e)} \rangle_{\mathbf{k}}}{\|\mathcal{D}\|(\mathbf{k}, \omega)^2} \right) f_M(v_{\parallel}), \quad (3.19)$$

where $\delta \Phi \doteq e\delta\varphi/T_e$. Very explicitly, the meaning of $\delta \tilde{\Phi}^{(e)}$, the electron contribution to the incoherent potential, can be understood from the general definition (2.8):

$$\begin{aligned} \delta \tilde{\Phi}_{\mathbf{k}}^{(e)} &= \hat{\Phi}_{\mathbf{k}}^e \delta \tilde{f}_{\mathbf{k}}^e \\ &= -\frac{1}{(k_{\perp} \rho_s)^2} \int d\tilde{v}_{\parallel} \delta \tilde{f}_{\mathbf{k}}^e. \end{aligned} \quad (3.20)$$

Furthermore, in the ballistic approximation the result (3.15) allows us to perform the $\tilde{v}_{\parallel}$ integral in Eq. (3.20) when that expression is used in Eq. (3.19). Thus, the result for $\Gamma_e^{(b)}$ is proportional to the real quantity

$$\frac{\langle [\delta \tilde{f}_e(v_{\parallel})]^2 \rangle_{\mathbf{k}}}{\|\mathcal{D}\|(\mathbf{k}, \omega)^2}. \quad (3.21)$$

This reality was used in obtaining Eq. (3.19).

In order to compare with the result of TDH, it is useful to use Eq. (3.18a) to write

$$\Gamma_e^{(b)} = -\left(\frac{cT_e}{eB}\right) \int d\omega \sum_{\mathbf{k}} k_y \delta(\omega - k_{\parallel} v_{\parallel}) [\text{Im} \chi_{\varphi\varphi}^{(i)}] \left(\frac{\langle [\delta \tilde{f}_e(v_{\parallel})/f_M] \delta \tilde{\Phi}^{(e)} \rangle_{\mathbf{k}}}{\|\mathcal{D}\|(\mathbf{k}, \omega)} \right) f_M(v_{\parallel}). \quad (3.22)$$

This should be compared to formula 8 of TDH. In our notation, they found

$$\begin{aligned} \Gamma_e^{(b)} &= -\left(\frac{cT_e}{eB}\right) \int d\omega \sum_{\mathbf{k}} k_y \delta(\omega - k_{\parallel} v_{\parallel}) [\text{Im} \chi_{\varphi\varphi}^{(i)}] \\ &\quad \times \text{Re} \left[\left(\frac{[1] + \chi_{AA}^{(e)}}{|k_{\parallel}|} \right) \left(\frac{\langle [\delta \tilde{h}(v_{\parallel})/f_M] \delta \tilde{\Phi}^{(e)} \rangle_{\mathbf{k}}}{\|\mathcal{D}\|(\mathbf{k}, \omega)} \right) \right] f_M(v_{\parallel}). \end{aligned} \quad (3.23)$$

This formula differs from Eq. (3.22) in several respects: the presence of $\delta \tilde{h}$ instead of $\delta \tilde{f}$; the presence of χ_{AA} ; and the factor of $|k_{\parallel}|$. Also, we placed the factor of 1 in brackets, because there seems to be some ambiguity in their description of their quantity $d^{A,A}$; however, it seems clear that it should be retained. In fact, $\delta \tilde{h} = \delta \tilde{f}$, because Eq. (2.6) can be decomposed into

$$\begin{aligned} \delta h^{(e)} &= \delta \tilde{f}^{(e)} + \left(\frac{q}{T}\right) \delta \psi f_M, \\ \delta \tilde{h} &= \delta \tilde{f} \end{aligned}$$

We believe their factor of $|k_{\parallel}|$ is an error (their formula is not dimensionally correct) and is related to the relationship (3.15) between $\tilde{C}$ and $\tilde{F}$. Finally, since we have shown that $\langle \delta \tilde{f} \delta \Phi^{(e)} \rangle / \|\mathcal{D}\|$ is entirely real and since $\chi_{AA}^{(e)}$ is purely imaginary in the resonance approximation, it does not contribute in Eq. (3.23), leaving just the factor of 1.

Thus, we have succeeded in recovering the essential features of the result of TDH. Note, however, that *nowhere did we use the "collective resonance" condition of TDH*. This is fortunate, since the approximation those authors make appears to be inconsistent. Namely, they set to zero both $\text{Re} \|\mathcal{D}\|$ and the real parts of individual elements of the dielectric tensor. Furthermore, had that approximation really been necessary, their final result should have contained in the denominator a factor of $\partial \|\mathcal{D}\| / \partial \omega$ rather than $\|\mathcal{D}\|$.

Returning to the form (3.19), because it fundamentally involves the quantity (3.21), it is covariant, and this is significant. Namely, because the $f^{\dagger}$ equation involves only ψ , it is clear that $\langle \delta \tilde{f}_e \delta \tilde{f}_e \rangle$ is a functional of the spectrum of $\delta \psi$, not of $\delta \varphi$ or of $\delta A_{\parallel}$ separately. Thus, the fact that Eq. (3.19) does not explicitly involve $\delta \psi$ does not in any sense mean that "magnetic transport... is not responsible for electron energy or momentum transport!" It is, of course, a difficult task to evaluate $\langle \delta \tilde{f}_e \delta \tilde{f}_e \rangle$ in practical situations, and we will defer such a calculation to a future paper. However, it is possible to crudely compare the result (3.19) to the ion contribution to Γ_e , which we do in the next section.

4. ION CONTRIBUTION TO ELECTRON FLUX

In this section we show that the ion contribution to the electron flux is significant.

4.1 Contribution from ion noise

The contribution $\Gamma_e^{(i)}$ to Γ_e due to ion noise can be found quite simply. From Eq. (3.3), this is obtained from

$$\begin{aligned} \langle \delta f_e^{(i)} \delta \psi \rangle &= -g_e (\partial \tilde{f})_e \cdot \nabla u^{\dagger} \cdot \mathbf{D}^{-1} \cdot \hat{\mathcal{A}}^{\dagger} \langle \delta \tilde{f}_i \delta \psi \rangle \\ &= -g_e (\partial \tilde{f})_e \cdot \nabla \langle \delta \psi^{(i)} \delta \psi \rangle. \end{aligned}$$

Writing this out explicitly,

$$\Gamma_e^{(i)} = \left(\frac{c I_e}{e j \beta} \right) \int d\omega \sum_k k_y \delta(\omega - k_{\parallel} v_{\parallel}) (\omega_* - \omega) \langle \delta \Psi(v_{\parallel}) \delta \Psi^{(i)}(v_{\parallel}) \rangle_{k, \omega} f_M(v_{\parallel}). \quad (4.1)$$

When the electron noise is negligible, this is, of course, just the quasilinear expression (3.1b).

4.2 Comparison of ballistic and ion contributions to electron transport

In order to assess the relative sizes of $\Gamma_e^{(b)}$ and $\Gamma_e^{(i)}$, we must estimate the size of $\text{Im } \chi_{\varphi\varphi}^{(i)}$. For definiteness, consider the case of saturation by primarily electrostatic ion nonlinearities. In this case $\text{Im } \chi_{\varphi\varphi}^{(i)} \approx \text{Im } \|\mathcal{D}\| \sim -\Delta\omega/\omega$, where $\Delta\omega$ is the nonlinear spectral broadening. [Here, it is important to understand that $\text{Im } \chi^{(i)}$ is not proportional to the classical ion dissipation (ion viscosity or ion Landau damping, for example). This dissipation may be required in order to establish a steady state; however, the macroscopic properties of that state may be independent of the classical dissipation when that dissipation is very small.] We will replace $\langle \delta\Psi \delta\Psi^{(i)} \rangle$ by $\langle \delta\Psi^2 \rangle$ in Eq. (4.1). Crudely, we simply argue that if the ions are playing an important role in the saturation, their contribution to $\delta\varphi$, hence $\delta\psi$, should be of the same order as the total contribution. More rigorously, this is an underestimate of $\langle \delta\Psi \delta\Psi^{(i)} \rangle$, because there may be cancellations between $\delta\Psi^{(e)}$ and $\delta\Psi^{(i)}$. For example, if the electron response is adiabatic, then from the dimensionless Poisson equation $\rho_s^2 \nabla_\perp^2 \delta\Phi = -\delta n_i + \delta\Phi$ we obtain $\delta n_i = [1 + (k_\perp \rho_s)^2] \delta\Phi$; since

$$\delta\Phi^{(i)} = \frac{\delta n_i}{(k_\perp \rho_s)^2},$$

we obtain

$$\frac{\delta\Phi^{(i)}}{\delta\Phi} = 1 + \frac{1}{(k_\perp \rho_s)^2} > 1.$$

The consequence for $\delta\Psi^{(i)}$ is similar, since $\delta\Psi^{(i)} \propto \delta\Phi^{(i)}$. We may use a mean value theorem to remove the wave number and frequency integrals in Eqs. (3.19) and (4.1) in favor of characteristic wave numbers, and note that $(\omega_* - \omega) \langle \delta\Psi^2 \rangle_{k,\omega} \sim \langle \delta\Psi^2 \rangle_k$ because the characteristic width $\Delta\omega$ of the spectrum is $\mathcal{O}(\omega)$. Also, it would appear that an upper bound on the electron incoherent fluctuations is

$$\frac{\langle \delta\tilde{f}_e \delta\tilde{\Phi} \rangle}{\|\|\mathcal{D}\|\|^2} < \langle \delta\Psi^2 \rangle_k.$$

(In this and the following inequality, we ignore numerical constants.) Noting that $\Delta\omega/\omega \lesssim 1$, we then find

$$\frac{\Gamma_e^{(b)}}{\Gamma_e^{(i)}} < 1.$$

The fact that we underestimated $\langle \delta\Psi \delta\Psi^{(i)} \rangle$ means that we have overestimated this ratio, so the inequality is strengthened. However, this estimate is very crude and uncertain and does not cover all possible physical situations. Our point is just that $\Gamma_e^{(i)}$ would appear to be at least as large or larger than $\Gamma_e^{(b)}$ for at least one plausible physical limit.

5. DISCUSSION

In summary, we have reconsidered the calculations of Terry, Diamond, and Hahm from the point of view of well-known general results about the statistical dynamics of nonlinear systems,⁹ and of recent insights into the nature of the clump approximation.¹⁶ We were motivated to do this by the following features of their result: It is not able to describe saturation through either quasilinear plateau formation or induced scattering; it applies simultaneously to all moments of the distribution function; it unphysically predicts that transport vanishes for one-component plasmas, or when the ions are linear and dissipation-free; and it was built around the clump approximation to the nonlinear interaction, which has been previously argued to be significantly incomplete.¹⁶ We obtained two principal results rigorously supported by mathematics:

- (1) We gave a more general and concise rederivation of the cancellations described by TDH. Our analysis showed that these cancellations arise from the well-known identity $\hat{\mathbf{A}}\hat{\mathbf{R}} = \mathbf{D}^{-1} \cdot \hat{\mathbf{A}}\mathbf{g}$, which relates the field of a bare test particle to the total observed response. In the course of the analysis, we introduced the notion of a tensor, indexed by velocity and species, "dual" to the dielectric. Our calculation was manifestly covariant, and did not use TDH's somewhat inconsistent approximation of collective resonance.
- (2) We showed that it is incorrect to ignore the ion nonlinearities, even when one is computing the magnetic fluctuations. The reason is that the ion incoherent noise is not primarily ballistic, but rather describes conventional renormalized mode-coupling effects. Thus, if ion nonlinearities play any role at all in the saturation, the ion contribution to the incoherent (and total) electrostatic potential will be finite. That potential can then drive parallel electron currents through linear mechanisms. (Electron nonlinearities need not be involved in this process at all.) This argument justifies the title of this paper.

Even when the electron nonlinearities are under discussion, it is not clear that TDH have selected the dominant term for analysis. By ambipolarity of the particle transport, the electron statistics are closely related to the ion statistics. Thus, conventional renormalized mode coupling should make an important contribution to the electron flux. TDH appear to have ignored that effect, appealing to their intuitive clump interpretation of the nonlinear interactions. However, as we explained, that interpretation is not totally compatible with the formally exact decomposition of the distribution function into coherent and incoherent parts: The clump interpretation ignores the conventional mode-coupling effects, half of which is described by the coherent response, the other half by the incoherent response.

That the result of TDH vanishes for pure electron plasma with density or temperature gradients is totally at odds with a mixing-length estimate of the transport. Fundamentally, such an estimate involves a prediction of the nonlinear phase shift between the density and

potential, or between magnetic field and current. It is difficult to understand why that phase shift should not be quite finite in a turbulent situation; we believe that it is. We shall give a more detailed analysis of the mixing-length approach to transport elsewhere. However, even in a situation less dramatic than a one-component plasma, the difficulty remains. Indeed, TDH criticize the mixing-length estimates of Kadomtsev and Pogutse⁵³ of the electron thermal conductivity χ_e , pointing out that their own analysis would predict that χ_e vanishes. While an assessment of the work of Kadomtsev and Pogutse is beyond the scope of this article, it should be clear from the previous discussions in this paper that in such a calculation it is not the mixing length philosophy itself which is fundamentally flawed. Rather, one must do a specific nonlinear calculation in order to understand the nonlinear cascade and determine the fluctuation spectrum.

TDH advance an argument that the electron contribution to electron transport should be small by drawing an analogy to the *one-dimensional* Balescu-Lenard operator. We are unable to understand this argument, since the nonlinearity in the present problem is not one-dimensional (the $v_{\parallel}$ nonlinearity is ignored) but rather two-dimensional (due to the $\mathbf{E} \times \mathbf{B}$ nonlinearity). We believe the situation should be similar to the conventional one for two- or three-dimensional classical transport. Namely, the particle transport is ambipolar, and the electron and ion contributions to the electron energy transport are comparable. We will explore the compatibility of this assertion with the predictions of statistical closure theory in a future publication.

The work of TDH represents an attempt to understand the effects of self-consistency on transport due to turbulent fluctuations, especially magnetic ones. We believe it was motivated, in part, by widespread estimates of energy transport due to magnetic fluctuations on the basis of the so-called Rechester-Rosenbluth formula.⁵⁰ [See Eqs. (3.2a,b).] TDH would conclude that that formula is completely unjustified in self-consistent situations. Though we have come to somewhat different conclusions than they did, it is clear that self-consistency is a crucial question deserving much further study, and the work of TDH represents an important step. In particular, we agree with those authors when they stress that formulas of the Rechester-Rosenbluth variety should be used with care, particularly in self-consistent situations. However, we must remark that it is unclear from where the idea arose that one should expect the quasilinear formula to be *literally* true when the fluctuations are self-consistent—if that is indeed the current belief. Certainly in the most complete work to date on magnetic transport,³⁷ the distinction was very carefully drawn between “test particle transport” (in a passive situation, in which the stochastic magnetic fields are specified) and self-consistent transport, and the Rechester-Rosenbluth and related formulas were discussed only for the passive case. The suggestion was made in that work (and in previous works cited in Ref. 37) that the *scaling* of the Rechester-Rosenbluth formula might apply to the self-consistent transport of energy. Specifically,³⁷ “we...explicitly study the transport of *test* particles inserted into a *specified* stochastic magnetic field. It is conventionally believed that the *scaling laws* for the *test* particle density diffusion coefficient of species *s* agree with those of the *local* *conduction* coefficient of species *s* for the plasma *fluid*...” (boldface added); it certainly was not stated or

implied that the numerical coefficients should be the same. In any event, formulas such as Eq. (4.1) seem to uphold, or at least do not contradict, the assertion that formulas of the Rechester-Rosenbluth variety have something to do with self-consistent transport, at least for some realizable physical situations. We believe the actual level of magnetic transport is problem-specific, and requires detailed numerical analysis of nontrivial statistical closures.

An interesting alternative to statistical closure is to derive *rigorous upper bounds* on the transport coefficient. This method, which has been explained in detail for passive situations,²³ can also be applied to self-consistent ones,⁵⁴ although research in that direction is still in progress.⁵⁵

Whether the method is precise computation of the transport coefficient *via* statistical closure, or derivation of a rigorous bound from above *via* variational principles, it is clear that self-consistency adds a considerable amount of complexity to the already-complicated passive problem. One is again reminded of the power of dimensional and scaling analysis.⁵⁶ Inasmuch as formula 8 of Ref. 1 involves detailed spectral sums over specific correlation functions, their work goes beyond dimensional analysis and may be thought of as an attempt to provide an expression for the numerical coefficient in the scaling laws. Since a good deal of the nonlinear dynamics is folded into that coefficient, it is not surprising that disagreement exists on its value. No doubt substantial further computations will be required in order to unambiguously determine any transport coefficient in a turbulent plasma.

ACKNOWLEDGMENTS

This work was supported by U.S.D.o.E. Contract No. DE-AC02-76CHO3073. One of us (JAK) wishes to acknowledge the kind hospitality of the organizing committees for the 1987 International Conference on Plasma Physics (Kiev, USSR, April 6-12, 1987) and the International Workshop on "Nonlinear and Turbulent Processes in Physics" (Kiev, USSR, April 13-26, 1987), during which some of this work was performed. We are grateful to Dr. T. S. Hahn and Dr. P. Terry for a number of illuminating explanations of their work, to Dr. W. Lee for many discussions on the gyrokinetic formalism, to Dr. T. O'Neil for insights into the physics of trapping and the clump approximation, to Dr. H. Rose for stimulating discussions on the methods and features of renormalized turbulence theory, and to Dr. P. Similon for reporting to us his own work on this subject in advance of publication. Especially, we are indebted to Mr. J. Bowman and Dr. H. Mynick, who read a draft of the manuscript with great care and offered numerous useful suggestions for its improvement.

A. AMBIPOLARITY

Here we establish that the particle transport predicted by our gyrokinetic system is intrinsically ambipolar. To this end, we note that

$$\begin{aligned}\Gamma_s^{(n)} &= n_s \int dv_{\parallel} \Gamma_s(v_{\parallel}) \\ &= \left(\frac{c}{B}\right) \left\langle \left(\frac{\partial}{\partial y} A_{\alpha} \right) \left(\int dv_{\parallel} n_s v^{\alpha} \delta f_s^{\dagger} \right) \right\rangle \\ &= \left(\frac{c}{e_s B}\right) \left\langle \left(\frac{\partial}{\partial y} A_{\alpha} \right) J_s^{\alpha} \right\rangle.\end{aligned}$$

Then

$$\begin{aligned}\Gamma_i^{(n)} - \Gamma_e^{(n)} &= \left(\frac{c}{eB}\right) \left\langle \left(\frac{\partial}{\partial y} A_{\alpha} \right) J^{\alpha} \right\rangle \\ &= -\frac{1}{4\pi} \left(\frac{c}{eB}\right) \left\langle \left(\frac{\partial}{\partial y} A_{\alpha} \right) (\widehat{M}^{\alpha\beta} A_{\beta}) \right\rangle.\end{aligned}$$

Because $\widehat{M}^{\alpha\beta}$ is diagonal, the electrostatic and electromagnetic terms decouple; each is of the form (no sum on r)

$$G_r \doteq \left\langle \left(\frac{\partial}{\partial y} A_r \right) (a \nabla_{\perp}^2 - \kappa^2) A_r \right\rangle$$

for certain constants a and κ . If the system is assumed to be homogeneous in x (that is, if the gradients of the mean density and temperature are constant everywhere), then our previous assumption of homogeneity in the y and z directions allows us to replace the ensemble average by a volume integral. We may then integrate by parts, obtaining

$$\begin{aligned}G_r &= -a \left\langle \left(\nabla_{\perp} \frac{\partial}{\partial y} A_r \right) \cdot (\nabla_{\perp} A_r) \right\rangle - \kappa^2 \left\langle \left(\frac{\partial}{\partial y} A_r \right) A_r \right\rangle \\ &= -\frac{1}{2} \left\langle \frac{\partial}{\partial y} \left(a |\nabla_{\perp} A_r|^2 + \kappa^2 A_r^2 \right) \right\rangle \\ &= 0\end{aligned}$$

by homogeneity in y . If the system is not homogeneous in x , the proper generalization of the calculation is to consider $\bar{\Gamma} \doteq \int_0^1 d\bar{x} \Gamma(\bar{x})$ and $\bar{G}$ (assuming the x direction to have unit extent); $\bar{G}$ vanishes as before. The flux through an inhomogeneous system is entirely known in terms of $\bar{\Gamma}$, as has been discussed at length in Ref. 23.

REFERENCES

- ¹ P. W. Terry, P. H. Diamond, and T. S. Hahm, *Phys. Rev. Lett.* **57**, 1899 (1986).
- ² P. W. Terry and P. H. Diamond, in *Statistical Physics and Chaos in Fusion Plasmas*, edited by W. Horton and L. Reichl (Wiley, New York, 1984), p. 335.
- ³ B. B. Kadomtsev, *Plasma Turbulence*, translated by L. C. Ronson, edited by M. G. Rusbridge (Academic, New York, 1965).
- ⁴ P. W. Terry, in *Proc. 1987 International Conference on Plasma Physics* (Kiev, USSR, 1987), to be published by World Scientific Publishing Co., Singapore.
- ⁵ T. H. Dupree, *Phys. Fluids* **15**, 334 (1972).
- ⁶ T. H. Dupree, *Phys. Fluids* **10**, 1049 (1967); *Phys. Fluids* **11**, 2680 (1968).
- ⁷ V. N. Tsytovich, *Theory of Turbulent Plasma*, translated by D. L. Burdick (Consultants Bureau, New York, 1977).
- ⁸ T. H. Dupree, *Phys. Rev. Lett.* **25**, 789 (1970).
- ⁹ J. A. Krommes, in *Handbook of Plasma Physics*, Vol. II, edited by A. A. Galeev and R. N. Sudan (North-Holland, Amsterdam, 1984), Chap. 5.5.
- ¹⁰ D. F. DuBois and M. Espedal, *Plasma Phys.* **20**, 1209 (1978).
- ¹¹ J. A. Krommes, in *Theoretical and Computational Plasma Physics* (IAEA, Vienna, 1978), p. 405.
- ¹² P. C. Martin, E. D. Siggia, and H. A. Rose, *Phys. Rev. A* **8**, 423 (1973).
- ¹³ D. F. DuBois, *Phys. Fluids* **19**, 1764 (1976).
- ¹⁴ J. A. Krommes and R. G. Kleva, *Phys. Fluids* **22**, 2168 (1979).
- ¹⁵ R. H. Kraichnan, *J. Fluid Mech.* **5**, 497 (1959).
- ¹⁶ J. A. Krommes, *Phys. Fluids* **29**, 2756 (1986); J. A. Krommes, in *Proc. of Fifth Statistical Plasma Physics Workshop of U.S.-Japan Joint Institute for Fusion Theory Program* (Nagoya, Japan, 1986), p. 226.
- ¹⁷ J. L. Lumley, *Stochastic Tools in Turbulence* (Academic, New York, 1970).

- ¹⁸ H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT Press, Cambridge, MA, 1972).
- ¹⁹ P. W. Terry and P. H. Diamond, *Phys. Fluids* **28**, 1419 (1985).
- ²⁰ T. H. Dupree, *Phys. Fluids* **25**, 277 (1982).
- ²¹ G. Holloway and M. C. Hendershott, *J. Fluid Mech.* **82**, 747 (1977).
- ²² R. H. Kraichnan, *J. Atmos. Sci.* **33**, 1521 (1976).
- ²³ J. A. Krommes and R. A. Smith, *Ann. Phys.* **177**, (1987) (in press).
- ²⁴ D. C. Leslie, *Developments in the Theory of Turbulence* (Clarendon, Oxford, 1973).
- ²⁵ R. H. Kraichnan, *Phys. Fluids* **7**, 1163 (1964).
- ²⁶ J. A. Krommes and P. Similon, *Phys. Fluids* **23**, 1553 (1980).
- ²⁷ N. Rostoker, *Nucl. Fusion* **1**, 101 (1961).
- ²⁸ J. A. Krommes, *Phys. Fluids* **19**, 649 (1976).
- ²⁹ See, for example, S. Ichimaru, *Basic Principles of Plasma Physics* (Benjamin, Reading, MA, 1973).
- ³⁰ P. Similon, Ph.D. thesis, Princeton University, 1981.
- ³¹ J. A. Krommes and M. T. Kotschenreuther, *J. Plasma Phys.* **27**, 83 (1982).
- ³² T. Boutros-Ghali and T. H. Dupree, *Phys. Fluids* **21**, 783 (1978).
- ³³ A. A. Galeev and R. Z. Sagdeev, in *Handbook of Plasma Physics*, Vol. 1, edited by A. A. Galeev and R. N. Sudan (North-Holland, Amsterdam, 1984), Chap. 4.
- ³⁴ We have verified this point by discussions with several of the authors of Ref. 1: P. W. Terry, private communications, 1987; T. S. Hahn, private communications, 1987.
- ³⁵ R. A. Smith, J. A. Krommes, and W. W. Lee, *Phys. Fluids* **28**, 1069 (1985).
- ³⁶ A. Hasegawa and K. Mima, *Phys. Fluids* **21**, 87 (1978).
- ³⁷ J. A. Krommes, C. Oberman, and R. G. Kleva, *J. Plasma Phys.* **30**, 11 (1983).

- ³⁸ P. N. Guzdar, C. S. Liu, J. Q. Dong, and Y. T. Lee, *Phys. Rev. Lett.* **57**, 2818 (1986).
- ³⁹ Indeed, transport in the pure electron laboratory plasmas studied at the University of California, San Diego is quite finite, and of major concern: T. M. O'Neil, *Phys. Rev. Lett.* **55**, 943 (1985); D. H. E. Dubin, T. M. O'Neil, and C. S. Driscoll, in *Proc. of Fifth Statistical Plasma Physics Workshop of U.S.-Japan Joint Institute for Fusion Theory Program* (Nagoya, Japan, 1986), p. 265.
- ⁴⁰ P. W. Terry and W. Horton, *Phys. Fluids* **26**, 106 (1983).
- ⁴¹ J. A. Krommes and C. Oberman, *J. Plasma Phys.* **16**, 229 (1976).
- ⁴² H. Okuda and J. M. Dawson, *Phys. Fluids* **16**, 408 (1973).
- ⁴³ D. H. E. Dubin, J. A. Krommes, C. Oberman, and W. W. Lee, *Phys. Fluids* **26**, 3524 (1983).
- ⁴⁴ J. A. Krommes, W. W. Lee, and C. Oberman, *Phys. Fluids* **29**, 2421 (1986).
- ⁴⁵ See, for example, A. A. Galeev, *Phys. Fluids* **21**, 1353 (1978).
- ⁴⁶ R. G. Littlejohn, *Phys. Fluids* **27**, 976 (1984).
- ⁴⁷ C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman and Co., San Francisco, 1973).
- ⁴⁸ H. A. Rose, Ph.D. thesis, Harvard University, 1974.
- ⁴⁹ J. R. Jokipii and E. N. Parker, *Ap. J.* **155**, 777 (1969).
- ⁵⁰ A. B. Rechester and M. N. Rosenbluth, *Phys. Rev. Lett.* **40**, 38 (1978).
- ⁵¹ H. P. Robertson, *Proc. Cambridge Philos. Soc.* **36**, 209 (1940).
- ⁵² P. Similon, private communication, 1987.
- ⁵³ B. B. Kadomtsev and O. P. Pogutse, in *Proceedings of the International Conference on Plasma Physics and Controlled Nuclear Fusion Research* (IAEA, Vienna, 1985), Vol. 2, p. 69.
- ⁵⁴ R. A. Smith, Ph.D. thesis, Princeton University, 1986.
- ⁵⁵ R. Smith and J. A. Krommes, *Bull. Am. Phys. Soc.* **31**, 1560 (1986).

⁵⁶ J. W. Connor and J. B. Taylor, *Nucl. Fusion* **17**, 1047 (1977); *Phys. Fluids* **27**, 2676 (1984).

EXTERNAL DISTRIBUTION IN ADDITION TO UC-20

Dr. Frank J. Paoloni, Univ of Wollongong, AUSTRALIA
 Prof. M.H. Brennan, Univ Sydney, AUSTRALIA
 Plasma Research Lab., Australian Nat. Univ., AUSTRALIA
 Prof. I.R. Jones, Flinders Univ., AUSTRALIA
 Prof. F. Cap, Inst Theo Phys, AUSTRIA
 Prof. M. Heindler, Institut für Theoretische Physik, AUSTRIA
 M. Goossens, Astronomisch Instituut, BELGIUM
 Ecole Royale Militaire, Lab de Phys Plasmas, BELGIUM
 Com. of European, Dg XII Fusion Prog, BELGIUM
 Prof. R. Bouclique, Laboratorium voor Natuurkunde, BELGIUM
 Dr. P.H. Sakanaka, Univ Estadual, BRAZIL
 Instituto De Pesquisas Espaciais-INPE, BRAZIL
 Library, Atomic Energy of Canada Limited, CANADA
 Dr. M.P. Bachynski, MPB Technologies, Inc., CANADA
 Dr. H.M. Skarsgard, Univ of Saskatchewan, CANADA
 Dr. W. Bernard, University of British Columbia, CANADA
 Prof. J. Teichmann, Univ. of Montreal, CANADA
 Prof. S.R. Sreenivasan, University of Calgary, CANADA
 Prof. Tudor W. Johnston, INRS-Energie, CANADA
 Dr. C.R. James, Univ. of Alberta, CANADA
 Dr. Peter Lukac, Komenského Univ, CZECHOSLOVAKIA
 The Librarian, Culham Laboratory, ENGLAND
 Mrs. S.A. Hutchinson, JET Library, ENGLAND
 C. Moutter, Lab. de Physique des Milieux Ionisés, FRANCE
 J. Redet, CEN/CADARACHE - Bat 306, FRANCE
 Dr. Tom Mui, Academy Bibliographic, HONG KONG
 Preprint Library, Cent Res Inst Phys, HUNGARY
 Dr. B. Dasgupta, Saha Inst, INDIA
 Dr. R.K. Chhajlani, Vikram Univ, INDIA
 Dr. P. Kaw, Institute for Plasma Research, INDIA
 Dr. Phillip Rosenau, Israel Inst Tech, ISRAEL
 Prof. S. Cuperman, Tel Aviv University, ISRAEL
 Librarian, Int'l Ctr Theo Phys, ITALY
 Prof. G. Rostagni, Univ Di Padova, ITALY
 Miss Clelia De Palo, Assoc EURATOM-ENEA, ITALY
 Biblioteca, del CNR EURATOM, ITALY
 Dr. H. Yamato, Toshiba Res & Dev, JAPAN
 Prof. I. Kawakami, Atomic Energy Res. Institute, JAPAN
 Prof. Kyoji Nishikawa, Univ of Hiroshima, JAPAN
 Direc, Dept. Lg. Tokamak Res. JAERI, JAPAN
 Prof. Satoshi Itoh, Kyushu University, JAPAN
 Research Info Center, Nagoya University, JAPAN
 Prof. S. Tanaka, Kyoto University, JAPAN
 Library, Kyoto University, JAPAN
 Prof. Nobuyuki Inoue, University of Tokyo, JAPAN
 S. Mori, JAERI, JAPAN
 M.H. Kim, Korea Advanced Energy Research Institute, KOREA
 Prof. D.-I. Choi, Adv. Inst Sci & Tech, KOREA
 Prof. B.S. Lilley, University of Waikato, NEW ZEALAND
 Institute of Plasma Physics, PEOPLE'S REPUBLIC OF CHINA
 Librarian, Institute of Phys., PEOPLE'S REPUBLIC OF CHINA
 Library, Tsing Hua University, PEOPLE'S REPUBLIC OF CHINA
 Z. Li, Southwest Inst. Physics, PEOPLE'S REPUBLIC OF CHINA
 Prof. J.A.C. Cabral, Inst Superior Tech, PORTUGAL
 Dr. Octavian Petrus, AL I CUZA University, ROMANIA
 Dr. Johan de Villiers, Plasma Physics, AEC, SO AFRICA
 Prof. M.A. Hallberg, University of Natal, SO AFRICA
 Fusion Div. Library, JEN, SPAIN
 Dr. Lennart Stenflo, University of UMEA, SWEDEN
 Library, Royal Inst Tech, SWEDEN
 Prof. Hans Wilhelmsson, Chalmers Univ Tech, SWEDEN
 Centre Phys des Plasmas, Ecole Polytech Fed, SWITZERLAND
 Bibliotheek, Fom-Inst Voor Plasma-Fysica, THE NETHERLANDS
 Dr. D.D. Ryutov, Siberian Acad Sci, USSR
 Dr. G.A. Eliseev, Kurchatov Institute, USSR
 Dr. V.A. Glukhikh, Inst Electro-Physical, USSR
 Dr. V.T. Tolok, Inst. Phys. Tech. USSR
 Dr. L.M. Kovrizhnykh, Institute Gen. Physics, USSR
 Prof. T.J.M. Boyd, Univ College N Wales, WALES
 Nuclear Res. Establishment, Jülich Ltd., W. GERMANY
 Bibliothek, Inst. für Plasmaforschung, W. GERMANY
 Dr. K. Schindler, Ruhr Universität, W. GERMANY
 ASDEX Reading Rm, IPP/Max-Planck-Institut für
 Plasmaphysik, W. GERMANY
 Librarian, Max-Planck Institut, W. GERMANY
 Prof. R.K. Janew, Inst Phys, YUGOSLAVIA