**APPLICATION OF THE GREEN'S FUNCTION METHOD TO
SOME NONLINEAR PROBLEMS OF AN ELECTRON STORAGE RING.**

**PART III. BEAM-SIZE ENHANCEMENT DUE TO THE
PRESENCE OF NONLINEAR MAGNETS IN A RING***

S. KHEIFETS

Stanford Linear Accelerator Center

Stanford University, Stanford, California 94305

1. INTRODUCTION

A perturbation method which allows one to find the distribution function and the beam size for a broad class of storage ring nonlinear problems is described in Part I of this work.¹ In present note I apply this method to a particular problem. Namely, I want to evaluate an enhancement of the vertical beam size of a bunch due to the presence in the ring of nonlinear magnetic fields. Main part of the work (Sections 3, 4 and 5) deals with sextupole magnets. Formula for the beam size in the presence of octupole fields are also developed to the first order in the octupole strength (Section 7), although octupole magnets are not widely used in present storage ring designs. This calculation is done mainly because the octupole field has the same symmetry as the beam-beam force for the head-on collision. This will give us the opportunity to compare the conduct of the bunch due to this two types of nonlinear kicks.

The general terms of the applicability of the Green's function method is discussed in the first part of this work. For the actual problem considered here, the method should be applicable in the wide range of machine parameters. The only exceptions are regions of the tune diagram close to the lines of the nonlinear

DISCLAIMER * Work supported by the Department of Energy, contract DE-AC03-78SF00515.

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Handwritten signature

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

resonances excited by the nonlinear forces. The vertical size of the bunch calculated here is presented as the power series in sextupole strength parameter.

If we consider the beam size in function of the machine tune, the curve will correctly describe the size enhancement only if tune is not too close to a resonance. In immediate vicinity of the resonance the beam size will deviate from correct value, since here breaks the assumption put into the base of the calculation. Namely, the perturbed distribution function will deviate strongly from its unperturbed value. The perturbation theory cannot be used near the resonance and other methods are needed to treat the problem.

On the other hand, far from resonance the assumption is valid and the method gives an estimate of the perturbed beam size. Since the increase in the beam size is an unwanted effect, it should be kept as small as possible. In principle, the derived formulae give an opportunity to develop such distribution of sextupole magnets around the ring, that will produce the minimum beam size enhancement and hence the 'best' ring performance.

For reader's convenience I reproduce in Section 2 the general formulae for the distribution function and the beam size derived in Part I to which I refer the reader for details and definitions.

In Section 6 one finds a numerical example of the beam enhancement due to the presence of sextupole magnets in PEP storage ring.

2. PERTURBATION FORCE OF A NONLINEAR MAGNET

To describe a particle motion in a storage ring we use the Courant-Snyder variables² u, ϕ ($u' \equiv du/d\phi$) for the horizontal and v, θ ($v' \equiv dv/d\theta$) for the vertical planes respectively. The sudden change in the particle velocity by a passage through a nonlinear magnet ('kick') in these variables is connected to

12-2000

the kick in variables x and y by the following relationship [cf. expressions (5.9) and (5.10) of Part I]:

$$\tilde{F}_x(u, v) = \nu \sqrt{\beta_x} F_x[x(u), y(v)] , \quad (2.1)$$

and

$$\tilde{F}_y(u, v) = \tau \sqrt{\beta_y} F_y[x(u), y(v)] , \quad (2.2)$$

where ν and τ are horizontal and vertical tunes of the machine correspondingly.

The simplest way to obtain the dependence of the field component of the k -th pole magnet on coordinate z and y is to use an expansion of the corresponding power of a complex number z :

$$F_x = \operatorname{Re} \{ S z^n \} \quad (2.3)$$

and

$$F_y = \operatorname{Im} \{ S z^n \} , \quad (2.4)$$

where S is the integrated strength of the magnet, $z = x - iy$ and $n = (k/2) - 1$.³ Thus, for a sextupole $F_x = S(x^2 - y^2)$, $F_y = -2Sxy$. We get in the variables u, v :

$$\tilde{F}_x = \nu(S^{(1)}u^2 - S^{(2)}v^2) \quad (2.5)$$

and

$$\tilde{F}_y = -2\tau S^{(2)}uv , \quad (2.6)$$

where

$$S^{(1)} = S\beta_x^{3/2} \quad (2.7)$$

and

$$S^{(2)} = S\beta_x^{1/2}\beta_y . \quad (2.8)$$

In the same way, for an octupole $F_x = T(x^3 - 3xy^2)$, $F_y = T(y^3 - 3x^2y)$ and we get

$$\tilde{F}_x = \nu(T^{(1)}u^3 - 3T^{(2)}uv^2) \quad (2.9)$$

and

$$\tilde{F}_y = \tau(T^{(3)}v^3 - 3T^{(2)}u^2v), \quad (2.10)$$

where

$$T^{(1)} = T\beta_z^2 \quad (2.11)$$

$$T^{(2)} = T\beta_x\beta_y \quad (2.12)$$

$$T^{(3)} = T\beta_y^2. \quad (2.13)$$

It is easy to write down similar expressions for any nonlinear magnetic field.

We rewrite now here formulae analogous to (3.8) and (3.9) from Part I for the first and the second order corrections for the distribution function:

$$\psi_1(V_1, s_k) = - \sum_{m < k} \int dV_0 G(V_1, s_k, V_0, s_m) \left(\tilde{F}_x \frac{\partial \psi_0}{\partial u^I} + \tilde{F}_y \frac{\partial \psi_0}{\partial v^I} \right)_{V_0} \quad (2.14)$$

and

$$\begin{aligned} \psi_2(V, s_l) = & \sum_{k < l} \int dV_1 G(V, s_l, V_1, s_k) \sum_{m < k} \left(\tilde{F}_x \frac{\partial}{\partial u^I} + \tilde{F}_y \frac{\partial}{\partial v^I} \right)_{V_1} \\ & \times \int dV_0 G(V_1, s_k, V_0, s_m) \left(\tilde{F}_x \frac{\partial \psi_0}{\partial u^I} + \tilde{F}_y \frac{\partial \psi_0}{\partial v^I} \right)_{V_0}, \end{aligned} \quad (2.15)$$

where $V = (u, u', v, v')$ and $V_0 = (u_0, u'_0, v_0, v'_0)$, are points in a four-dimensional phase space of the transverse motion,

$$\psi_0 = \exp \left\{ -\frac{u^2}{2\epsilon_x} - \frac{u'^2}{2\epsilon_x\nu^2} - \frac{v^2}{2\epsilon_y} - \frac{v'^2}{2\epsilon_y\tau^2} \right\} / (2\pi)^2 \epsilon_x \nu \epsilon_y \tau \quad (2.16)$$

and

$$G(V, s_k, V_0, s_m) = G_u(u, u', \phi_k | u_0, u'_0, \phi_m) G_v(v, v', \theta_k | v_0, v'_0, \theta_m)$$

is the Green's function as it is discussed in Part I. The summations in expressions (2.14) and (2.15) for any given 'moment' s_k are performed over all the 'moments' $s_m < s_k$ at which a particle experiences a kick from the side of a nonlinear element in the lattice. ϵ_x and ϵ_y are the horizontal and the vertical unperturbed emittances of the bunch.

The reader has noticed that in order to avoid excessive indexing the tunes of the machine are denoted ν and τ for horizontal and vertical planes respectively. In the same manner α and δ mean the corresponding damping constants.

It is worth to mention also, that average value of both ψ_1 and ψ_2 over whole phase space is zero. This follows from the fact, that there is no particle loss and the normalization of ψ should be 1. On the other hand the normalization of ψ_0 is also 1. Certainly, the same can be found by a direct integration of ψ_1 over V_1 in expression (2.14) and ψ_2 over V in expression (2.15).

Tedious but straightforward calculations show that both v and v' are zeros by averaging over ψ_1 and ψ_2 .

3. ENHANCEMENT OF THE VERTICAL EMITTANCE DUE TO SEXTUPOLE FIELD

The perturbed distribution function $\psi = \psi_0 + \psi_1 + \psi_2$ which is found in the previous section allows us to calculate the perturbed beam emittances. As an example, I will perform the calculations for a vertical beam emittance E_y :

$$E_y = \int dV \psi(V) (v^2 + v'^2/\tau^2)/2 \quad (3.1)$$

Since the distribution function ψ is found in the form of a series expansion, the vertical beam emittance E_y is also an expansion. The zeroth order term of this series is the unperturbed beam emittance ϵ_y . It is easy to see that due to

symmetry of the sextupole field, the first order term in E_y is zero. Hence

$$\frac{E_y}{\epsilon_y} = 1 + \Delta_2, \quad (3.2)$$

where

$$\begin{aligned} \Delta_2 = & \frac{1}{2\epsilon_y} \sum_{k \leq l} \int dV (v^2 + v'^2/\tau^2) \int dV_1 G(V, V_1, s_{lk}) \\ & \times \sum_{m \leq k} \left(\bar{F}_x \frac{\partial}{\partial u^i} + \bar{F}_y \frac{\partial}{\partial v^j} \right)_{V_1} \int dV_0 G(V_1, V_0, s_{km}) \\ & \times \left(\bar{F}_x \frac{\partial \psi_0}{\partial u^i} + \bar{F}_y \frac{\partial \psi_0}{\partial v^j} \right)_{V_0}. \end{aligned} \quad (3.3)$$

Here s denotes ϕ in G_u and θ in G_v respectively. It is simpler to perform the space integrations first over V , then over V_1 and at last over V_0 .

The integral of v^2 over V is the second Green's function moment $P_2 = p_0 + p_1^2 v_1^2 + p_2^2 v_1'^2 + 2p_3 v_1 v_1'$, which has been evaluated in Appendix B of part I¹ [see formulae (B.12) through (B.15) for coefficients $p_i(\theta)$]. The second Green's function moment (of v'^2) $Q_2 = q_0 + q_1^2 v_1^2 + q_2^2 v_1'^2 + 2q_3 v_1 v_1'$ is found in Appendix B of Part II⁴ [see formulae (B.10) through (B.13) for coefficients $q_i(\theta)$]. Since neither P_2 , nor Q_2 depend on u_1' , only the term containing $\bar{F}_y$ contributes to the integral over V_1 . In addition to this, only terms in P_2 and Q_2 which depend on v_1' contribute to the value of the integral. To see this perform the integration by parts over v_1' :

$$\begin{aligned} \Delta_2 = & -\frac{1}{\epsilon_y} \sum_{k \leq l} \int dV_1 \left[\left(p_2^2 + \frac{q_2^2}{\tau^2} \right) v_1' + \left(p_3 + \frac{q_3}{\tau^2} \right) v_1 \right] \bar{F}_y(u_1, v_1) \\ & \times \sum_{m \leq k} \int dV_0 G(V_1, V_0, s_{km}) \left(\bar{F}_x \frac{\partial \psi_0}{\partial u^i} + \bar{F}_y \frac{\partial \psi_0}{\partial v^j} \right)_{V_0}. \end{aligned} \quad (3.4)$$

Combining formulae (B.14) and (B.15) from¹ and (B.12) and (B.13) from⁴ one finds [with the accuracy $(\delta/\tau)^2$]:

$$p_2^2 + \frac{q_2^2}{\tau^2} \equiv f_2 = e^{-2\delta\bar{\theta}} \left(1 - \frac{\delta}{\tau} \sin 2r\bar{\theta} \right) / \tau^2, \quad (3.5)$$

$$p_3 + \frac{q_3}{r^2} \equiv f_3 = \delta e^{-2\delta\bar{\theta}} (1 - \cos 2r\bar{\theta}) / r^2, \quad (3.6)$$

where $\bar{\theta}$ stands for $\theta_l - \theta_k$.

Since $\tilde{F}_y(u_1, v_1)$ is proportional to $u_1 v_1$ [compare expression (2.6)], the integral over V_1 is the product of the first moment of G_u (P_1 in the notation of Ref. 1) and the sum of two second moments of G_v ($\dot{P}_2$ and P_2 in the same notation):

$$P_1 = \bar{p}_1 u_0 + \bar{p}_2 u_0', \quad (3.7)$$

where

$$\bar{p}_1 = e^{-\alpha\phi} \left(\cos \nu\phi + \frac{\alpha}{\nu} \sin \nu\phi \right), \quad (3.8)$$

$$\bar{p}_2 = e^{-\alpha\phi} \sin \nu\phi / \nu. \quad (3.9)$$

Here ϕ stands for $\phi_k - \phi_m$. Further,

$$P_2 = p_0 + p_1^2 v_0^2 + p_2^2 v_0'^2 + 2p_3 v_0 v_0' \quad (3.10)$$

and

$$\dot{P}_2 = r_0 + r_1 v_0^2 + r_2 v_0'^2 + 2r_3 v_0 v_0'. \quad (3.11)$$

Expressions for coefficients $p_i(\theta)$ and $r_i(\theta)$, where $\theta = \theta_k - \theta_m$, may be found in Appendix B, Part I.¹

The last integration in (3.4) over V_0 is very simple. The table of relevant integrals can be found in Appendix to this note. The result of the calculation is:

$$\begin{aligned} \Delta_2 = & 2r \sum_k S_k^{(2)} \left\{ f_2 \sum_m \left[\nu \bar{p}_2 S_m^{(2)} \epsilon_y \left(\frac{r_0}{\epsilon_y} + 3r_1 + r_2 r^2 \right) \right. \right. \\ & - \nu \bar{p}_2 S_m^{(1)} \epsilon_x \left(\frac{r_0}{\epsilon_y} + r_1 + r_2 r^2 \right) + 4r r_3 \bar{p}_1 S_m^{(2)} \epsilon_x \Big] \\ & + f_3 \sum_m \left[\nu \bar{p}_2 S_m^{(2)} \epsilon_y \left(\frac{p_0}{\epsilon_y} + 3p_1^2 + p_2^2 r^2 \right) \right. \\ & \left. \left. - \nu \bar{p}_2 S_m^{(1)} \epsilon_x \left(\frac{p_0}{\epsilon_y} + p_1^2 + p_2^2 r^2 \right) + 4r p_3 \bar{p}_1 S_m^{(2)} \epsilon_x \right] \right\}. \end{aligned} \quad (3.12)$$

The subscript (k or m) in the notation of sextupole magnets numbers them in the order in which they are seen by a bunch.

It is easy to check that to the order of $(\delta/r)^2$:

$$\frac{r_0}{\epsilon_y} + r_1 + r_2 r^2 = 0 \quad (3.13)$$

$$\frac{p_0}{\epsilon_y} + p_1^2 + p_2^2 r^2 = 1. \quad (3.14)$$

For the sake of completeness I give here also those coefficients r_i and p_i , which appear in the result:

$$r_1 = -\frac{\tau}{2} e^{-2\delta\theta} \left[\sin 2\tau\theta + \frac{\delta}{\tau} (1 - \cos 2\tau\theta) \right] \quad (3.15)$$

$$r_3 = \frac{1}{2} e^{-2\delta\theta} \cos 2\tau\theta \quad (3.16)$$

$$p_1^2 = \frac{1}{2} e^{-2\delta\theta} \left(1 + \cos 2\tau\theta + \frac{2\delta}{\tau} \sin 2\tau\theta \right) \quad (3.17)$$

$$p_3 = \frac{1}{2\tau} e^{-2\delta\theta} \left[\sin 2\tau\theta + \frac{\delta}{\tau} (1 - \cos 2\tau\theta) \right]. \quad (3.18)$$

Here again θ denotes $\theta_k - \theta_m$.

Substitute now expressions (3.8), (3.9) and (3.13) through (3.18) into (3.12) and get:

$$\begin{aligned} \Delta_2 = & 2 \sum_{k < \ell} S_k^{(2)} \left\{ \tau^2 f_2 \sum_{m < k} \left[2\epsilon_x S_m^{(2)} e^{-\alpha\phi - 2\delta\theta} \cos 2\tau\theta \left(\cos \nu\phi + \frac{\alpha}{\nu} \sin \nu\phi \right) \right. \right. \\ & - \epsilon_y S_m^{(2)} e^{-\alpha\phi - 2\delta\theta} \sin \nu\phi \left(\sin 2\tau\theta + \frac{\delta}{\tau} - \frac{\delta}{\tau} \cos 2\tau\theta \right) \Big] \\ & + \tau f_3 \sum_{m < k} \left[(\epsilon_y S_m^{(2)} - \epsilon_x S_m^{(1)}) e^{-\alpha\phi} \sin \nu\phi \right. \\ & + \epsilon_y S_m^{(2)} e^{-\alpha\phi - 2\delta\theta} \sin \nu\phi \left(1 + \cos 2\tau\theta + 2 \frac{\delta}{\tau} \sin 2\tau\theta \right) \\ & \left. \left. + 2\epsilon_x S_m^{(2)} e^{-\alpha\phi - 2\delta\theta} \left(\cos \nu\phi + \frac{\alpha}{\nu} \sin \nu\phi \right) \left(\sin 2\tau\theta + \frac{\delta}{\tau} - \frac{\delta}{\tau} \cos 2\tau\theta \right) \right] \right\}. \quad (3.19) \end{aligned}$$

4. ONE SEXTUPOLE IN THE RING

It is instructive to consider first the simplest case when there is only one sextupole magnet in the ring. Let us first evaluate the sum over m in (3.19). In this case ϕ and θ stand for $\phi_k - \phi_m$ and $\theta_k - \theta_m$ respectively, which in this case are:

$$\begin{aligned}\phi_k - \phi_m &= 2\pi m, \\ \theta_k - \theta_m &= 2\pi m, \quad m = 0, 1, 2, \dots\end{aligned}\tag{4.1}$$

and

$$S_m^{(1)} = S^{(1)}, \quad S_m^{(2)} = S^{(2)}.$$

The sums over m can be found now using the rule of summation described in Ref. 4 (table of corresponding sums are given in Appendix C of Part II⁴ of this work). To the order of α/ν (or δ/τ) one gets:

$$\begin{aligned}\Delta_2 &= 2S^{(2)2} \sum_k e^{-2\delta\bar{\theta}} \left(1 - \frac{\delta}{\tau} \sin 2\tau\bar{\theta}\right) \left\{ -\frac{\epsilon_y \delta}{2\tau \tan \pi\nu} \right. \\ &+ \frac{\epsilon_y \delta}{4\tau \tan \pi(2\tau + \nu)} - \frac{\epsilon_y \delta}{4\tau \tan \pi(2\tau - \nu)} + \frac{\pi(2\epsilon_z + \epsilon_y)(\delta + \alpha/2)}{1 - \cos 2\pi(2\tau - \nu)} \\ &+ \frac{\pi(2\epsilon_z - \epsilon_y)(\delta + \alpha/2)}{1 - \cos 2\pi(2\tau + \nu)} + \frac{\epsilon_z \alpha}{2\nu \tan \pi(2\tau + \nu)} - \frac{\epsilon_z \alpha}{2\nu \tan \pi(2\tau - \nu)} \Big\} \\ &+ 2S^{(2)} \sum_k \frac{\delta}{\tau} e^{-2\delta\bar{\theta}} (1 - \cos 2\tau\bar{\theta}) \left\{ -\frac{\epsilon_x S^{(1)} - \epsilon_y S^{(2)}}{2\tan \pi\nu} \right. \\ &+ \epsilon_y S^{(2)} \left[\frac{1}{2\tan \pi\nu} + \frac{1}{4\tan \pi(2\tau + \nu)} - \frac{1}{4\tan \pi(2\tau - \nu)} \right] \\ &+ \epsilon_x S^{(2)} \left[\frac{1}{2\tan \pi(2\tau + \nu)} + \frac{1}{2\tan \pi(2\tau - \nu)} \right] \Big\}.\end{aligned}\tag{4.2}$$

Here $\bar{\theta}$ means $\theta_k - \theta_k = 2\pi k$, $k = 0, 1, 2, \dots$. Applying again the same rule of

summation we find (neglecting terms of the order of α/ν or δ/τ):

$$\begin{aligned} \Delta_2 = & \frac{S^{(2)^2}}{4\pi} \left\{ \frac{\pi(2\epsilon_x + \epsilon_y)(1 + \frac{\alpha}{2\delta})}{\sin^2 \pi(2\tau + \nu)} + \frac{\pi(2\epsilon_x - \epsilon_y)(1 + \frac{\alpha}{2\delta})}{\sin^2 \pi(2\tau - \nu)} \right. \\ & + \frac{\epsilon_x (\frac{\alpha}{\nu\delta} + \frac{1}{\tau})}{\tan \pi(2\tau + \nu)} - \frac{\epsilon_x (\frac{\alpha}{\nu\delta} - \frac{1}{\tau})}{\tan \pi(2\tau - \nu)} \\ & \left. + \frac{\epsilon_y}{\tau \tan \pi\nu} + \frac{\epsilon_y}{\tau \tan \pi(2\tau + \nu)} - \frac{\epsilon_y}{\tau \tan \pi(2\tau - \nu)} \right\} - \frac{S^{(1)} S^{(2)} \epsilon_x}{4\pi\tau \tan \pi\nu} \end{aligned} \quad (4.3)$$

Substitute now expressions (2.7) and (2.8) for $S^{(1)}$ and $S^{(2)}$:

$$\begin{aligned} \Delta_2 = & \frac{1}{2} S^2 \beta_x \beta_y^2 \epsilon_x \left\{ \frac{(1 + \frac{\epsilon_y}{2\epsilon_x})(1 + \frac{\alpha}{2\delta})}{\sin^2 \pi(2\tau + \nu)} + \frac{(1 - \frac{\epsilon_y}{2\epsilon_x})(1 + \frac{\alpha}{2\delta})}{\sin^2 \pi(2\tau - \nu)} \right. \\ & + \frac{(\frac{\alpha\tau}{\nu\delta} + 1)}{2\pi\tau \tan \pi(2\tau + \nu)} - \frac{(\frac{\alpha\tau}{\nu\delta} - 1)}{2\pi\tau \tan \pi(2\tau - \nu)} \\ & \left. + \frac{\epsilon_y/\epsilon_x}{2\pi\tau} \left[\frac{1}{\tan \pi\nu} + \frac{1}{\tan \pi(2\tau + \nu)} - \frac{1}{\tan \pi(2\tau - \nu)} \right] \right\} \\ & - \frac{S^2 \beta_x^2 \beta_y \epsilon_x}{4\pi\tau \tan \pi\nu} \end{aligned} \quad (4.4)$$

Usually $\epsilon_y \ll \epsilon_x$ and $\alpha = \delta$. Then

$$\begin{aligned} \Delta_2 = & \frac{1}{4} S^2 \beta_x \beta_y^2 \epsilon_x \left\{ \frac{3}{\sin^2 \pi(2\tau + \nu)} + \frac{3}{\sin^2 \pi(2\tau - \nu)} \right. \\ & \left. + \frac{1 + \frac{1}{\nu}}{\pi\tau \tan \pi(2\tau + \nu)} + \frac{1 - \frac{1}{\nu}}{\pi\tau \tan \pi(2\tau - \nu)} \right\} - \frac{S^2 \beta_x^2 \beta_y \epsilon_x}{4\pi\tau \tan \pi\nu} \end{aligned} \quad (4.5)$$

Since $\beta_x \epsilon_x = \sigma_x^2$, where σ_x is horizontal rms size of the bunch, Δ_2 far from any resonance is by the order of magnitude $(S\sigma_x \beta_y)^2$ or square of the product of β_y times the strength of the 'effective quadrupole' $S\sigma_x$.

Expression (4.4) exhibits resonance behavior close to the lines $2\tau \pm \nu$ of the (ν, τ) plane. One should remember of course, that the validity of expression (4.4)

breaks in the vicinity of resonance lines for two reasons. First of all here the change of the distribution function cannot be found by perturbation method. In addition to that, expression (4.4) is calculated with assumption, that all the denominators are not zeros. Discussion of this in more detail can be found in Ref. 4.

5. ANY NUMBER OF SEXTUPOLES IN THE RING

Consider now the most general case of n arbitrarily positioned sextupoles in the ring. Taking as an example the typical sum over m of cosine terms in expression (3.19), we present it in the following form:

$$\begin{aligned}
 R_1^+ &= \sum_{m>k} S_m^{(2)} \exp[-\alpha(\phi_m - \phi_k) - 2\delta(\theta_m - \theta_k)] \\
 &\quad \times \cos[2\tau(\theta_m - \theta_k) + \nu(\phi_m - \phi_k)] \\
 &= S_1^{(2)} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \exp(-2\pi\alpha m - 4\pi\delta m) \cos(4\pi\tau m + 2\pi\nu m) \right] \\
 &\quad + S_2^{(2)} \exp[-\alpha(\phi_2 - \phi_1) - 2\delta(\theta_2 - \theta_1)] \sum_{m=0}^{\infty} \exp(-2\pi\alpha m - 4\pi\delta m) \\
 &\quad \times \cos[4\pi\tau m + 2\pi\nu m + 2\tau(\theta_2 - \theta_1) + \nu(\phi_2 - \phi_1)] + \dots \\
 &\quad + S_n^{(2)} \exp[-\alpha(\phi_n - \phi_1) - 2\delta(\theta_n - \theta_1)] \sum_{m=0}^{\infty} \exp(-2\pi\alpha m - 4\pi\delta m) \\
 &\quad \times \cos[4\pi\tau m + 2\pi\nu m + 2\tau(\theta_n - \theta_1) + \nu(\phi_n - \phi_1)] .
 \end{aligned} \tag{5.1}$$

Here quite arbitrarily the starting sextupole is numbered as S_1 . In the first term of expression (5.1) only half pulse at 'time' zero is taken into account in accord with the summation rule developed in Ref. 4. Performing some algebra we get:

$$\begin{aligned}
 R_1^+ = & \frac{\pi(2\delta + \alpha)}{1 - \cos 2\pi(2\tau + \nu)} \sum_{i=1}^n S_i^{(2)} \cos [2\tau(\theta_i - \theta_1) + \nu(\phi_i - \phi_1)] \\
 & + \frac{1}{2\sin \pi(2\tau + \nu)} \sum_{i=2}^n S_i^{(2)} \exp [-\alpha(\phi_i - \phi_1) - 2\delta(\theta_i - \theta_1)] \\
 & \times \sin [\pi(2\tau + \nu) - 2\tau(\theta_i - \theta_1) - \nu(\phi_i - \phi_1)] .
 \end{aligned} \tag{5.2}$$

There are in general n different sums R_j^+ , $j = 1, 2, \dots, n$, similar to R_1^+ . They appear, when one consider similar sums starting from the sextupole, positioned at j -th place in the ring. Any sum evaluated for the sextupole shifted by $n + 1$ positions from the j -th one is equal to the j -th sum:

$$R_{j+n}^+ = R_j^+ . \tag{5.3}$$

Exponents in each of the second sum in expression (5.2) are very close to 1, since α and δ are usually so small, that even at the maximum value for one period of differences $(\phi_i - \phi_1)_{max} = 2\pi$ and $(\theta_i - \theta_1)_{max} = 2\pi$

$$2\pi\alpha + 4\pi\delta \ll 1 . \tag{5.4}$$

Hence, the exponential factors may be expanded and all the terms of the order $(\alpha\Delta\phi)^2$ and $(\delta\Delta\theta)^2$ and higher may be omitted.

Let us introduce notations:

$$A_j^+ = \sum_{i=1}^n S_{j+i-1}^{(2)} \cos [2\tau(\theta_{j+i-1} - \theta_j) + \nu(\phi_{j+i-1} - \phi_j)] \quad (5.5)$$

$$B_j^+ = \sum_{i=2}^n S_{j+i-1}^{(2)} \sin [\pi(2\tau + \nu) - 2\tau(\theta_{j+i-1} - \theta_j) - \nu(\phi_{j+i-1} - \phi_j)] \quad (5.6)$$

$$C_j^+ = \sum_{i=1}^n S_{j+i-1}^{(2)} \left[2(\theta_{j+i-1} - \theta_j) + \frac{\alpha}{\delta}(\phi_{j+i-1} - \phi_j) \right] \times \sin [\pi(2\tau + \nu) - 2\tau(\theta_{j+i-1} - \theta_j) - \nu(\phi_{j+i-1} - \phi_j)]$$

$$(j = 1, 2, \dots, n) . \quad (5.7)$$

These values have the same periodic property (5.3) as R_j^+ . The latter can be now written as follows:

$$R_j^+ = \frac{\pi(2\delta + \alpha)}{2\sin^2\pi(2\tau + \nu)} A_j^+ + \frac{1}{2\sin\pi(2\tau + \nu)} B_j^+ - \frac{\delta}{2\sin\pi(2\tau + \nu)} C_j^+ . \quad (5.8)$$

It is easy to find now the double sum over k and m of the considered term:

$$F_c^+ = \sum_{k>l} S_k^{(2)} \exp[-2\delta(\theta_k - \theta_l)] R_k^+$$

$$= \frac{(1 + \alpha/2\delta)}{4\sin^2\pi(2\tau + \nu)} a^+ - \frac{1}{8\pi \sin\pi(2\tau + \nu)} c^+ \quad (5.9)$$

where a new notations are introduced:

$$a^+ = \sum_{j=1}^n S_j^{(2)} A_j^+ \quad (5.10)$$

$$c^+ = \sum_{j=1}^n S_j^{(2)} C_j^+ . \quad (5.11)$$

Notice the absence in expression (5.9) of the term proportional to $b^+ = \sum_{j=1}^n S_j^{(2)} B_j^+$. Indeed, it is easy to check, that this quantity is equal to zero.

Introduce now similar notations for all other types of terms:

$$a^- = \sum_{j=1}^n S_j^{(2)} A_j^- , \quad (5.12)$$

$$c^- = \sum_{j=1}^n S_j^{(2)} C_j^- \quad (5.13)$$

where

$$A_j^- = \sum_{i=1}^n S_{j+i-1}^{(2)} \cos [2\tau(\theta_{j+i-1} - \theta_j) - \nu(\phi_{j+i-1} - \phi_j)] , \quad (5.14)$$

$$C_j^- = \sum_{i=1}^n S_{j+i-1}^{(2)} \left[2(\theta_{j+i-1} - \theta_j) + \frac{\alpha}{\delta}(\phi_{j+i-1} - \phi_j) \right] \\ \times \sin [\pi(2\tau - \nu) - 2\tau(\theta_{j+i-1} - \theta_j) + \nu(\phi_{j+i-1} - \phi_j)] \quad (5.15)$$

$$a_{1,2}^0 = \sum_{j=1}^n S_j^{(2)} A_j^{(1,2)} , \quad (5.16)$$

where

$$A_j^{(1,2)} = \sum_{i=1}^n S_{j+i-1}^{(1,2)} \cos \nu(\phi_j - \phi_{j+i-1}) . \quad (5.17)$$

The terms of $a_{1,2}^0$ types are of higher order of magnitude in α and do not enter the final result.

In a similar way one can calculate also the sine type terms in expression (3.10). In this case one needs to introduce quantities

$$d^\pm = \sum_{j=1}^n S_j^{(2)} D_j^\pm \quad (5.18)$$

and

$$d_{1,2}^0 = \sum_{j=1}^n S_j^{(2)} D_j^{(1,2)} , \quad (5.19)$$

where

$$D_j^{\pm} = \sum_{i=1}^n S_{j+i-1}^{(2)} \cos [\pi(2\tau \pm \nu) - 2\tau(\theta_{j+i-1} - \theta_j) \mp \nu(\phi_{j+i-1} - \phi_j)] \quad (5.20)$$

and

$$D_j^{(1,2)} = \sum_{i=2}^n S_{j+i-1}^{(1,2)} \cos [\pi\nu - \nu(\phi_{j+i-1} - \phi_j)] \quad (5.21)$$

In terms of the quantities a, c and d the final result for Δ_2 is:

$$\begin{aligned} \Delta_2 = & \frac{1}{4\pi} \left\{ \frac{2\pi(\epsilon_x + \epsilon_y/2)(1 + \alpha/2\delta)}{\sin^2 \pi(2\tau + \nu)} a^+ - \frac{\epsilon_x + \epsilon_y/2}{\sin \pi(2\tau + \nu)} c^+ \right. \\ & + \frac{2\pi(\epsilon_x - \epsilon_y/2)(1 + \alpha/2\delta)}{\sin^2 \pi(2\tau - \nu)} a^- - \frac{\epsilon_x - \epsilon_y/2}{\sin \pi(2\tau - \nu)} c^- \\ & + \frac{(\epsilon_x + \epsilon_x \frac{\alpha\tau}{\nu\delta} + \epsilon_y)}{\tau \sin \pi(2\tau + \nu)} d^+ + \frac{(\epsilon_x - \epsilon_x \frac{\alpha\tau}{\nu\delta} - \epsilon_y)}{\tau \sin \pi(2\tau - \nu)} d^- \\ & \left. + \frac{\epsilon_y d_2^0}{\tau \sin \pi\nu} - \frac{\epsilon_x d_1^0}{\tau \sin \pi\nu} \right\} \quad (5.22) \end{aligned}$$

For one sextupole in the ring expression (5.22) goes into expression (4.4) as it should be. For a ring with M identical superperiods expression (5.22) is invariant under the following transformation:

$$\Delta_2(\nu, \alpha, \tau, \delta, n) = \Delta_2\left(\frac{\nu}{M}, \frac{\alpha}{M}, \frac{\tau}{M}, \frac{\delta}{M}, \frac{n}{M}\right) \quad (5.23)$$

The quantities $a^{\pm,0}$, $c^{\pm}$ and $d^{\pm,0}$ can be named the (sextupole) distribution factors. For a given ring only the distribution factors change, when parameters of the distribution of the sextupole magnets, i.e. their number, strengths and positions in the ring, are changed. In general there are eight distribution factors altogether.

It is interesting to consider a ring, for which the sextupole distribution is built from the second order achromats.⁵ The most general conditions sufficient

for the sextupole distribution to be the second order achromat is:

$$\sum_{i=1}^m S_i \exp(\theta_i) = 0 \quad (5.24)$$

$$\sum_{i=1}^m S_i \exp(\phi_i) = 0 \quad (5.25)$$

$$\sum_{i=1}^m S_i \exp(3\theta_i) = 0 \quad (5.26)$$

$$\sum_{i=1}^m S_i \exp(3\phi_i) = 0 \quad (5.27)$$

for $\phi_m - \phi_1 = 2\pi$ and $\theta_m - \theta_1 = 2\pi$ (or any integer number of 2π 's). Applying this condition to expressions (5.10) through (5.19) we find that the distribution factors are equal zero simultaneously only for integer tunes. That means that for a storage ring build out of the second order achromats the geometric aberrations are not zero (in contrast to a transport line).

The reason for the emittance growth in this case is the fact, that for noninteger tune it is impossible to achieve condition when all the subsequent betatron phase advances between any two subsequent sextupoles belonging to the same sextupole family are multiples of an odd number of π 's. The system which has zero geometric aberrations is still possible. Apart from the integer values of tunes, such a system might be designed for example, by using much longer than one turn period of the sextupole distribution (many turns periodicity).⁶

6. NUMERICAL EXAMPLE FOR PEP STORAGE RING

Here I present the result of the evaluation of the vertical beam size growth for the current PEP configuration (assuming optimal coupling):

$$\nu = 21.25$$

$$\tau = 18.19$$

$$\beta_x^* = 3.0 \text{ m}$$

$$\beta_y^* = 0.11 \text{ m} \quad (6.1)$$

$$\alpha/\delta = 1.0$$

$$\sqrt{\epsilon_y/\epsilon_x} = 0.19$$

$$\sigma_x = 0.58 \text{ mm} .$$

Table I contains $\phi/2\pi$, $\theta/2\pi$, β_x , β_y and S for all sextupoles in one half of the mirror-symmetric superperiod.

The calculation by formula (5.22) yields:

$$\frac{\Delta\epsilon_y}{\epsilon_y} = 1.3\% . \quad (6.2)$$

The beam emittance enhancement of the same magnitude is found also for other points of the tune diagram which are not too close to the resonance lines $2\tau \pm \nu = \text{integer}$ and $\nu = \text{integer}$.

The beam increase $\Delta\epsilon_y/\epsilon_y$ goes as cube of the beta function value at the sextupole positions. Hence its value might be much larger for a larger size ring. The invariance of the result (6.2) under the transformation (5.23) has been confirmed by the numerical calculation.

TABLE I

Sext.	$\phi/2\pi$	$\theta/2\pi$	β_x (m)	β_y (m)	S (m ⁻²)
SD	0.74679	0.81079	5.87170	31.72110	-0.23973
SF	0.83853	0.86941	24.38300	14.53700	+0.25885
SD	0.91770	0.91702	10.45390	37.12470	-0.23973
SF	0.99003	0.96352	21.97980	18.71960	+0.25885
SD	1.07330	1.00559	10.45390	37.12460	-0.23973
SF	1.14563	1.05209	21.97980	18.71970	+0.25885
SD5	1.23057	1.09464	10.58500	36.70370	-0.05168
SF6	1.30124	1.14067	21.98000	18.71970	+0.22408
SD6	1.38617	1.18322	10.58500	36.70380	-0.78580
SF	1.45684	1.22925	21.97990	18.71970	+0.25885
SD7	1.54177	1.27180	10.58500	36.70380	-1.00800
SF	1.61244	1.31782	21.97980	18.71970	+0.25885
SD	1.69836	1.36070	10.92310	35.55290	-0.23973
SF9	1.75990	1.44829	33.30500	6.07620	+0.05629

7. ENHANCEMENT OF THE VERTICAL BEAM EMITTANCE DUE TO OCTUPOLE FIELD

The same technique which was applied to calculate the enhancement of the beam emittance due to sextupole field in the ring can easily be used to evaluate the effect due to octupole field as well. Since the first order term (in the octupole strength parameter) is not zero in this case, I will restrict myself to the first order calculation. Use now expression (2.10) for the perturbation force and (2.14) for the first order correction to the distribution function.

The first order correction to the beam emittance $E_y/\epsilon_y = 1 + \Delta_1$ is now:

$$\Delta_1 = \frac{r}{\epsilon_y} \sum_{k \leq t} \int dV \left(f_2 v' v^3 T^{(3)} + f_3 v^4 T^{(3)} - 3f_2 u^2 v' v T^{(2)} - 3f_3 u^2 v^2 T^{(2)} \right) \psi_0, \quad (7.1)$$

where $T^{(2)}$ and $T^{(3)}$ are defined in (2.12) and (2.13). The meaning of the coefficients f_2 and f_3 are the same as in all previous calculations, see expressions (3.5) and (3.6).

Due to symmetry of ψ_0 only terms proportional to f_3 in (7.1) are nonzero:

$$\Delta_1 = 3\tau \sum_{k \leq l} f_3 (T^{(3)}_{\epsilon_y} - T^{(2)}_{\epsilon_x}) . \quad (7.2)$$

Performing summation over k in the same way we have done before we get (in zeroth order with respect to δ/τ):

$$\Delta_1 = -\frac{3}{4\pi\tau} (e_2\epsilon_x - e_3\epsilon_y) , \quad (7.3)$$

where

$$e_2 = \sum_{i=1}^n T_i^{(2)} = \sum_{i=1}^n T_i \beta_{xi} \beta_{yi} \quad (7.4)$$

$$e_3 = \sum_{i=1}^n T_i^{(3)} = \sum_{i=1}^n T_i \beta_{yi}^2 . \quad (7.5)$$

ACKNOWLEDGEMENTS

Discussions with many people helped me in the course of preparing this paper. To all of them I am very grateful. My special gratitude goes to A. Chao, who helped me to resolve several troublesome points, and to H. Shoaee, who helped me with numerical calculations.

APPENDIX

Table of Integrals Relevant to Present Note

Consider a normalized unperturbed distribution function:

$$\psi(u, u') = \frac{1}{2\pi\epsilon\nu} \exp \left\{ -\frac{u^2}{2\epsilon} - \frac{u'^2}{2\epsilon\nu^2} \right\}$$

The values of $I = \int_{-\infty}^{\infty} f(u, u') \psi(u, u') du du'$ for several different functions $f(u, u')$ are given in the following table.

TABLE

$f(u, u')$	I
u	0
u'	0
u^2	ϵ
u'^2	$\epsilon\nu^2$
uu'	0
u^4	$3\epsilon^2$

REFERENCES

1. S. Kheifets, "Application of the Green's Function Method to Some Nonlinear Problems of an Electron Storage Ring. Part I. The Green's Function for the Fokker-Planck Equation," SLAC, Stanford University, PEP-NOTE-372, July 1982.
2. E. Courant and H. Snyder, Ann. Phys. 3, 1-48 (1958).
3. S. Kheifets and H. Wiedemann, SLAC, Stanford University, PTM-151, May 1978.
4. S. Kheifets, "Application of the Green's Function Method to Some Nonlinear Problems of an Electron Storage Ring. Part II. Checking the Method by a Quadrupole Perturbation," SLAC, Stanford University, PEP-NOTE-377, October 1982.
5. K. Brown, Nucl. Sci. NS-26, No. 3, p. 3460 (1979).
6. K. Brown, private communication.