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Two-Stream Theory

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RELATIVISTIC ELECTRON BEAM PROPAGATION: TWO-STREAM
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by

Barry S. Newberger

ABSTRACT

Progress achieved in the theoretical understanding of the two-stream instability as it pertains to the propagation of relativistic electron beams through low pressure neutral gases is presented.

Research has been performed on both aspects of the stability issue: linear stabilization of the two-stream mode and the nonlinear state of an unstable system. Linear stability for the high frequency two-stream mode requires the background plasma electrons to be collisional and the beam particles to have a finite distribution of momenta. The linear theory has been specifically carried out for a beam with a scattered distribution in which all particles have the same energy but have their momenta distributed in direction about the mean direction of beam propagation. Scaling laws obtained from the linear theory differ from earlier results based on essentially nonrelativistic models and underscore the need for a rigorous linear theory. In a nonlinear theory, the mode saturation amplitude is an important quantity. In the two-stream instability, this is determined by particle trapping and can be obtained by modeling the background plasma as a linear dielectric medium while treating the beam particles nonlinearly. This scheme is typically referred to as partial numerical simulation. A generalized partial numerical simulation formalism is described and results from its implementation in a computer code, PNS, are discussed. The course of future work is presented.

I. INTRODUCTION

For several years, there has existed an interest in the propagation of intense, relativistic electron beams through neutral gases.¹ While such a description is somewhat misleading as any gas through which such a beam is moving does not remain neutral very long, it is useful in that it calls attention to the fact that there may be physical processes which should be included in

any attempt to model these systems. It is the presence of charged bodies over almost the entire lifetime of the beam, however, that appears to be responsible for the destruction of the beam after traveling relatively short distances.

Experimental results over fairly large spans of parameter space are consistent with beam disruption by way of hydromagnetic instabilities at high and moderate values of initial neutral gas pressure and microinstabilities at low to moderate values of the pressure. In the work to be described below, we will focus our attention on the latter instabilities.

In general terms, there are three issues which are important in considering an instability mechanism as it applies to REB propagation in neutral gases. The fundamental issue is whether there are any linearly unstable modes for the model under consideration. An integral part of this issue is, of course, the description of the unperturbed state, or equilibrium, whose stability in which one is interested. Given that there are one or more unstable modes, the question of the non-linear state to which these evolve must then be addressed. If the nonlinear state is weak, the instabilities may have little or no effect over the lifetime of the beam while a strong nonlinear state may essentially mean the destruction of the beam altogether. If indeed the nonlinear state is such that the beam is severely disrupted, one must then investigate the possibility of modifying the equilibrium in such a manner as to weaken the nonlinear state to the point where the beam remains, effectively, intact.

None of the issues above is simple and, to date, no complete theory including even the most fundamental of the physical considerations has been carried out. Research undertaken over the past year has been aimed at addressing some of these issues with the intent of developing a theoretical base with sufficient content to allow both comparison with past experiments and to be useful in the design of future experiments. To this end, a general hybrid computer code based on the method of "partial numerical simulation"^{2,3} has been developed. Treating the motion of the beam particles nonlinearly and the background plasma linearly, these methods are useful in examining the nonlinear saturation of two-stream (beam-plasma) modes especially in regions of parameter space where full numerical simulations are too costly. By generalizing this method, physical effects heretofore ignored have been included. Nonlinear models are of little use, however, if the linear theories on which they rest are weak. While the linear theory of instability in beam-plasma systems has been the subject of study for many years,^{4,5} a rigorous, complete theory for a

relativistic electron beam with velocity spread passing through plasma was not available. We have made significant progress in developing the theory. Some results that have been obtained show substantial disagreement with conclusions based on simplified models.⁶ As experiments are currently being planned on the basis of these models, the importance of a complete theory is obvious.

In Secs. II and III below, the nonlinear model and the linear theory will be described and some results presented. Details of the calculations will be given in the Appendices.

II. NONLINEAR MODEL - PARTIAL NUMERICAL SIMULATIONS

A fundamental question in the plasma physics of any system which admits unstable small amplitude motions is the question of the final, finite amplitude state to which the system evolves. Various techniques have evolved to examine this issue but, to date, there is no complete picture, although several texts have been written outlining the state of the subject.^{7,8,9}

In the configuration of interest, the interaction of an intense relativistic electron beam with a plasma, we can utilize the fact that the most unstable waves interact with the background plasma nonresonantly. That is, the phase velocity of the most unstable waves is in the vicinity of the beam velocity which, in turn, is about the speed of light. Consequently, the speed of any motion of the plasma particles is substantially less than the phase velocity of the wave and the plasma interaction is sufficiently weak that it can be treated in the linear limit. Thus, the only nonlinear motion that needs to be followed is that of the beam particles in the self-consistent field of the wave. While this is a considerable simplification and alleviates one of the difficulties of full numerical simulations, it still results in a formalism which must be solved numerically. Furthermore, no guarantee is provided that the plasma response will remain linear for all time and, indeed, such is not the case. To account for the heating observed both in numerical simulations¹⁰ and experiments¹¹, nonlinear effects in the plasma response must be included. However, justification for the method at least until the time at which the beam particles become trapped is provided by full numerical simulations^{3,10} and laboratory systems such as the traveling wave tube. In the latter, the medium with which the beam interacts is manifestly linear and the response is consistent with theoretical expectations.¹² We will outline below the derivation of

the formalism and present a few results. Details of the calculations will be given in Appendix A.

The system under consideration consists of a background plasma of density n_p , interpenetrated by a relativistic electron beam of density n_b . The beam energy is γ_0 and both the beam and plasma will be taken to be infinite in extent. As described above, the plasma response is linear and thus conventionally described by a dielectric tensor $\bar{\epsilon}$ which may be complex. Note that $\bar{\epsilon} = \bar{\epsilon}(\omega)$ is a frequency domain quantity. We will limit our attention to electrostatic modes. The beam electrons respond nonlinearly to the electrostatic fields in the system which are generated self-consistently via Poisson's equation. Beam particle motion is followed by integrating their equations of motion directly.

Specifically, we follow the evolution of the beam and a superposition of test waves. The test waves are prescribed by

$$\vec{E}(t) = \frac{1}{2i} \left\{ \sum_{j=1}^L \vec{E}_j(t) \exp[i\vec{k}_j \cdot \vec{x} - \omega_j t + \alpha_j(t)] - \text{c.c.} \right\} , \quad (\text{II.1})$$

where c.c. denotes the complex conjugate. The $E_j(t)$ and $\alpha_j(t)$ are assumed to be slowly varying functions of time, i.e.,

$$\frac{1}{\omega_p} \frac{d}{dt} [\vec{E}_j(t), \alpha_j(t)] \ll 1 . \quad (\text{II.2})$$

The growth rate of the unstable modes is reflected in $d\vec{E}_j/dt$ and $d\alpha_j/dt$ allows for a frequency shift away from the linear mode frequency ω_j and is essentially a nonlinear effect. The mode wave vector is $\vec{k}_j$. A set of equations for the evolution of the E_j 's and α_j 's are derived by substitution into Poisson's equation.

Since we are treating the plasma as a linear dielectric medium, Poisson's equation takes the form

$$\nabla \cdot \vec{D} = 4\pi\rho_{\text{ext}} \quad (\text{II.3})$$

where $\vec{D}$ is the electric displacement and ρ_{ext} are the external charge densities in the system. In our case, these will be the beam particles. Note that

Eq. (II.3) is in the time domain, thus $\vec{D}(t)$ is defined by

$$\vec{D}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \vec{\epsilon}(\omega) \cdot \int_{-\infty}^{\infty} \vec{E}(\tau) e^{i\omega\tau} d\tau \quad . \quad (\text{II.4})$$

Substituting our definition of $\vec{E}(t)$ into Eq. (II.4) gives

$$\begin{aligned} \vec{D}(t) = \sum_j \frac{1}{2i} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \vec{\epsilon}(\omega) e^{-i\omega t} d\omega \cdot \int_{-\infty}^{\infty} \{ \vec{E}_j(\tau) \exp[i\vec{k}_j \cdot \vec{x} \right. \\ \left. - \omega_j \tau + \alpha_j(\tau)] - \text{c.c.} \} e^{i\omega\tau} d\tau \right) \quad . \end{aligned} \quad (\text{II.5})$$

Further substitution into Poisson's equation, gives

$$\begin{aligned} \sum_j \frac{\vec{k}_j}{4\pi} \cdot \int_{-\infty}^{\infty} \vec{\epsilon} e^{-i\omega t} d\omega \cdot \int_{-\infty}^{\infty} \{ \vec{E}_j \exp[i\vec{k}_j \cdot \vec{x} - \omega_j \tau + \alpha_j(\tau)] e^{i\omega\tau} d\tau \\ + \text{c.c.} \} = 4\pi \rho_{\text{ext}} \quad . \end{aligned} \quad (\text{II.6})$$

The complex conjugate now refers to the entire expression in the brackets and follows from the relation $\vec{\epsilon}(\omega) = \vec{\epsilon}^*(-\omega)$ for ω real.¹³

Equation (II.6) is an integral equation for the electric field. One can reduce this to an equivalent infinite order system of differential equations which is, of course, no simpler to solve than the original integral equation. However, in the limit

$$\frac{1}{(\omega_j)^m} \frac{d^m E_j}{dt^m}, \quad \frac{1}{(\omega_j)^m} \frac{d^m \alpha_j}{dt^m} \ll 1$$

for any m , the system can be truncated at order n yielding essentially an asymptotic theory in $(\gamma_j/\omega_j)^n \cong (\gamma/\omega_p)^n \ll 1$, where γ is a typical two-stream growth rate and ω_p the plasma frequency. The details of this calculation are given in Appendix A. Retaining only the lowest order terms Poisson's equation, Eq. (II.3) becomes

$$\begin{aligned}
& \sum_{j=1}^L \left\{ \vec{k}_j \cdot \text{Re } \vec{\bar{\epsilon}}_j' \cdot \dot{\vec{E}}_j \sin \psi_j + \vec{k}_j \cdot \text{Im } \vec{\bar{\epsilon}}_j \cdot \dot{\vec{E}}_j \cos \psi_j \right. \\
& + \vec{k}_j \cdot \text{Re } \vec{\bar{\epsilon}}_j \cdot \dot{\alpha}_j \vec{E}_j \cos \psi_j - \vec{k}_j \cdot \text{Im } \vec{\bar{\epsilon}}_j' \cdot \dot{\alpha}_j \vec{E}_j \sin \psi_j \\
& \left. - \vec{k}_j \cdot \text{Re } \vec{\bar{\epsilon}}_j \cdot \vec{E}_j \cos \psi_j + \vec{k}_j \cdot \text{Im } \vec{\bar{\epsilon}}_j \cdot \vec{E}_j \sin \psi_j \right\} \\
& = -8\pi^2 \rho_{\text{ext}}
\end{aligned} \tag{II.7}$$

where ψ_j is the phase factor of the j th mode, $\psi_j = \vec{k}_j \cdot \vec{x} - \omega_j t + \alpha_j(t)$, primes denote differentiation with respect to ω , superimposed dots differentiation with respect to time and the subscript on $\vec{\bar{\epsilon}}$ and $\vec{\bar{\epsilon}}'$ remind us that they are evaluated at ω_j .

A set of equations for the $E_j(t)$ and $\alpha_j(t)$ can be obtained by projection using the operator

$$\int_{-\lambda_i/2}^{\lambda_i/2} f(\psi_j) \begin{Bmatrix} \cos \psi_i \\ \sin \psi_i \end{Bmatrix} dx_i$$

where dx_i is in the direction of $\vec{k}_i$ and $\lambda_i = 2\pi/k_i$.

The results of this operation lead to a system of $2L$ equations.

$$\begin{aligned}
& \sum_{j=1}^L M_{ij}^1 \vec{k}_j \cdot \text{Re } \vec{\bar{\epsilon}}_j' \cdot \dot{\vec{E}}_j + \sum_{j=1}^L C_{ij} \vec{k}_j \cdot \text{Im } \vec{\bar{\epsilon}}_j' \cdot \dot{\vec{E}}_j \\
& + \sum_{j=1}^L C_{ij} \vec{k}_j \cdot \text{Re } \vec{\bar{\epsilon}}_j' \cdot \vec{E}_j \dot{\alpha}_j - \sum_{j=1}^L M_{ij}^1 \vec{k}_j \cdot \text{Im } \vec{\bar{\epsilon}}_j' \cdot \vec{E}_j \dot{\alpha}_j \\
& - \sum_{j=1}^L C_{ij} \vec{k}_j \cdot \text{Re } \vec{\bar{\epsilon}}_j \cdot \vec{E}_j + \sum_{j=1}^L M_{ij}^1 \vec{k}_j \cdot \text{Im } \vec{\bar{\epsilon}}_j \cdot \vec{E}_j \\
& = -4\pi \int_{-\pi}^{\pi} \rho_{\text{ext}} \frac{\vec{k}_i \cdot \vec{\theta}}{k_i^2} \cos(\theta - \omega_i t + \alpha_i(t)) d\theta
\end{aligned} \tag{II.7a}$$

and

$$\begin{aligned}
& \sum_{j=1}^L S_{ij} \vec{k}_j \cdot \text{Re } \bar{\vec{\epsilon}}_j' \cdot \dot{\vec{E}}_j + \sum_{j=1}^L M_{ij}^2 \vec{k}_j \cdot \text{Im } \bar{\vec{\epsilon}}_j' \cdot \dot{\vec{E}}_j \\
& + \sum_{j=1}^L M_{ij}^2 \vec{k}_j \cdot \text{Re } \bar{\vec{\epsilon}}_j' \cdot \dot{\vec{E}}_j \dot{\alpha}_j - \sum_{j=1}^L S_{ij} \vec{k}_j \cdot \text{Im } \bar{\vec{\epsilon}}_j' \cdot \dot{\vec{E}}_j \dot{\alpha}_j \\
& - \sum_{j=1}^L M_{ij}^2 \vec{k}_j \cdot \text{Re } \bar{\vec{\epsilon}}_j \cdot \dot{\vec{E}}_j + \sum_{j=1}^L S_{ij} \vec{k}_j \cdot \text{Im } \bar{\vec{\epsilon}}_j \cdot \dot{\vec{E}}_j \\
& = -4\pi \int_{-\pi}^{\pi} \rho_{\text{ext}} \left(\frac{\vec{k}_i \theta}{k_i^2} \right) \sin(\theta - \omega_i t + \alpha_i(t)) d\theta
\end{aligned} \tag{II.7b}$$

where the coefficients C_{ij} , S_{ij} , M_{ij}^1 , and M_{ij}^2 are defined and evaluated in Appendix A, Eqs. A.7 through A.12.

We must now consider the beam particles which, in this problem, comprise the external charges. This will allow us to evaluate the right sides of Eqs. 7a and 7b above. The beam particle distribution is taken to be of the Klimontovich type

$$f(\vec{x}, \vec{v}) = \frac{n_{b0}}{N} \sum_{\ell=1}^N \prod_{n=1}^3 \delta[v - v_n^\ell(t)] \delta[x - x_n^\ell(t)] \tag{II.8}$$

where N is the number of particles in the beam, v_n^ℓ is the n th component of velocity of the ℓ th particle along its trajectory parameterized by t , x_n^ℓ is its corresponding position and n_{b0} the mean beam density. The particle density $n_b(\vec{x})$ is then

$$n_b = \frac{n_{b0}}{N} \sum_{\ell=1}^N \prod_{n=1}^3 \delta[x_n - x_n^\ell(t)] \tag{II.9a}$$

and the current density

$$j_{bm}(\vec{x}) = -e \frac{n_{bo}}{N} \sum_{\ell=1}^N v_m^{\ell}(t) \prod_{n=1}^3 \delta[x_n - x_n^{\ell}(t)] \quad . \quad (II.9b)$$

The charge density ρ_{ext} is then

$$\rho_{ext} = -e n_b \quad (II.9c)$$

where e is the magnitude of the electronic charge. The contributions to Eqs. II.8a and II.8b due to the beam charge are thus

$$- \frac{en_{bo}}{N} \sum_{\ell=1}^N \begin{pmatrix} \cos [\vec{k}_i \cdot \vec{x}^{\ell} - \omega_i t + \alpha_i(t)] \\ \sin [\vec{k}_i \cdot \vec{x}^{\ell} - \omega_i t + \alpha_i(t)] \end{pmatrix} , \quad (II.10)$$

respectively.

Using the fact that the modes are electrostatic, $\vec{E}_j = \vec{k}_j E_j / k_j$, introducing the dimensionless quantities $\varepsilon_j = E_j / (4\pi n_e m_e c^2)^{1/2}$, $\tau = \omega_e t$, $\kappa_j = (c/\omega_e) k_j$, $\vec{\chi}_\ell = (\omega_e/c) \vec{x}_\ell$ and $\tilde{\omega}_i = \omega_i / \omega_e$ where n_e is the plasma density and ω_e the corresponding electron plasma frequency, m_e , the electron rest mass and c , the speed of light and introducing the column vectors

$$\vec{\alpha} \equiv \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_L \end{pmatrix} \quad (II.11a)$$

$$\vec{\varepsilon} \equiv \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_L \end{pmatrix} \quad (II.11b)$$

$$\bar{\rho}_{s,c} = \frac{n_b}{n_e N} \begin{bmatrix} \sum_{\ell=1}^N \frac{\sin}{\cos} (\vec{\kappa}_J \cdot \vec{\chi}^\ell - \tilde{\omega}_J \tau + \alpha_J) \\ \vdots \\ \sum_{\ell=1}^N \frac{\sin}{\cos} (\vec{\kappa}_L \cdot \vec{\chi}^\ell - \tilde{\omega}_L \tau + \alpha_L) \end{bmatrix} \quad (\text{II.11c})$$

and the diagonal matrices

$$\bar{\varepsilon} = \begin{bmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & 0 \\ & & \ddots & \\ 0 & & & \varepsilon_L \end{bmatrix} \quad (\text{II.11d})$$

$$\bar{\bar{K}}_{R,I} = \begin{bmatrix} \frac{\vec{\kappa}_1}{\kappa_1} \cdot \begin{pmatrix} \text{Re} \bar{\varepsilon}_1 \\ \text{Im} \bar{\varepsilon}_1 \end{pmatrix} \cdot \vec{\kappa}_1 & & 0 \\ & \ddots & \\ & & \frac{\vec{\kappa}_L}{\kappa_L} \cdot \begin{pmatrix} \text{Re} \bar{\varepsilon}_L \\ \text{Im} \bar{\varepsilon}_L \end{pmatrix} \cdot \vec{\kappa}_L \\ 0 & & & \end{bmatrix} \quad (\text{II.11e})$$

and

$$\bar{\bar{K}}'_{R,I} = \begin{bmatrix} \frac{\vec{k}_1}{\kappa_1} \cdot \begin{Bmatrix} \text{Re} \bar{\bar{\epsilon}}'_1 \\ \text{Im} \bar{\bar{\epsilon}}'_1 \end{Bmatrix} \cdot \vec{k}_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ 0 & & & \frac{\vec{k}_L}{\kappa_L} \cdot \begin{Bmatrix} \text{Re} \bar{\bar{\epsilon}}'_L \\ \text{Im} \bar{\bar{\epsilon}}'_L \end{Bmatrix} \cdot \vec{k}_L \end{bmatrix} \quad (II.11f)$$

With these definitions, and the projection matrices $\bar{\bar{C}}$, $\bar{\bar{S}}$, $\bar{\bar{M}}^1$, $\bar{\bar{M}}^2$ we can write Poissons equation in very compact notation,

$$\bar{\bar{T}}'_c \cdot \dot{\bar{\epsilon}} + \bar{\bar{T}}_c^2 \cdot \dot{\bar{\epsilon}}^d \cdot \dot{\bar{\alpha}} - \bar{\bar{C}} \cdot \bar{\bar{K}}_R \cdot \bar{\epsilon} + \bar{\bar{M}}^1 \cdot \bar{\bar{K}}_I \cdot \bar{\epsilon} = \bar{\rho}_c \quad (II.12a)$$

and

$$\bar{\bar{T}}'_s \cdot \dot{\bar{\epsilon}} + \bar{\bar{T}}_s^2 \cdot \dot{\bar{\epsilon}}^d \cdot \dot{\bar{\alpha}} + \bar{\bar{M}}^2 \cdot \bar{\bar{K}}_R \cdot \bar{\epsilon} - \bar{\bar{S}} \cdot \bar{\bar{K}}_I \cdot \bar{\epsilon} = -\bar{\rho}_s \quad (II.12b)$$

where

$$\bar{\bar{T}}'_c = \bar{\bar{C}} \cdot \bar{\bar{K}}'_I + \bar{\bar{M}}^1 \cdot \bar{\bar{K}}'_R \quad (II.13a)$$

$$\bar{\bar{T}}_c^2 = \bar{\bar{C}} \cdot \bar{\bar{K}}'_R - \bar{\bar{M}}^1 \cdot \bar{\bar{K}}'_I \quad (II.13b)$$

$$\bar{\bar{T}}'_s = \bar{\bar{S}} \cdot \bar{\bar{K}}'_R + \bar{\bar{M}}^2 \cdot \bar{\bar{K}}'_I \quad (II.13c)$$

and

$$\bar{\bar{T}}_s^2 = \bar{\bar{S}} \cdot \bar{\bar{K}}'_I - \bar{\bar{M}}^2 \cdot \bar{\bar{K}}'_R \quad (II.13d)$$

It is now a simple matter to solve, formally, for the derivatives $\bar{\varepsilon}$ and $\bar{\alpha}$ and thus cast Eqs. (12a) and (12b) in a form suitable for integration by any of several numerical techniques. The results are

$$\begin{aligned} \dot{\bar{\varepsilon}} = & \left((\bar{T}_c^2)^{-1} \cdot \bar{T}_c^1 + (\bar{T}_s^2)^{-1} \cdot \bar{T}_s^1 \right)^{-1} \cdot \\ & \{ [(\bar{T}_c^2)^{-1} \cdot \bar{C} + (\bar{T}_s^2)^{-1} \cdot \bar{M}^2] \cdot \bar{K}_R \cdot \bar{\varepsilon} \\ & - [(\bar{T}_c^2)^{-1} \cdot \bar{M}^1 + (\bar{T}_s^2)^{-1} \cdot \bar{S}] \cdot \bar{K}_I \cdot \bar{\varepsilon} \\ & + (\bar{T}_c^2)^{-1} \cdot \bar{\rho}_c + (\bar{T}_s^2)^{-1} \cdot \bar{\rho}_s \} \end{aligned} \quad (\text{II.14a})$$

and

$$\begin{aligned} \dot{\bar{\alpha}} = & \{ [(\bar{T}_c^1)^{-1} \cdot \bar{T}_c^2 + (\bar{T}_s^1)^{-1} \cdot \bar{T}_s^2] \cdot \bar{\varepsilon}^d \}^{-1} \cdot \\ & \{ [(\bar{T}_c^1)^{-1} \cdot \bar{C} - (\bar{T}_s^1)^{-1} \cdot \bar{M}^2] \cdot \bar{K}_R \cdot \bar{\varepsilon} \\ & - [(\bar{T}_c^1)^{-1} \cdot \bar{M}^1 - (\bar{T}_s^1)^{-1} \cdot \bar{S}] \cdot \bar{K}_I \cdot \bar{\varepsilon} \\ & + (\bar{T}_c^1)^{-1} \cdot \bar{\rho}_c - (\bar{T}_s^1)^{-1} \cdot \bar{\rho}_s \} \end{aligned} \quad (\text{II.14b})$$

In the special case that the $\bar{T}$ matrices are singular, the inversion of Eq. (12a,b) are simple. The result can be obtained by formally passing to the limit $(\bar{T})^{-1} \rightarrow \infty$. Numerical techniques which test for singularity can be employed and the analytic results found as described above used as appropriate.

The equations for the beam particle orbits are trivial to describe as they are simply Newton's laws. The beam particles are, of course, relativistic and thus, in terms of our dimensionless quantities, become

$$\frac{d\vec{\chi}^{\ell}}{d\tau} = \frac{\vec{u}^{\ell}}{\gamma^{\ell}} \quad (11.15a)$$

and

$$\begin{aligned} \frac{d\vec{u}^{\ell}}{d\tau} = & - \sum_{j=1}^L \frac{\vec{\kappa}_j}{\kappa_j} \varepsilon_j \sin(\vec{\kappa}_j \cdot \vec{\chi}^{\ell} - \tilde{\omega}_j \tau + \alpha_j(\tau)) \\ & + \frac{\vec{u}^{\ell} \times \vec{B}_0}{\gamma^{\ell}}, \end{aligned} \quad (11.15b)$$

$$\ell = 1, \dots, N.$$

A computer code, PNS, based on the model, has been written. The details of the numerics will not be described. It is sufficient to point out that standard subroutine packages provided as part of the FORTRAN compiler-operating system by the Los Alamos Scientific Laboratory Central Computing Facility have been used throughout. While this allows fairly efficient programming, it could mean reduced running efficiency and some code optimization may be anticipated in the future.

Some results will now be given. Obviously, it is not the intent to present a comprehensive study of a particular case but to present a few cases which exercise the different capabilities of the model and to familiarize the reader with the kind of information which is obtained.

The first example is a single mode calculation and exhibits the classical characteristics of the partial numerical simulation. The beam is propagating through a cold collisionless plasma, $\gamma_0 = 10.0$ and the beam to plasma density ratio is, $n_b/n_p = 1.0 \times 10^{-2}$. The fastest growing mode is at $kc/\omega_p \equiv \tilde{k} = 1.00553$ and has a frequency $\omega_o/\omega_p = \tilde{\omega}_o = 0.99124$. The growth rate scaled from the plot of ε as a function of $\omega_o t$, Fig. 1 is $\delta/\omega_p \equiv \tilde{\delta} = 1.49 \times 10^{-2}$ which agrees quite favorably with that predicted by linear theory, $\tilde{\delta} = 1.48 \times 10^{-2}$. Thus, the wave grows at the linear growth rate over about 4.5 e-folds, until it very rapidly saturates by beam particle trapping. This has occurred at $\omega_p t = 414.0$ in Fig. 1. Subsequently, there persists a finite amplitude wave in the

system, with the beam particles undergoing trapping oscillations in the field of the wave. Presumably the trapping oscillations would persist, however, full numerical simulations show that in fact the wave grows to the saturation amplitude predicted by the partial numerical simulation and then decays. In the full numerical simulations, nonlinear processes involving the background plasma are responsible for the decay of one wave and the related appearance of heating of the background plasma.³ Since the plasma nonlinearities are not considered in partial numerical simulations, the decay of the finite amplitude cannot be seen, at least as it results from nonlinear processes in the plasma. In traveling wave tube geometries, where a helical conductor replaces the plasma as the slow wave medium, and where this medium is manifestly linear in its response, persistent finite amplitude waves have been seen. Furthermore these waves were shown to be capable of being destroyed by resistivity in the medium or the presence of other unstable waves in the system.¹² Such behavior can be accessible to partial numerical simulation. In Fig. 2 the wave frequency shift due to the onset of nonlinear effects is seen. During the time the wave is growing at the linear rate, there is no frequency shift, i.e., $d\alpha/dt = 0$. The local maximum in $d\alpha/dt$ which occurs early in time, here about $\omega_p t = 94$, is typical and certainly spurious. Presumably for early times the expansion in $\bar{\delta} \ll 1$ is breaking down. This could be related to the fact that if the beam particles are loaded, initially, uniformly in space and the electric has a finite, though small, initial value, the system is physically inconsistent although mathematically consistent. The initial growth of the electric field at a rate substantially slower than the linear growth rate is probably related. We have attempted to test this possibility by holding the electric fixed in amplitude until the particle motions have become consistent with the applied field but preliminary results indicate that the behavior of $d\alpha/dt$ is essentially unaffected. If this is the case, the breakdown is more fundamental and not understood. It is important to note that the results are unaffected, however this behavior can present difficulties for the numerical integration packages in multi-wave problems. The mean kinetic energy, $\langle \gamma \rangle$ is shown as a function of $\tau \equiv \omega_p t$ in Fig. 3. One can estimate the energy and momentum the beam would give up to the plasma from $\gamma_0 - \langle \gamma \rangle$ at the time of saturation. The phase space (momentum, p , versus position, x ,) is shown in Fig. 4 at the particular time $\tau = 461.7$ which is just after the onset of trapping. The vortex structure in phase space is

typical² and the trapping oscillations seen in Fig. 1 are reflected as a rotation, with time, of the phase space vortices.

The second example is a two-wave calculation. The beam in this case has a kinetic energy, $\gamma_0 = 2.0$ and $n_b/n_p = 0.1$. The first wave has $\tilde{k} = 0.6$ and the second wave is the second spatial harmonic with $\tilde{k} = 1.2$. This mode is essentially at the linear growth peak and has a linear growth rate of $\tilde{\delta} = 0.146$. The field amplitudes are initialized differently so that the effect of the more slowly growing fundamental on the more rapidly growing harmonic can be seen. The fundamental amplitude, ε_1 , was initialized at $\varepsilon_1(0) = 1.0 \times 10^{-4}$ while the harmonic amplitude, ε_2 , was initially $\varepsilon_2(0) = 1.0 \times 10^{-6}$. Figures 5 and 6 are, respectively, the values of ε_1 and ε_2 as a function of the time, τ . The very early behavior of both field amplitudes are markedly effected by the presence of the other wave, however in view of our previous remarks this is probably not correct in detail. ε_2 grows very rapidly until it is about equal in amplitude with ε_1 and then growth at the linear rate ensues. The growth rate scaled from Fig. 6, $\tilde{\delta} = 0.148$ is in good agreement with that from linear theory. From this point it is clear that the evolution of the harmonic is independent of the fundamental. The converse is equally clearly not true. No region of linear growth is observed in the fundamental. This contrasts with single mode calculations made for this case which shows the canonical behavior described in our first example. The saturation amplitude of the fundamental is about eighty times smaller in the case when the harmonic is present than when it alone is present in the system. The saturation amplitude of the harmonic is the same in both cases, however. The frequency shifts of the waves are illustrated in Fig. 7 and 8. The frequency shift of the fundamental is strongly perturbed away from the canonical single wave behavior consistent with the observation that no region of exponential growth in the field amplitude exists. When the amplitude of the two waves are comparable, there is a positive shift in the frequency of the harmonic (a negative value of $d\alpha/d\tau$ is a positive frequency shift since the phase function is $\vec{k} \cdot \vec{x} - \tilde{\omega}\tau + \alpha$) which approaches zero as the harmonic grows above the fundamental. There is an oscillation in the frequency shift at a frequency approximately one-half that of the fundamental during this period, $23 < \tau < 43$. The result for the average kinetic energy and the phase space, Fig. 9 and 10 respectively, are little effected since their features only develop after the beam particle motion becomes non-linear and occurs only upon saturation of the harmonic, ε_2 .

Having illustrated the features typical in partial numerical simulation models and described in some detail the development of the formalism which defines the model, we will now return to the issue of linear stability in relativistic electron beam-plasma systems. While of fundamental interest, this issue bears significantly on the partial numerical simulation in that our test waves are chosen to oscillate in space and time at the normal mode frequencies of the linear system and thus the solution of the linear dispersion relation is needed as input to the nonlinear model. Discussion of future application of the model will be deferred to the last section which will briefly describe those issues which are to be addressed as we progress from the current state of knowledge in the microinstability of relativistic electron beams.

III. LINEAR THEORY OF A SCATTERED BEAM

No physically realizable relativistic electron beam is monochromatic in the particle momenta. The spectrum in momenta can come from several sources which may be operating simultaneously. For example, in diode generated beams scattering in the anode foil as the electrons traverse the anode lead to a finite spread in momenta even though the particles are, to a good approximation, monochromatic in energy. In a self pinched beam, the particle motion in the self-consistent fields, the so called betatron motion, can lead to a distribution of momenta at a fixed point in space. The Bennett beam¹⁴ is one such case.

The presence of a finite momentum spread can alter the nature of the two-stream instability. As the spread increases, the nature of the instability changes from a hydrodynamic mode with all the beam particles acting together to, ultimately, a bump-on-tail instability⁵ where only those particles in a small range, Δv , around the wave phase velocity participate. The latter is, of course much weaker. The two-stream is persistent, however, and no amount of spread, alone, will stabilize the mode. If the plasma through which the beam is propagating is collisional or warm, however, a spread in beam momenta can lead to stabilization of the mode.^{5,7} In the propagation of a relativistic electron beam through an initially neutral gas, the plasma produced is cold but the fractional ionization is small so the plasma can be very collisional. It is important then to address the problem of a relativistic electron beam propagating through a collisional plasma with respect to the two stream instability.

The stabilization of the two-stream instability in a warm beam by collisions on the plasma particles was first considered by Singhaus.⁷ Wave propagation along the beam was considered and the particles had no motion transverse to the beam. Relativistic effects only appeared in the mass of the beam particles where the rest mass, m_0 , was enhanced by the factor γ_0^3 , where γ_0 is essentially the beam kinetic energy. In the high temperature limit, a collisional stabilization was found and a simple criterion for the magnitude of the collision frequency required for stabilization was established. Due to the simplified model for relativistic effects which are inherent in the calculation, we will see, shortly, that this result can be substantially optimistic. In several cases, a Lorentzian distribution in velocity space has been used to estimate the effect of beam thermal spread on the two stream.^{15,16} While this model makes for ease in analysis, the strong mass anisotropy in highly relativistic beams casts doubt on their validity in the experimental regimes which are currently of interest. Here more realistic distribution functions may project onto the direction of beam propagation with a longitudinal mass which is reduced from $\gamma_0^3 m_0$.

A distribution function which does have this kind of behavior is that of the so-called scattered beam.^{17,18} This beam is monochromatic in energy with all particles having an energy $\gamma = \gamma_0$. (Since the system is assumed to be charge neutral and infinite in extent the potential energy is constant and taken to be zero). The beam particles are not monochromatic in momenta, however, and have a spread in momenta both transverse to and parallel to the mean momentum of the beam which is in the propagation direction. The distribution function is given by

$$f_b(p) = \frac{1}{2\pi p_0^2} \delta(p-p_0) h(\cos \theta) \quad (\text{III.1a})$$

$$h(\cos \theta) = \frac{a}{2} \text{csch } a \exp(a \cos \theta) \quad (\text{III.1b})$$

where $p_0 = (\gamma_0^2 - 1)^{\frac{1}{2}}$ and a is a characteristic width of the momentum spread related to the mean angle in momentum space, $\bar{\theta}$, by $a = 2/\bar{\theta}^2$. While this distribution in itself provides a good test of the effect of spread of velocities in the parallel direction and hence the effect of reduction of the mean parallel mass from $\gamma_0^3 m_0$ on the growth rates of the instability,¹⁸ it

is also a reasonable model of the actual distribution function in a high energy beam generated in a foil diode.¹⁸ Furthermore, for relatively small scattering angles the distribution is approximately Gaussian and we can use the model to estimate the effect of the betatron motion on the two-stream instability¹⁹ although it must be emphasized that the model is in no way a rigorous linear analysis of the normal modes in a self-pinch equilibrium. This pertains even if a system of finite extent transverse to the beam were considered but the model were otherwise unchanged.

Specifically, we will consider the linear stability of a scattered relativistic electron beam in which both the beam and plasma are taken to be infinite and spatially homogeneous. The plasma is collisional and cold and the collisions enter the plasma contribution to the dispersion function through a mean collision frequency ν_c . The collisional stabilization of the mode will be an important result of the theory. The beam distribution is as in Eq. (I.1). Results presented here will be limited to parallel propagation however the general dispersion function will be given and a representation for a particular integral arising in the beam contribution to the dispersion function will be derived in Appendix B. This representation will be essential in extending the results to the important case of non-parallel propagation.

The dispersion function for a scattered beam in the limit of a collisionless plasma is¹⁷

$$D_o(\tilde{\omega}) \equiv 1 - 1 + \frac{m_e}{m_i} \frac{1}{\tilde{\omega}^2} - \varepsilon(\tilde{\omega}, \tilde{k}) \quad (\text{III.2})$$

where ε describes the beam contribution, $\tilde{\omega}$ is the frequency in terms of the plasma frequency, ω_p and $\tilde{k}$ the wavenumber in units of ω_p/c . The beam contribution $\varepsilon(\tilde{\omega})$ is in the case of propagation along the beam direction given by

$$\begin{aligned} \varepsilon(\tilde{\omega}, \tilde{k}) = & \frac{\alpha a}{\gamma_o^3} \frac{\coth a + \tilde{\omega}/\beta_o \tilde{k}}{\tilde{\omega}^2 - \beta_o^2 \tilde{k}^2} \\ & + \frac{\alpha \text{acscha}}{2\gamma_o (\beta_o \tilde{k})^2} \int_{-1}^1 \frac{(a - 2\beta_o^2 x - a\beta_o^2 x^2) e^{ax} dx}{(x - \tilde{\omega}/\beta_o \tilde{k})} \end{aligned} \quad (\text{III.3})$$

where $\alpha \equiv \frac{n_b}{n_p}$ and $\beta_o^2 = (1 - 1/\gamma_o^2)$.

In the case of oblique propagation, the beam contribution, $\varepsilon(\tilde{\omega})$ is substantially more complicated.

$$\begin{aligned} \varepsilon(\tilde{\omega}) = & \frac{-\alpha}{\gamma_o \tilde{k}^2} - \frac{\alpha \operatorname{acoth} a}{\gamma_o \tilde{k}^2} \frac{(\tilde{k}^2 - \tilde{\omega}^2)}{\beta_o^2 \tilde{k}^2 - \tilde{\omega}^2} \\ & + \frac{\alpha a}{2\gamma_o^2 \tilde{k}^2 \sinh a} \int_{-1}^1 \left[2\tilde{\omega}/\beta_o \tilde{k} + a(\tilde{k}^2 - \tilde{\omega}^2) \left(\frac{\tilde{\omega}x}{\beta_o \tilde{k}} - \cos\psi \right) / (\beta_o^2 \tilde{k}^2 - \tilde{\omega}^2) \right] \\ & \cdot \frac{e^{ax} dx}{\left(x - \frac{2\tilde{\omega}x}{\beta_o \tilde{k}} \cos\psi + \frac{\tilde{\omega}^2}{\beta_o^2 \tilde{k}^2} - \sin^2\psi \right)^{1/2}} . \end{aligned} \quad (\text{III.4})$$

We will defer discussion of the evaluation of the integrals appearing in Eqs. (III.3,4) temporarily.

We are interested in the collisional stabilization of the two-stream modes which can exist in the beam-plasma system described, in the collision free limit, by the linear dispersion function $D_o(\tilde{\omega})$. The temporal growth of the collisionless modes are given by the imaginary parts of the complex zeros, $\tilde{\omega}_o$ of D_o for fixed $\tilde{k}$. The dispersion function including the effect of collisions is little modified from D_o . In that case, the contribution from the plasma electrons is modified from

$$\frac{1}{\tilde{\omega}^2} \text{ to } \frac{1}{\tilde{\omega}(\tilde{\omega} + i\tilde{\nu}_c)} , \text{ where}$$

$\tilde{\nu}_c$ is the mean collision frequency in units of ω_p . All other terms remain the same and our dispersion function, $D(\tilde{\omega})$ is then

$$D(\tilde{\omega}) = 1 - \frac{1}{\tilde{\omega}^2(1 + i\tilde{\nu}_c/\tilde{\omega})} - \frac{m_e}{m_i} \frac{1}{\tilde{\omega}^2} - \varepsilon(\tilde{\omega}) \quad (\text{III.5})$$

The complex zeros of $D(\tilde{\omega})$ can be found as a function of $\tilde{k}$ precisely as in the collisionless case. We note however that the modes that are of most interest to us are those that are little modified by the effect of collisions. That is, we are interested in those high frequency modes which have $\text{Re}(\tilde{\omega}_0) \gg 1$. These are the most unstable waves in the collisionless case and are potentially the most dangerous. Furthermore these modes have $I_m(\tilde{\omega}_0) \ll 1$. Thus we might expect that collisions with $\tilde{v}_c \ll 1$ may be able to reduce the growth rates to zero while leaving the mode frequencies relatively unchanged. It is thus appropriate to expand $D(\tilde{\omega})$ around $\tilde{v}_c = 0$ and $\tilde{\omega} = \tilde{\omega}_0$. This procedure is particularly useful in that it yields stability boundaries directly. That is, we will be able to directly calculate the values of $\tilde{v}_c$ to stabilize the mode as a function of the beam and plasma parameters. In carrying out the calculation, it is no less convenient to expand about $\tilde{v}_c = \tilde{v}_0$ rather than about $\tilde{v}_c = 0$. The reason for which we may want to expand about such solutions to the dispersion function D , will be indicated shortly. In most instances we will set $\tilde{v}_0 = 0$.

Letting $\tilde{\omega} = \tilde{\omega}_0 + \delta\omega$ and $\tilde{v} = \tilde{v}_0 + \delta v$ and simply expanding D in a Taylor series, we have

$$\begin{aligned}
 D(\tilde{\omega}, \tilde{k}; \tilde{v}) &= D(\tilde{\omega}_0, \tilde{k}; \tilde{v}_0) + \delta\omega \left. \frac{\partial D}{\partial \tilde{\omega}} \right|_{\tilde{\omega}_0, \tilde{k}; \tilde{v}_0} \\
 &+ \delta v \left. \frac{\partial D}{\partial \tilde{v}} \right|_{\tilde{\omega}_0, \tilde{k}; \tilde{v}_0} + \dots
 \end{aligned}
 \tag{III.6}$$

Now $\tilde{\omega}_0$ is a root of $D(\tilde{\omega}, \tilde{k}; \tilde{v}_0)$ and hence the first term vanishes. Thus to first order in the small quantities, the correction to $\tilde{\omega}_0$ due to the added collisions δv is given by

$$\delta\omega = - \frac{\delta v \left. \frac{\partial D}{\partial \tilde{v}} \right|_{\tilde{\omega}_0, \tilde{k}; \tilde{v}_0}}{\left. \frac{\partial D}{\partial \tilde{\omega}} \right|_{\tilde{\omega}_0, \tilde{k}; \tilde{v}_0}}
 \tag{III.7}$$

The stability boundary is defined by $\text{Im}(\tilde{\omega}) = 0$, or $\text{Im}\delta\omega = -\text{Im}\tilde{\omega}_0$ and we can thus find the increment in the collisions necessary to stabilize the modes, $\delta\nu_{\text{ST}}$, simply by taking the imaginary part of Eq. (III.7), substituting $-\text{Im}\tilde{\omega}_0$ for $\text{Im}\delta\omega$, and solving for $\delta\nu$. Thus

$$\delta\nu_{\text{ST}} = \frac{\text{Im}\tilde{\omega}_0}{\text{Im} \frac{\partial D / \partial \tilde{\nu}}{\partial D / \partial \tilde{\omega}}}, \quad (\text{III.8})$$

where it is understood that the partial derivatives are evaluated at $(\tilde{\omega}_0, \tilde{k}; \tilde{\nu}_0)$. These are trivial to evaluate for the plasma contribution to the dispersion function, and of course, $\frac{\partial \varepsilon}{\partial \nu} = 0$. The contribution to $\frac{\partial D}{\partial \tilde{\omega}}$ from the beam is a little more complicated and we will defer writing down the result until evaluation of the integral appearing in $\varepsilon(\tilde{\omega})$ is discussed. The plasma part is

$$\left. \frac{\partial D}{\partial \tilde{\omega}} \right|_{\tilde{\omega}_0, \tilde{\nu}_0} = \frac{\left(2 + \frac{i\tilde{\nu}_0}{\tilde{\omega}_0} \right)}{\left(\tilde{\omega}_0^3 1 + \frac{i\tilde{\nu}_0}{\tilde{\omega}_0} \right)^2} + \frac{2m_e}{m_i \tilde{\omega}_0^3} \quad (\text{III.9a})$$

and

$$\frac{\partial D}{\partial \tilde{\nu}} = \frac{i}{\tilde{\omega}_0^3 \left(1 + \frac{i\tilde{\nu}_0}{\tilde{\omega}_0} \right)^2} \quad (\text{III.9b})$$

The beam contribution to the dispersion function $\varepsilon(\tilde{\omega}, \tilde{k})$ (see Eq. (III.3)) is, in part, defined as a function of a definite integral,

$$I_{||} = \int_{-1}^1 \frac{a - 2\beta_o^2 x - a\beta_o^2 x^2}{x - \tilde{\omega}/\beta_o \tilde{k}} e^{ax} dx \quad (\text{III.10a})$$

The numerator is simply expressed as an operator independent of x and taken outside of the integral

$$I_{||} = \left(a - 2\beta_o^2 \frac{d}{da} - a\beta_o^2 \frac{d^2}{da^2} \right) \int_1^\infty \frac{e^{ax}}{x - \frac{\tilde{\omega}}{\beta_o \tilde{k}}} dx \quad (III.10b)$$

The remaining integral can be represented in terms of the exponential integral, $E_1(z)$,²⁰

$$I_{||} = \left(a - 2\beta_o^2 \frac{d}{da} - a\beta_o^2 \frac{d^2}{da^2} \right) \left\{ E_1 \left[a \left(\frac{\tilde{\omega}}{\beta_o \tilde{k}} + 1 \right) \right] - E_1 \left[a \left(\frac{\tilde{\omega}}{\beta_o \tilde{k}} - 1 \right) \right] \right\} \quad (III.10c)$$

In the case $\psi \neq 0$, the definite integral appearing in Eq. (III.4) is much more difficult to represent and the results for that case are given in Appendix B. The application of the operator is straightforward and we have the result

$$\begin{aligned} \varepsilon(\tilde{\omega}) = & -\frac{\alpha a}{\beta_o \tilde{k} \gamma_o^3} \frac{\beta_o \tilde{k} \cosh a + \tilde{\omega}}{\tilde{\omega}^2 - \beta_o^2 \tilde{k}^2} \\ & + \frac{\alpha \text{acscha}}{2 \gamma_o \beta_o \tilde{k}} \left(\left[a \left(1 - \frac{\tilde{\omega}^2}{\tilde{k}^2} \right) - \frac{2\beta_o \tilde{\omega}}{\tilde{k}} \right] e^{\frac{a\tilde{\omega}}{\beta_o \tilde{k}}} \right. \\ & \left. \left\{ E_1 \left[a \left(\frac{\tilde{\omega}}{\beta_o \tilde{k}} + 1 \right) \right] - E_1 \left[a \left(\frac{\tilde{\omega}}{\beta_o \tilde{k}} - 1 \right) \right] \right\} \right. \\ & \left. - 2\beta_o^2 \left[\cosh a + \left(\frac{1}{a} + \frac{\tilde{\omega}}{\beta_o \tilde{k}} \right) \sinh a \right] \right) \quad (III.11a) \end{aligned}$$

and

$$\begin{aligned}
\varepsilon'(\tilde{\omega}) = & \frac{-\alpha a}{\beta_o \tilde{k} \gamma_o^3} \left(\frac{\tilde{\omega}^2 + \beta_o^2 \tilde{k}^2 + 2\tilde{\omega} \beta_o \tilde{k} \coth a}{(\tilde{\omega}^2 - \beta_o^2 \tilde{k}^2)^2} \right) \\
& + \frac{\alpha \operatorname{acscha}}{2 \gamma_o \beta_o^2 \tilde{k}^2} \frac{1}{\beta_o \tilde{k}} \left[a^2 (1 - \omega^2) - 2 \beta_o^2 \left(1 + \frac{2a\tilde{\omega}}{\beta_o \tilde{k}} \right) \right] e^{\frac{a\tilde{\omega}}{\beta_o \tilde{k}}} \\
& \left\{ E_1 \left[a \left(\frac{\tilde{\omega}}{\beta_o \tilde{k}} + 1 \right) \right] - E_1 \left[a \left(\frac{\tilde{\omega}}{\beta_o \tilde{k}} - 1 \right) \right] \right\} \\
& + \frac{2}{\beta_o \tilde{k}} \left[a \left(1 - \frac{\tilde{\omega}}{\tilde{k}} \right) - \frac{2\beta_o \omega}{\tilde{k}} \right] \frac{\cosh a + \frac{\tilde{\omega}}{\beta_o \tilde{k}} \sinh a}{\frac{\tilde{\omega}^2}{\beta_o^2 \tilde{k}^2} - 1} \\
& - \frac{2\beta_o}{\tilde{k}} \sinh a .
\end{aligned} \tag{III.11b}$$

Before progressing to the results on the collisional stabilization of the two-stream in the scattered beam, we must briefly address the question of the validity of the Taylor series representation of $D(\tilde{\omega}, \tilde{k}; \tilde{v})$, Eq. (III.6). This question had to be deferred until we had our expression, III.11, for $\varepsilon(\tilde{\omega})$. The presence of singularities in $D(\tilde{\omega})$ impose limitations on expansion in the perturbation theory. The singularities of D in the complex $\tilde{\omega}$ -plane are shown in Fig. 11. From the plasma part there are poles at $\tilde{\omega} = 0$, $-i\tilde{v}$ and from the beam part there are branch points at $\tilde{\omega} = 0$, ∞ and two simple poles at $\tilde{\omega} = \pm \beta_o \tilde{k}$. The expansion about the point $\tilde{\omega} = \tilde{\omega}_o$ has then a radius of convergence R given by

$$R = [(Re(\tilde{\omega}_0) - \beta_0 \tilde{k})^2 + (Im(\tilde{\omega}_0))^2]^{\frac{1}{2}} \quad (III.12)$$

If $|\tilde{\omega} - \tilde{\omega}_0| = |\delta\tilde{\omega}| \geq R$ the Taylor series is no longer a representation of the function $D(\tilde{\omega})$. Solutions based on this method must be tested to insure that $|\delta\tilde{\omega}| < R$ at the least. If one is calculating stability boundaries, this criterion becomes

$$\left| \frac{Im(\tilde{\omega})}{Im \left[\frac{\partial D}{\partial \tilde{v}} / \frac{\partial D}{\partial \tilde{\omega}} \right]_{\tilde{\omega}_0}} \right| < R \quad (III.13)$$

If the expansion around $\tilde{v} = 0$ does not yield a result for $\delta\tilde{\omega}$ satisfying $|\delta\tilde{\omega}| < R$ or the zeroth order solution $\tilde{\omega}_0$ is such that III.13 is not satisfied it is possible to expand around some other value of $\tilde{v} = \tilde{v}_0$. The values of $\delta\tilde{\omega}$ about the zeroth order solution $\tilde{\omega}_0$ ($\tilde{v}_0 \neq 0$) may then satisfy our criteria set by the finite radius of convergence. For this reason we have carried out the more general expansion above.

Because of the simple nature of the troublesome singularity, there is another way to treat the breakdown of the Taylor series representation. One can simply remove the pole at $\tilde{\omega} = \beta_0 \tilde{k}$. Consider the function $G(\tilde{\omega})$ defined by

$$G(\tilde{\omega}) = (\tilde{\omega} - \beta_0 \tilde{k}) D(\tilde{\omega}) \quad (III.14)$$

G has no pole at $\tilde{\omega} = \beta_0 \tilde{k}$ and has the same zeros as $D(\tilde{\omega})$. Thus expanding G around $\tilde{\omega}_0$ we have

$$G(\tilde{\omega}) = \delta\tilde{\omega} G_{\tilde{\omega}}(\tilde{\omega}_0, \tilde{v}_0) + \delta\tilde{v} G_{\tilde{v}}(\tilde{\omega}_0, \tilde{v}_0) + \dots \quad (III.15)$$

where, $G_{\tilde{\omega}} \equiv \frac{\partial G}{\partial \tilde{\omega}}$ and $G_{\tilde{v}} = \frac{\partial G}{\partial \tilde{v}}$,

and thus

$$\delta\tilde{\omega} = - \delta\tilde{v} \frac{G_{\tilde{v}_0}(\tilde{\omega}_0, \tilde{v}_0)}{G_{\tilde{\omega}_0}(\tilde{\omega}_0, \tilde{v}_0)} . \quad (\text{III.16})$$

and for generality we have retained the expansion about $v = v_0$ which may or may not be taken to be zero.

Expression III.8 has been used to calculate stability boundaries for several cases of interest. Furthermore, with the availability of a simple representation for $\varepsilon(\tilde{\omega})$, Eq. III.11, in terms of transcendental functions whose properties are tabulated, it has proven to be a relatively simply matter to solve for the zeros of the dispersion function (Eq. III.5) numerically. In this way we can observe the behavior of the roots as the collision frequency is varied and determine the values necessary for stabilization as well as test values calculated via the perturbation theory. We will now discuss some of the results we have obtained. Figure 12 shows the stability boundaries calculated for the Lawrence Livermore Laboratory FX-25 device experiments.^{1,19} The beam current was 10kA in a 1 cm radius and the energy was equivalent to $\gamma_0 = 3.0$. Stabilization of the two-stream is predicted to occur for pressures in excess of those on the stability boundaries. The three curves correspond to the three different values of scattering angle shown. The smallest angle corresponds to scattering introduced in the anode foil alone with the beam electrons following ballistic trajectories thereafter. The intermediate value of $\bar{\theta}$ is the angle calculated on the basis of the betatron motion alone. The highest value of the scattering angle is the rms average of the other two. The three data points correspond to those experiments where a complete set of data were available. The Ne and N₂ experiments are consistent with theory although the scattering angle is larger than that expected due to betatron motion alone and agrees well with rms average value. There are several possible sources for this difference but in any case either of the two curves are likely to be within reasonable experimental error bars. The He data is clearly inconsistent and not understood at this time. At the lower values of n_b/n_p , the instability is in the kinetic regime, $\tilde{\delta} \propto n_b/n_p$, and thus $P_c \propto (n_b/n_p)^{-\frac{1}{2}} \tilde{v}_c^p \propto (n_b/n_p)^{\frac{1}{2}}$, Fig. 12. As n_b/n_p increases the instability becomes quasi-hydrodynamic. (This is noticeable for $\bar{\theta} = 13.54^\circ$ and $\bar{\theta} = 17.85^\circ$). It is clear that the FX-25 experiments are in the kinetic regime. Here we can compare with other results⁶ and find that there is little difference between the two theories for the parameters used here

although the simpler theory in Ref. 6 overestimates the effect of the momentum scatter slightly. As we shall see shortly, the scaling with parameters can differ substantially, even in the kinetic regime. In the quasi-hydrodynamic regime, the result of Ref. 6 is not valid.

In interpreting the results based on the perturbative calculation of stability boundaries, one caveat must be observed. Recall that the perturbation theory is based on a fixed value of the wavenumber, $\tilde{k}$. It is found, in the solutions to the full dispersion function, that the wavenumber at which the peak value of the linear growth rate occurs shifts in value as $\tilde{v}$ is increased from zero. This effect is most pronounced in the quasi-hydrodynamic regime. Based on comparisons between the two approaches, it appears that in the kinetic regime the perturbative results are likely to be accurate to within ten or fifteen percent. In the hydrodynamic regime it has been found that by iterating the two methods an accurate result is obtained after about three or four iterations despite the fact that the peak growth rate drops slowly with increasing $\tilde{v}$ until the value of $\tilde{v}$ is very near to $\tilde{v}_c$. This is shown in Fig. 13, where the growth rate, $\tilde{\delta}$ is plotted versus the collision frequency, $\tilde{v}$. The parameters are as in the $\bar{\theta} = 13.54^\circ$ case of Fig. 12 and $n_b/n_p = 0.01$, but the behavior is typical. Note that the value of $\tilde{v}_c \cong 0.47$ is well above that calculated from the perturbation theory in one pass. The wavenumber at which the maximum growth occurs is $\tilde{k} \cong 0.7$ which is about a thirty percent decrease from that value on which the perturbation theory is based. The spectrum, generally, is substantially modified from the typical two-stream spectrum,⁵ as can be seen in Fig. 14a. Here the growth rate $\tilde{\delta}$ is plotted as a function of $\beta_o \tilde{k}$. There is a large shift to lower values of $\beta_o \tilde{k}$ of the wavenumber at which the peak growth rate occurs and the spectrum is broadened. Furthermore the group velocity $v_g = \partial \text{Re}(\tilde{\omega}) / \partial \tilde{k}$, has increased from $v_g \cong 0.67 \beta_o c$ to $v_g \cong 0.73 \beta_o c$. The real part of the spectrum is shown in Fig. 14b and, in this case, is linear in $\tilde{k}$. The qualitative features discussed here are typical and are more or less pronounced depending on the parameters. It should be noted that the reduction in growth rate and increase in group velocity which occurs generally in the interaction of a scattered beam with a collisional plasma act in concert to complicate the interpretation of experiments done in finite length systems. Both effects act to increase the growth length of the instability and hence, in an experiment of finite length, one could be led to interpret the data to imply an

absence of instability when, in fact, the instability had not grown in amplitude to the threshold of some diagnostic.

As we pointed out earlier, the difference between our results for the FX-25 experiments and those of other theories are sufficiently small as to be experimentally indistinguishable. The scalings are different however. This is illustrated in Fig. 15 where the growth rate is plotted against γ_0 for an instability which is in the kinetic regime. The results of Singhaus are valid only in this regime. Two features are evident. First, for large γ_0 , the growth rate for a scattered beam are proportional to γ_0^{-1} while Singhaus' model has the growth rate scaling as γ_0^{-3} . Second, at the lowest values of γ_0 the growth rates are almost the same. The importance of experiments at high values of γ_0 is clear. This is illustrated further in Table I which contains a comparison of the critical collision frequencies required to stabilize the instability. Tabulated for several values of γ_0 are the critical collision frequency based on Singhaus' theory ($\tilde{\nu}_s$) and that calculated from the linear theory for a scattered beam ($\tilde{\nu}_c$). The scattering angles shown are based on the betatron motion. The differences are obvious as is the need for experiments to test these scalings. Also needed are extensions of the theoretical research in this problem. These will be briefly outlined in the following section.

TABLE I
STABILIZING COLLISION FREQUENCY FOR
TWO-STREAM INSTABILITY IN A SCATTERED BEAM

Relativistic Factor, γ_0	Mean Scattering Angle, $\bar{\theta}$	Beam-to-Plasma Density Ratio, n_b/n_p	Singhaus Collision Frequency, ^a ν_s	Stabilizing Collision Frequency, ^b ν_c
11	9.29°	0.01	3.3×10^{-2}	8.7×10^{-2}
100	3.10°	0.01	3.6×10^{-3}	5.8×10^{-2}
1000	0.98°	0.01	3.6×10^{-4}	4.7×10^{-2}

^aCollisions to stabilize based on Singhaus theory $\nu_s = 3.04/\gamma_0^3 n_b/n_p \beta_0^2/\bar{\theta}^4$.

^bAll frequencies are in units of the plasma frequency.

IV. CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

Oscillation and instability has been a traditional field of research since the earliest days of plasma physics. The beam-plasma system itself has been a subject of study for approximately thirty years and despite its relative simplicity there remain open issues.

In large part these issues involve the nonlinear states of the instabilities in the beam-plasma interaction and hence have had to wait for the development of new analytic techniques and, to an even greater extent, numerical techniques and the computational facilities necessary to make these techniques viable. Even with these powerful numerical approaches, cost and storage considerations make simulations prohibitive in large regions of parameter space. To this end the partial numerical simulation model has been developed and used to examine the nonlinear saturation level of the two-stream instability. Our extension and generalization of this technique has been described in Section II and several examples to illustrate the method were given. Quite recently we have developed a particle initialization algorithm which allows us to follow the evolution of the instability in a beam which has a finite momentum scatter. Outside of testing of the method in this configuration, this essentially completes the developmental effort of the partial numerical simulation research. It is likely that some optimization in the implementation of the algorithms will take place at some future time. It must be noted that investigation of the nonlinear state of the two stream using partial numerical simulation requires fairly accurate values of the eigenfrequencies of the linear system. To this end, as well as in its own right, we have undertaken research into the linear theory of the two-stream instability for a relativistic electron beam with a finite momentum scatter.

The linear theory of unstable oscillations parallel to a scattered beam interacting with a cold, collisional plasma was developed in Section III. Some results were presented and we discussed the scaling of the solutions with various parameters. An extension of the theory to two dimensions, that is, to waves propagating at an angle relative to the mean motion of the beam is of the highest priority. Two competing effects come into play in two dimensions. Relativistic effects tend to increase the growth rate of the oblique modes over the parallel modes in the quasi-hydrodynamic regime. However, for a given scattering angle, $\bar{\theta} < 1$, the parallel velocity spread is smaller than the parallel velocity spread by a factor $\bar{\theta}$. Theories based on a balancing of a

damping rate due to phase mixing (of order $\tilde{k} \cdot \delta \tilde{v}$) against some linear growth rate seem to indicate that the latter can dominate the former. A difficulty with such theories is that as one passes, continuously, to the $\tilde{k} = 0$ limit these results go over to the Singhaus stability criterion to within numerical factors of order unity, and, we have seen in Section III that the scaling of this result is not correct. It is difficult, then, to see how such theories can contain the correct scaling for finite k and somehow lose their physical content when k is allowed to vanish. This issue can only be resolved by carrying out the full linear theory of the two-stream in a scattered beam in two dimensions. Several of the calculations necessary to the development of the theory have already been completed and we hope to have made substantial progress on this problem in the near future.

After completing the linear theory in two dimensions, the theory in a system of finite radial extent needs to be carried out in order to determine the effects of finite geometry in tightly pinched beams. Any such effects are likely to be in the direction of stabilization. This research will be an extension of the linear theory already completed and the linear theory of a cold beam in finite geometry which we have had for several years. In the interim, it is likely that the latter will be modified to include the effects of collisions on the plasma electrons as this is a fairly simple generalization to do. Finally, a linear theory including the effects of fully pinched beam dynamics and spatial gradients perpendicular to the beam direction will be developed. An approach to this problem involving numerical integration of the first order fields along the unperturbed particle orbits (the characteristics of the linearized Vlasov equation) is currently being investigated in another context. Such methods have historically been used to provide analytic solutions in the linear theory of plasma waves in geometries where the unperturbed orbits are reasonably simple.²¹

APPENDIX A

In this appendix, we will fill in the details of the derivation of the partial numerical simulation model presented in section II. Beginning with Eq. (II.6) we will now demonstrate the reduction of this integral equation to an infinite order differential equation. Equation (II.6) states

$$\sum_j \frac{\vec{k}_j}{2\pi} \cdot \int_{-\infty}^{\infty} \bar{\bar{\epsilon}} e^{-i\omega t} d\omega \cdot \int_{-\infty}^{\infty} \vec{E}_j(\tau) \exp i(\vec{k}_j \cdot \vec{x} - \omega_j \tau + \alpha_j(\tau)) e^{i\omega t} d\tau$$

$$= 4\pi\rho_b ,$$

which is simply Poissons equation in the time domain, expressed in terms of the electric displacement. Let us consider one term in the summation on the left hand side. The operators are clearly linear so there is no difficulty in doing this.

$$I \equiv \int_{-\infty}^{\infty} \bar{\bar{\epsilon}}(\omega) e^{-i\omega t} d\omega \cdot \int_{-\infty}^{\infty} \vec{E}_j(\tau) \exp i(\vec{k}_j \cdot \vec{x} - \omega_j \tau + \alpha_j(\tau)) e^{i\omega t} d\tau \quad (\text{A.1})$$

Putting $\bar{\omega} \equiv \omega - \omega_j$, we have

$$I = \int_{-\infty}^{\infty} \bar{\bar{\epsilon}}(\bar{\omega} + \omega_j) e^{-i(\bar{\omega} + \omega_j)t} d\bar{\omega} \cdot \int_{-\infty}^{\infty} \vec{E}_j(\tau) \exp i(\vec{k}_j \cdot \vec{x} - \omega_j \tau + \alpha_j(\tau))$$

$$\times e^{i(\bar{\omega} + \omega_j)\tau} d\tau \quad (\text{A.2})$$

We now expand the dielectric tensor $\bar{\bar{\epsilon}}$ about the mode frequency ω_j . The physical motivation is simple. If there were no growth in the wave, the field would be monochromatic and the spectrum would be a delta function. We are interested in unstable systems so the field will have a slow modulation in its amplitude and phase, resulting in a broadening in the spectrum so there exists a wave packet of width, in the Fourier space, δ/ω_j where δ is the linear growth rate. Thus $\bar{\omega} \delta/\omega_j \ll 1$ by assumption, and the expansion follows, thus

$$I = e^{-i\omega_j t} \int_{-\infty}^{\infty} \left[\bar{\bar{\epsilon}}(\omega_j) + \bar{\omega} \left. \frac{\partial \bar{\bar{\epsilon}}}{\partial \omega} \right|_{\omega_j} + \dots + e^{-i\bar{\omega} t} \right] d\bar{\omega}$$

$$\cdot \int_{-\infty}^{\infty} \vec{E}_j(\tau) \exp i(\vec{k}_j \cdot \vec{x} + \alpha_j(\tau)) e^{i\bar{\omega} \tau} d\tau . \quad (\text{A.3})$$

Interchanging orders of integration we have

$$I = e^{-i\omega_j t} \int_{-\infty}^{\infty} \exp(i(\vec{k}_j \cdot \vec{x} - \alpha_j(\tau))) d\tau \int_{-\infty}^{\infty} \left[\bar{\varepsilon}(\omega_j) + \bar{\omega} \frac{\partial \bar{\varepsilon}}{\partial \omega} \Big|_{\omega_j} + \dots + \right] \cdot \vec{E}_j(\tau) e^{-i\bar{\omega}(t-\tau)} d\bar{\omega} . \quad (A.4)$$

The integration over $\bar{\omega}$ is now trivial.

$$I = e^{-i\omega_j t} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \frac{\partial^n \bar{\varepsilon}}{\partial \omega^n} \Big|_{\omega_j} \frac{d^n}{d\tau^n} \delta(t-\tau) \cdot \vec{E}_j(\tau) \exp(i(\vec{k}_j \cdot \vec{x} + \alpha_j(\tau))) d\tau . \quad (A.5)$$

The τ integrations are also trivial now. Partial integration places the derivatives in τ on $\vec{E}_j$ and α_j and we have a series which proceeds in orders of

$$\frac{1}{\omega_j} \frac{d\vec{E}_j}{dt} , \frac{1}{\omega_j} \frac{d\alpha_j}{d\tau} = O(\delta/\omega_j) \ll 1$$

by assumption and we can truncate the system at some low order. The technique of partial numerical simulation is an asymptotic theory in inverse powers of ω_j . In section II, we retained only the first order terms

$$I \cong \vec{E}_j(t) \exp(i(\vec{k}_j \cdot \vec{x} - \omega_j t + \alpha_j(t))) + i e^{-i\omega_j t} \frac{\partial \bar{\varepsilon}}{\partial \omega} \Big|_{\omega_j} \cdot \frac{d}{dt} [\vec{E}_j(t) \exp(i(\vec{k}_j \cdot \vec{x} + \alpha_j(t)))] . \quad (A.6)$$

Straightforward substitution of Eq. (A.6) into Eq. (II.6) yields Eq. (II.7).

We now proceed to consider the evaluation of the matrices C_{ij} , M_{ij}^1 , M_{ij}^2 and S_{ij} resulting from the projection by the operator

$$\int_{-\lambda_i/2}^{\lambda_i/2} \begin{bmatrix} \cos \psi_i \\ \sin \psi_i \end{bmatrix} dx_i .$$

Consider first the matrix C_{ij} . This matrix is defined by

$$C_{ij} = \frac{1}{\lambda_i \lambda_j} \int_{-\lambda_j/2}^{\lambda_j/2} dx_j \int_{-\lambda_i/2}^{\lambda_i/2} \cos(\vec{k}_j \cdot \vec{x} - \omega_j t + \alpha_j(t)) \cos(\vec{k}_i \cdot \vec{x} - \omega_i t + \alpha_i(t)) dx_i , \quad (A.7)$$

$$\text{where } dx_i = \frac{\vec{k}_i \cdot d\vec{x}}{k_i} \text{ and } dx_j = \frac{\vec{k}_j}{k_j} \cdot \left(d\vec{x} - \frac{\vec{k}_i \vec{k}_i}{k_i^2} \cdot d\vec{x} \right)$$

$$\text{and } \lambda_i \equiv 2\pi/k_i, \lambda_j = 2\pi/k_j \sin \theta_{ij} \text{ and } \cos \theta_{ij} = \frac{\vec{k}_i \cdot \vec{k}_j}{k_i k_j}$$

$$\text{Setting } d\theta = \frac{2\pi dx_i}{\lambda_i} \text{ and } d\phi = \frac{2\pi dx_j}{\lambda_j}$$

$$C_{ij} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} \cos(\theta - \omega_i t + \alpha_i(t)) \cos\left[\frac{k_j}{k_i} \cos \theta_{ij} \theta - \omega_j t + \alpha_j(t) + \phi\right] d\theta \quad (A.8)$$

Note that if $\theta_{ij} = 0$, ($\vec{k}_j || \vec{k}_i$) then the integral is independent of ϕ . Furthermore, it is clear that unless $\vec{k}_i$ and $\vec{k}_j$ are parallel, the matrix elements vanish since by performing the ϕ integration first it is easy to see that this integration involves the average of either $\sin\phi$ or $\cos\phi$ over a period. Thus it is sufficient to consider the integral

$$\begin{aligned}
C_{ij} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \cos[\theta - \omega_i t + \alpha_i(t)] \cos \left[\frac{k_j}{k_i} \theta - \omega_j t + \alpha_j(t) \right] \\
&= \frac{1}{\pi} \frac{\sin(\pi k_i/k_j)}{1 - k_j^2/k_i^2} \left\{ \frac{k_j}{k_i} \cos[-\omega_i t + \alpha_i(t)] \cos[\omega_j t - \alpha_j(t)] \right. \\
&\quad \left. - \sin[-\omega_i t + \alpha_i(t)] \sin[\omega_j t - \alpha_j(t)] \right\} \tag{A.9}
\end{aligned}$$

The latter follows from straightforward trigonometric identities and integration. The remaining matrices follow in precisely the same way:

$$\begin{aligned}
S_{ij} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \sin[\theta - \omega_i t + \alpha_i(t)] \sin \left[\frac{k_j}{k_i} \theta - \omega_j t + \alpha_j(t) \right] \\
&= \frac{1}{\pi} \frac{\sin(\pi k_i/k_j)}{1 - k_j^2/k_i^2} \left\{ \frac{k_j}{k_i} \sin[-\omega_i t + \alpha_i(t)] \cos[-\omega_j t + \alpha_j(t)] \right. \\
&\quad \left. - \sin[-\omega_j t + \alpha_j(t)] \cos[-\omega_i t + \alpha_i(t)] \right\} , \tag{A.10}
\end{aligned}$$

$$\begin{aligned}
M_{ij}^1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \cos[\theta - \omega_i t + \alpha_i(t)] \sin \left[\frac{k_j}{k_i} \theta - \omega_j t + \alpha_j(t) \right] \\
&= \frac{1}{\pi} \frac{\sin(\pi k_j/k_i)}{1 - k_j^2/k_i^2} \left\{ \sin[-\omega_i t + \alpha_i(t)] \cos[-\omega_j t + \alpha_j(t)] \right. \\
&\quad \left. - \frac{k_j}{k_i} \sin[-\omega_j t + \alpha_j(t)] \cos(\omega_i t - \alpha_i) \right\} , \tag{A.11}
\end{aligned}$$

and

$$\begin{aligned}
M_{ij}^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \sin[\theta - \omega_i t + \alpha_i(t)] \cos\left[\frac{k_j}{k_i} \theta - \omega_j t + \alpha_j(t)\right] \\
&= \frac{1}{\pi} \frac{\sin(\pi k_j/k_i)}{1 - k_j^2/k_i^2} \left\{ \frac{k_j}{k_i} \sin[-\omega_i t + \alpha_i(t)] \cos[-\omega_j t + \alpha_j(t)] \right. \\
&\quad \left. - \sin[-\omega_j t + \alpha_j(t)] \cos[-\omega_i t + \alpha_i(t)] \right\}. \tag{A.12}
\end{aligned}$$

The values of the diagonal elements are easily seen to be $C_{ii} = S_{ii} = \frac{1}{2}$ and $M_{ii}^1 = M_{ii}^2 = 0$. It is important to note here that it is in defining the matrices above that any periodicity requirement has been explicitly introduced. This is reflected, ultimately, in the initial positions of the beam particles in that the model now requires that the system length in each of the three spatial directions be periodic for all the projected wavelengths in the corresponding directions. This effectively puts a lower limit on the projected wavenumber difference which can be tolerated in an actual calculation. Otherwise the system size becomes so large that the number of beam particles required to fill the system and still maintain good statistics becomes prohibitive. We have, in effect, traded off simplicity in the model for an increased computational burden.

With the calculation of the matrices $\bar{C}$, $\bar{S}$, $\bar{M}^1$ and $\bar{M}^2$ above and the reduction of Poisson's equation in the asymptotic limit, ω_j large, the derivation of the model in Section II is complete.

APPENDIX B

The beam contribution to the dispersion function in the two-dimensional linear theory of the two-stream instability in a scattered beam is considerably more complicated than that for parallel wave propagation. The beam contribution is, in part, a function of $\tilde{\omega}$ defined by a definite integral which has singularities in the complex $\tilde{\omega}$ plane. Standard numerical integration procedures are typically not well suited to the evaluation of such functions.¹⁷ In this appendix we will derive a representation for this function in terms of certain transcendental functions whose properties are well known. Subsidiary expansions

can then be made as appropriate to specific values of the parameters. Furthermore, the analytic continuation of the function to the lower half $\tilde{\omega}$ plane will be obtained directly from the calculation.

The function, $g(\tilde{\omega})$, which we need to consider is

$$g(\tilde{\omega}) \equiv \int_{-1}^1 \frac{e^{ax} dx}{\left(x^2 - \frac{2\tilde{\omega}}{b} x \cos \psi + \frac{\tilde{\omega}^2}{b^2} - \sin^2 \psi \right)^{1/2}} \quad (\text{B.1})$$

where

$$a = 2/\bar{\theta}^2$$

$$\tilde{\omega} = \omega/\omega_p$$

$$b = \frac{k\beta_o c}{\omega_p} = \beta_o \tilde{k}$$

and $\psi = \arctan(k_1/k_z)$. If we denote the two roots of the denominator by x_1 and x_2 we have

$$x_{1,2} = \frac{\tilde{\omega}}{b} \cos \psi \pm \left(1 - \frac{\tilde{\omega}^2}{b^2} \right)^{1/2} \sin \psi \quad (\text{B.2})$$

and hence

$$g(\tilde{\omega}) = \int_{-1}^1 \frac{e^{ax} dx}{(x-x_1)^{1/2}(x-x_2)^{1/2}}.$$

The denominator can be made "symmetric" by defining the change of variables

$$y = x - \frac{1}{2}(x_1+x_2). \quad \text{Then } g(\tilde{\omega}) \text{ is}$$

$$g(\tilde{\omega}) = \exp(a/2(x_1+x_2)) \cdot \int_{x_a}^{x_b} \frac{e^{ay} dy}{(y-\bar{y})^{\frac{1}{2}}(y+\bar{y})^{\frac{1}{2}}} \quad (\text{B.3})$$

where

$$x_a = -(1+\frac{1}{2}(x_1+x_2)) ,$$

$$x_b = 1-\frac{1}{2}(x_1+x_2) \text{ and}$$

$$\bar{y} = \frac{x_1-x_2}{2} = (1-\tilde{\omega}^2/b^2)^{\frac{1}{2}} \sin\psi .$$

We will now consider only

$$I(\tilde{\omega}) \equiv \int_{x_a}^{x_b} \frac{e^{ay} dy}{(y-\bar{y})^{\frac{1}{2}}(y+\bar{y})^{\frac{1}{2}}} \quad (\text{B.4})$$

One could simply expand I for large a and small a and use the results in the dispersion relation. Typically, the problems of interest do not have $a \ll 1$ and a difficulty arises for $a \gg 1$. While one could formally expand I for $a \gg 1$, the result contains the product $a\bar{y}$ and $|\bar{y}| \approx 0(\delta \sin\psi)$ where δ is the growth rate. Thus, it is possible that in parameter regimes of interest we could have $a \gg 1$ while $|a\bar{y}| \sim 0(1)$. The asymptotic expansion then obtained is not uniform and not useful in this case. In taking the approach below this difficulty does not arise. In those cases where $|\bar{y}a| \gg 1$, the appropriate subsidiary expansion can be used to obtain the proper asymptotic result.

We proceed by breaking the integral as follows

$$I(\tilde{\omega}) = \int_{x_a}^{-\bar{y}} \frac{dy e^{ay}}{(y-\bar{y})^{\frac{1}{2}}(y+\bar{y})^{\frac{1}{2}}} + \int_{-\bar{y}}^{\bar{y}} \frac{dy e^{ay}}{(y-\bar{y})^{\frac{1}{2}}(y+\bar{y})^{\frac{1}{2}}} + \int_y^{x_b} \frac{dy e^{ay}}{(y-\bar{y})^{\frac{1}{2}}(y+\bar{y})^{\frac{1}{2}}} , \quad (\text{B.5})$$

where these integrals will be denoted by I_1 , I_2 and I_3 respectively. We remark that the branch cuts can be so chosen that they need not be considered for $\text{Im}(\tilde{w}) > 0$. As indicated above the analytic continuation to $\text{Im}(\tilde{w}) < 0$ will follow from the calculation fairly simply. Another change of variables gives

$$I_1 = \int_{x_a/\bar{y}}^{-1} \frac{du e^{a\bar{y}u}}{(u-1)^{\frac{1}{2}}(u+1)^{\frac{1}{2}}} , \quad (\text{B.6a})$$

$$I_2 = \int_{-1}^1 \frac{du e^{a\bar{y}u}}{(u+1)^{\frac{1}{2}}(u-1)^{\frac{1}{2}}} , \quad (\text{B.6b})$$

and

$$I_3 = \int_1^{x_b/\bar{y}} \frac{du e^{a\bar{y}u}}{(u+1)^{\frac{1}{2}}(u-1)^{\frac{1}{2}}} . \quad (\text{B.6c})$$

I_2 can be evaluated directly:

$$I_2 = \pm i\pi I_0(-a\bar{y}) , \quad (\text{B.7})$$

where I_0 is the modified Bessel function of the first kind, of order zero. The choice of branch will be discussed later. It will provide the analytic continuation to the lower half $\tilde{w}$ plane. Note too that I_0 is an entire function of its argument. We now consider I_1 .

$$\begin{aligned} I_1 &= \int_{x_a/\bar{y}}^{-1} \frac{du e^{a\bar{y}u}}{(u+1)^{\frac{1}{2}}(u-1)^{\frac{1}{2}}} = \int_{-\infty}^{-1} \frac{du e^{a\bar{y}u}}{(u+1)^{\frac{1}{2}}(u-1)^{\frac{1}{2}}} \\ &\quad + \int_{x_a/\bar{y}}^{-\infty} \frac{du e^{a\bar{y}u}}{(u+1)^{\frac{1}{2}}(u-1)^{\frac{1}{2}}} \end{aligned} \quad (\text{B.8a})$$

$$= \int_{-\infty}^1 \frac{du e^{a\bar{y}u}}{u(1 - 1/u^2)^{\frac{1}{2}}} + \int_{x_a/\bar{y}}^{\infty} \frac{du e^{a\bar{y}u}}{u(1 - 1/u^2)^{\frac{1}{2}}} \quad (\text{B.8b})$$

$$= -E_1(a\bar{y}) - \sum_{n=1}^{\infty} \frac{(1/2)(3/2) \dots (n-1/2) E_{2n+1}(a\bar{y})}{n!} + \int_1^{\infty} \frac{dz e^{ax_a z}}{z(1 - \bar{y}^2/x_b^2 z^2)^{\frac{1}{2}}} \quad (\text{B.8c})$$

$$= -E_1(a\bar{y}) - \sum_{n=1}^{\infty} \frac{(1/2)(3/2) \dots (n-1/2) E_{2n+1}(a\bar{y})}{n!} + E_1(-ax_a) + \sum_{n=1}^{\infty} \frac{(1/2)(3/2) \dots (n-1/2) E_{2n+1}(-ax_a)}{n! (x_b^2/\bar{y}^2)^n} . \quad (\text{B.8d})$$

which is the desired representation for $I_1(\tilde{\omega})$. The E_m are the exponential integrals of order m ²⁰.

The function $I_3(\tilde{\omega})$ must be considered in two cases. If the propagation angle, ψ , is not too small then $|x_b/\bar{y}| \approx 0(1)$. Then under a simple change of variable

$$I_3 = \exp(a\bar{y}) \cdot \int_1^{\frac{x_b}{\bar{y}}-1} \frac{dz e^{a\bar{y}z}}{z^{\frac{1}{2}}(z+2)^{\frac{1}{2}}} \quad (\text{B.9a})$$

$$= -\frac{ie^{a\bar{y}}}{\sqrt{2}} \frac{\gamma(1/2, a(\bar{y}-x_b))}{a\bar{y}^{\frac{1}{2}}} + \sum_{n=1}^{\infty} \frac{(1/2)(3/2) \dots (n-1/2) \gamma(1/2+n, a(\bar{y}-x_b))}{2^n n! (a\bar{y})^n} . \quad (\text{B.9b})$$

where $\gamma(\mu, z)$ is the incomplete gamma function.²² This is a valid representation when ψ is large enough that $|x_b - \bar{y}| < 2|\bar{y}|$ although this is a valid asymptotic expansion whenever $|\bar{a}\bar{y}| \gg 1$. If ψ is small, then $|x_b/\bar{y}| \gg 1$ and I_3 must be represented differently. It now suffices to expand:

$$I_3 = \int_1^{x_b/\bar{y}} \frac{du e^{ayu}}{u(1 - 1/u^2)^{1/2}} \quad (\text{B.10a})$$

$$= \int_1^{x_b/\bar{y}} \frac{du e^{a\bar{y}u}}{u} + \int_1^{\infty} \frac{du e^{a\bar{y}u}}{u} \sum_{n=1}^{\infty} \frac{(1/2)(3/2)\dots(n-1/2)}{n! u^{2n}} \\ + \int_{\infty}^{x_b/\bar{y}} \frac{du e^{a\bar{y}u}}{u} \sum_{n=1}^{\infty} \frac{(1/2)(3/2)\dots(n-1/2)}{n! u^{2n}} \quad (\text{B.10b})$$

$$= -E_1(-ax_b) + E_1(-a\bar{y}) \\ + \sum_{n=1}^{\infty} \frac{(1/2)(3/2)\dots(n-1/2)E_{2n+1}(-a\bar{y})}{n!} \\ + \int_{\infty}^1 \frac{dz e^{ax_b z}}{z} \sum_{n=1}^{\infty} \frac{(1/2)(3/2)\dots(n-1/2)}{n!} \frac{x_b^{2n}}{\bar{y}^{2n} z^{2n}} \quad (\text{B.10c})$$

$$= -E_1(-ax_b) + E_1(-a\bar{y}) \\ + \sum_{n=1}^{\infty} \frac{(1/2)(3/2)\dots(n-1/2)}{n!} \left[E_{2n+1}(-a\bar{y}) - \left(\frac{\bar{y}}{x_b}\right)^{2n} E_{2n+1}(-ax_b) \right] \quad (\text{B.10d})$$

which is the desired result, and we now have the representation of $I(\tilde{\omega})$ we sought. Now consider the limit $\psi \rightarrow 0$. Equation B.10d is the appropriate representation of I_3 to use and noting that

$$\lim_{z \rightarrow 0} E_{1+n}(z) = 0 \quad (\text{B.11a})$$

$$\lim_{z \rightarrow 0} I_0(z) = 1 \quad \text{and} \quad (B.11b)$$

$$\begin{aligned} \lim_{z \rightarrow 0} -E_1(z) + E_1(-z) &= \lim_{\rho \rightarrow 0} [c + \ln \rho e^{\pm i \phi} \\ &\quad - c - \ln \rho e^{\pm i(\pi - \phi)}] = \pm i\pi \end{aligned} \quad (B.11c)$$

where c is a constant and the $\begin{bmatrix} \text{upper} \\ \text{lower} \end{bmatrix}$ sign holds if $\text{Im}(a\bar{y})$ is $\begin{bmatrix} > \\ < \end{bmatrix} 0$, since E_1 is to be restricted to its principle branch. Since $I(\tilde{w})$ must pass continuously to the one-dimensional result as $\psi \rightarrow 0$, we find that the choice of sign in Eq. (B.7) is negative for $\text{Im}(a\bar{y}) > 0$ and positive if $\text{Im}(a\bar{y}) < 0$. Thus I_2 becomes

$$I_2 = -i \, \text{sgn} [\text{Im}(a\bar{y})] I_0(a\bar{y}) \quad (B.12)$$

where the signum function $\text{sgn}(z)$ selects the sign of its argument. This completes the determination of I including its analytic continuation throughout the complex $\tilde{w}$ plane.

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Fig. 1.

Single wave electric field amplitude as a function of time for a beam with $\gamma_0 = 10$ and $n_b/n_p = 0.01$. This is the fastest growing mode, $\tilde{k} = 1.0055$, $\tilde{\omega} = 0.9912$.

Fig. 2.

Mode frequency shift as a function of time for the same case as Fig. 1.

Fig. 3.

The mean kinetic energy of the beam as a function of time for the case as in Fig. 1.

Fig. 4.

Snapshots of the phase space (p, x) at $\varepsilon = 461.7$ (just after onset of trapping) for the case as in Fig. 1.

Fig. 5.

Fundamental field amplitude as a function of time for a two-wave simulation. The beam parameters were $\gamma_0 = 2.0$ and $n_b/n_p = 0.1$. The mode was at $\tilde{k} = 0.6$.

Fig. 6.

First harmonic field amplitude as a function of time for the two-wave simulation as in Fig. 5. The harmonic was at $\tilde{k} = 1.2$ which is essentially the linear growth peak for the parameters used.

Fig. 7.

Frequency shift for the fundamental mode as a function of time for the two-wave partial numerical simulation.

Fig. 8.

Frequency shift of the harmonic mode as a function of time for the two-wave partial numerical simulation.

Fig. 9

Mean beam kinetic energy as a function of time for the two-wave calculation.

Fig. 10a.

Phase space snapshot for the two-wave calculation.

Fig. 10b.

Phase space snapshot for the two-wave calculation.

Fig. 11.

Singularities of the linear dispersion function in the complex $\tilde{\omega}$ -plane. There are poles at $\pm b$, $b \equiv \beta_0 k$, $-i\nu$, and 0.

Fig. 12.

Stability boundaries expressed as a critical ambient pressure for stabilization as a function of beam-to-plasma density ratio. The curves are given for several values of the mean scattering angle $\bar{\theta}$ and the other parameters are appropriate to the Livermore FX-25 device. Three experimental data points are indicated.

Fig. 13.
Instability growth rate as a function of collision frequency. The parameters are typical of the FX-25 experiment, $\gamma_0 = 3$ and $n_b/n_p = 0.01$.

Fig. 14.
(a) Growth rate spectrum ($\tilde{\delta}$ vs. $\beta_0 \tilde{k}$) for the FX-25 experiment, including only scattering in the anode foil.
(b) frequency spectrum ($\tilde{\omega}$ vs. $\beta_0 \tilde{k}$) for the case of (a).

Fig. 15.

Gamma scaling of the two-stream growth rates in the kinetic regime: comparison of the Singhaus and scattered beam results. The background plasma is collisionless.