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The Representative-Vector Method for Calculating Eigenvalues

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and the same expected value of the loss function is obtained when the optimal control is used and the initial condition is that the population is equal to the average population of the previous period with a constant population growth rate. The stability over a relatively long delay.

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$\hat{C}_2^2$, spin-projection $\hat{C}_2^2$, isospin $(\hat{T}_x^2)$, isospin-projection $\hat{T}_x^2$, and parity $\hat{P}$. We note immediately that $\hat{C}_2^2$, and $\hat{T}_x^2$, and $\hat{P}$ are treated by a different footing from $\hat{S}^2$ and $\hat{T}^2$; see below.

In the first published report of this method¹⁰ the emphasis was placed on the following procedure: an algorithm developed to take advantage of the important feature of this approach, namely that (i) after the possibility of generating a distribution of particles giving the partition characterized by $\hat{C}_2^2$, $\hat{T}_x^2$, and $\hat{P}$ is exhausted, any of the operators $\hat{C}_2^2$, $\hat{T}_x^2$, or $\hat{P}$ may be applied to the particles which are already distributed on the basis of $\hat{S}^2$ and $\hat{T}^2$ to give a new distribution of the system, which will be characterized by $\hat{C}_2^2$, $\hat{T}_x^2$, $\hat{P}$, $\hat{S}^2$, and $\hat{T}^2$. This procedure is repeated until the desired partition is reached. The algorithm is analogous to the QMC method, except that the operators $\hat{C}_2^2$, $\hat{T}_x^2$, and $\hat{P}$ are used instead of $\hat{H}$. The operators $\hat{C}_2^2$, $\hat{T}_x^2$, and $\hat{P}$ are used in the following sense: $\hat{C}_2^2$ is used to generate a new distribution of the system, which will be characterized by $\hat{C}_2^2$, $\hat{T}_x^2$, $\hat{P}$, $\hat{S}^2$, and $\hat{T}^2$; $\hat{T}_x^2$ is used to generate a new distribution of the system, which will be characterized by $\hat{C}_2^2$, $\hat{T}_x^2$, $\hat{P}$, $\hat{S}^2$, and $\hat{T}^2$; and $\hat{P}$ is used to generate a new distribution of the system, which will be characterized by $\hat{C}_2^2$, $\hat{T}_x^2$, $\hat{P}$, $\hat{S}^2$, and $\hat{T}^2$. The operators $\hat{C}_2^2$, $\hat{T}_x^2$, and $\hat{P}$ are used in the following sense: $\hat{C}_2^2$ is used to generate a new distribution of the system, which will be characterized by $\hat{C}_2^2$, $\hat{T}_x^2$, $\hat{P}$, $\hat{S}^2$, and $\hat{T}^2$; $\hat{T}_x^2$ is used to generate a new distribution of the system, which will be characterized by $\hat{C}_2^2$, $\hat{T}_x^2$, $\hat{P}$, $\hat{S}^2$, and $\hat{T}^2$; and $\hat{P}$ is used to generate a new distribution of the system, which will be characterized by $\hat{C}_2^2$, $\hat{T}_x^2$, $\hat{P}$, $\hat{S}^2$, and $\hat{T}^2$.

II. THE STATISTICAL PROPERTIES OF THE RANDOM REPRESENTATIVE VECTOR (RRV)

The most economical way to define the prototype RRV as well as to derive its statistical properties is in terms of the signal, i.e., characterizing the Hamiltonian in the n -d-mode space. We will call the set of multiplet labels constituting the eigenstates $\{e_i\}$. The representation is

$$|e_i\rangle = \sum_k I_{ik} |x_k\rangle \quad (1)$$

where $|x_k\rangle$ is a basis that although the analysis of the $|x_k\rangle$ will be very similar to that of $|e_i\rangle$, in general, $|x_k\rangle$ is a pure state, i.e., the time $t=0$ state $|x_k\rangle$ does not rotate w/ time. In our case, $|x_k\rangle$ is the state of the k th mode, i.e., the k th component of the spin $\mathbf{S}$, $|x_k\rangle = |S_k, S_k\rangle$. The I_{ik} are the components of the unitary transformation U that carries the $|x_k\rangle$ into the $|e_i\rangle$. The I_{ik} are dependent on the Hamiltonian, i.e., $I_{ik} = I_{ik}(H)$. The $|x_k\rangle$ are dependent on the multiplet labels, i.e., $|x_k\rangle = |x_k(e_i)\rangle$. For the same $|e_i\rangle$, there will be different $|x_k\rangle$ for different i . For example, $|x_1\rangle = |S_1, S_1\rangle$ for $i=1$ but $|x_1\rangle = |S_2, S_2\rangle$ for $i=2$ and I_{21} is the value of I_{21} for $i=2$ and $k=1$. The $|x_k\rangle$ are the $|x_k(e_i)\rangle$ for the prototype RRV, i.e., $k=1, 2, \dots, n$.

$$r = \frac{\sum_k a_k x_k}{\sum_k (a_k)^2} \quad (2a)$$

$$\sum_k (a_k)^2 = 1 \quad (2b)$$

where the shell-model basis amplitude a_k is completely random except for the restrictions that all a_k be real, we confine ourselves to bound states) and that they obey the normalization condition eq. (2b). (There will be no change in these conditions when we introduce random sampling; see below). It is then clear that we can re-express $|r\rangle$ as follows,

$$|e\rangle = \sum_i (1 + c_i) |e_i\rangle / (N_e)^{1/2} \quad (3a)$$

where, as noted, N_e is the dimension of $|e_i\rangle$ and $\{c_i\}$ is a set of real coefficients which must satisfy the following normalization conditions,

$$\sum_i (1 + c_i)^2 = \sum_i (1 + d_i) = N_e \quad (3b)$$

$$d_i = c_i^2 + 2c_i \quad (3c)$$

$$\sum_i d_i = 0 \quad (3d)$$

Now let us calculate the matrix element $U_e(r|e\rangle$, where U is any operator that commutes with H ,

$$U_e(r|e\rangle = \sum_i (1 + d_i) \langle e_i | \quad (4)$$

and $\langle e_i |$ is the eigenvalue of $\hat{O}$ corresponding to the eigenvalue e_i of $\hat{O}$. Since the element $\langle e |$ in the axial eq. is well defined in the limit $\epsilon \rightarrow 0$, we have

$$\langle e | \hat{O} = \frac{1}{N_e} \sum_i \langle e_i | \quad (5a)$$

$$\delta \langle e_i | = \langle e_i | - \langle e | \quad (5b)$$

Substituting (5b) into (4) and (5a) we have,

$$U_e(r|e\rangle = \sum_i (\langle e | + d_i) \langle e_i | + d_i \langle e_i | \quad (6a)$$

$$r|e\rangle = \langle e | + \left(\sum_i d_i \right) \langle e | / N_e \quad (6b)$$

where we have used Eq. (3d) in deriving Eq. (6b). Now let us consider the results of averaging Eq. (6b) over a set of N_r RRV's. We will call this set of $\{r_i\}$ ($|r_j\rangle$). (Typically such a set will consist of ~ 20 RRV's; i.e., $N_r \sim 20$.) Let us denote the set average over $\{r_j|0\rangle|r_j\rangle$ by the symbol $\langle 0 \rangle_r$. The result for $\langle 0 \rangle_r$ is as follows,

$$\langle 0 \rangle_r = \langle 0 \rangle + \left(\sum_i \sum_j d_{ji} \right) \langle 0_i \rangle / N_e N_r \quad (7)$$

where we have added the suffix j to d_i to make d_{ji} , since we now have N_r sets $\{d_{ji}\}$. As the sets $\{d_{ji}\}$ and $\{0_i\}$ are uncorrelated the second term of Eq. (7) must approach zero as N_r gets larger. Thus we obtain the result,

$$\langle 0 \rangle_r = \langle 0 \rangle \quad (8)$$

which is to say that on the average the matrix element $\langle r|0\rangle_r$ must approach the operator-moment $\langle 0 \rangle$. This is the statistical result which forms the basis for the RRV method.

The utility of Eq. (8) depends on the magnitude of the dispersion of the set $\{r_j|0\rangle|r_j\rangle$ around the mean $\langle 0 \rangle_r$. This dispersion yields the probable error of $\langle 0 \rangle_r$, which we denote by $R(0)$. Assuming a gaussian distribution we have,

$$R(0) = 0.6745 \left[\sum_i (r_i|0\rangle|r_i\rangle - \langle 0 \rangle_r)^2 / N_r (N_r - 1) \right]^{1/2} \quad (9)$$

Thus the complete statistical result is given by combining Eqs. (6) and (9), which yields,

$$\langle 0 \rangle = \langle 0 \rangle_r + R(0) \quad (10)$$

Since $R(0)$ will depend on the character of the model space and the operator $\hat{O}$, as well as other things having no direct relation with the moments $\langle \hat{O}^n \rangle$, in general it is not possible to give a form for it which corresponds to Eq. (7), which, for example, leads directly to Eq. (8). The $R(0)$ has to be evaluated for each model using the set $\{|r_j\rangle\}$, (i.e., we cannot use the moments of $\hat{O}$.) Since the set $\{|r_j\rangle\}$ is available, $R(0)$ is calculated along with $\langle \hat{O}^n \rangle$. The remaining question of a gaussian distribution for the individual deviations is then answered by direct inspection of the set $\{|r_j\rangle\}$. Initial results indicate that the dispersion is approximately gaussian in the test case of ^{21}Ne , as is discussed in Section III.

Several remarks concerning the above derivation are in order at this point:

(1) The full shell-model basis $\{|x_j\rangle\}$ is not required in Eq. (2a) in order for the results of Eqs. (7) and (8) to remain valid. In fact only a small portion of the total model space may be needed to get accurate results for at least the lowest moments, so long as each $|r_j\rangle$ is built by randomly sampling the full basis $\{|x_j\rangle\}$. The dimensions of these truncated sets will be denoted by k_j in the following, where $N_d \ll N_b$. Results based on such a statistical sampling of the full model space in the cases of ^{21}Ne and ^{28}Si are presented in the next section. These results clearly demonstrate the utility of sampling, which thus makes it possible to study extremely large model spaces. At the moment the largest for which we have results has a dimension of $2 \cdot 10^8$. This is the case of ^{28}Si .

(2) The detailed character of the operator $\hat{O}$ also does not affect the validity of Eqs. (7) and (8). The moments of one-body, two-body, three-body, or more complex operators can all be studied provided the operators are available in suitable form. To date we have worked only with the first two types of operators. Of course, the more complex the operator the bigger the demand for computer memory storage and time.

(3) The symmetries characterizing the set $\{|r_j\rangle\}$ will also characterize the model space for which $\langle\hat{O}\rangle_r$ is calculated. This model space could well be a subspace of the full basis $\{|x_j\rangle\}$. In any case, this statement is exact so long as $\hat{O}$ is a member of the commuting set of operators which correspond to the symmetries in question. (Several examples were given earlier). However, we note immediately that Eqs. (7) and (8) are not necessarily valid if the RRV set $\{|r_j\rangle\}$ is symmetry limited. This will depend on the character of the symmetries. These could introduce a non-statistical bias into the set $\{|r_j\rangle\}$. It is for this reason that the correction algorithm mentioned earlier and discussed fully in Ref. 6 was developed. These remarks do not apply to the axial-type symmetries corresponding to J_z and T_z , or to parity (μ). This is because the Fock-space representation is inherently constituted with these symmetries; i.e., each Slater determinant individually has good J_z , T_z , and μ . This is explicitly not true for such operators as J^2 , T^2 , and of course H . Thus, in general it is not possible to work in subspaces of good J or T when calculating moments of H , for one example, without a correction algorithm of some kind.

Although, as discussed above, there are symmetry limitations to the RRV method it is perfectly feasible to use m-scheme subtraction to get the moments for good J or T . This of course requires a minimum of two calculations. Thus

to calculate, for example, the moment of the operator $\mathcal{O}$, (which we assume commutes with J_z) for $J = J_0$, one needs the moments for $J_z = J_0$ and $J_z = J_0 + 1$. In obvious notation, we immediately can say

$$\langle \mathcal{O}(J = J_0) \rangle = \frac{N_e(J_0) \langle \mathcal{O}(J_z = J_0) \rangle - N_e(J_0 + 1) \langle \mathcal{O}(J_z = J_0 + 1) \rangle}{N_e(J_0) - N_e(J_0 + 1)} \quad (11)$$

where $N_e(M)$ is the dimension of the space with $J_z = M$.

III. RESULTS FOR $A = 21$

In the following we present numerical results for ^{21}Ne as modeled by 5 nucleons in the $1d5/2$, $2s1/2$, $1d3/2$ shells using (mainly) the Chung-Wilentz (CW) matrix elements for our hamiltonian.⁹ We also present a few numerical results for ^{28}Si as modeled by 12 nucleons in the $1d5/2$, $2s1/2$, $1d3/2$, $1f7/2$ shells. Here we use the PMM (Petrovich-McManus-Madsen) hamiltonian,¹⁰ a local variety of the Kallio-Koltveit interaction.¹¹ We begin with the ^{21}Ne case, for which we calculated the moments $\langle H^n \rangle_\rho$, $n = 1$ to 6. (Some results for $\langle J^2 H^n \rangle_\rho$, $n = 1$ to 3 are given in Ref. 6). Since exact shell model calculations for the $1113, T = 1/2$ states in this case were available using the CW matrix elements it was also possible to calculate the true error in $\langle H^n \rangle$ and compare this with the probable error calculated from the dispersion in the set $\{\langle H^n \rangle_{\rho_j}\}$ (see Eq. (9)). The $T = 1/2$ result was gotten by using the m -subtraction scheme as given in Eq. (11). We used 50 random vectors in each of our sets $\{\rho_j\}$. The full basis for $J_z = T_z = 1/2$ has 1935 Slater determinants and calculations were carried out for five different random sets with $N_\rho = 1935, 963, 490, 245$, and 50. The successive reduction in the basis size for each random set is for the purpose of testing the sampling technique described in Section II. It is to be emphasized that the sampling was from the full basis as is required for Eqs. (7) and (8) to retain their validity.

In Table 1 we give the full basis ($N_\rho = 1935$) results for $J_z = 1/2$ and $T_z = 3/2, 1/2$. We also give the subtracted results for $T = 1/2$. The true errors or deviations from the exact-diagonalization results are shown in the fourth row. The probable errors as calculated from the observed dispersions

(see above) are given with each result. As can be seen the RRV results for $T = 1/2$ have probable errors of the same magnitude as the true errors, which is in reasonable accord with the gaussian assumption mentioned earlier. The more-or-less systematic variation in the true error as a function of n is due to the fact that the same set $\{H_j\}$ is used for the calculation of all the moments, $n = 1$ to 8. Thus, whatever bias characterizes this set for the first moment $\langle H_j \rangle$ will also influence the higher moments. Besides this there is also a roughly linear increase in the fractional probable error as a function of n . It is possible to understand these regular changes on the basis of simple models for the amplitude distributions in such expressions for $\langle H_j \rangle$ as Eq. (3a), however, the actual magnitude of the increase will depend to some extent on the particular properties of the bell-model space and distribution as well as the sampling fraction. In this case the sampling fraction is 100 (full basis) and the errors range from 40.2% for H_1 to 1.0% for H_8 . It will be seen shortly that both the sign and magnitude of this behavior depends on truncation. We also emphasize that these figures pertain to a set of 20 RRV's. Thus the range of the probable error for any single full-basis RRV is $\sqrt{20}$ bigger or 11.0% to 4.5%. For smaller truncations, e.g., 21, it is relatively easy to cut down the statistical error drastically and quickly by simply enlarging the number of members in the set of RRV's.

In Table 2 we present results for the same model as in Table 1, but $\tau_{1/2}$ as a function of the random truncation of the full-basis $\tau_{1/2}$ (see Section II). As already noted, five truncations were studied for the moments $\langle H_j \rangle$, $j = 1$ to 8: 100 (full basis), 50, 10, 5, and 2.5. The results of Table 2 are for $J_z = I_z = 1/2$ and the sets of RRV's used to obtain them were distinct from those used for Table 1. For this reason we were able to use the results of

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in the sets $\{r_j\}$. Furthermore, the magnitude of these moments for our studies in this area are continuing and will be presented in the near future.

The last set of results, which we present pertains to the eigenvalues and moments of H_0^n in the eigen-basis $\{r_j\}$. In order to present the results to the information necessary to calculate the hyperfine interaction, the results are for all J and all I ; they are shown in Table 1. They are calculated based on the results obtained with the RRV interaction, and are shown in Table 1₁. In order to obtain the eigenvalues and moments with the CW interaction, the dispersion in the moments, which, for the present, we call ΔH_r^n , is given by

$$H_r^n = (H_0^n + \Delta H_r^n) / \sqrt{N_r} \quad (12)$$

For the sake of comparison we also present moments calculated with the RRV interaction using the sets of RRV's in order to study the effect of the choice of the distribution in this case compared to the "physical" hyperfine interaction distributions. We call this dispersion ΔH_r^n , and it is given by

$$H_r^n = (\sum_j (r_j^n H_r^n / r_j^n - H_r^n / r_j^n) / \sqrt{N_r})^{1/2} \quad (13)$$

Only results for $n = 1$ to 5 are presented. The calculation of H_r^6 requires the evaluation of H_0^{2n} , and as only calculation up to the 5th moment for the CW interaction and the 10th moment for the RRV interaction, in the case of ^{21}Ne . In the case of ^{29}Si , calculations have (so far) been made only up to the 3rd moment, so only results for H_r^3 can be presented at this

[illegible]

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NOTICE

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	$\langle H^1 \rangle_r$ $\times 1$	$\langle H^2 \rangle_r$ $\times 10^2$	$\langle H^3 \rangle_r$ $\times 10^4$	$\langle H^4 \rangle_r$ $\times 10^5$	$\langle H^5 \rangle_r$ $\times 10^7$	$\langle H^6 \rangle_r$ $\times 10^8$	$\langle H^7 \rangle_r$ $\times 10^{10}$	$\langle H^8 \rangle_r$ $\times 10^{11}$
$T_z = 3/2$	-17.60	3.714	-0.8665	2.175	-0.5796	1.614	-0.467	1.392
	$\pm .05$	$\pm .015$	$\pm .0051$	$\pm .017$	$\pm .0059$	$\pm .021$	$\pm .008$	$\pm .041$
$T_z = 1/2$	-20.10	4.788	-1.262	3.593	-1.082	3.474	-1.158	4.011
	$\pm .04$	$\pm .013$	$\pm .005$	$\pm .016$	$\pm .007$	$\pm .027$	$\pm .010$	$\pm .044$
$T = 1/2$	-21.95	5.580	1.553	4.638	1.453	4.847	1.669	5.945
	$\pm .07$	$\pm .028$	$\pm .009$	$\pm .027$	$\pm .013$	$\pm .053$	$\pm .020$	$\pm .080$
TRUE ERROR,	$\pm .11$	$\pm .050$	$\pm .014$	$\pm .051$	$\pm .001$	$\pm .019$	$\pm .003$	$\pm .054$
$T = 1/2$								

TABLE 1.

Moments of H^n , i.e. $\langle H^n \rangle_r$, as calculated with the RRV method for the case of 5 nucleons in the $45/2$, $51/2$, $d3/2$ shells (^{21}Ne). The units of H are MeV. Calculations were made for $T_z = 3/2$ ($N_d = 322$) and $T_z = 1/2$ ($N_d = 1935$), $T = 1/2$ for all cases. Full basis RRV's were used in both cases. The result for $T = 1/2$ was calculated using Eq. (11), Sect. II. The difference between the $T = 1/2$ results and the exact diagonalization results, i.e. the true error, is given on row 4. The Chung-Wildenthal matrix elements were used in these calculations (Ref. 9) and each set of j_j had 20 RRV's. Since $J_z = 1/2$ these results are averages over all J .

f	n →	1	2	3	4	5	6	7	8
1.0	r	0.80	1.34	1.79	2.32	2.77	3.58	4.56	5.50
	R	0.18	0.30	0.40	0.52	0.62	0.80	1.02	1.23
	ERROR	-0.20	-0.36	-0.32	0.00	+0.83	+0.46	+0.78	+1.02
0.5	r	1.16	2.10	3.00	3.85	4.79	5.56	7.07	8.50
	R	0.26	0.47	0.67	0.86	1.07	1.31	1.58	1.90
	ERROR	-0.20	-0.50	-0.71	-0.72	0.00	-0.37	+0.86	+0.82
0.1	r	1.43	2.68	4.02	5.32	6.89	8.59	10.5	12.8
	R	0.32	0.60	0.90	1.19	1.54	1.92	2.35	2.85
	ERROR	-0.40	-0.96	-1.43	-1.78	-1.48	-2.30	-2.45	-2.57
.05	r	3.50	5.95	8.09	10.2	12.7	14.7	16.90	19.5
	R	0.79	1.33	1.81	2.29	2.83	3.28	3.78	4.35
	ERROR	-0.66	-1.52	-2.42	-3.14	-3.31	-4.46	-4.84	-5.05
.025	r	4.56	7.07	9.08	10.8	12.3	13.6	15.2	16.1
	R	1.02	1.58	2.03	2.41	2.74	3.05	3.39	3.60
	ERROR	+1.09	+1.60	+1.95	+2.33	+3.42	+3.31	+4.32	+4.48

TABLE 2. Probable errors and true errors in χ in H^n , $n = 1$ to 8 , as a function of the basis truncation (f) in the sets r, R, ERROR for the case $J_z = T_z = 1/2$ for ^{21}Ne . See the caption to Table 1 and Sect. II for further details. Five sets of results are given corresponding to a full basis ($N_d = 1935$), 50 basis ($N_d = 968$), 10 basis ($N_d = 194$), 5 basis ($N_d = 97$), and 2.5 basis ($N_d = 48$). Three results are given for each value of f , and n : $r \equiv$ the probable error for a single RRV's, $R \equiv$ the probable error in the mean of 20 RRV's, and the error or deviation between the mean and the "true" result given in Table 1 for $T_z = 1/2$.

n	1	2	3
$\langle H^n \rangle$	-1.13×10^2	1.33×10^4	-2.99×10^6
n, %	2.3%	4.2%	0.75%

TABLE 3. Values of $\langle H^n \rangle$, as calculated with the RRV method for ^{28}Si using the PMM hamiltonian (Refs. 10,11) and a shell-model basis consisting of the $1d5/2$, $2s1/2$, $1d3/2$, and $1f7/2$ shells with 12 nucleons. The moments are averages over all positive-parity states. The full-basis dimension is 5.5×10^4 , but the dimension (N_d) of each number of the set $\{|r_n\rangle$ is 60, in this case. These results are preliminary as they are based on only 5 RRV's.

	$\leftarrow \text{---} ^{21}\text{Ne} \text{---} \rightarrow$					^{28}Si
f, Basis fraction	1.0	0.5	0.1	.05	.025	10^{-6}
r,	0.8	1.2	1.43	3.5	4.56	2.3
$\sqrt{N_d} r,$	36	36	20	34	32	18

TABLE 4. Summary of probable errors r in $\langle H \rangle$ for ^{21}Ne and ^{28}Si . The models used for each case are described in Sect. II and Tables 1-3. The dimensions of the members of the sets $\{|r_n\rangle$ are called N_d (see Sect. II). As seen in the third row multiplying r by $\sqrt{N_d}$ brings the variation in all entries to less than a factor of 2.

n	ΔH_e^n	ΔH_r^n	ρ_{er}	f	Ham.	Nucleus
1	8.7	.23	37.6	1.0	CW	^{21}Ne
2	360	8.9	40.4	1.0	CW	^{21}Ne
3	1.4×10^4	320	42.8	1.0	CW	^{21}Ne
4	5.2×10^5	1.2×10^4	43.5	1.0	CW	^{21}Ne
5	5.2×10^7	9.6×10^5	43.2	1.0	PMM	^{21}Ne
1	8.7	1.3	6.7	.025	CW	^{21}Ne
2	360	50	7.2	.025	CW	^{21}Ne
3	1.4×10^4	1.7×10^3	8.2	.025	CW	^{21}Ne
4	5.2×10^5	5.6×10^4	9.3	.025	CW	^{21}Ne
5	5.2×10^7	5.8×10^6	9.0	.025	PMM	^{21}Ne
1	23.0	3.4	6.8	10^{-6}	PMM	^{28}Si

TABLE 5. Comparative results for rms dispersions in moments of H^n as calculated in the eigenbasis (column 2) and in the RRV sets (column 3); see Sect. III. The eigenbasis results are actually taken from the RRV results for the average values $\langle H^n \rangle$ in Sect. III. The units are $(\text{MeV})^n$. Column 4 (ρ_{er}) is the ratio of column 2 to column 1. Column 5 gives the truncation fraction (f) in the RRV's. Column 6 gives the hamiltonians: CW $\equiv$ Chung-Wildenthal (Ref. 9), PMM is given in Refs. 10, 11. Column 7 gives the nucleus; the shell models for ^{21}Ne and ^{28}Si are described in the captions to Tables 1 and 3 and in Sect. III.

FIGURE CAPTIONS

- Figure 1: Plot of R , the probable error, from Table 2, as a function of n (the power of H in $\langle H^n \rangle$) and the degree of truncation of the basis. See the caption to Table 1 and Sect. II for the description of the shell-model space and hamiltonian. Note that the probable error for a given RRV (or $\langle r_j^2 \rangle$) is $(20)^{1/2}$ times bigger than shown in this figure. Also note the change in scale at the truncation fraction = 0.1.
- Figure 2: Plots of the ratio of the probable errors R_n/R_1 as a function of n , from Table 2, where n is the power of the hamiltonian in the moment $\langle H^n \rangle$. There are five curves for each truncation fraction. See the captions to Table 2 and Fig. 1.
- Figure 3: Plot of the product $(R_n(f)/R_n(0)) \times f^{1/2}$, as a function of f for each value of n (1 to 8). This is a normalized dispersion; i.e., it has most of the variation due to basis size removed. See Sect. III.

FIGURE 1

FIGURE 2

FIGURE 3