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A Quenched $c = 1$ Critical Matrix Model

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Abstract:

We study a variant of the Penner-Distler-Vafa model, proposed as a $c = 1$ quantum gravity: 'quenched' matrix model with logarithmic potential. The model is exactly soluble, and exhibits a two-cut branching as observed in multicritical unitary matrix models and multicut Hermitian matrix models. Using analytic continuation of the power in the conventional polynomial potential, we also show that both the Penner-Distler-Vafa model and our 'quenched' matrix model satisfy Virasoro algebra constraints.

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1. Recently, there has been significant progress in understanding integrability of noncritical strings for $c < 1$ ¹. It is also believed that there exists a phase transition across $c = 1$ ² to a strong coupling regime of quantum gravity. Precisely at $c = 1$, it is possible to give an interpretation to the noncritical strings of propagating in two-dimensional spacetime: the Liouville mode provides one extra coordinate in target space, thus the string is propagating effectively in two-dimensions³. Furthermore, logarithmic scaling violations of the free energy at genus zero and one were observed². Its origin was attributed to the target space translational invariance respected at such orders³. Still, a complete understanding of the $c = 1$ noncritical string, such as the integrability and the logarithmic scaling violation, is missing.

Distler and Vafa⁴ observed quite important points that might shed light to deeper origin of the logarithmic scaling violation in the $c = 1$ noncritical strings. They convincingly argued that $c = 1$ noncritical string at integer radii can be described by the Penner matrix model^{5,6}: a Hermitian one-matrix model with a *logarithmic potential*. (Hereafter, we call the one discussed by Distler and Vafa⁴ as Penner-Distler-Vafa (PDV) model.) This description is in contrast to the conventional formulation in terms of a matrix quantum mechanics². The logarithmic potential in the PDV model might be interpreted as an 'effective' potential after integrating out all matrices but a stationary one from the conventional matrix chain. (Curiously, the PDV logarithmic potential turns out to be quite relevant also to the study of universal conductance fluctuations in disordered condensed matter systems⁷). Distler and Vafa⁴ observed that the logarithmic scaling violation is correctly reproduced in the PDV model formulation too. However, the origin of such violations is now due to the *worldsheet* conformal Killing symmetries on genus zero and one rather than the target space translational invariances. Thus, PDV model provides another important connection between the worldsheet symmetries and the spacetime symmetries of $c = 1$ noncritical strings.

Is the PDV model a unique one-matrix model which exhibits the logarithmic scaling vio-

lations? In this Letter, we find another exactly soluble one-matrix model (which is 'quenched' in the sense that is described below) with a logarithmic potential and observe again the logarithmic scaling violations at genus zero and one. However, its double scaling limit behavior is more akin to multicritical unitary matrix models or multicut Hermitian matrix models⁸, in contrast to the PDV model. We also show that both the PDV and our new 'quenched' matrix models can be reinterpreted in terms of analytic continuation of polynomial power in the conventional one-matrix model¹ potential.

2. We start with a reinterpretation of the PDV model, which makes the model less exotic than it appears. Consider a *complex* matrix model defined by

$$Z_{FP}[k; t] \equiv e^{N^2 t} \int d\phi d\bar{\phi} \exp Nt \text{Tr} \left(-\frac{1}{k} + \phi\bar{\phi} + \frac{1}{k}(\phi\bar{\phi})^k \right). \quad (1)$$

We generalize the model in such a way that the polynomial power index k takes a continuous positive real value by analytic continuation. The PDV model is identical to a limit $k \rightarrow 0^+$

$$Z_{PDV} = \lim_{k \rightarrow 0} Z_{FP} = e^{N^2 t} \int d\phi d\bar{\phi} \exp Nt \text{tr}(\ln \phi\bar{\phi} + \phi\bar{\phi}). \quad (2)$$

Since the complex matrix ϕ appears in a Hermitian combination $M = \phi\bar{\phi}$, one can diagonalize it in a standard way, $M_d = e^{i\omega} \cdot M \cdot e^{-i\omega}$ where ω is a diagonalizing Hermitian matrix. Then, 'fractional power' of the matrix $M = \phi\bar{\phi}$ can be defined unambiguously: $M^k = e^{-ik\omega} (M_d)^k e^{ik\omega}$. The eigenvalues are manifestly positive definite, and the Jacobian in the integral measure upon change of variables from $\phi, \bar{\phi}$ into M is an eigenvalue independent constant. Therefore, the 'fractional power' model Eq.(1) is well-defined which includes the PDV model as a limit, and has an advantage of restricting eigenvalues to an interval $[0, \infty)$ in a natural way unlike the original PDV model⁴.

The free energy in the doubling scaling limit for a fixed $\mu = -N(t - t_c)$, where $t_c = -1$ is⁴

$$F(\mu) = \mu^2 \ln \mu - \frac{1}{12} \ln \mu + \sum_{g=1} \chi_{2g} \mu^{-2g+2} = \int^{\mu} dx x \psi(x) \quad (3)$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$, $\chi_{2g} = \frac{B_{2g}}{2g(2g-2)}$, and χ_{2g} is the Bernoulli number. We recall that the ψ function satisfies a nice difference equation $\psi(x+1) - \psi(x) = \frac{1}{x}$. The free energy is quite reminiscent of the topological quantum gravity⁹.

At integer values of k , the ‘fractional power’ model Eq.(1) defines a discretization of two dimensional Riemann surfaces by k -gons. Thus, geometrical interpretation of the ‘fractional power’ model Eq.(1) calls for an analytic continuation of the polygons we use to discretize the two-dimensional Riemann surfaces. The limit $k \rightarrow 0$ amounts to no regular polygons. However, the logarithmic potential corresponds to an infinitely many polygons with equal weights. Thus, we find that discretization of the Riemann surfaces by carefully arranged infinitely many polygons is equivalent to that with no polygons at all!

3. It is known that one-matrix models with conventional polynomial potentials obey Virasoro constraints¹⁰. One may ask whether this holds true for PDV model too. We now show that, using the above ‘fractional power’ model, it is indeed the case. Consider the operators

$$\hat{L}_n = \text{Tr}[(\phi\bar{\phi})^n \phi \frac{\partial}{\partial \phi} + (\phi\bar{\phi})^n \bar{\phi} \frac{\partial}{\partial \bar{\phi}}]$$

and polynomially perturbed generating function $Z[t_0, t_1, \dots]$

$$Z_{FP}[k; t_0, t_1, \dots] = \int d\phi d\bar{\phi} e^V, \quad V = \left[\frac{t_0}{k} \{(\phi\bar{\phi})^k - 1\} + \sum_{a \geq 1} t_a (\phi\bar{\phi})^a \right]. \quad (4)$$

The Ward identity is derived from

$$0 = \int d\phi d\bar{\phi} \text{Tr} \left(\frac{\partial}{\partial \phi} \phi (\phi\bar{\phi})^n + \frac{\partial}{\partial \bar{\phi}} \bar{\phi} (\phi\bar{\phi})^n \right) e^V. \quad (5)$$

In order for the surface term to be absent, we need to restrict $n \geq -1$. Explicitly expanding Eq.(5), we find that

$$L_n Z_{FP}[k; t_0, t_1, \dots] = 0, \quad L_n = \sum_m \frac{\partial}{\partial t_{m+n}} \frac{\partial}{\partial t_m} + t_0 \frac{\partial}{\partial t_n} + \sum_{m \geq 1} m t_m \frac{\partial}{\partial t_{m+n}}. \quad (6)$$

It is easy to check that the algebra of L_n 's is closed and form the Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n}$ without a central extension. Since PDV model is an analytic limit $k \rightarrow 0^+$

of the 'fractional power' generating function Eq.(4), and the operators L_n 's are independent of k , the Virasoro constraint Eq.(6) and the Virasoro algebra certainly holds for the PDV model too, $L_n Z_{\text{PDV}}[t_0, t_1, \dots] = 0$. This completes the proof that the PDV model satisfies a discrete version of the Virasoro constraint. Whether the constraints hold true after the continuum limit is taken, however, remains to be seen.

4. Is there any other one-matrix model which is as simple as the PDV's, but with significant qualitative differences? The answer is yes; let us consider a 'quenched' Hermitian matrix model defined by

$$Z = e^F \equiv \int dM \exp(Nt - \frac{1}{2}) \text{Tr} \ln(1 - M^2) \quad (7)$$

in which $M \equiv \frac{1}{2}(U + U^\dagger)$, made out of a unitary matrix U . The model is 'quenched' in the sense that, as will be shown below, the eigenvalues of M are bounded on a finite interval $[-1, 1]$. The measure in Eq.(1) is defined by $dM = \prod_a d\text{Re}M_{aa} \prod_{a>b} d\text{Re}M_{ab} d\text{Im}M_{ab}$ as in the standard Hermitian matrix models.

One can again define our 'quenched' model through the 'fractional power trick', from which it is easy to prove that 'quenched' model again obeys discrete Virasoro constraints.

Diagonalizing the Hermitian matrix M ,

$$M_D = \frac{1}{2} \text{diag}(e^{i\theta_1} + e^{-i\theta_1}, e^{i\theta_2} + e^{-i\theta_2}, \dots),$$

the partition function reads

$$F = \ln \int_0^\pi \prod_a d\theta_a \prod_{a<b} |\cos \theta_a - \cos \theta_b|^2 \exp(2Nt \sum_a \ln \sin \theta_a). \quad (8)$$

We now introduce orthogonal polynomials

$$P_n^\lambda(\cos \theta) = [\cos^n \theta + \dots],$$

$$\int_0^\pi d\theta e^{\lambda \ln \sin \theta} P_n(\cos \theta) \cdot P_m(\cos \theta) = h_n^\lambda \delta_{n,m}. \quad (9)$$

where $\lambda = Nt$. It is easy to show that the polynomials P_n^λ satisfy a recursion relation

$$xP_n^\lambda(x) = P_{n+1}^\lambda(x) + R_n P_{n-1}^\lambda(x) \quad (10)$$

where

$$R_n = \frac{h_n}{h_{n-1}} = \frac{n(n+2\lambda-1)}{4(n+\lambda)(n+\lambda-1)}, \quad h_0 = 2\pi \cdot 2^{-2\lambda} \cdot \frac{\Gamma(2\lambda)}{\lambda\Gamma^2(\lambda)}. \quad (11)$$

In fact, this recursion relation is a special case of the relations between contiguous Riemann $\mathcal{P}$ -functions¹¹.

In terms of these polynomials, the free energy is written as

$$\begin{aligned} F &= \ln \int \prod_a d\theta_a \det P_a(\cos \theta_b) \prod_a (\sin \theta_a)^\lambda \\ &= \ln N! + N \ln h_0 + \sum_{k=1}^{N-1} k \ln R_{N-k} \end{aligned} \quad (12)$$

The expression of free energy is quite similar to the PDV model, except that now we have several terms that contribute. As in the PDV model, the formal sum does not make sense as it stands. Thus, we consider convergent, third derivatives of the free energy with respect to a scaling parameter related to λ . The potential in Eq.(8) has its maximum at $\theta = \pi/2$, and deep wells at near $\theta = 0$ and π . Thus, there are two possibilities of eigenvalue distributions: one-cut eigenvalue filling and two-cut eigenvalue fillings.

First, we study the one-cut eigenvalue filling. Keeping the scaling parameter $\mu = -N - (\lambda - 1/2)$ fixed,

$$\begin{aligned} \frac{\partial^3 F}{\partial \mu^3} &= -N \frac{\partial^3 \ln h_0}{\partial \mu^3} + \sum_{k=1}^{N-1} k \left[\frac{2}{(N-k+\lambda)^3} + \frac{2}{(N-k+\lambda-1)^3} - \frac{2 \cdot 2^3}{(N-k+2\lambda-1)^3} \right] \\ &\rightarrow - \sum_{k=1}^{\infty} \left[\frac{2k}{(k+\mu-\frac{1}{2})^3} + \frac{2k}{(k+\mu+\frac{1}{2})^3} \right]. \end{aligned} \quad (13)$$

Thus, an unregularized free energy reads

$$F(\mu) = - \sum_k k \left[\ln(k+\mu+\frac{1}{2}) + \ln(k+\mu-\frac{1}{2}) \right]$$

$$= - \sum_{k=\frac{1}{2}+Z_+} 2k \ln(k + \mu) \quad (14)$$

In terms of the ψ function, the free energy is a sum of two contributions of different 'Fermi energy'

$$F(\mu) = - \left(\int^{\mu+\frac{1}{2}} + \int^{\mu-\frac{1}{2}} \right) dx \, x \psi(x) \quad (15)$$

The free energy is quite analogous to the PDV model, but now, consists of two terms with different Fermi energies, $\mu \pm 1/2$. Therefore, we may interpret that the theory is a sum of two separate PDV models. After the analytic continuation of $\mu \rightarrow i\mu$, the free energy is real but *negative* definite. This indicates that the model might be related to some nonunitary $c = 1$ matter coupled to gravity. This is traced back to the fact that the denominator of R_n in Eq.(11) provides the single-well filling critical point. When $\mu \rightarrow \infty$ (high temperature limit), it is easy to see that the free energy behaves as $F(\mu) \approx -2\mu^2 \ln \mu - \frac{1}{3} \ln \mu + \dots$. Thus it exhibits logarithmic divergences at genus zero and one, the characteristic of $c = 1$ matrix quantum mechanics models. Taylor expansion of Eq.(15) have only even powers of μ due to the symmetric splitting of the two different Fermi energy levels around μ , $\mu \pm 1/2$. Furthermore, in the high-temperature limit $\mu \rightarrow \infty$, the higher genus contributions are dominated by the leading terms $\chi_{2g} \propto (2g)!$. Corrections due to Fermi energy level difference is negligible for higher genus. Therefore, with all these right features, we expect the 'quenched' matrix model related to a $c = 1$ topological gravity.

Next, let us consider the two-cut eigenvalue fillings of the potential. This is the case analogous to the two eigenvalue cuts into one eigenvalue cut critical behavior observed in the multicritical unitary matrix models and in the multicut Hermitian matrix models. Let us keep the scaling parameter $\mu = -N - 2(\lambda - 1/2)$ fixed. Then, Eq.(13) becomes

$$\frac{\partial^3 F}{\partial \mu^3} = \sum_{k=1}^{N-1} 2k \left[\frac{1}{(k + \mu)^3} - \frac{(1/2)^3}{(k + \frac{\mu}{2} - \frac{N}{2} - \frac{1}{2})^3} - \frac{(1/2)^3}{(k + \frac{\mu}{2} - \frac{N}{2} + \frac{1}{2})^3} \right] \quad (16)$$

Subtracting out bulk contributions from Eq.(16) by Cauchy principal prescription, we find

the unregularized free energy

$$F(\mu) \rightarrow \sum_{k=\mathbb{Z}_+} k \ln(k + \mu) - \sum_{k=\frac{1}{2}+\mathbb{Z}} 2k [\ln(k + \frac{\mu}{2}) + \ln(k - \frac{\mu}{2})] \quad (17)$$

in the double scaling limit. Again, this can be expressed in terms of the ψ -function

$$F(\mu) = \left(\int^{\mu} - \int^{\frac{\mu+1}{2}} - \int^{\frac{\mu-1}{2}} - \int^{\frac{-\mu+1}{2}} - \int^{\frac{-\mu-1}{2}} \right) dx x \psi(x) \quad (18)$$

Therefore, the model can be interpreted as products of a unitary PDV model and four nonunitary PDV-like models. The high-temperature expansion shows $F(\mu) \approx 0 + \frac{3}{4} \ln \mu + \dots$. Due to cancellations between the positive and the negative parts, there is no tree-level contribution of free energy. Again, the symmetric splitting of the Fermi energy levels in Eq.(18) ensures that Taylor expansion contains only even powers of μ^2 . Higher genus contributions to the free energy are dominated by the nonunitary part, and is proportional to $2^{2g} \cdot (2g)!$. Again, all these features of the double scaling limit for two-cut eigenvalue fillings imply that it describes another topological phase of $c = 1$ two-dimensional quantum gravity.

The reason why there exist nonunitary contributions in the two-cut filling into one-cut filling critical behavior can be traced again back to formula R_n in Eq.(11). When we take the continuum limit by keeping $\mu = -N - 2(\lambda - 1/2)$ fixed, we have to pass through the near-zeros of the denominators of R_n . They generate poles to the free energy at all integer values along the real axis, and create the aforementioned nonunitary behaviors.

We found that our 'quenched' matrix model exhibits essentially the same behavior as the PDV model, but the free energy contains both positive and negative contributions. They originate from the zeros and poles of R_n with respect to the scaling parameter. Still, the genus expansion is well-defined, and exhibits the characteristic $(2g)!$ behavior, a hallmark of topological gravity. In addition, the model is quite reminiscent of the unitary matrix model in that the eigenvalue distribution shows a critical transition from two-cut into one-cut filling

at double scaling limit. Thus, we are tempted to conjecture that these models describe a system which includes spacetime fermions¹². (This conjecture is hinted further by the fact that the sign of one-loop free energy of the 'quenched' matrix model is negative to that of the PDV-model.) Generic models that contain both spacetime bosons and fermions may be described by one matrix models whose R_n has both zeros and poles with respect to the scaling parameter. It would be extremely interesting if any such connection can be made between our 'quenched' theory and, for example, supersymmetric $\hat{c} = 1$ matrix model¹³.

5. The two topological matrix models, PDV and 'quenched' ones, are special in the sense that they are related to the associated Laguerre and the associated Gegenbauer polynomials, familiar from the study of Sturm-Liouville second-order ordinary differential operators¹¹ and the theory of orthogonal polynomials¹⁴. It may hint a possible connection of the $c = 1$ noncritical strings to the Sturm-Liouville theory. It is also known that the associated Gegenbauer polynomials are a subset of Jacobi polynomials which are solutions of hypergeometric differential equations. We are currently investigating critical behaviors of the most general 'quenched' one-matrix models associated with the hypergeometric differential equations, and will report elsewhere. Certainly, it remains to study these models more to learn about the topological phase of $c = 1$ string theory.

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