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THEORY OF FIELD LINE RESONANCES OF STANDING SHEAR ALFVEN
WAVES IN THREE-DIMENSIONAL INHOMOGENEOUS PLASMAS

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Theory of Field Line Resonances of Standing Shear Alfvén Waves in Three-Dimensional Inhomogeneous Plasmas

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Abstract

We have analyzed field line resonances of Alfvén waves in a rectangular box model with a straight uniform magnetic field but three dimensionally varying density. Field line resonances are shown to exist even with this three-dimensional nonuniformity. For a given wave frequency, we can construct the surface on which the resonance occurs and derive the local form of the singular solution. Magnetic perturbations are found to lie predominantly in the resonant surface. In the presence of azimuthal inhomogeneities, the present theory could explain why some satellite measurements show geomagnetic pulsations of comparable magnitude in radial and azimuthal components.

I. Introduction

The theory of field line resonances (FLR) has been developed in various magnetic field geometries and with different levels of asymmetry. Originally, in the papers of Southwood [1974] as well as Chen and Hasegawa [1974], only one-dimensional inhomogeneity perpendicular to the straight magnetic field $\mathbf{B}$ was assumed. Later inhomogeneities along $\mathbf{B}$ have been accounted for by both Southwood and Kivelson [1986] and Chen and Cowley [1989], the latter using a dipole magnetic field model. As in the original papers, the eigenfrequencies for the transverse (shear) Alfvén waves varied spatially. On the surface where the wave frequency is equal to the shear Alfvén frequency the wave becomes singular. One therefore says that the Alfvén wave is resonant at this surface. The observed spectrum of Alfvén waves at a point is expected to be dominated by the resonant component (i.e., the component with the local shear Alfvén wave frequency). The perturbed magnetic field will lie predominantly in the azimuthal direction.

All of the models described above neglect nonaxisymmetric inhomogeneities in the magnetosphere such as the day-night asymmetry, the plasma bulge, etc. As it is not clear, how to generalize to these situations without continuous symmetries, we chose to investigate a three-dimensional inhomogeneous plasma in our work. Thus accounting for the previously mentioned azimuthal asymmetries, we examine whether the existence of field line resonances persists in this altered geometry. As in the earlier two-dimensional treatments, the resonant surfaces and the field line eigenfunctions are defined and the form of the singularity is calculated. An additional motivation for introducing a third dimension of asymmetry is given by the observation of field line resonances with $|\delta B_r| \sim |\delta B_\theta|$. A possible explanation for this kind of satellite measurements is provided.

II. Theoretical Model

A cold plasma in a straight uniform magnetic field in z direction will be considered. The density, and therefore the Alfvén speed v_A , is assumed to vary in x , y and z directions.

corresponding to the radial, azimuthal, and field-aligned directions in the magnetosphere. The field lines are considered to have fixed end points at $z = \pm\chi_0$. No boundary conditions are imposed in x and y directions. We will use ideal magnetohydrodynamic (MHD) equations to describe the plasma, and the perturbed quantities are assumed to vary as $\exp(-i\omega t)$.

Linearizing Ohm's law, force balance, Faraday's law, and the low-frequency Ampere's law and defining $\boldsymbol{\eta} \equiv \boldsymbol{\xi} \times \mathbf{B}$, where $\boldsymbol{\xi}$ is the usual one-fluid displacement vector, we get the following equation

$$\left\{ \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v_A^2} \right\} \boldsymbol{\eta} = -\hat{\mathbf{z}} \times \nabla \frac{\delta B_{\parallel}}{B}. \quad (1)$$

Here

$$\frac{\delta B_{\parallel}}{B} = \hat{\mathbf{z}} \cdot (\nabla \times \boldsymbol{\eta}) \quad (2)$$

is the perturbed compressional component of the magnetic field and $v_A = v_A(x, y, z) = [B^2/\mu_0\rho(x, y, z)]^{1/2}$ is the Alfvén velocity.

Now replacing x and y with two arbitrary coordinates p and q and with $\nabla p \cdot \nabla q = 0$ (we will make an appropriately motivated choice later), we can express $\boldsymbol{\eta}$ as

$$\boldsymbol{\eta} = \eta_p \nabla p + \eta_q \nabla q. \quad (3)$$

The Jacobian of this transformation is

$$J(p, q) = \nabla p \times \nabla q \cdot \hat{\mathbf{z}}. \quad (4)$$

Taking the ∇p and ∇q components of Eq. (1) and defining $|\nabla q|/|\nabla p| = \Gamma$, the set of equations to be solved is

$$\frac{\delta B_{\parallel}}{B} = J(p, q) \left(\frac{\partial \eta_q}{\partial p} - \frac{\partial \eta_p}{\partial q} \right), \quad (5)$$

$$\left\{ \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v_A^2} \right\} \eta_p = \Gamma \frac{\partial}{\partial q} \frac{\delta B_{\parallel}}{B}, \quad \text{and} \quad (6)$$

$$\left\{ \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v_A^2} \right\} \eta_q = -\frac{1}{\Gamma} \frac{\partial}{\partial p} \frac{\delta B_{\parallel}}{B}. \quad (7)$$

III. FLR Eigenfunctions

Next let us define field line resonance (FLR) eigenfunctions $a_i(p, q, z)$ with eigenvalues $\omega_i(p, q)$ and standing wave boundary conditions,

$$a_i(p, q, z = \pm \chi_0) = 0, \quad (8)$$

by the equation

$$\left\{ \frac{\partial^2}{\partial z^2} + \frac{\omega_i^2}{v_A^2} \right\} a_i(p, q, z) = 0, \quad (9)$$

where the ω_i are functions of p and q . Up to now, p and q are not defined yet, but as the operator $\{\dots\}$ depends on z only, we can solve Eq. (9) in our initial coordinate system and obtain $\omega_i(x, y)$ and $a_i(x, y, z)$. Once we will have decided on how to choose p and q , we then replace x and y by $x(p, q)$ and $y(p, q)$.

Comparing Eq. (9) with Eq. (1) we see that these would be the eigenfunctions if we were able to set $\delta B_{\parallel}$ to zero thus eliminating the compressional Alfvén wave. These a_i —the shear Alfvén waves with their corresponding eigenfrequencies ω_i —provide a convenient base in which to expand η_p , η_q , and $\delta B_{\parallel}/B$. Due to the smooth spatial variation of the Alfvén frequency, they form the so-called shear Alfvén continuum. On any given field line, these eigenfunctions are orthogonal, normalizable, and form a complete base, with the orthogonality condition given by

$$\int_{-\chi_0}^{\chi_0} \frac{a_i a_j}{v_A^2(p, q, z)} dz = \delta_{ij}, \quad (10)$$

where δ_{ij} is the Kronecker delta. Now we can express η_p , η_q , and $\delta B_{\parallel}/B$ in terms of a_i as:

$$\eta_p = \sum_{m=0}^{\infty} \lambda_m(p, q) a_m(p, q, z), \quad (11)$$

$$\eta_q = \sum_{m=0}^{\infty} \mu_m(p, q) a_m(p, q, z), \quad \text{and} \quad (12)$$

$$\frac{\delta B_{\parallel}}{B} = \sum_{m=0}^{\infty} \gamma_m(p, q) a_m(p, q, z). \quad (13)$$

The coefficients λ_m , μ_m , and γ_m can be singular, but the FLR eigenfunctions are smoothly varying.

If we express Eq. (6) in terms of Eqs. (11) – (13) and then examine the a_i -th component, we have

$$(\omega^2 - \omega_i^2) \lambda_i = \Gamma \int_{-x_0}^{x_0} a_i \frac{\partial}{\partial q} \frac{\delta B_{\parallel}}{B} dz. \quad (14)$$

We see that λ_i will be singular and there will be a resonance close to a surface where $\omega = \omega_i$. Variations across this surface will be much faster than variations along the surface, which now motivates our final choice of p and q : for $\omega \sim \omega_i$, p is chosen so that $\omega_i = \omega_i(p)$, q will denote the direction of slow variation along the resonant surface. It should be noted that in general $\nabla \omega_j$ is not parallel to $\nabla \omega_i$ for $j \neq i$. Therefore, once we have defined p so that $\omega_i = \omega_i(p)$, the other surfaces will have to be expressed as $\omega_j = \omega_j(p, q)$ for $j \neq i$. An exception is given by the case when the z dependence of the density can be factored out, i.e., $\rho = \rho_{xy}(x, y) \rho_z(z)$.

A frequency ω matching an eigenfrequency ω_i will excite the corresponding resonant surface which we have now defined to be given by a surface of constant p . Close to this surface we expect energy absorption and mode conversion. The solutions of Eqs. (5) – (7) will exhibit singular behaviour.

In the following, we shall consider one specific ω_i and, for this purpose, we adopt the notation $\omega = \omega_i(p_i)$.

IV. Singular Solutions Near the Resonant Surfaces

In order to examine singular solutions near a resonant surface $\omega = \omega_i(p_i)$, we define a smallness parameter ϵ to be given by $\epsilon \equiv |p - p_i|/|p_i| \ll 1$. Note that now

$$\left| \frac{\partial}{\partial p} \right| \sim O\left(\frac{1}{\epsilon}\right) \gg 1, \quad (15)$$

and

$$\left| \frac{\partial}{\partial q} \right| \sim O(1). \quad (16)$$

Equation (7) and the observation that the compressional Alfvén wave can propagate across B suggest the following expansions:

$$\frac{\delta B_{\parallel}}{B} = \frac{\delta B_{\parallel}^{(0)}}{B} + \epsilon \frac{\delta B_{\parallel}^{(1)}}{B} + \dots, \quad (17)$$

$$\eta_p = \frac{1}{\epsilon} \eta_p^{(-1)} + \eta_p^{(0)} + \epsilon \eta_p^{(1)} + \dots, \quad (18)$$

and

$$\eta_q = \eta_q^{(0)} + \epsilon \eta_q^{(1)} + \dots. \quad (19)$$

If we look at the lowest order of Eq. (7), the ϵ^{-1} order,

$$0 = -\frac{1}{\Gamma} \frac{\partial}{\partial p} \frac{\delta B_{\parallel}^{(0)}}{B}, \quad (20)$$

we see that $\delta B_{\parallel}/B$, in lowest order, is smooth across the singular surface. This is consistent with the construction of Eq. (17), and none of the γ_m in Eq. (13) will be singular.

Next let us examine Eq. (6) which can be rewritten as

$$\left\{ \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v_A^2} \right\} \eta_p = \left\{ \frac{\partial^2}{\partial z^2} + \frac{\omega_i^2}{v_A^2} \right\} \eta_p + \left\{ \frac{\omega^2 - \omega_i^2}{v_A^2} \right\} \eta_p = \Gamma \frac{\partial}{\partial q} \frac{\delta B_{\parallel}}{B}. \quad (21)$$

Note that close to a resonant surface p_i , we can make the Taylor expansion

$$(\omega^2 - \omega_i^2)_p \approx (p - p_i) \frac{\partial}{\partial p} (\omega^2 - \omega_i^2)_{p_i} \equiv c(p - p_i). \quad (22)$$

Now the ϵ^{-1} order is given by

$$\left\{ \frac{\partial^2}{\partial z^2} + \frac{\omega_i^2}{v_A^2} \right\} \eta_p^{(-1)} = 0, \quad (23)$$

from which we see that (we assume a nondegenerate set of eigenfunctions)

$$\eta_p^{(-1)} = \lambda_i^{(-1)}(p, q) a_i(p, q, z). \quad (24)$$

The next order is given by

$$\left\{ \frac{\partial^2}{\partial z^2} + \frac{\omega_i^2}{v_A^2} \right\} \eta_p^{(0)} + \left\{ \frac{\omega^2 - \omega_i^2}{v_A^2} \right\} \eta_p^{(-1)} = \Gamma \frac{\partial}{\partial q} \frac{\delta B_{\parallel}^{(0)}}{B}. \quad (25)$$

Multiplying Eq. (25) by a_m and integrating from $-\chi_0$ to χ_0 in dz , we have to consider two cases separately:

$$m \neq i \quad (\omega_i^2 - \omega_m^2) \lambda_m^{(0)} = \Gamma \int_{-\chi_0}^{\chi_0} a_m \frac{\partial}{\partial q} \frac{\delta B_{\parallel}^{(0)}}{B} dz, \quad (26)$$

$$m = i \quad (\omega^2 - \omega_i^2) \lambda_i^{(-1)} = \Gamma \int_{-\chi_0}^{\chi_0} a_i \frac{\partial}{\partial q} \frac{\delta B_{\parallel}^{(0)}}{B} dz. \quad (27)$$

As both Γ and $\delta B_{\parallel}/B$ are nonsingular functions and as $\partial/\partial q$ is operating on the smoothly varying variable, the right-hand side of Eqs. (26) and (27) is nonsingular. Therefore λ_m will be nonsingular for all $m \neq i$, and λ_i will be the only singular λ as $(\omega^2 - \omega_i^2)$ is small and vanishes on the resonant surface itself. Thus the main contribution to $\eta_p = \sum_m \lambda_m a_m$ will come from the $m = i$ term.

Now taking the ϵ^{-1} order of Eq. (5),

$$0 = J(p, q) \left(\frac{\partial \eta_q^{(0)}}{\partial p} - \frac{\partial \eta_p^{(-1)}}{\partial q} \right), \quad (28)$$

and noting that, because of the way we defined the new coordinate system, $J(p, q)$ does not vanish anywhere, we see that

$$\left| \frac{\eta_p}{\eta_q} \right| \sim O\left(\frac{1}{\epsilon}\right) \gg 1. \quad (29)$$

Therefore also $|\lambda_m/\mu_m| \sim O(1/\epsilon)$ and μ_m can be singular only for $m = i$.

If we continue to examine Eqs. (5) - (7) systematically order by order, we find the form of λ_i and μ_i to be

$$\lambda_i = \frac{\bar{\lambda}_i^{(-1)}}{p - p_i} + \bar{\lambda}_i^{(0)} + \dots \quad \text{and} \quad (30)$$

$$\mu_i = \bar{\mu}_i^{(0)} + \bar{\mu}_{i, \ln}^{(0)} \ln(p - p_i) + \dots \quad (31)$$

where, comparing Eqs. (27) and (30) and using Eq. (22),

$$\bar{\lambda}_i^{(-1)} = (p - p_i) \lambda_i^{(-1)} = \frac{\Gamma}{c} \int_{-\chi_0}^{\chi_0} a_i \frac{\partial}{\partial q} \frac{\delta B_{\parallel}^{(0)}}{B} dz \quad (32)$$

and from Eq. (28)

$$\bar{\mu}_{i,m}^{(0)} = \frac{\partial \bar{\lambda}_i^{(-1)}}{\partial q} . \quad (33)$$

Here all $\bar{\lambda}_i^a$ and $\bar{\mu}_i^a$ are functions of q only. For $m \neq i$, both λ_m and μ_m will be nonsingular.

We see that both η_p and η_q will get singular close to the resonant surface p_i . As we approach the singularity $[(p - p_i) \rightarrow 0]$, η_p will go like $1/(p - p_i)$, and η_q will exhibit a logarithmic singularity $\sim \ln(p - p_i)$. In both directions, the eigenfunction a_i whose eigenfrequency ω_i matches the incoming frequency ω will be dominant. Recalling the definition $\eta \equiv \xi \times \mathbf{B}$, we see that ξ_q , the displacement (or, equivalently, the magnetic perturbation) in the resonant surface, is more singular than ξ_p , the displacement (magnetic perturbation) perpendicular to the resonant surface. Furthermore, the resonant surface is not in general parallel to the x - y surface. A probe measuring the magnetic perturbations in this geometry will therefore find components in both x and y direction, representing the two projections of δB_q .

Taking the analogy to the Earth's magnetosphere, let us first consider the axisymmetric case [Chen and Cowley, 1989]. Here the most singular magnetic component will be δB_θ , parallel to the Earth's surface in the azimuthal direction. Now taking into account azimuthal inhomogeneities, a nonaxisymmetric change in density, e.g., due to the plasma bulge or the day-night asymmetry, will cause the resonant surfaces to be tilted with respect to the undisturbed axisymmetric shells. In this case, perturbations measured to be in r and θ directions are the two components of δB_q respectively.

V. Summary and Discussion

We have demonstrated analytically that field line resonances of Alfvén waves persist in a three dimensionally varying density. Although we have simplified the problem by using a rectangular box model with a straight uniform magnetic field, we believe that our results can be generalized and applied to the Earth's dipole field. The form of the singularity is similar to that of the previous one- and two-dimensional models. The displacement eigenvector is found to still lie predominantly in the resonant surface, but the location of the resonant surface itself

is now influenced by azimuthal inhomogeneities in the density. Whereas the two-dimensional calculations predict the magnetic perturbation at the resonance to be mainly in the θ direction, the inclusion of nonaxisymmetric features allows for an additional radial component of the perturbed field. Note that indeed most observations show a dominant δB_θ component [Anderson, 1990], but some measurements indicate δB_r and δB_θ to be of comparable magnitude. Additional insight could be gained by further analysis of the available data. For example, it could be investigated whether observations of predominantly transverse waves with an appreciable radial component in addition to an azimuthal component occur during periods of either anticipated or observed azimuthal asymmetry in the plasma density. For future work, a more realistic model could consider a dipole magnetic field as well as finite pressure effects, although the physical picture is not expected to be altered by these changes.

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References

- ¹B. Anderson, et al., *A statistical study of Pc 3-5 pulsations observed by the AMPTE/CCE magnetic fields experiment, 1. Occurrence distribution*, J. Geophys. Res. **95** (A7), 10495 (1990).
- ²L. Chen and S. Cowley, *On field line resonances of hydromagnetic Alfvén waves in dipole magnetic field*, Geophys. Res. Lett. **16** (8), 895 (1989).
- ³L. Chen and A. Hasegawa, *A theory of long period magnetic pulsations, 1. Steady state excitations of field line resonances*, J. Geophys. Res. **79**, 1024 1974.
- ⁴D.J. Southwood, *Some features of field line resonances in the magnetosphere*, Planet. Space Sci. **22**, 483 (1974).
- ⁵D.J. Southwood and M.G. Kivelson, *The effect of parallel inhomogeneity on magnetospheric wave coupling*, J. Geophys. Res. **91**, (A6), 6871 (1986).