VARIATION OF THE DAMPING PARTITION NUMBERS WITH PARTICLE ENERGY DEVIATION*

J. P. DELAHAYE[§] AND J. JÄGER
*Stanford Linear Accelerator Center
Stanford University, Stanford CA 94305*

1. Summary

Until now, calculation of the rates of change of the damping partition numbers^(1,2) with particle energy took into account only the effects of quadrupole fields⁽³⁾. It has been found⁽⁴⁾, however, by tracking particles⁽⁵⁾ in the LEP Electron Positron Accumulator⁽⁶⁾ that the actual variation of the damping partition numbers with energy can be increased up to a factor three by introduction of sextupole fields.

The present note gives a complete analytical expression for the variation of the damping partition numbers with particle energy to first order in $\frac{\Delta E}{E}$.

This expression is then used to illustrate in three different machines the order of magnitude of each of the contributing terms. We shall see that there are terms driven mainly by small radius bending magnets which cannot be neglected.

* Work supported by the Department of Energy, contract DE-AC03-76SF00515.

§ Permanent address: CERN, Geneva 23, Switzerland.

NOTICE
PORTIONS OF THIS REPORT ARE ILLERIBLE.
 It has been reproduced from the best available copy to permit the broadest possible availability.

MASTER

2. Introduction

For circular machines whose closed orbit lies in a horizontal plane, the transverse and longitudinal damping partition numbers J_x , J_y and J_z can be defined^(1,2) in terms of the second (I_2) and fourth (I_4) synchrotron integrals as:

$$J_x = 1 - \frac{I_4}{I_2} \qquad J_y = 1 \qquad J_z = 2 + \frac{I_4}{I_2}$$

with

$$I_2 = \int k^2 ds$$

and

$$I_4 = \int (h_0 k^2 - 2hk) D ds$$

and where

- $h_0(s) = 1/\rho_0$ is the curvature of the design orbit,
- $h(s) = B_y/B\rho$ is the curvature of the actual orbit with B_y the vertical component of the magnetic field and $B\rho$ the particle rigidity,
- $k(s) = -(1/B\rho) \left(\partial B_y / \partial x \right)$ is the normalised quadrupole strength,
- $D(s)$ is the dispersion function.

The variation of the longitudinal damping partition number J_z with respect to an energy deviation $\delta = \Delta p/p$ is given by:

$$\frac{dJ_z}{d\delta} = \frac{1}{I_{20}} \frac{dI_4}{d\delta} - \frac{I_{40}}{I_{20}^2} \frac{dI_2}{d\delta},$$

with $I_{20} = I_2$ and $I_{40} = I_4$ for $\delta = 0$.

JSW

3. Expansion of the synchrotron integrals: I_2 and I_4

In order to evaluate this expression we must first find the dependence of I_2 and I_4 on δ by expanding $h(s)$, $k(s)$, $D(s)$ and θ through first order in δ .

$$\underline{h(s) = B_y/B\rho}$$

Expanding B_y in a Taylor series and substituting $x = D\delta$ we get

$$B_y = B \Big|_{x=0} + \frac{\partial B_y}{\partial x} \Big|_{x=0} x = B \Big|_{x=0} + \frac{\partial B_y}{\partial x} \Big|_{x=0} D\delta .$$

Thus

$$h(s) = \frac{1}{B\rho(1+\delta)} \left(B \Big|_{x=0} + \frac{\partial B_y}{\partial x} \Big|_{x=0} D\delta \right) = h_0 - (h_0 + k_0 D) \delta .$$

$$\underline{k(s) = - (1/B\rho) (\partial B_y/\partial x)}$$

Expanding $\partial B_y/\partial x$ gives

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_y}{\partial x} \Big|_{x=0} + \frac{\partial^2 B_y}{\partial x^2} \Big|_{x=0} x = \frac{\partial B_y}{\partial x} \Big|_{x=0} + \frac{\partial^2 B_y}{\partial x^2} \Big|_{x=0} D\delta ,$$

which yields:

$$k(s) = k_0 - (k_0 - r_0 D) \delta ,$$

where $r_0(s) = - (1/B\rho) \left(\partial^2 B_y/\partial x^2 \right)$ is the normalized sextupole component.

Dispersion Function $D = D_0 + D_1\delta$

The first order Dispersion function, D_0 , is the periodic solution of the well known differential equation:

$$D_0'' + (h_0^2 - k_0)D_0 = h_0.$$

Replacing the different parameters in this equation by their expansions in δ would lead to an incomplete differential equation for the Dispersion function D_1 to first order in δ . In order to get all of the involved terms, the differential equation for D_1 must be deduced from the general equation of motion to second order⁽⁷⁾:

$$D_1'' + (h_0^2 - k_0)D_1 = -h_0 - (k_0 - 2h_0^2)D_0 + (2h_0k_0 - h_0^3 + \frac{1}{2}r_0)D_0^2 + \frac{1}{2}h_0D_0'^2 + h_0'D_0'D_0.$$

In a series of reports^(8,9) the equation for D_1 has been derived for machines with large bending radius ρ_0 where high order terms in h_0 as well as the terms coming from combined function magnets h_0k_0 and from the variation of the dispersion function with azimuth D_0' were neglected. For small machines these terms cannot be neglected. The periodic solution of D_1 can be found either by the harmonic analysis method or by integration using Green functions⁽⁸⁾.

Edge Angle θ

For a magnet whose pole-faces are inclined by θ_{01} at the entrance and by θ_{02} at the exit the pole-face angles are given by:

$$\theta_1 = \theta_{01} + \langle D'_{01} \rangle \delta,$$

$$\theta_2 = \theta_{02} - \langle D'_{02} \rangle \delta,$$

with

$$\langle D'_{01} \rangle = D'_{01} + \frac{1}{2}hD_{01}\tan\theta_{01},$$

$$(D'_{02}) = D'_{02} - \frac{1}{2} h D_{02} \tan \theta_{02} .$$

The suffix "1" denotes the beginning of the fringe field region at the magnet entrance and the suffix "2" the end of the fringe field region at the magnet exit.

These expansions enable us to write the synchrotron integrals I_2 and I_4 to first order in δ . Omitting the suffix "0" of h , k , r and θ we have

$$I_2 = \int h^2 ds - 2\delta \int (h^2 + hkD_0) ds$$

and

$$I_4 = \int (h^3 - 2hk) D_0 ds \\ + 2\delta \int \left[k^2 D_0^2 - (h^3 - 2hk) D_0 - h^2 k D_0^2 - hr D_0^2 + \left(\frac{1}{2} h^3 - hk \right) D_1 \right] ds$$

These formulations are completely general and give the contribution to the second and fourth synchrotron integrals for any magnetic element, from simple dipoles and quadrupoles to combined function magnets with sextupole. The contribution of an element's edges to these integrals can be deduced directly from the characteristics of this element, which is the subject of the next section.

4. Contribution of the edges

For magnetic elements with inclined boundaries whose entrance and/or exit faces are not normal to the design trajectory, the local quadrupole $k(s)$ and sextupole $r(s)$ fields experienced by a particle traversing the element and its fringe field along the design trajectory is given by⁽¹⁰⁾:

$$k(s) = k_m + k' \tan \theta - h k' D_0 \tan \theta \delta,$$

$$r(s) = r_m - 2k'_m \tan \theta \mp h'' \tan^2 \theta + h h' \tan^3 \theta,$$

where θ is the edge angle with the sign convention defined in ⁽⁷⁾, ' and '' the first and second derivatives with respect to the azimuth s , k_m and r_m the quadrupole and sextupole components of the element. (The upper sign corresponds to the entrance edge, the lower sign to an exit edge.)

Introducing these relations for the quadrupole and sextupole fields into the equations for the second and fourth synchrotron integrals and integrating over the edges using a "hard edge" approximation gives:

$$I_2 = I_{20} - \delta \left(2I_{20} + 2 \int h k D_0 + \sum h^2 D_0 \tan \theta \right)$$

and

$$\begin{aligned} I_4 = I_{40} - 2\delta \left(I_{40} - \int \left[(k^2 - h^2 k - h r) D_0^2 + \left(\frac{1}{2} h^3 - h k \right) D_1 \right] ds \right) \\ + \delta \sum \left(\left[4k + \frac{5}{6} h^2 (1 - \tan^2 \theta) \right] D_0^2 - h D_1 \right) h \tan \theta \\ \mp \delta \sum h^2 (1 + 3 \tan^2 \theta) D_0 D_0', \end{aligned}$$

where the index "m" has been omitted.

5. Variation of the damping partition number with energy

Using the results obtained so far, we can differentiate I_2 and I_4 with respect to δ and obtain the variation to first order of the damping partition numbers with particle energy deviation:

$$\begin{aligned} \frac{dJ_\epsilon}{d\delta} = \frac{1}{I_{20}} \left\{ 2 \frac{I_{40}}{I_{20}} \int h k D_0 ds + 2 \int (k^2 - h^2 k - h r) D_0^2 ds + \int (h^3 - 2 h k) D_1 ds \right. \\ \left. + \sum \left(\frac{I_{40}}{I_{20}} h D_0 + \left[4k + \frac{5}{6} h^2 (1 - \tan^2 \theta) \right] D_0^2 - h D_1 \right) h \tan \theta \right. \\ \left. \mp \sum h^2 (1 + 3 \tan^2 \theta) D_0 D_0' \right\}. \end{aligned}$$

$$\frac{dJ_x}{d\delta} = -\frac{dJ_\epsilon}{d\delta},$$

$$\frac{dJ_y}{d\delta} = 0.$$

where the upper sign corresponds to an entrance edge and the lower sign corresponds to an exit edge.

The different contributions can be separated into four categories according to where they originate:

- the quadrupole magnets (terms in k);
- the bending magnets (terms in h);
- the combined function magnets including quadrupole (terms in hk) and/or sextupole components (terms in hr);

- the variation of the dispersion function with the azimuth (terms in D'_0) and/or with particle momentum (terms in D_1).

The contribution of the quadrupoles responsible for the term

$$\frac{2}{J_{20}} \int k^2 D_0^2 ds$$

is normally dominant in large machines and therefore is the only one which is calculated in various optics programs⁽³⁾. Nevertheless, for rings with small bending radius or with combined function magnets the other terms cannot be neglected, as shown in the following examples (the perturbation of the Dispersion function D_1 to first order in δ has been calculated by the general optics program MAD⁽¹¹⁾).

6. Applications

LEP Electron-Positron Accumulator (EPA)⁽⁶⁾

The EPA is a good example of a machine for which no terms in the equations for the variation of the damping partition numbers with energy should be neglected. In order to increase horizontal damping for fast betatron stacking the bending magnets have been designed with:

- very small curvature, $\lambda = 0.701 \text{ m}^{-1}$
- combined quadrupole function, $k = 0.500 \text{ m}^{-2}$
- parallel pole-ends, $\theta_1 = \theta_2 = \pi/16$

Moreover, the EPA ring includes long, dispersion-free straight sections in order to minimize the beam dimensions at injection and extraction and hence the corresponding kicker's requirements. As a consequence, chromaticity correction by sextupoles localised in the arcs strongly perturbs the Dispersion function so that the D_1 terms are no longer negligible.

The table below summarizes the contributions of the different terms in the equations both before and after the ring's natural chromaticities have been corrected. The variation of the longitudinal damping partition number with energy after chromaticity correction is increased by a factor of two due mainly to the perturbation of the Dispersion function in the combined function magnets by the sextupoles.

SLC Damping Ring (DR)

In the SLAC North Damping Ring, the perturbation of the variation of the damping partition numbers with particle energy is minimal since this ring will

be equipped with rectangular, separated-function bending magnets, albeit their curvature is quite small:

$$h = 0.491 \text{ m}^{-1}, \quad k = 0 \text{ m}^{-2}, \quad \theta_1 = \theta_2 = \frac{\pi}{40}.$$

The contribution to the perturbation of the dispersion function D_1 with and without sextupoles also cannot be very important as the two terms driven by the bending magnets cancel each other thanks to their parallel pole-faces.

CERN Proton Synchrotron (PS)

The PS, initially designed as a proton accelerator, is horizontally antidamped because of its FODO lattice based on combined-function wedge bending magnets:

$$h = 0.0142 \text{ m}^{-1}, \quad k_F = -0.056734 \text{ m}^{-2}, \quad k_D = +0.056771 \text{ m}^{-2}.$$

The addition of Robinson wigglers, reversal of the quadrupole fields in the F and D combined-function bending magnets and displacement of the beam horizontally have all been proposed in order to decrease its longitudinal damping partition number. As already shown⁽¹²⁾, the variation of the longitudinal damping partition number with particle energy is reduced by 1.6% due to the Dispersion function D_0 in the combined-function bending magnets. In fact, in the absence of sextupoles there is another 1% reduction due to the Dispersion function D_1 . This correction would be even larger if the perturbation of the Dispersion function by the sextupoles were taken into account.

Table 1

	LEP EPA <i>sect. off (on)</i>	SLC DR <i>sect. off</i>	CERN PS <i>sect. off</i>
I_2	4.406	3.084	0.089
I_4	-4.259	-0.006	0.173
J_c	1.033	1.998	3.933
$2/I_{20} \int k^4 D_0^2 ds$	14.009	15.071	230.671
$2I_{40}/I_{20}^2 \int h k D_0 ds$	-0.916	0.0	-3.662
$-2/I_{20} \int h^2 k D_0^2 ds$	-0.844	0.0	0.146
$I_{40}/I_{20}^2 \sum h^2 D_0 \tan \theta$	-0.469	0.0	0.0
$4/I_{20} \sum h k D_0^2 \tan \theta$	1.792	0.0	0.0
$5/(6I_{20}) \sum h^3 D_0^2 \tan \theta (1 - \tan^2 \theta)$	0.353	0.0	0.0
$\mp 1/I_{20} \sum h^2 D_0 D_0' (1 + 3 \tan^2 \theta)$	1.382	0.060	0.0
$1/I_{20} \int h^3 D_1 ds$	0.88 (-5.18)	0.04	0.03
$-2/I_{20} \int h k D_1 ds$	-1.78 (10.53)	0.0	-2.21
$-1/I_{20} \sum h^2 D_1 \tan \theta$	-0.89 (5.22)	- 0.04	0.0
$dJ_c/d\delta$	13.52 (25.88)	15.13	224.97

References

1. M. Sands, *"The Physics of Electron Storage Rings - An Introduction,"* SLAC 121.
2. R. H. Helm, M. J. Lee, P. L. Morton, M. Sands, *"Evaluation of Synchrotron Radiation Integrals,"* IEEE Trans. Nucl. Science **NS 20**, 900 (1973).
3. M. Hanney and E. Keil, *BEAMPARAM - A Program for Computing Beam Dynamics and Performance of e^+e^- Storage Rings,* CERN-ISR/TH/79-29.
4. M. Bell and H. Kugler, private communication.
5. M. Bell and J. P. Delahaye, *"Particle Tracking in the EPA Bending Magnet,"* LEP Note 459.
6. J. P. Delahaye and A. Krusche, *"The Lattice Design of the LEP Electron Positron Accumulator (EPA),"* IEEE Trans. on Nucl. Science, **NS 30**, NO.4, 2050 (August 1983).
7. K. L. Brown and R. V. Servranckx, *"First and Second Order Charged Particle Optics,"* SLAC-PUB 3381.
8. H. Wiedemann, *"Chromaticity Correction in Large Storage Rings,"* PEP Note 220.
9. B. Autin, *"Simultaneous Correction of Chromaticity and Orbit Dispersion in a Strong Focussing Machine,"* XIth Int. Conf. on High Energy Accelerators, CERN, 7-12 July 1980.
10. W. Hardt, J. Jäger, D. Möhl, *"A General Analytical Expression for the Chromaticity of Accelerator Rings,"* PS/LEAR/Note 82-5.
11. F. C. Iselin, *The MAD Program,* CERN-LEP/TH/83-30.
12. T. Risselada, *Calcul de J_e and $dJ_e/(d\delta)$,* LEP Note 488.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.