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General Theory of Kinetic Ballooning Modes

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ABSTRACT

The "ballooning mode formalism", previously developed for the ideal MHD problem, is applied here to the kinetic problem in tokamaks. The general two-dimensional equation governing drift and trapped-electron eigenmodes reduces to a one-dimensional integral equation along the lines of force with the radial structure determined by a WKB procedure. Comparisons made between the present one-dimensional code and a previous two-dimensional code embodying identical physical assumptions indicate reasonable agreement. This correspondence holds both for the structure along the field line and for the radial structure in the special case of closely spaced turning points.

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1. INTRODUCTION

The recent success¹⁻³ in the analytic and numerical treatment of high toroidal mode number ideal MHD stability theory points toward the utility of similar techniques in the treatment of kinetic stability problems. The general features of the "ballooning representation," which automatically satisfies the requirements in tokamaks of poloidal periodicity and of small effective wave-number $k_{\perp}$ along the equilibrium magnetic field lines, in the presence of finite magnetic shear, apply to the various types of drift modes in axisymmetric geometry in the kinetic regime. The MHD theory reduces the general two-dimensional eigenfunction equation to a one-dimensional differential equation along the lines of force with a simple treatment of the radial structure utilizing a WKB procedure. We show in this paper that these simplifying features are also present in the kinetic problem where a one-dimensional integral equation for the behavior of the eigenfunction along the lines of force results when the effects of trapped particles and magnetic drifts are included.

In the past, a general two-dimensional code^{4,5} has been used to generate numerical solutions in the low- β limit. It has been demonstrated that this procedure, while valid, leads to an impractically large expenditure of computer time when all relevant physical features (e.g., spatial variations in the equilibrium gradients) are included in the calculation. However, as will be shown, a code implementing the one-dimensional ballooning representation produces results which are in reasonable agreement with those

generated by a two-dimensional code containing identical physical assumptions.

This paper will be organized as follows: The fundamental equations and the basic formalism used are presented in Sec. II. In Sec. III the problem of determining the radial structure of the eigenfunctions is considered. The lowest order one-dimensional equation, which to a good approximation determines the relevant eigenvalues, is then cast in a computationally useful form in the electrostatic limit in Sec. IV, and the numerical procedure implemented is described here. In Sec. V, the one-dimensional numerical results are presented and comparisons are made with the two-dimensional results computed for similar cases. Finally, in Sec. VI some general conclusions are drawn and implications of this type of analysis are discussed.

II. FORMULATION OF THE BASIC EQUATIONS

We use the familiar axisymmetric ψ, χ, ζ coordinate system,⁶ where ψ is the poloidal flux and acts as a radial coordinate, χ is a poloidal coordinate and ζ is the toroidal angle. The equilibrium magnetic field is given by

$$\underline{B} = \nabla\psi \times \nabla\zeta + I(\psi)\nabla\zeta, \quad (1)$$

where I is a prescribed function, with the line element

$$(d\lambda)^2 = \frac{(d\psi)^2}{(RB_\lambda)^2} + (JB_\lambda)^2 (d\lambda)^2 + r^2 (d\lambda)^2, \quad (2)$$

and the volume element

$$d\tau = J d\psi d\lambda d\lambda, \quad (3)$$

where R is the major radius and J is the Jacobian

$$J = (\mathbf{V}_\psi \cdot \mathbf{V}_\zeta \times \mathbf{V}_\lambda)^{-1}.$$

The linear stability problem is governed by the standard gyrokinetic equation⁷

$$\begin{aligned} \frac{\partial h}{\partial t} + v_{\parallel} \mathbf{n} \cdot \nabla^{(1)} h + \mathbf{v}_d \cdot \nabla^{(0)} h + \frac{e}{m} \frac{\partial}{\partial E} \langle \mathbf{r} - \mathbf{v} \cdot \mathbf{A}/c \rangle_{\xi} \cdot \frac{\partial F^{(0)}}{\partial E} \\ + \frac{c}{B} \mathbf{n} \cdot \nabla^{(0)} \langle \Phi - \mathbf{v} \cdot \mathbf{A}/c \rangle_{\xi} \times \nabla^{(1)} F^{(0)} = \langle C \rangle_{\xi} \cdot \end{aligned} \quad (4)$$

Here $v_{\parallel}$ is the component of the velocity $\mathbf{v}$ along $\mathbf{n}$, $\mathbf{n} = \mathbf{B}/B$; $\mathbf{v}_d$ is the magnetic gradient and curvature drift velocity; $\nabla^{(0)}$ and $\nabla^{(1)}$ indicate that the gradient operators are of order $1/\rho_i$ and $1/r_n$ respectively, where ρ_i is the ion gyroradius and r_n is the radial density gradient scale length; Φ is the perturbed electrostatic potential and $\mathbf{A}$ is the perturbed vector potential; $F^{(0)}$ is the lowest order part (in ρ_i/r_n) of the equilibrium distribution function; E is the particle kinetic energy per unit mass; and C is a collision operator to be specified later. Also

$$\langle \cdot \rangle_{\xi} = \frac{1}{2\pi} \int_0^{2\pi} d\xi$$

indicates a gyrophase average in the gyro-center coordinate system. We have temporarily suppressed species subscripts.

The perturbed distribution function is given by

$$f = \frac{e}{m} \psi \frac{\partial F^{(0)}}{\partial E} + h \quad (5)$$

and the gyrokinetic equation is written in terms of the velocity variables E , $\mu = v_{\perp}^2/2B$, the particle magnetic moment per unit mass, where $v_{\perp}$ indicates the velocity component perpendicular to B , and ξ' the gyrophase in the gyro-center coordinate system. The spatial gyro-center coordinates are

$$\begin{aligned} \psi' &= \psi + \frac{v_{\perp} RB}{\Omega} \sin \xi, \\ \chi' &= \chi + \frac{v_{\perp}}{\Omega} \frac{1}{jB_{\chi}} \cos \xi, \\ \xi' &= \xi - \frac{v_{\perp} B}{\Omega RB} \cos \xi, \end{aligned} \quad (6)$$

with the unprimed variables ψ, χ, ξ, ζ representing the original particle coordinates, with $\Omega = eB/mc$. Note that $\xi' = \xi$.

In arriving at Eq. (4), the usual expansion of the Fokker-Planck equation was used in which

$$\frac{\omega}{\Omega} \sim k_{\parallel} \rho_i \sim \epsilon, \quad k_{\perp} \rho_i \sim 1, \quad (7)$$

where ω is the mode frequency, $k_{\perp}$ is the effective wave-number perpendicular to B , and ϵ is our basic small ordering parameter. The zeroth order equation in this expansion not only requires the gyro-center transformation given in Eq. (6), but also that

$$\underline{n} \cdot \nabla^{(0)} h = \underline{n} \cdot \nabla^{(0)} \phi = \underline{n} \cdot \nabla^{(0)} A = 0 . \quad (8)$$

Thus

$$\underline{n} \cdot \nabla^{(0)} \phi = \frac{1}{BJ} \left(\frac{\partial \phi}{\partial \chi} + \frac{IJ}{R^2} \frac{\partial \phi}{\partial \zeta} \right) = 0 , \quad (9)$$

which yields

$$\phi = \psi \exp \left(i\ell\zeta - i\ell \int^{\chi} d\chi' IJ/R^2 \right), \quad (10)$$

with

$$q = \frac{1}{2\pi} \oint d\chi' IJ/R^2 \quad (11)$$

being the usual "safety factor" and ℓ the toroidal mode number, upon Fourier analyzing in ζ . If the fundamental period in χ is taken as 2π , then ϕ can be written using the "ballooning representation"

$$\phi = \sum_{p=-\infty}^{\infty} \hat{\phi}(\chi - 2\pi p) \exp(i\ell q 2\pi p) , \quad (12)$$

which makes ϕ automatically periodic in χ . Alternatively, we can seek an infinitely degenerate set of aperiodic solutions, in the infinite χ domain, of the form given in Eq. (10), and superpose them at the end of the calculation to form a periodic eigenfunction. This latter approach will be adopted in Sec. II and III.

It is further assumed that the radial structure of the eigenfunction is given by a WKB form; i.e.,

$$\phi = \hat{\phi}(\chi) \exp \left[i\ell\zeta - i\ell \int^{\chi} d\chi' IJ/R^2 + i\ell S(\psi) \right] . \quad (13)$$

We next transform to the gyro-center variables of Eq. (6) and carry out the gyro-phase average to get

$$\langle \phi \rangle_{\xi} = \hat{\phi}(\chi) \exp [i l \xi + i l S(\psi) - i l \int^{\chi} d\chi' I J / R^2] \\ \times J_0 \left\{ \frac{l v_{\perp}}{\Omega} \left[\left(\frac{B}{R B_{\chi}} \right)^2 + \left(R B_{\chi} \frac{\partial S}{\partial \psi} \right)^2 \right]^{1/2} \right\} \quad (14)$$

where J_0 is the usual Bessel function. Similar considerations apply for the perturbed vector potential $\underline{A}$. Since we are primarily interested in tokamaks for which $\beta \equiv 8\pi n(T_e + T_i)/B^2 \approx a/Rq^2$ by ideal MHD stability considerations, we retain only $A_{\parallel}$, the component along $\underline{B}$.

After considerable algebra, the kinetic equation [Eq. (4)] may be written as

$$\frac{v_{\parallel}}{J B} \frac{\partial \hat{h}}{\partial \chi} - i(\omega - \omega_d) \hat{h} = - \frac{q}{T} i(\omega - \omega_{*}^T) F^{(0)} J_0 (\hat{\phi} - v_{\parallel} \hat{\lambda}_{\parallel} / c) + \langle C \rangle_{\xi} \quad (15)$$

where

$$\omega_d \equiv - \frac{l v_{\parallel}}{\Omega} \left[\frac{1}{J} \frac{\partial}{\partial \chi} \left(\frac{v_{\parallel}}{B} \right) \left(\frac{\partial S}{\partial \psi} - \frac{\partial}{\partial \psi} \int^{\chi} d\chi' \frac{I J}{R^2} \right) + \frac{1}{J} \frac{\partial}{\partial \psi} \left(\frac{v_{\parallel} J B^2}{B} \right) + \frac{1}{R^2} \frac{\partial}{\partial \psi} \left(\frac{v_{\parallel}}{B} \right) \right] \quad (16)$$

and

$$\omega_{*}^T \equiv - \frac{l c T}{e} \frac{\partial}{\partial \psi} \ln F^{(0)} \quad (17)$$

The lowest order equilibrium distribution function $F^{(0)}$ has been taken to be a Maxwellian with temperature T , and $A_{\parallel}$ and h have been factored like ϕ in Eq. (13). For the moment, we

neglect the averaged collision operator, but it will be restored later using a Krook model. We also drop the primes on the coordinates.

For the circulating particles, the appropriate boundary conditions for Eq. (15) are that $\hat{h}_{\pm}(\chi) \rightarrow 0$ as $\chi \rightarrow \pm\infty$. (The $\pm$ subscript on $\hat{h}$ refers to the sign of $v_{||}$ for the particles in question.) We assume that ω has a positive imaginary part corresponding to an unstable eigenmode. This automatically guarantees convergence at one end of the χ range. Thus we find

$$\hat{h}_{+}(\chi) = - \int_{-\infty}^{\chi} d\chi' \frac{e}{T} i(\omega - \omega_{*}^T) F^{(0)} J_0 \frac{BJ}{|v_{||}|} (\hat{\phi} + |v_{||}| \hat{A}_{||}/c) \\ \times \exp \left[-i \int_{\chi'}^{\chi} d\chi'' (\omega - \omega_d) BJ/|v_{||}| \right],$$

and

$$\hat{h}_{-}(\chi) = - \int_{\chi}^{\infty} d\chi' \frac{e}{T} i(\omega - \omega_{*}^T) F^{(0)} J_0 \frac{BJ}{|v_{||}|} (\hat{\phi} + |v_{||}| \hat{A}_{||}/c) \\ \times \exp \left[+i \int_{\chi}^{\chi'} d\chi'' (\omega - \omega_d) BJ/|v_{||}| \right]. \quad (18)$$

For the trapped particles, the standard boundary conditions at the turning points $\pm\chi_0$ must be applied, i.e.,

$$\hat{h}_{+}(\chi_0) = \hat{h}_{-}(\chi_0) \quad , \\ \hat{h}_{+}(-\chi_0) = \hat{h}_{-}(-\chi_0) \quad . \quad (19)$$

After some algebra, we find

$$\begin{aligned} \frac{1}{2}(\hat{n}_+ + \hat{n}_-) = & \left(\sin I_{-X_0}^{X_0} \right)^{-1} \left[\int_{X_0}^{X_0} dX' \left(\gamma \cos I_{X_0}^{X'} \cos I_{-X_0}^{X'} + i\delta \sin I_{X_0}^{X'} \cos I_{-X_0}^{X'} \right) \right. \\ & \left. + \int_{-X_0}^X dX' \left(\gamma \cos I_{-X_0}^{X'} \cos I_{X_0}^{X'} + i\delta \sin I_{-X_0}^{X'} \cos I_{X_0}^{X'} \right) \right], \quad (20) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}(\hat{n}_+ - \hat{n}_-) = & \left(\sin I_{-X_0}^{X_0} \right)^{-1} \left[\int_{X_0}^{X_0} dX' \left(i\gamma \cos I_{X_0}^{X'} \sin I_{-X_0}^{X'} - \delta \sin I_{X_0}^{X'} \sin I_{-X_0}^{X'} \right) \right. \\ & \left. + \int_{-X_0}^X dX' \left(i\gamma \sin I_{X_0}^{X'} \cos I_{-X_0}^{X'} - \delta \sin I_{-X_0}^{X'} \sin I_{X_0}^{X'} \right) \right]. \quad (21) \end{aligned}$$

Here we have defined

$$\begin{aligned} \gamma &= \frac{e}{T} (\omega - \omega_*^T) F^{(0)} J_0 \frac{BJ}{|v_{||}|} \hat{\phi}, \\ \delta &= \frac{e}{T} (\omega - \omega_*^T) F^{(0)} J_0 B J \hat{A}_{||} / c, \end{aligned} \quad (22)$$

and the integrals in the trigonometric functions are

$$I_a^b \equiv \int_a^b dX' \gamma (\omega - \omega_d) BJ / |v_{||}|. \quad (23)$$

Finally, the basic pair of equations governing the eigenmodes of the system are determined by the quasineutrality condition,

$$0 = - \sum \frac{ne^2}{T} \hat{\phi} + \sum e \int d\omega d\epsilon d\zeta \frac{B}{|v_{||}|} J_0 (\hat{n}_+ + \hat{n}_-), \quad (24)$$

and by the component of Ampere's law along $\hat{z}$,

$$2 \left[R^2 S^2 \left(\frac{\partial}{\partial \psi} \right)^2 + \frac{1}{R^2} \left(\frac{\partial S}{\partial \psi} \right)^2 + \frac{B^2}{B_V^2 P^2} \right] J_{\parallel} = \frac{4\pi}{c} J_{\parallel} \quad (25)$$

with

$$J_{\parallel} = \sum_s q_s \int d\mu dE d^3 B J_{\parallel s} (h_+ - h_-) .$$

Here the summations are over species. In terms of the perturbed distribution functions derived in the preceding analysis, these equations become:

$$\begin{aligned} 0 = & - \sum_s \frac{n_s e^2}{T} \phi - \sum_s \frac{c^2}{T} i \int_0^{\infty} d\mu dE d^3 B \frac{B}{|V_{\parallel}|} J_0 F^{(0)}(\omega - \omega_{*s}^{\pm}) \\ & \times \left(\int_{-\infty}^{\infty} d\lambda \frac{J_0 B J}{|V_{\parallel}|} \left(1 - |V_{\parallel}| \hat{A}_{\parallel} / c \right) \exp(-i\lambda \hat{\phi}) \right. \\ & \left. + \int_{-\infty}^{\infty} d\lambda \frac{J_0 B J}{|V_{\parallel}|} \left(1 + |V_{\parallel}| \hat{A}_{\parallel} / c \right) \exp(+i\lambda \hat{\phi}) \right) \\ & - \sum_s \frac{2J_0}{\omega_{*s}^{\pm}} d\mu dE d^3 B \frac{B}{|V_{\parallel}|} J_0 (\sin I_{-\chi_0}^{\pm})^{-1} (\omega - \omega_{*s}^{\pm}) F^{(0)} \\ & \times \left[\int_{\chi}^{\chi_0} d\chi' J_0 B J \left(\frac{\hat{\phi}}{|V_{\parallel}|} \cos I_{-\chi_0}^{\chi'} \cos I_{-\chi_0}^{\chi} + i \frac{\hat{A}_{\parallel}}{c} \sin I_{-\chi_0}^{\chi'} \cos I_{-\chi_0}^{\chi} \right) \right. \\ & \left. + \int_{-\chi_0}^{\chi} d\chi' J_0 B J \left(\frac{\hat{\phi}}{|V_{\parallel}|} \cos I_{-\chi_0}^{\chi'} \cos I_{\chi_0}^{\chi} + i \frac{\hat{A}_{\parallel}}{c} \sin I_{-\chi_0}^{\chi'} \cos I_{\chi_0}^{\chi} \right) \right] , \quad (26) \end{aligned}$$

and

$$\begin{aligned}
 & 2 \left[R^2 B_{\chi}^2 \left(\frac{\partial}{\partial \psi} \right)^2 d\chi \cdot \frac{IJ}{R^2} - \frac{\partial S}{\partial \psi} \right]^2 + \frac{B^2}{B_{\chi}^2 R^2} \hat{A}_{||} \\
 &= \frac{4\pi}{c} \left\{ \sum \frac{e^2}{T} i \int_C d\mu dE d\xi B J_0 F^{(0)} (\omega - \omega_*^T) \right. \\
 &\quad \times \left(- \int_{-\infty}^{\chi} d\chi' \frac{J_0 B J}{|v_{||}|} (\hat{\phi} - |v_{||}| \hat{A}_{||} / c, \exp(-i I_{\chi}^{\chi'})) \right. \\
 &\quad \left. + \int_{\chi}^{\infty} d\chi' \frac{J_0 B J}{|v_{||}|} (\hat{\phi} + |v_{||}| \hat{A}_{||} / c) \exp(i I_{\chi}^{\chi'}) \right) \\
 &\quad + \sum 2 \frac{e^2}{T} \int_T d\mu dE d\xi B J_0 F^{(0)} (\omega - \omega_*^T) (\sin I_{-\chi_0}^{\chi_0})^{-1} \\
 &\quad \times \left[\int_{\chi}^{\chi_0} d\chi' J_0 B J \left(i \frac{\hat{\phi}}{|v_{||}|} \cos I_{\chi_0}^{\chi'} \sin I_{-\chi_0}^{\chi} - \frac{\hat{A}_{||}}{c} \sin I_{\chi_0}^{\chi'} \sin I_{-\chi_0}^{\chi} \right) \right. \\
 &\quad \left. + \int_{-\chi_0}^{\chi} d\chi' J_0 B J \left(i \frac{\hat{\phi}}{|v_{||}|} \sin I_{\chi_0}^{\chi} \cos I_{-\chi_0}^{\chi'} - \frac{\hat{A}_{||}}{c} \sin I_{-\chi_0}^{\chi'} \sin I_{\chi_0}^{\chi} \right) \right] \Bigg\}. \quad (27)
 \end{aligned}$$

Here T and C indicate that the corresponding velocity space integrals are only over trapped and circulating particles respectively. A Krook collision operator could be added at this point by changing the definition of I_a^b to

$$I_a^b = \int_a^b d\chi' (\omega + i\nu - \omega_d) B J / |v_{||}|, \quad (28)$$

where ν is some prescribed function of F and μ .

Equations (26) and (27) constitute a closed set of coupled, one-dimensional integral equations, with independent variable x and dependent variables $\hat{\psi}$ and $\hat{A}_{||}$. They determine the poloidal mode structure of the perturbations. They are analogous to the ordinary differential equations, Eqs. (5) and (6) of Ref. 8, governing the poloidal mode structure of ideal MHD ballooning modes. The essential mathematical difference is that these are non-Hermitian integral equations, rather than Hermitian differential equations.

Just as in Ref. 8, there are two types of eigenvalues in the problem. The quantity $\partial S / \partial \psi$ acts as a local radial wave number and determines the rapid radial variation. It is chosen on each surface as an eigenvalue of the poloidal equations. In the ideal MHD case, it is chosen to make $\hat{\psi}$ and $\hat{A}_{||}$ vanish as $\psi \rightarrow \psi_0$, as boundary conditions on the differential equations. For the integral equations treated here, the method of solution is to expand in basis functions which themselves satisfy these conditions. This converts the problem into a homogeneous matrix equation, and then $\partial S / \partial \psi$ is chosen to make the determinant of the matrix vanish.

The complex mode frequency ω , on the other hand, is a global constant, and is chosen to satisfy radial boundary conditions. Equations (26) and (27), however, do not determine the slow, or equilibrium length scale, radial variation of the eigenfunction. To determine this, and also ω , it is necessary to

proceed to higher order in the expansion. This is done in Sec. III. In practice, it is often useful, as an initial attack on the problem, to let $\partial S / \partial \psi = 0$, and to determine ω as the local eigenvalue of Eqs. (26) and (27). This is interpreted and justified in Sec. III.

III. THE RADIAL STRUCTURE PROBLEM

The procedure described in Sec. II leaves the slow radial, or ψ , structure of the eigenmodes undetermined, and it is therefore necessary to pursue the analysis to higher order in the basic expansion in ϵ . In the analogous ideal MHD stability problem,⁸ a higher order analysis in $\ell^{-1/2}$ was found to be necessary for the case of closely spaced turning points, appropriate for nearly marginally stable states. A Weber equation then resulted for the radial mode structure. It was also found that,⁸ far from marginal stability, the WKB turning points are, in general, widely separated, and an expansion in $\ell^{-1/3}$ is needed in the vicinity of the turning points. The standard WKB connection formula is normally obtained for determining the eigenfrequency, while the radial structure of the eigenmode has the familiar WKB form. Dewar has suggested,⁹ for the ideal MHD problem, that this procedure can be cast into a rather general form applicable to a Lagrangian system with Hermitian (or self-adjoint) governing equations, and that the condition for the determination of the WKB turning points is closely related to Whitham's results¹⁰ on conservation of action. The turning points are determined by

$$\left(\psi^*, \frac{\partial L}{\partial (\partial S / \partial \psi)} \psi \right) = 0. \quad (29)$$

where L is the Lagrangian operator, ψ is the eigenfunction, and S is the WKB eikonal.

In the present kinetic analysis, the Hermitian property is lost, because of the presence of resonant particles and/or collisions, and it is therefore not clear, a priori, that the radial structure problem can be handled in the same manner as for the ideal MHD analysis. In this section, therefore, we demonstrate directly, for a model slab problem, that a WKB analysis¹¹ is indeed applicable in the kinetic regime with $k_{\perp} \rho_i \sim 1$. We then go on to speculate that the inclusion of terms arising from the toroidal geometry can be handled in the same manner. The case of closely spaced turning points in toroidal geometry will be treated explicitly.

The demonstration of the symmetry property (weaker than a Hermitian or self-adjoint property) of the fundamental equations follows by rewriting Eq. (26) and (27) in the compact operator form

$$K_{11}\hat{\phi} + K_{12}\hat{A} = 0, \quad (30)$$

$$K_{21}\hat{\phi} + K_{22}\hat{A} = 0. \quad (31)$$

Defining the column vector $\underline{\hat{b}}$ as

$$\underline{\hat{b}} = \begin{Bmatrix} \hat{\phi} \\ \hat{A} \end{Bmatrix}, \quad (32)$$

Eqs. (30) and (31) become

$$K_{ij}b_j = 0. \quad (33)$$

The quadratic form

$$\int_{-\infty}^{\infty} d\chi J b_i K_{ij} b_j = 0 \quad (34)$$

is found to be variational if

$$K_{ij} = K_{ji} . \quad (35)$$

The symmetry property, Eq. (35), can be demonstrated by a number of integrations by parts.

The conjectured condition for the existence of WKB turning points is

$$\int_{-\infty}^{\infty} d\chi J b_i \frac{\partial K_{ij}}{\partial [\partial S / \partial (\epsilon \psi)]} b_j = 0 , \quad (36)$$

and the WKB connection formula is

$$\left\{ \begin{array}{l} \epsilon \psi_{\sim} \\ \epsilon \psi_1 \end{array} \right. d(\epsilon \psi) \frac{\partial S}{\partial (\epsilon \psi)} = (p + \frac{1}{2})\pi , \quad (p = 0, 1, 2, \dots) . \quad (37)$$

A. The Model Slab WKB Problem

To demonstrate the procedure for determining the radial structure, we consider the kinetic low frequency stability problem for a plane slab in the electrostatic limit. In order to model the physical aspects of the toroidal problem, shear is included as well as spatial variation in the equilibrium fields. In the E , μ , ξ coordinate system, the linearized Vlasov equation is

$$\begin{aligned} & \frac{\partial f}{\partial t} + \underline{v} \cdot \left[\nabla f - (\mu \nabla B + v_{\parallel} \nabla n \cdot \underline{v}_{\perp}) \frac{1}{B} \frac{\partial f}{\partial \mu} + \left(\frac{v_{\parallel} \nabla n \cdot \underline{v}_{\perp}}{v_{\perp}^2} + \underline{v}_B \cdot \underline{p} \right) \frac{\partial f}{\partial \xi} \right] \\ & - \frac{e}{m} v_{\parallel} \nabla n \cdot \nabla \left(\frac{\partial F}{\partial E} - \frac{e}{m} \underline{v}_{\perp} \cdot \nabla \left(\frac{\partial F}{\partial E} + \frac{1}{B} \frac{\partial F}{\partial \mu} \right) - \frac{e}{m} \frac{n \times \underline{v}_{\perp}}{v_{\perp}^2} \cdot \nabla \left(\frac{\partial F}{\partial \xi} - \Omega \frac{\partial f}{\partial \xi} \right) = 0. \end{aligned} \quad (38)$$

Here, $\underline{n}$, $\underline{p}$, $\underline{b}$ is the orthogonal unit vector triplet in velocity space and

$$\underline{v} = v_{\parallel} \underline{n} + v_{\perp} (\underline{p} \cos \xi + \underline{b} \sin \xi). \quad (39)$$

We denote by x and ϵx the length scales ρ_i and r_n and perform a multiple space scale analysis, with $\epsilon \equiv \rho_i/r_n$ being the ordering parameter.

The equilibrium geometry is specified by $B = (B_z^2 + B_y^2)^{1/2}$ and the unit tangent vector

$$\underline{n} = \underline{e}_z \frac{B_z}{B} + \underline{e}_y \frac{B_y}{B}, \quad (40)$$

where B , B_z , and B_y are functions of ϵx . The familiar plane slab representation assumes that the shear-producing field B_y is of order ϵB_z . Thus, correct to second order in ϵ ,

$$\begin{aligned} B &= B_z \left[1 + \frac{1}{2} \left(\frac{B_y}{B_z} \right)^2 \right], \\ \underline{n} &= \underline{e}_z \left[1 - \frac{1}{2} \left(\frac{B_y}{B_z} \right)^2 \right] + \underline{e}_y \frac{B_y}{B_z}, \\ \underline{p} &= \underline{e}_x, \\ \underline{b} &= \underline{e}_y \left[1 - \frac{1}{2} \left(\frac{B_y}{B_z} \right)^2 \right] - \underline{e}_z \frac{B_y}{B_z}. \end{aligned} \quad (41)$$

Since x , E , and ξ are fixed velocity space coordinates, $v_{\perp}$ and $v_{\parallel}$ must be expanded in ϵ to maintain their constancy. We easily find

$$v_{\perp} = v_{\perp}^{(0)} + v_{\perp}^{(2)}, \quad (42)$$

with

$$\mu = \frac{[v_{\perp}^{(0)}]^2}{2B_z}, \quad (43)$$

and

$$v_{\perp}^{(2)} = \frac{v_{\perp}^{(0)}}{4} \left(\frac{B_y}{B_z} \right)^2. \quad (44)$$

Similarly,

$$v_{\parallel} = v_{\parallel}^{(0)} + v_{\parallel}^{(2)}, \quad (45)$$

with

$$\frac{v_{\parallel}^{(0)2}}{2} = E - \mu B_z, \quad (46)$$

and

$$v_{\parallel}^{(2)} = - \frac{\mu B_z}{v_{\parallel}^{(0)}} \frac{1}{2} \left(\frac{B_y}{B_z} \right)^2. \quad (47)$$

The Vlasov equation for the equilibrium distribution function, correct to second order in ϵ , is

$$\begin{aligned} v_{\perp} \cos \xi \frac{\partial F}{\partial \epsilon x} - \left[\mu v_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} + v_{\perp}^2 \cos \xi \sin \xi \frac{\partial}{\partial \epsilon x} \left(\frac{B_y}{B_z} \right) \right] \frac{1}{B_z} \frac{\partial F}{\partial \mu} \\ - v_{\parallel} \cos^2 \xi \frac{\partial}{\partial \epsilon x} \left(\frac{B_y}{B_z} \right) \frac{\partial F}{\partial \xi} - \Omega \frac{\partial F}{\partial \xi} = 0. \end{aligned} \quad (48)$$

Here and in the following we drop the (0) superscript on $v_{\perp}$ and $v_{\parallel}$. Applying standard procedures, this equation readily yields the result

$$F = F^{(0)} + F^{(1)} + F^{(2)} \quad , \quad (49)$$

with $F^{(0)}$ being a Maxwellian

$$F^{(0)} = n \left(\frac{m}{2\pi T} \right)^{3/2} \exp(-mE/T) \quad , \quad (50)$$

$$F^{(1)} = \frac{v_{\perp}}{v} \sin \zeta \frac{\partial F^{(0)}}{\partial v_{\perp}} \quad , \quad (51)$$

and

$$F^{(2)} = -\frac{\cos 2\zeta}{4} \frac{v_{\perp}^2}{v} \frac{\partial}{\partial v_{\perp}} \left(\frac{1}{v} \frac{\partial F^{(0)}}{\partial v_{\perp}} \right) \quad . \quad (52)$$

The linearized Vlasov equation, governing perturbations, correct to two orders in ϵ is

$$\begin{aligned} & \frac{\partial f}{\partial t} + v_{\parallel} \left(\frac{\partial f}{\partial z} + \frac{B}{B_z} \frac{\partial f}{\partial Y} \right) + \cos \zeta \left[v_{\perp} + v_{\perp}^{(2)} \right] \frac{\partial f}{\partial v_{\perp}} + \sin \zeta \left[v_{\perp} + v_{\perp}^{(2)} - \frac{1}{2} \left(\frac{B_Y}{B_z} \right)^2 v_{\perp} \right] \\ & \times \frac{\partial f}{\partial v_{\parallel}} - v_{\perp} \sin \zeta \left[\frac{B_Y}{B_z} \frac{\partial f}{\partial z} - \left(v_{\perp} \cos \zeta \frac{\partial B_z}{\partial X} + v_{\parallel} v_{\perp}^2 \cos \zeta \sin \zeta \frac{\partial}{\partial X} \left(\frac{B_Y}{B_z} \right) \right) \right] \\ & \times \frac{1}{B_z} \frac{\partial f}{\partial t} - v_{\parallel} \cos^2 \zeta \left[\frac{\partial}{\partial X} \frac{B_Y}{B_z} \frac{\partial f}{\partial v_{\perp}} - \frac{e}{m} v_{\parallel} \left(\frac{\partial f}{\partial z} + \frac{B_Y}{B_z} \frac{\partial f}{\partial Y} \right) \frac{\partial F}{\partial v_{\perp}} \right. \\ & \left. - \frac{e}{m} \left[\cos \zeta \left(v_{\perp} + v_{\perp}^{(2)} \right) \frac{\partial f}{\partial X} + \sin \zeta \left[v_{\perp} + v_{\perp}^{(2)} - \frac{1}{2} v_{\perp} \left(\frac{B_Y}{B_z} \right)^2 \right] \right. \right. \\ & \left. \left. \times \frac{\partial f}{\partial v_{\parallel}} - v_{\perp} \sin \zeta \left[\frac{B_Y}{B_z} \frac{\partial f}{\partial z} \right] \left(\frac{\partial F}{\partial v_{\perp}} + \frac{1}{v} \frac{\partial F}{\partial v_{\perp}} \right) + \frac{e}{m} \right. \right. \\ & \left. \left. \times \left(\frac{\partial f}{\partial X} \sin \zeta - \frac{\partial f}{\partial Y} \cos \zeta \right) \frac{1}{v_{\perp}} \frac{\partial F}{\partial v_{\perp}} - v \left(1 + \frac{1}{2} \frac{B_Y^2}{B_z^2} \right) \frac{\partial f}{\partial v_{\perp}} \right] = 0 \quad . \quad (53) \end{aligned}$$

To determine the perturbed distribution function, the usual low frequency, finite $k_{\perp} \rho_i$, gyrokinetic expansion⁷ is followed through three orders in ϵ . Decomposing $\mathbf{r}$ in the same way as $\mathbf{F}$ in Eq. (49), this procedure leads to:

$$v_{\perp} \cos \xi \frac{\partial f^{(0)}}{\partial x} + v_{\perp} \sin \xi \frac{\partial f^{(0)}}{\partial y} - \frac{e}{m} v_{\perp} \left(\cos \xi \frac{\partial \Phi}{\partial x} + \sin \xi \frac{\partial \Phi}{\partial y} \right) \frac{\partial F^{(0)}}{\partial E} - \Omega \frac{\partial f^{(0)}}{\partial \xi} = 0, \quad (54)$$

$$\begin{aligned} & \frac{\partial f^{(0)}}{\partial t} + v_{\parallel} \left(\frac{\partial f^{(0)}}{\partial z} + \frac{B_y}{B_z} \frac{\partial f^{(0)}}{\partial z} \right) + v_{\perp} \cos \xi \left(\frac{\partial f^{(0)}}{\partial x} + \frac{\partial f^{(0)}}{\partial \epsilon x} \right) \\ & + v_{\perp} \sin \xi \frac{\partial f^{(0)}}{\partial y} - \mu v_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial f^{(0)}}{\partial \mu} - \frac{e}{m} v_{\parallel} \\ & \cdot \left(\frac{\partial f^{(0)}}{\partial z} + \frac{B_y}{B_z} \frac{\partial f^{(0)}}{\partial y} \right) \frac{\partial F^{(0)}}{\partial E} - \frac{e}{m} v_{\perp} \left(\cos \xi \frac{\partial \Phi}{\partial x} + \sin \xi \frac{\partial \Phi}{\partial y} \right) \left(\frac{\partial F^{(1)}}{\partial E} + \frac{1}{B} \frac{\partial F^{(1)}}{\partial \mu} \right) \\ & + \frac{e}{m} \frac{1}{v_{\perp}} \left(\frac{\partial f^{(0)}}{\partial x} \sin \xi - \frac{\partial f^{(0)}}{\partial y} \cos \xi \right) \frac{\partial F^{(1)}}{\partial E} - \frac{e}{m} v_{\perp} \cos \xi \frac{\partial f^{(0)}}{\partial \epsilon x} \frac{\partial F^{(0)}}{\partial E} - \frac{\partial \partial f^{(1)}}{\partial \epsilon} = 0, \end{aligned} \quad (55)$$

and

$$\begin{aligned}
 & \frac{\partial f^{(1)}}{\partial t} + v_{\parallel} \left(\frac{\partial f^{(1)}}{\partial z} + \frac{B_y}{B_z} \frac{\partial f^{(1)}}{\partial y} \right) + v_{\perp} \cos \xi \left(-\frac{\partial f^{(2)}}{\partial x} + \frac{\partial f^{(1)}}{\partial \epsilon x} \right) \\
 & + v_{\perp} \sin \xi \frac{\partial f^{(2)}}{\partial y} + v_{\perp}^{(2)} \cos \xi \frac{\partial f^{(0)}}{\partial x} + \left[v_{\perp}^{(2)} - \frac{1}{2} \left(\frac{B_y}{B_z} \right)^2 v_{\perp} \right] \sin \xi \frac{\partial f^{(0)}}{\partial y} \\
 & - v_{\perp} \sin \xi \frac{B_y}{B_z} \frac{\partial f^{(0)}}{\partial z} - \mu v_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial f^{(1)}}{\partial \mu} \\
 & - v_{\parallel} v_{\perp}^2 \cos \xi \sin \xi \frac{\partial}{\partial \epsilon x} \left(\frac{B_y}{B_z} \right) \frac{1}{B_z} \frac{\partial f^{(0)}}{\partial \mu} - v_{\parallel} \cos^2 \xi \frac{\partial}{\partial \epsilon x} \left(\frac{B_y}{B_z} \right) \frac{\partial f^{(0)}}{\partial \xi} \\
 & - \frac{e}{m} v_{\parallel} \left(\frac{\partial \phi}{\partial z} + \frac{B_y}{B_z} \frac{\partial \phi}{\partial y} \right) \frac{\partial F^{(1)}}{\partial E} - \frac{e}{m} v_{\perp} \left(\cos \xi \frac{\partial \phi}{\partial x} + \sin \xi \frac{\partial \phi}{\partial y} \right) \\
 & \times \left(-\frac{\partial F^{(2)}}{\partial E} + \frac{1}{B_z} \frac{\partial F^{(2)}}{\partial \mu} \right) - \frac{e}{m} v_{\perp} \cos \xi \frac{\partial \phi}{\partial \epsilon x} \left(\frac{\partial F^{(1)}}{\partial E} + \frac{1}{B_z} \frac{\partial F^{(1)}}{\partial \mu} \right) - \frac{e}{m} \\
 & \times \left[v_{\perp}^{(2)} \cos \xi \frac{\partial \phi}{\partial x} + \left[v_{\perp}^{(2)} - \frac{1}{2} \left(\frac{B_y}{B_z} \right)^2 v_{\perp} \right] \sin \xi \frac{\partial \phi}{\partial y} - v_{\perp} \sin \xi \frac{B_y}{B_z} \frac{\partial \phi}{\partial z} \right] \frac{\partial F^{(0)}}{\partial E} \\
 & + \frac{e}{m} \frac{1}{v_{\perp}} \left(\frac{\partial \phi}{\partial x} \sin \xi - \frac{\partial \phi}{\partial y} \cos \xi \right) \frac{\partial F^{(2)}}{\partial E} + \frac{e}{m} \frac{1}{v_{\perp}} \frac{\partial \phi}{\partial \epsilon x} \sin \xi \frac{\partial F^{(1)}}{\partial E} \\
 & - \frac{\partial f^{(2)}}{\partial t} - \frac{1}{2} \left(\frac{B_y}{B_z} \right)^2 \frac{\partial f^{(0)}}{\partial \epsilon} = 0. \quad (56)
 \end{aligned}$$

To solve this set of equations, the ansatz adopted for the perturbed distribution function, in the region between the turning points, has the form

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \dots$$

$$= (f^{(0)} + f^{(1)} + f^{(2)} + \dots) \exp(-i(\omega t + ik_y y + ik_z z + i\psi(\epsilon - x)/v_{\perp})), \quad (57)$$

while for ψ we assume

$$\Phi = \tilde{\Phi}(\epsilon x) \exp \{-i\omega t + ik_y y + ik_z z + i[S(\epsilon x)/\epsilon]\} \quad (58)$$

The solution of Eq. (54) is then found to be

$$f^{(0)} = \frac{e}{m} \Phi \frac{\partial F^{(0)}}{\partial E} + h^{(0)} \quad (59)$$

with

$$h^{(0)} = \hat{h}^{(0)} e^{iL} \quad (60)$$

$$L = \frac{v_1}{\Omega} (\sin \xi \frac{\partial S}{\partial \epsilon x} - k_y \cos \xi) \quad (61)$$

and

$$\frac{\partial \hat{h}^{(0)}}{\partial \xi} = 0 \quad (62)$$

Upon substituting $f^{(0)}$ and $F^{(1)}$ into Eq. (55), we find

$$f^{(1)} = \frac{e\Phi}{m} \frac{\partial F^{(1)}}{\partial E} + \hat{g}^{(1)} e^{iL} \quad (63)$$

and

$$\begin{aligned} & \left(-i\omega + ik_{||} v_{||} + ik_y \frac{\mu}{\Omega} \frac{\partial B_z}{\partial \epsilon x} \right) \hat{h}^{(0)} - i \frac{e\Phi}{m} \left(\omega \frac{\partial F^{(0)}}{\partial E} + \frac{k_y}{\Omega} \frac{\partial F^{(0)}}{\partial \epsilon x} \right) e^{-iL} \\ & + \cos \xi \left(v_1 \frac{\partial \hat{h}^{(0)}}{\partial \epsilon x} - \mu v_1 \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial \hat{h}^{(0)}}{\partial \mu} \right) \\ & + i \hat{h}^{(0)} \sin 2\xi B_z \frac{\partial}{\partial \epsilon x} \left(\frac{\mu}{\Omega} \frac{\partial S}{\partial \epsilon x} \right) + i \hat{h}^{(0)} \cos 2\xi k_y \frac{\mu}{\Omega} \frac{\partial B_z}{\partial \epsilon x} = i \frac{\partial \hat{g}^{(1)}}{\partial \xi} \quad (64) \end{aligned}$$

Averaging over ξ in Eq. (64) then yields the lowest order kinetic equation

$$\left(-i\omega + ik_{||} v_{||} + ik_y \frac{\mu}{\Omega} \frac{\partial B_z}{\partial \epsilon x} \right) \hat{h}^{(0)} = i \frac{e}{m} \Phi \left(\omega \frac{\partial F^{(0)}}{\partial E} + \frac{k_y}{\Omega} \frac{\partial F^{(0)}}{\partial \epsilon x} \right) e^{-iL} \quad (65)$$

with

$$(\Lambda)^{-1} = \frac{1}{2^{\text{B}}} \int_{\epsilon_0}^{2^{\text{B}}} d\epsilon \Lambda^{-1}. \quad (66)$$

The solution of Eq. (65) determines $\psi^{(0)}$, which then allows the lowest order charge density to be calculated using Eq. (59).

This, in turn, yields the familiar dispersion relation:

$$0 = -\frac{1}{2} \frac{m_e^2}{\pi} \left(1 - \frac{v^2}{c^2} \right) \int d^3v \Gamma^{(0)} \frac{(v_{\parallel} - v_{\text{A}}^{\text{p}})}{(v_{\parallel}^2 - v_{\text{A}}^2) v_{\parallel}} + e^{-1} \frac{1}{4\pi} \frac{1}{\epsilon_0} \frac{1}{v_{\text{A}}^2} \quad (67)$$

with the summation again being over species. The result then determines the rational function $\psi^{(0)}(z)$ as a function of the complex frequency $\omega = z$. It is *not* necessary to proceed to next order to determine the slow $\psi^{(1)}(z)$ variation of the perturbed potential $\psi^{(1)}(t, z)$.

We begin by substituting Eq. (65) into Eq. (64) to get

$$\begin{aligned} \psi^{(1)} &= -1 \left[\frac{e^{-1}}{m} \left(\frac{d^2 \psi^{(0)}}{dz^2} + \frac{1}{2} \frac{d^2 \psi^{(0)}}{dz^2} \right) \right] \left(e^{-1} \frac{1}{4\pi} \frac{1}{\epsilon_0} \frac{1}{v_{\text{A}}^2} \right) \\ &+ \text{term} \propto \left(v_{\text{A}} \frac{d\psi^{(0)}}{dz} + v_{\text{A}}^2 \frac{d^2 \psi^{(0)}}{dz^2} + \frac{d\psi^{(0)}}{dz} \right) \\ &+ \psi^{(0)} \frac{d}{dz} \left(\frac{v_{\text{A}}^2}{2} \right) + \frac{v_{\text{A}}^2}{2} \left(\frac{d^2 \psi^{(0)}}{dz^2} + \frac{d^2 \psi^{(0)}}{dz^2} \right) + \frac{d\psi^{(0)}}{dz} \left[\psi^{(1)} \right], \quad (68) \end{aligned}$$

with

$$\frac{d\psi^{(1)}}{dz} = 0, \quad (69)$$

and

$$q^{(1)} = q^{(1)} e^{-ikz}, \quad (70)$$

We then substitute $f^{(0)}$, $f^{(1)}$ and $F^{(1)}$ from Eqs. (59), (63) and (51) into Eq. (56). Multiplying the resultant expression by e^{-ikz} and z -averaging yields

$$\begin{aligned} & \left\langle e^{-ikz} \left(-i\omega + ik \right) v_{\parallel} \right\rangle + v_{\perp} \cos \theta \left(\frac{\partial}{\partial x} + \nu v_{\perp} \cos \theta \left(\frac{\partial}{\partial x} + \frac{1}{R} \frac{\partial}{\partial \theta} \right) \right) q^{(1)} \rangle \\ & = i\omega \frac{e}{m} \Phi \left\langle \frac{\partial F^{(1)}}{\partial E} e^{-ikz} \right\rangle + \frac{e\Phi}{m} \left(\frac{\partial}{\partial E} + \nu v_{\perp}^2 \left(\frac{\partial}{\partial x} + \frac{1}{R} \frac{\partial}{\partial \theta} \right) \right) \frac{\partial F^{(0)}}{\partial x} \left\langle \sin \frac{2\theta}{2} e^{-ikz} \right\rangle \\ & + \frac{e}{m} \left\langle e^{-ikz} \left[\left(\cos \theta \frac{\partial \Phi}{\partial x} + \sin \theta \frac{\partial \Phi}{\partial \theta} \right) v_{\perp} \left(\frac{\partial F^{(2)}}{\partial E} + \frac{1}{R} \frac{\partial F^{(2)}}{\partial \theta} \right) \right. \right. \\ & \left. \left. + \left(\frac{\partial^2}{\partial \theta^2} \cos \theta - \frac{\partial^2}{\partial x^2} \sin \theta \right) \frac{1}{2} \frac{\partial F^{(0)}}{\partial x} \right] \right\rangle = 0. \end{aligned} \quad (71)$$

Using the properties of $F^{(0)}$ obtained by Eq. (61), we find that the first term in brackets in Eq. (71) may be written in the form,

$$\frac{e\Phi}{m} \left\langle \left(\frac{\partial}{\partial E} + \frac{1}{R} \frac{\partial}{\partial \theta} \right) \right\rangle \left\langle e^{-ikz} \frac{\partial F^{(2)}}{\partial E} \right\rangle, \quad (72)$$

We next substitute $q^{(1)}$ and $F^{(2)}$ from Eqs. (68), (52) and (72) into Eq. (71) and, after a considerable amount of calculation, we obtain

$$\begin{aligned}
 & \left(-i\omega + ik_{||} v_{||} + ik_Y \frac{\mu}{\Omega} \frac{\partial B_z}{\partial \epsilon x} \right) \\
 & \times \left[\hat{h}^{(1)} - \frac{e\Phi i}{m\Omega} \left(\omega \frac{\partial F^{(0)}}{\partial E} + \frac{k_Y}{\Omega} \frac{\partial F^{(0)}}{\partial \epsilon x} \right) \langle G(\xi) \rangle \right] \\
 & + v_{\perp} \left(\omega \frac{\partial F^{(0)}}{\partial E} + \frac{k_Y}{\Omega} \frac{\partial F^{(0)}}{\partial \epsilon x} \right) \frac{\partial}{\partial \epsilon x} \left(\frac{e\Phi}{m} \frac{i}{\Omega} \langle \sin \xi e^{-iL} \rangle \right) \\
 & + \frac{e\Phi}{m\Omega^2} \left(\omega \frac{\partial F^{(0)}}{\partial E} + \frac{k_Y}{\Omega} \frac{\partial F^{(0)}}{\partial \epsilon x} \right) \\
 & \times \left[\frac{v_{\perp}^2}{4} \frac{\partial^2 S}{\partial (\epsilon x)^2} \langle \cos 2\xi e^{-iL} \rangle - \mu \frac{\partial B_z}{\partial \epsilon x} \frac{1}{2} \frac{\partial S}{\partial \epsilon x} \langle e^{-iL} \rangle \right] = 0, \quad (73)
 \end{aligned}$$

where

$$G(\xi) = \int^{\xi} d\xi' (e^{-iL} - \langle e^{-iL} \rangle).$$

Since Eq. (63) implies the result

$$\langle f^{(1)} \rangle = \langle \hat{g}^{(1)} e^{iL} \rangle, \quad (74)$$

with $\hat{g}^{(1)}$ given by Eq. (68) in terms of $\hat{h}^{(1)}$, it follows that

$$\begin{aligned}
 \langle f^{(1)} \rangle &= -v_{\perp} \Gamma \frac{\partial}{\partial \epsilon x} \left(\frac{e\Phi}{m\Omega} \langle \sin \xi e^{-iL} \rangle \right) \langle e^{iL} \rangle \\
 &+ i \frac{e\Phi}{m} \Gamma \frac{\mu}{\Omega^2} \frac{\partial B_z}{\partial \epsilon x} \left(-\frac{\partial S}{\partial \epsilon x} \langle e^{iL} \rangle - i \frac{\Omega}{v_{\perp}} \langle \sin \xi e^{iL} \rangle \right) \langle e^{-iL} \rangle \\
 &+ \frac{v_{\perp}}{\Omega} \langle \sin \xi e^{iL} \rangle \frac{\partial}{\partial \epsilon x} \left(\frac{e\Phi}{m} \Gamma \langle e^{-iL} \rangle \right) \\
 &- \frac{\mu v_{\perp}}{\Omega} \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \langle \sin \xi e^{iL} \rangle \frac{\partial}{\partial \mu} \left(\frac{e\Phi}{m} \Gamma \langle e^{-iL} \rangle \right), \quad (75)
 \end{aligned}$$

with

$$\Gamma = \frac{\{\omega [\partial F^{(0)} / \partial E] + (k_y / \Omega) [\partial F^{(0)} / \partial \epsilon x]\}}{-\omega + k_{||} v_{||} + k_y (u / \Omega) (\partial B_z / \partial \epsilon x)} . \quad (76)$$

Using $f^{(1)}$ to construct the first order charge neutrality condition yields

$$\begin{aligned} 2 \frac{\partial \phi}{\partial \epsilon x} \int \frac{e^2}{m} \int \frac{B d\mu dE d\xi}{v_{||}} \frac{v_{\perp}}{\Omega} \Gamma \langle \sin \xi e^{-iL} \rangle \langle e^{iL} \rangle \\ + \frac{\psi \partial}{\partial \epsilon x} \int \frac{e^2}{m} \int \frac{B d\mu dE d\xi}{v_{||}} \frac{v_{\perp}}{\Omega} \Gamma \langle \sin \xi e^{-iL} \rangle \langle e^{iL} \rangle = 0. \end{aligned} \quad (77)$$

Equation (77) then leads to the familiar WKB structure,

$$\phi \propto \Lambda^{-1/2}, \quad (78)$$

with

$$\Lambda = \int \frac{e^2}{m} \int \frac{B d\mu dE d\xi}{v_{||}} \frac{v_{\perp}}{\Omega} \Gamma \langle \sin \xi e^{-iL} \rangle \langle e^{iL} \rangle . \quad (79)$$

It is useful to note that multiplying Eq. (77) by ϕ yields the form

$$\frac{\partial}{\partial \epsilon x} (\phi \Lambda \phi) = 0 , \quad (80)$$

which in turn can be written as

$$\frac{\partial}{\partial \epsilon x} \left[\phi \frac{\partial}{\partial (\partial / \partial \epsilon x)} \left(\int \frac{e^2 n}{T} - \int \frac{e^2}{T} \int \frac{B d\mu dE d\xi}{v_{||}} \frac{(u - v_{*}^T) F^{(0)}}{\omega - k_{||} v_{||} - \omega_d} \langle e^{-iL} \rangle \langle e^{iL} \rangle \right) \phi \right] = 0 . \quad (81)$$

The term in parentheses in Eq. (81) is simply the zeroth order dispersion relation, Eq. (67). From the preceding analysis we see that for this kinetic model problem, the standard SW theory holds in the region between the turning points and that relations analogous to those given in Eqs. (29) and (36) may be exhibited.

We next turn to the analysis in the vicinity of the turning points. As noted earlier, an expansion in powers of $\epsilon^{1/3}$ is appropriate in this region. Thus, we assume the multiple length-scale expansion

$$f = f^{(0)} + \epsilon^{1/3} f^{(1/3)} + \dots = [f^{(0)}(\epsilon^{1/3} x, x) + f^{(1/3)}(\epsilon^{1/3} x, x) + \dots] \exp(-i\omega t + i k_y y + i k_z z), \quad (82)$$

and

$$\phi = \phi(\epsilon^{1/3} x, x) \exp(-i\omega t + i k_y y + i k_z z).$$

The governing equations, which follow from the expansion of Eq. (53), are

$$v_{\perp} \sin \xi \frac{\partial f^{(0)}}{\partial y} - \frac{e}{m} \frac{\partial \phi}{\partial y} \sin \xi - \omega \frac{\partial f^{(0)}}{\partial \epsilon} = 0, \quad (83)$$

$$v_{\perp} \cos \xi \frac{\partial f^{(0)}}{\partial \epsilon^{1/3} x} + v_{\perp} \sin \xi \frac{\partial f^{(1/3)}}{\partial y} - \frac{e}{m} v_{\perp} \cos \xi \frac{\partial \phi}{\partial \epsilon^{1/3} x} \frac{\partial F^{(0)}}{\partial \epsilon} - \omega \frac{\partial f^{(1/3)}}{\partial \epsilon} = 0, \quad (84)$$

$$v_{\perp} \cos \xi \frac{\partial f^{(1/3)}}{\partial \epsilon^{1/3} x} + v_{\perp} \sin \xi \frac{\partial f^{(2/3)}}{\partial y} - \omega \frac{\partial f^{(2/3)}}{\partial \epsilon} = 0, \quad (85)$$

$$\begin{aligned}
 & -i\omega f^{(0)} + ik_{||} v_{||} f^{(0)} + v_{\perp} \cos \xi \left(\frac{\partial f^{(0)}}{\partial \epsilon x} + \frac{\partial f^{(2/3)}}{\partial \epsilon 1/3 x} \right) + v_{\perp} \sin \xi \frac{\partial f^{(1)}}{\partial y} \\
 & - \mu v_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial f^{(0)}}{\partial \mu} - \frac{e}{m} ik_{||} v_{||} \phi \frac{\partial F^{(0)}}{\partial E} - \frac{e}{m} v_{\perp} \cos \xi \frac{\partial \phi}{\partial \epsilon x} \frac{\partial F^{(0)}}{\partial E} \\
 & - \frac{e}{m} v_{\perp} \sin \xi \frac{\partial \phi}{\partial y} \left(\frac{\partial F^{(1)}}{\partial E} + \frac{1}{B_z} \frac{\partial F^{(1)}}{\partial \mu} \right) - \frac{e}{m} \frac{1}{v_{\perp}} \frac{\partial \phi}{\partial y} \cos \xi \frac{\partial F^{(1)}}{\partial \xi} - \Omega \frac{\partial F^{(1)}}{\partial \xi} = 0 ,
 \end{aligned} \tag{86}$$

$$\begin{aligned}
 & -i\omega f^{(1/3)} + ik_{||} v_{||} f^{(1/3)} + v_{\perp} \cos \xi \left[\frac{\partial f^{(1)}}{\partial \epsilon 1/3 x} + \frac{\partial f^{(1/3)}}{\partial \epsilon x} \right] + v_{\perp} \sin \xi \frac{\partial f^{(4/3)}}{\partial y} \\
 & - \mu v_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial f^{(1/3)}}{\partial \mu} - \frac{e}{m} v_{\perp} \cos \xi \frac{\partial \phi}{\partial \epsilon 1/3 x} \left(\frac{\partial F^{(1)}}{\partial E} + \frac{1}{B_z} \frac{\partial F^{(1)}}{\partial \mu} \right) \\
 & + \frac{e}{m} \frac{1}{v} \frac{\partial F^{(1)}}{\partial \xi} \frac{\partial \phi}{\partial \epsilon 1/3 x} \sin \xi - \Omega \frac{\partial f^{(4/3)}}{\partial \xi} = 0 ,
 \end{aligned} \tag{87}$$

and

$$\begin{aligned}
 & -i\omega f^{(2/3)} + ik_{||} v_{||} f^{(2/3)} + v_{\perp} \cos \xi \left(\frac{\partial f^{(4/3)}}{\partial \epsilon 1/3 x} + \frac{\partial f^{(2/3)}}{\partial \epsilon x} \right) + v_{\perp} \sin \xi \frac{\partial f^{(5/3)}}{\partial y} \\
 & - \mu v_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial f^{(2/3)}}{\partial \mu} - \Omega \frac{\partial f^{(5/3)}}{\partial \xi} = 0 .
 \end{aligned} \tag{88}$$

The calculation now proceeds in much the same manner as in the main region. From Eq. (83) we immediately find

$$f^{(0)} = \frac{e}{m} \phi \frac{\partial F^{(0)}}{\partial E} + \hat{h}^{(0)} e^{iL} , \tag{89}$$

with L now written as

$$L = - \frac{k_y v_{\perp}}{\Omega} \cos \xi . \tag{90}$$

Next, Eq. (84) yields

$$\hat{f}(1/3) = f(1/3) e^{-iL} = \frac{v_1}{\omega} \sin \xi \frac{\partial \hat{h}^{(0)}}{\partial (\epsilon^{1/3} x)} + \hat{h}^{(1/3)} . \quad (91)$$

Equation (85) then gives

$$\hat{f}(2/3) = f(2/3) e^{-iL} = -\frac{v_1^2}{4\omega^2} \cos 2\xi \frac{\partial^2 \hat{h}^{(0)}}{\partial (\epsilon^{1/3} x)^2} + \frac{v_1}{\omega} \sin \xi \frac{\partial \hat{h}^{(1/3)}}{\partial (\epsilon^{1/3} x)} + \hat{h}^{(2/3)} . \quad (92)$$

From Eq. (86) we immediately find

$$f(1) = \frac{e\phi}{m} \frac{\partial F(1)}{\partial E} + \hat{g}(1) e^{iL} + \hat{k}(1) e^{-iL} \quad (93)$$

where $\hat{g}(1)$ is given by Eq. (68) and $\hat{k}(1)$ is

$$\begin{aligned} \hat{k}(1) = & \frac{v_1}{\omega} \sin \xi \frac{\partial \hat{h}^{(2/3)}}{\partial (\epsilon^{1/3} x)} - \frac{v_1^2}{4\omega^2} \cos 2\xi \frac{\partial^2 \hat{h}^{(1/3)}}{\partial (\epsilon^{1/3} x)^2} \\ & - \frac{v_1^3}{8\omega^3} \left(\frac{\sin 3\xi}{3} + \sin \xi \right) \frac{\partial^3 \hat{h}^{(0)}}{\partial (\epsilon^{1/3} x)^3} . \end{aligned} \quad (94)$$

The equation governing $\hat{h}^{(0)}$ is again Eq. (65), so that the zeroth order dispersion relation is given by Eq. (67) with $\partial S / \partial \epsilon x = 0$. Thus, this agrees with the results obtained in the main region in the limit as $\partial S / \partial \epsilon x$ tends to zero.

To next order, Eq. (87) leads to

$$\hat{h}^{(1/3)} = 0 . \quad (95)$$

We also obtain from Eq. (87)

$$\begin{aligned}
 & \left(-i\omega + ik_{\parallel} v_{\parallel} + v_{\perp} \cos \xi \frac{\partial}{\partial \epsilon x} - uv_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial}{\partial \mu} \right. \\
 & \quad \left. + v_{\perp} \cos \xi \frac{\partial iL}{\partial \epsilon x} - uv_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial iL}{\partial \mu} \right) \frac{v_{\perp}}{\Omega} \sin \xi \frac{\partial \hat{h}^{(0)}}{\partial \epsilon^{1/3} x} \\
 & \quad + v_{\perp} \cos \xi \frac{\partial}{\partial \epsilon^{1/3} x} \left[\frac{ik_y \mu}{\Omega^2} \frac{\partial B_z}{\partial \epsilon x} \frac{\sin 2\xi}{2} \hat{h}^{(0)} \right] \\
 & \quad - \frac{e\Phi i}{m\Omega} \left(\omega \frac{\partial F^{(0)}}{\partial \epsilon} + \frac{k_y}{\Omega} \frac{\partial F^{(0)}}{\partial \epsilon x} \right) G(\xi) + \frac{v_{\perp}}{\Omega} \sin \xi \frac{\partial \hat{h}^{(2/3)}}{\partial \epsilon^{1/3} x} \\
 & \quad - \frac{v_{\perp}^3}{8\Omega^3} \left(\frac{\sin 3\xi}{3} + \sin \xi \right) \frac{\partial^3 \hat{h}^{(0)}}{\partial (\epsilon^{1/3} x)^3} + \frac{\hat{k}^{(1)}}{k^{(1)}} \Bigg] = \Omega \frac{\partial \hat{f}^{(4/3)}}{\partial \xi}. \quad (96)
 \end{aligned}$$

Upon substituting Eq. (96) into Eq. (88) and carrying out the ξ average, we get

$$\begin{aligned}
 & \left(-i\omega + ik_{\parallel} v_{\parallel} + ik_y \frac{\mu}{\Omega} \frac{\partial B_z}{\partial \epsilon x} \right) \left(\hat{h}^{(2/3)} - \frac{v_{\perp}^2}{2\Omega^2} \frac{\partial^2 \hat{h}^{(0)}}{\partial (\epsilon^{1/3} x)^2} \right) \\
 & = - \frac{v_{\perp}^2}{4\Omega^2} \frac{ei}{m} \left(\omega \frac{\partial F^{(0)}}{\partial \epsilon} + \frac{k_y}{\Omega} \frac{\partial F^{(0)}}{\partial \epsilon x} \right) \frac{\partial^2 \phi}{\partial (\epsilon^{1/3} x)^2} \langle \cos 2\xi e^{-iL} \rangle. \quad (97)
 \end{aligned}$$

From Eqs. (92) and (97) we then get

$$\begin{aligned}
 \langle \hat{f}^{(2/3)} \rangle & = \frac{e}{m} \frac{\{ \omega [\partial F^{(0)} / \partial \epsilon] + (k_y / \Omega) [\partial F^{(0)} / \partial \epsilon x] \}}{-\omega + k_{\parallel} v_{\parallel} + \omega_d} \\
 & \quad \times \frac{v_{\perp}^2}{\Omega^2} \frac{\partial^2 \phi}{\partial (\epsilon^{1/3} x)^2} \langle e^{iL} \rangle \langle \sin^2 \xi e^{-iL} \rangle. \quad (98)
 \end{aligned}$$

The charge neutrality condition, correct to order $\epsilon^{2/3}$, follows from Eq. (98) and the Taylor expansion of Eq. (67). Therefore, we find

$$\begin{aligned} & \epsilon^{1/3} (x-x_0) \partial/\partial x \left[-\frac{1}{2} (ne^2/T) + \frac{1}{2} (e^2/T) \int d^3 v F^{(0)} \left[\frac{(\omega - k_{\parallel} v_{\parallel})}{(\omega - k_{\parallel} v_{\parallel} - \omega_d)} \right] \langle e^{iL} \rangle \langle e^{-iL} \rangle \right] \cdot x_0 \\ & \frac{\int d^3 v (v_{\perp}^2/\Omega^2) [(\omega - \omega_{*})/(\omega - k_{\parallel} v_{\parallel} - \omega_d)] F^{(0)} \langle e^{iL} \rangle \langle \sin^2 \xi e^{-iL} \rangle}{\int d^3 v (v_{\perp}^2/\Omega^2) [(\omega - \omega_{*})/(\omega - k_{\parallel} v_{\parallel} - \omega_d)] F^{(0)} \langle e^{iL} \rangle \langle \sin^2 \xi e^{-iL} \rangle} \\ & + \frac{\partial^2 \psi}{\partial (\epsilon^{1/3} x)^2} = 0 \quad , \end{aligned} \quad (99)$$

which is the required Airy equation in the region of the turning point x_0 where $\partial S/\partial x = 0$.

The calculation is completed by matching the asymptotic solutions of the Airy equation [Eq. (99)] to the solution in the main region given by Eqs. (58) and (78). The details of the matching procedure are well-known,¹¹ and result in

$$\int_{\epsilon x_1}^{\epsilon x_2} d(\epsilon x) \frac{\partial S}{\partial \epsilon x} = (p + 1/2)\pi \quad , \quad (p = 0, 1, 2, \dots) \quad , \quad (100)$$

which determines the eigenfrequency ω for the case of two widely spaced turning points. Of course, the Stokes diagram for the problem must be examined in detail to insure the appropriate convergence properties. We also conjecture here, for the ψ structure of the eigenmodes of the toroidal problem, that Eq. (37) is the analog of Eq. (100).

The next aspect of the radial problem examined is the case of closely spaced turning points. In this circumstance the appropriate expansion is in powers of $\epsilon^{1/2}$. The calculation closely follows that given above with the exception of minor conceptual differences. Since the eikonal $S(\epsilon x)$ does not appear, the ϵx dependence of the equilibrium enters through a

local eigenfrequency $\omega^{(0)}(x)$ which must be related to the actual frequency ω in the course of the expansion

The appropriate equations again follow from Eq. (53) and are:

$$v_1 \sin \xi \frac{\partial f^{(0)}}{\partial y} - \frac{e}{m} \frac{\partial \phi}{\partial y} v_1 \sin \xi \frac{\partial F^{(0)}}{\partial E} - \frac{\partial f^{(0)}}{\partial \xi} = 0, \quad (101)$$

$$v_1 \cos \xi \frac{\partial f^{(0)}}{\partial \epsilon^{1/2} x} + v_1 \sin \xi \frac{\partial f^{(1/2)}}{\partial y} - \frac{e}{m} v_1 \cos \xi \frac{\partial \phi}{\partial \epsilon^{1/2} x} \frac{\partial F^{(0)}}{\partial E} - \Omega \frac{\partial f^{(1/2)}}{\partial \xi} = 0, \quad (102)$$

$$\begin{aligned} & -i\omega^{(0)} f^{(0)} + ik_{||} v_{||} f^{(0)} + v_1 \cos \xi \left(\frac{\partial f^{(1/2)}}{\partial \epsilon^{1/2} x} + \frac{\partial f^{(0)}}{\partial \epsilon x} \right) + v_1 \sin \xi \frac{\partial f^{(1)}}{\partial y} \\ & - \mu v_1 \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial f^{(0)}}{\partial \mu} - \frac{e}{m} ik_{||} v_{||} \phi \frac{\partial F^{(0)}}{\partial E} - \frac{e}{m} v_1 \sin \xi \frac{\partial \phi}{\partial y} \\ & \times \left(\frac{\partial F^{(1)}}{\partial E} + \frac{1}{B_z} \frac{\partial F^{(1)}}{\partial \mu} \right) - \frac{e}{m} v_1 \cos \xi \frac{\partial \phi}{\partial \epsilon x} \frac{\partial F^{(0)}}{\partial E} \\ & - \frac{e}{m} \frac{1}{v_1} \frac{\partial F^{(1)}}{\partial \xi} \frac{\partial \phi}{\partial y} \cos \xi - \Omega \frac{\partial f^{(1)}}{\partial \xi} = 0, \end{aligned} \quad (103)$$

$$\begin{aligned} & -i\omega^{(0)} f^{(1/2)} + ik_{||} v_{||} f^{(1/2)} + v_1 \cos \xi \left(\frac{\partial f^{(1)}}{\partial \epsilon^{1/2} x} + \frac{\partial f^{(1/2)}}{\partial \epsilon x} \right) \\ & + v_1 \sin \xi \frac{\partial f^{(3/2)}}{\partial y} - \mu v_1 \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial f^{(1/2)}}{\partial \mu} - \frac{e}{m} v_1 \cos \xi \frac{\partial \phi}{\partial \epsilon^{1/2} x} \\ & \times \left(\frac{\partial F^{(1)}}{\partial E} + \frac{1}{B_z} \frac{\partial F^{(1)}}{\partial \mu} \right) + \frac{e}{m} \frac{1}{v_1} \frac{\partial F^{(1)}}{\partial \xi} \frac{\partial \phi}{\partial \epsilon^{1/2} x} \sin \xi - \Omega \frac{\partial f^{(3/2)}}{\partial \xi} = 0, \end{aligned} \quad (104)$$

and

$$\begin{aligned}
 & -i\omega^{(0)} f^{(1)} - i\omega^{(1)} f^{(0)} + ik_{||} v_{||} f^{(1)} + v_{\perp} \cos \xi \left(\frac{\partial f^{(3/2)}}{\partial \epsilon^{1/2} x} + \frac{\partial f^{(1)}}{\partial \epsilon x} \right) \\
 & + v_{\perp} \sin \xi \frac{\partial f^{(2)}}{\partial y} + \left[v_{\perp}^{(2)} - \frac{1}{2} \left(\frac{B_y}{B_z} \right)^2 v_{\perp} \right] \sin \xi \frac{\partial f^{(0)}}{\partial y} - v_{\perp} \sin \xi \frac{B_y}{B_z} \frac{\partial f^{(0)}}{\partial z} \\
 & - \mu v_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial f^{(1)}}{\partial \mu} - v_{||} v_{\perp}^2 \cos \xi \sin \xi \frac{\partial}{\partial \epsilon x} \left(\frac{B_y}{B_z} \right) \frac{1}{B_z} \frac{\partial f^{(0)}}{\partial \mu} \\
 & - v_{||} \cos^2 \xi \frac{\partial}{\partial \epsilon x} \left(\frac{B_y}{B_z} \right) \frac{\partial f^{(0)}}{\partial \xi} - \frac{e}{m} ik_{||} v_{||} \phi \frac{\partial F^{(1)}}{\partial E} - \frac{e}{m} v_{\perp} \cos \xi \frac{\partial \phi}{\partial \epsilon x} \\
 & \times \left(\frac{\partial F^{(1)}}{\partial E} + \frac{1}{B_z} \frac{\partial F^{(1)}}{\partial \mu} \right) - \frac{e}{m} v_{\perp} \sin \xi \frac{\partial \phi}{\partial y} \left(\frac{\partial F^{(2)}}{\partial E} + \frac{1}{B_z} \frac{\partial F^{(2)}}{\partial \mu} \right) \\
 & - \frac{e}{m} \left[v_{\perp}^{(2)} - \frac{1}{2} \left(\frac{B_y}{B_z} \right)^2 v_{\perp} \right] \sin \xi \frac{\partial \phi}{\partial y} \frac{\partial F^{(0)}}{\partial E} - \frac{e}{m} v_{\perp} \sin \xi \frac{B_y}{B_z} \frac{\partial \phi}{\partial z} \frac{\partial F^{(0)}}{\partial E} \\
 & + \frac{e}{m} \frac{1}{v_{\perp}} \frac{\partial F^{(1)}}{\partial \xi} \frac{\partial \phi}{\partial \epsilon x} \sin \xi - \frac{e}{m} \frac{1}{v_{\perp}} \frac{\partial F^{(2)}}{\partial \xi} \frac{\partial \phi}{\partial y} \cos \xi \\
 & - \Omega \frac{\partial F^{(2)}}{\partial E} - \Omega \frac{1}{2} \left(\frac{B_y}{B_z} \right)^2 \frac{\partial f^{(0)}}{\partial \xi} = 0 \quad . \quad (105)
 \end{aligned}$$

It then follows from Eq. (101) that

$$f^{(0)} = \frac{e}{m} \phi \frac{\partial F^{(0)}}{\partial E} + \hat{h}^{(0)} e^{iL} \quad , \quad (106)$$

while Eq. (102) yields

$$f^{(1/2)} = \left(\frac{v_{\perp}}{\Omega} \sin \xi \frac{\partial \hat{h}^{(0)}}{\partial \epsilon^{1/2} x} + \hat{h}^{(1/2)} \right) e^{iL} \quad . \quad (107)$$

From Eq. (103) we find

$$f^{(1)} = \frac{e}{m} \phi \frac{\partial F^{(1)}}{\partial E} + \hat{g}^{(1)} e^{iL} + \hat{k}^{(1)} e^{iL} \quad , \quad (108)$$

with $\hat{g}^{(1)}$ given by Eq. (68) and $\hat{k}^{(1)}$ given by

$$\hat{k}^{(1)} = -\frac{v_{\perp}^2}{4\Omega^2} \cos 2\xi \frac{\partial^2 \hat{h}^{(0)}}{\partial (\epsilon^{1/2} x)^2} + \frac{v_{\perp}}{\Omega} \sin \xi \frac{\partial \hat{h}^{(1/2)}}{\partial \epsilon^{1/2} x} . \quad (109)$$

This yields the same lowest order dispersion relation, Eq. (67), as found earlier.

Next, Eq. (104) reduces to

$$\begin{aligned} & \left\{ -i\omega^{(0)} + ik_{\parallel} v_{\parallel} + v_{\perp} \cos \xi \frac{\partial}{\partial \epsilon x} - \mu v_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial}{\partial \mu} + v_{\perp} \cos \xi \frac{\partial iL}{\partial \epsilon x} \right. \\ & \quad \left. - \mu v_{\perp} \cos \xi \frac{\partial B_z}{\partial \epsilon x} \frac{1}{B_z} \frac{\partial iL}{\partial \mu} \right\} \left\{ \hat{h}^{(1/2)} + \frac{v_{\perp}}{\Omega} \sin \xi \frac{\partial \hat{h}^{(0)}}{\partial \epsilon^{1/2} x} \right\} \\ & \quad + v_{\perp} \cos \xi \frac{\partial}{\partial \epsilon^{1/2} x} [\hat{g}^{(1)} + \hat{k}^{(1)}] - \Omega \frac{\partial \hat{f}^{(3/2)}}{\partial \xi} = 0 . \end{aligned} \quad (110)$$

Upon averaging over ξ , Eq. (110) yields

$$\hat{h}^{(1/2)} = 0 , \quad (111)$$

and from Eq. (107) we then find

$$\langle \hat{f}^{(1/2)} \rangle = 0 . \quad (112)$$

The charge neutrality condition, which to zeroth order is

$$\begin{aligned} L^0 \Phi &= L^0(\epsilon x, \omega^{(0)}) \Phi \\ &= \left(\int \frac{n e^2}{T} - \int \frac{e^2}{T} \int \frac{E d\mu dE d\xi}{v_{\parallel}} F^{(0)} \frac{\omega^{(0)} - \omega_{*}^T}{\omega^{(0)} - k_{\parallel} v_{\parallel} - \omega_d} \langle e^{iL} \rangle \langle e^{-iL} \rangle \right) \Phi = 0 , \end{aligned} \quad (113)$$

determines the lowest order local eigenfrequency $\omega^{(0)}(x)$. To order $\epsilon^{1/2}$, we pick up the first term in a Taylor series expansion about the point $x = x_0$, where

$$x = x_0 + \frac{z}{\epsilon^{1/2}}, \quad (114)$$

and write

$$z \left[\left(\frac{\partial L^0}{\partial \omega^{(0)}} \right) \frac{\partial \omega^{(0)}}{\partial \epsilon x} + \left(\frac{\partial L^0}{\partial \epsilon x} \right) \right] \bigg|_{\epsilon x_0} \phi = 0, \quad (115)$$

since there is no contribution from $f^{(1/2)}$. Equation (115) is satisfied by requiring

$$\frac{\partial L^0}{\partial \epsilon x} \bigg|_{\epsilon x_0} = 0, \quad \frac{\partial \omega^{(0)}}{\partial \epsilon x} \bigg|_{\epsilon x_0} \equiv \frac{\partial \omega^{(0)}}{\partial \epsilon x_0} = 0, \quad (116)$$

which determines the location x_0 .

Equation (105) then becomes, after substitution for $f^{(0)}$, $f^{(1)}$, $f^{(3/2)}$, $F^{(1)}$, and $F^{(2)}$, multiplication by e^{-iL} , and ξ averaging,

$$\begin{aligned} & \left(-i\omega^{(0)} + ik_{||}v_{||} + ik_y \frac{\mu}{\Omega} \frac{\partial B_z}{\partial \epsilon x} \right) \\ & \times \left[\hat{h}^{(1)} - \frac{e\phi}{m} \frac{i}{\Omega} \left(\omega^{(0)} \frac{\partial F^{(0)}}{\partial E} + \frac{k_y}{\Omega} \frac{\partial F^{(0)}}{\partial \epsilon x} \right) \langle G(\xi) \rangle - \frac{v_{\perp}^2}{2\Omega^2} \frac{\partial^2 \hat{h}^{(0)}}{\partial (\epsilon^{1/2} x)^2} \right] \\ & + \frac{v_{\perp}^2}{2\Omega} \frac{\partial^2 \hat{h}^{(0)}}{\partial (\epsilon^{1/2} x)^2} \frac{e}{m} \frac{i}{\Omega} \left(\omega^{(0)} \frac{\partial F^{(0)}}{\partial E} + \frac{k_y}{\Omega} \frac{\partial F^{(0)}}{\partial \epsilon x} \right) \langle \cos 2\xi e^{-iL} \rangle \\ & - i\omega^{(1)} \left(\frac{e}{m} \phi \frac{\partial F^{(0)}}{\partial E} \langle e^{-iL} \rangle + \hat{h}^{(0)} \right) = 0. \end{aligned} \quad (117)$$

From Eq. (108) we then find

$$\begin{aligned}
 f^{(1)} = & \frac{e}{m} \phi \frac{\partial F^{(1)}}{\partial E} - \frac{e}{m} \frac{i\phi}{\Omega} \left(\omega^{(0)} \frac{\partial F^{(0)}}{\partial E} + \frac{k_y}{\Omega} \frac{\partial F^{(0)}}{\partial \epsilon x} \right) e^{iL} \langle G - \langle G \rangle \rangle \\
 & + ik_y \frac{\mu}{\Omega^2} \frac{\partial B_z}{\partial \epsilon x} \hat{h}^{(0)} \frac{\sin 2\xi}{2} e^{iL} \\
 & + \frac{v_z^2}{2\Omega^2} e^{iL} \frac{\partial^2 \hat{h}^{(0)}}{\partial (\epsilon^{1/2} x)^2} \left(1 - \frac{\cos 2\xi}{2} \right) + \frac{v_z^2}{4\Omega^2} \frac{\partial^2 \phi}{\partial (\epsilon^{1/2} x)^2} \\
 & \times \frac{e}{m} \frac{\{ \omega^{(0)} [\partial F^{(0)} / \partial E] + (k_y / \Omega) (\partial F^{(0)} / \partial \epsilon x) \}}{\omega^{(0)} - k_{||} v_{||} - \omega_d} \langle \cos 2\xi e^{-iL} \rangle e^{iL} \\
 & - \omega^{(1)} \frac{\{ (e/m) \phi [\partial F^{(0)} / \partial E] e^{-iL} + \hat{h}^{(0)} \}}{\omega^{(0)} - k_{||} v_{||} - \omega_d} e^{iL} .
 \end{aligned}$$

and

$$\begin{aligned}
 \langle f^{(1)} \rangle = & \frac{v_z^2}{\Omega^2} \frac{\partial^2 \hat{h}^{(0)}}{\partial (\epsilon^{1/2} x)^2} \langle \sin^2 \xi e^{-iL} \rangle \\
 & - \omega^{(1)} \frac{\{ (e/m) \phi [\partial F^{(0)} / \partial E] \langle e^{iL} \rangle e^{-iL} + \hat{h}^{(0)} \langle e^{iL} \rangle \}}{\omega^{(0)} - k_{||} v_{||} - \omega_d} . \quad (118)
 \end{aligned}$$

The charge neutrality condition then leads to

$$\begin{aligned}
 \left(\frac{\partial^2}{\partial z^2} + \omega^{(1)} \right) & \frac{\partial L^{(0)} / \partial \omega^{(0)}}{\int d^3 v [F^{(0)} (\omega - \omega_{*T}) / (\omega^{(0)} - k_{||} v_{||} - \omega_d)] (v_z^2 / \Omega^2) \langle \sin^2 \xi e^{-iL} \rangle \langle e^{iL} \rangle} \\
 \times \phi & = 0 . \quad (119)
 \end{aligned}$$

Now, the first order frequency correction $\omega^{(1)}$ may be written

$$\begin{aligned}\omega^{(1)} &= \frac{\omega - \omega^{(0)}(\epsilon x)}{\epsilon} \\ &= \frac{\omega - \omega^{(0)}(\epsilon x_0)}{\epsilon} - \frac{1}{\epsilon^{1/2}} z \frac{\partial \omega^{(0)}}{\partial \epsilon x_0} - \frac{z^2}{2} \frac{\partial^2 \omega^{(0)}}{\partial (\epsilon x_0)^2}\end{aligned}\quad (120)$$

and

$$\omega = \omega^{(0)}(\epsilon x) + \epsilon \delta \omega \quad (121)$$

Thus, Eq. (120) becomes

$$\left[\frac{d^2}{dz^2} + \left(\delta \omega - \frac{z^2}{2} \frac{\partial^2 \omega^{(0)}}{\partial (\epsilon x)^2} \right) \frac{\partial L^0 / \partial \omega^{(0)}}{H} \right] \phi = 0 \quad (122)$$

where H is the denominator in Eq. (119).

From the structure of the main results, Eqs. (81), (99), and (122), we note that only for the case involving the behavior in the region between the turning points is it actually required that the calculation be carried to second order in ϵ . For both the $\epsilon^{1/2}$ and $\epsilon^{1/3}$ expansion, it is sufficient to use the usual kinetic equation results.⁷ In the next subsection, we carry this out in toroidal geometry for the case of closely spaced turning points.

B. Toroidal Geometry Calculation for Closely Spaced Turning Points

The usual low frequency expansion⁷ of the Vlasov equation [Eq. (38)] through first order in ϵ yields the results

$$F^{(1)} = \frac{e\phi}{m} \frac{\partial F^{(0)}}{\partial E} + h^{(1)} \quad (123)$$

with the lowest order equations being

$$\underline{\underline{n}} \cdot \nabla^{(0)} h^{(0)} = 0 , \quad (124)$$

and

$$\underline{\underline{n}} \cdot \nabla^{(0)} \phi = 0 , \quad (125)$$

and

$$\underline{\underline{v}}_1 \cdot \nabla^{(0)} h^{(0)} - \Omega \frac{\partial h^{(0)}}{\partial \xi} = 0 . \quad (126)$$

Restricting ourselves for simplicity to the low- β , electrostatic, collisionless limit, the governing first-order equation in toroidal geometry is

$$\begin{aligned} \frac{\partial h^{(0)}}{\partial t} + \underline{\underline{v}}_{||} \underline{\underline{n}} \cdot \nabla^{(1)} h^{(0)} + \underline{\underline{v}}_d \cdot \nabla^{(0)} h^{(0)} + \frac{e}{m} \frac{\partial \langle \phi \rangle}{\partial E} \frac{\partial F^{(0)}}{\partial E} \\ + \frac{c}{B} \underline{\underline{n}} \cdot [\nabla^{(0)} \langle \phi \rangle \times \nabla^{(1)} F^{(0)}] = 0 . \end{aligned} \quad (127)$$

Collisions will be reinstated later. We also specialize to the standard, low- β tokamak geometry with circular, concentric magnetic surfaces and large aspect ratio, r being the minor radius, θ the poloidal angle, ζ the toroidal angle, R the major radius of the magnetic axis, and $q(\epsilon r)$ the usual safety factor. From Eq. (125) we have

$$\phi = \hat{\phi}(\theta, r) \exp [i \ell \zeta - i \ell q(\epsilon r) (\theta - \theta_0) - i \omega t] , \quad (128)$$

and, upon satisfying Eqs. (124) and (126) we have

$$h^{(0)} = \hat{h}(\theta', r') \exp [i l \zeta' - i l q(r') (\theta' - \theta_0) - i \omega t] , \quad (129)$$

where

$$r' = r + \frac{v_{\perp}}{\Omega} \sin \xi , \quad (130)$$

$$\theta' = \theta - \frac{v_{\perp}}{\Omega r} \cos \xi , \quad (131)$$

and

$$\zeta' = \zeta - \frac{v_{\perp} r}{\Omega q R^2} \cos \xi . \quad (132)$$

Equation (127) may then be written as

$$\begin{aligned} -i \omega \hat{h} + \frac{v_{\parallel}}{qR} \frac{\partial \hat{h}}{\partial \theta} - \frac{1}{\Omega R} (\mu B + v^2) \sin \theta \left(\frac{\partial \hat{h}}{\partial r} - i l q' (\theta' - \theta_0) \hat{h} \right) \\ + \frac{1}{\Omega R} (\mu B + v_{\parallel}^2) \cos \theta \cdot \frac{i l q}{r} \hat{h} + i \omega \frac{e}{T} F^{(0)} \langle \hat{\phi} \rangle \\ + \frac{c}{B} \frac{i l q}{r} \langle \hat{\phi} \rangle \frac{\partial F^{(0)}}{\partial \epsilon r} = 0 , \end{aligned} \quad (133)$$

where $q' \equiv dq/dr$. We write this equation in the form

$$\frac{v_{\parallel}}{qR} \frac{\partial \hat{h}}{\partial \theta} - i (\omega - \omega_d) \hat{h} - \frac{1}{\Omega R} (\mu B + v_{\parallel}^2) \sin \theta \frac{\partial \hat{h}}{\partial r} = -i \frac{e}{T} F^{(0)} \langle \hat{\phi} \rangle (\omega - \omega_{*}) . \quad (134)$$

Note that the relation

$$\frac{v_{\parallel}}{r} \frac{\partial}{\partial \theta} \left(\frac{v_{\parallel}}{\Omega} \right) = - \frac{\mu B + v_{\parallel}^2}{\Omega R} \sin \theta \quad (135)$$

allows us to make the "banana-center" transformation, which removes the explicit radial derivative terms,

$$r'' = r' - \frac{qR}{r} \frac{v_{||}}{\Omega}, \quad (136)$$

so that Eq. (134) becomes

$$\frac{v_{||}}{qR} \frac{\partial h}{\partial v_{||}} - i(\omega - \omega_d) \hat{h} = -i \frac{e}{T} F^{(0)}(\omega - \omega_*^T) \langle \hat{\phi} \rangle, \quad (137)$$

with ω_d defined as

$$\omega_d \equiv \frac{(\mu B + v_{||}^2)}{\Omega R} \frac{q}{r} [\cos \theta' + \left(\frac{q'r}{q}\right) (\theta' - \theta_0) \sin \theta']. \quad (138)$$

Proceeding as in Sec. II, the solution of Eq. (137) for trapped and circulating particles is

$$\begin{aligned} [\hat{h}_+(\theta') + \hat{h}_-(\theta')] = & \frac{e}{T} F^{(0)}(\omega - \omega_*^T) qR \left[H_C \int_{-\infty}^{\theta'} d\theta_1' \frac{i \langle \hat{\phi} \rangle}{|v_{||}|} \right. \\ & \times \exp(-i I_{\theta_1'}^{\theta'}) - H_C \int_{\theta'}^{\infty} d\theta_1' \frac{i \langle \hat{\phi} \rangle}{|v_{||}|} \exp(i I_{\theta_1'}^{\theta'}) \\ & + \frac{2H_T}{\sin I_{-\theta_r}^{\theta_r}} \left(\int_{-\theta_r}^{\theta_r} d\theta_1' \frac{\langle \hat{\phi} \rangle}{|v_{||}|} \cos I_{\theta_1'}^{\theta_r} \cos I_{-\theta_r}^{\theta_1'} \right. \\ & \left. \left. + \int_{\theta_r}^{\theta_r} d\theta_1' \frac{\langle \hat{\phi} \rangle}{|v_{||}|} \cos I_{-\theta_r}^{\theta_r} \cos I_{\theta_r}^{\theta_1'} \right) \right]. \quad (139) \end{aligned}$$

Here H_C is a Heaviside step function which is one in the circulating particle part of velocity space and zero in the trapped particle part, and $H_T \equiv 1 - H_C$ is the corresponding

function for trapped particles. Also $\pm\theta_r$ labels the trapped particle reflection points (or orbit turning points), and

$$I_a^b \equiv \int_a^b d\theta'_2 (\omega - \omega_d) qR / |v_{||}| \quad . \quad (140)$$

Collisions have been omitted here but will be included later.

We transform ψ into the primed coordinate system and introduce the $\epsilon^{1/2}$ expansion to give

$$\begin{aligned} \psi = & \int d\theta'_2 \epsilon^{1/2} (r' - \frac{v_{\perp}}{\Omega} \sin \xi) \exp [i l \xi' - i l q (er') (\theta' - \theta_0) - i \omega t] \\ & \times \exp \{-i l \frac{v_{\perp}}{\Omega} [\frac{q(er')}{r} \cos \xi - q'(\theta' - \theta_0) \sin \xi]\} \quad . \end{aligned} \quad (141)$$

Introducing the banana-center transformation and carrying out the θ' average yields

$$\begin{aligned} \langle \hat{\psi} \rangle = & \int d\theta'_2 \epsilon^{1/2} r'' \langle e^{iL} \rangle + \epsilon^{1/2} (\frac{qR}{r''} \frac{v_{||}}{\Omega} \langle e^{iL} \rangle - \frac{v_{\perp}}{\Omega} \langle \sin \xi e^{iL} \rangle) \\ & \times \frac{\partial \hat{\phi}}{\partial (\epsilon^{1/2} r'')} + \epsilon \left[\frac{1}{2} (\frac{qR}{r''} \frac{v_{||}}{\Omega})^2 \langle e^{iL} \rangle - \frac{qR}{r''} \frac{v_{||}}{\Omega} \frac{v_{\perp}}{\Omega} \langle \sin \xi e^{iL} \rangle \right. \\ & \left. + \frac{v_{\perp}^2}{2\Omega^2} \langle \sin^2 \xi e^{iL} \rangle \right] \frac{\partial^2 \hat{\phi}}{\partial (\epsilon^{1/2} r'')^2} \quad . \end{aligned} \quad (142)$$

Defining $f \equiv \hat{\psi} \exp [i l \xi - i l q (\theta - \theta_0) - i \omega t]$, and noting that $\hat{h}$ is given in the transformed system, we simply transform back to obtain

$$\begin{aligned}
 \langle \hat{f} \rangle = & -\frac{e\phi}{T} F(0) + \hat{h}(\theta, \epsilon^{1/2} r) \langle e^{-iL} \rangle \\
 & + \epsilon^{1/2} \left(\frac{v_{\perp}}{\Omega} \langle \sin \xi e^{-iL} \rangle - \frac{qR}{r} \frac{v_{\parallel}}{\Omega} \langle e^{-iL} \rangle \right) \frac{\partial \hat{h}}{\partial (\epsilon^{1/2} r)} \\
 & + \epsilon \left[\frac{1}{2} \left(\frac{qR}{r} \frac{v_{\parallel}}{\Omega} \right)^2 \langle e^{-iL} \rangle - \frac{v_{\perp}}{\Omega} \frac{qR}{r} \frac{v_{\parallel}}{\Omega} \langle \sin \xi e^{-iL} \rangle \right. \\
 & \left. + \frac{v_{\perp}^2}{2\Omega^2} \langle \sin^2 \xi e^{-iL} \rangle \right] \frac{\partial^2 \hat{h}}{\partial (\epsilon^{1/2} r)^2} . \quad (143)
 \end{aligned}$$

Evaluating

$$L = \frac{v_{\perp}}{\Omega} \frac{q}{r} [\cos \xi - (\frac{q'r}{q}) (\theta - \theta_0) \sin \xi] , \quad (144)$$

and $\langle \exp(iL) \rangle$ as in Sec. II, we find

$$\langle e^{iL} \rangle = \langle e^{-iL} \rangle = J_0 \left\{ \frac{v_{\perp}}{\Omega} \frac{q}{r} [1 + (\frac{q'r}{q})^2 (\theta - \theta_0)^2]^{1/2} \right\} . \quad (145)$$

The expressions $\langle \sin \xi \exp(\pm iL) \rangle$ and $\langle \sin^2 \xi \exp(\pm iL) \rangle$ may be obtained from this result by suitable differentiation.

We next construct the charge quasineutrality condition

$$0 = - \int \frac{ne^2}{T} \hat{\phi} + \int e \int d^3v \left\langle \hat{h} \left[\theta, \epsilon^{1/2} \left(r + \frac{v_{\perp}}{\Omega} \sin \xi - \frac{qR}{r} \frac{v_{\parallel}}{\Omega} \right) \right] e^{-iL} \right\rangle . \quad (146)$$

To generate the expansion, we substitute the expanded potential, Eq. (142), into Eq. (139) and then use the expansion in Eq. (143) and subsequently in Eq. (146). To lowest order in $\epsilon^{1/2}$, the result is

$$\begin{aligned}
 0 = L^{(0)}_{\phi}(\theta) = & -\sum \frac{n e^2}{T} \phi^{(0)} + \sum \frac{e^2}{T} \int d^3 v F^{(0)}(\omega^{(0)} - \omega_{\star}) q R J_0(\theta) \\
 & \cdot \left[-H_C \int_{-\infty}^{\infty} d\theta_1 \frac{i \phi^{(0)}(\theta_1) J_0(\theta_1)}{|v_{\parallel}|} \exp(-i I_{\theta_1}^0) - H_C \int_{\theta_1}^{\infty} d\theta_1 \right. \\
 & \left. \frac{i \phi^{(0)}(\theta_1) J_0(\theta_1)}{|v_{\parallel}|} \exp(i I_{\theta_1}^0) + \frac{2 n_T}{\sin I_{-\theta_1}^0} \right. \\
 & \left. \cdot \left(\int_{-\theta_1}^{\theta} d\theta' \frac{\phi^{(0)}(\theta') J_0(\theta')}{|v_{\parallel}|} \cos I_{\theta'}^0 \cos I_{-\theta_1}^0 \right. \right. \\
 & \left. \left. + \int_{\theta}^{\theta_1} d\theta' \frac{\phi^{(0)}(\theta') J_0(\theta')}{|v_{\parallel}|} \cos I_{-\theta'}^0 \cos I_{\theta_1}^0 \right) \right] , \quad (147)
 \end{aligned}$$

where

$$J_0(\theta) = J_0 \left(\frac{v_{\perp}}{u} \frac{q}{r} [1 + (\frac{r}{q})^2 (\theta - \theta_0)^2]^{1/2} \right) , \quad (148)$$

and

$$I_a^b = \int_a^b d\theta \frac{1}{2} [\omega^{(0)} + i\nu - \omega_d] q R / |v_{\parallel}| . \quad (149)$$

We have now included a Krook collision term, with ν being some prescribed function of E and μ . Equation (147) determines the local eigenfrequency $\omega^{(0)}(er, \theta_0)$, as demonstrated earlier in Eq. (113) in the slab case.

To next order in $\epsilon^{1/2}$, the result is

$$L^{(0)}_{\phi} \hat{\phi}^{(1/2)} + L^{(1/2)} \frac{\partial \hat{\phi}^{(0)}}{\partial (e^{1/2} r)} + \omega^{(1/2)} \frac{\partial L^{(0)}}{\partial \omega^{(0)}} \hat{\phi}^{(0)} = 0 , \quad (150)$$

where $L^{(1/2)}$ is the higher order analog of the expression for $L^{(0)}$ given in Eq. (147). We also note here that Eq. (147) can be used to show that

$$\frac{\partial L^{(0)}}{\partial \psi} = i \cdot q \cdot L^{(1/2)} + \frac{\partial L^{(0)}}{\partial \omega} \frac{\partial \omega}{\partial \psi} . \quad (151)$$

Writing

$$\psi^{(0)} = \psi^{(0)}(1/2r) \varphi^{(0)}(\theta, \theta_0) , \quad (152)$$

and substituting Eqs. (151) and (152) in Eq. (150) yields the result

$$\begin{aligned} L^{(0)}(1/2) - \frac{i}{2q} \frac{\partial \eta^{(0)}}{\partial (1/2r)} - \left(\frac{\partial L^{(0)}}{\partial \psi} - \frac{\partial L^{(0)}}{\partial \omega} \frac{\partial \omega}{\partial \psi} \right) \varphi^{(0)} \\ + \omega(1/2) \frac{\partial L^{(0)}}{\partial \omega} \eta^{(0)} \varphi^{(0)} = 0 . \end{aligned} \quad (153)$$

With the definition

$$(A, LB) = \int_{-\infty}^{\infty} d\theta A LB , \quad (154)$$

we have

$$\begin{aligned} \left(\varphi^{(0)}, L^{(0)} \varphi^{(1/2)} \right) = \frac{i}{2q} \frac{\partial \eta^{(0)}}{\partial (1/2r)} \\ \times \left[\left(\varphi^{(0)}, \frac{\partial L^{(0)}}{\partial \psi} \varphi^{(0)} \right) - \left(\varphi^{(0)}, \frac{\partial L^{(0)}}{\partial \omega} \varphi^{(0)} \right) \frac{\partial \omega}{\partial \psi} \right] \\ + \omega(1/2) \left(\varphi^{(0)}, \frac{\partial L^{(0)}}{\partial \omega} \varphi^{(0)} \right) \eta^{(0)} = 0 . \end{aligned} \quad (155)$$

From Eq. (147) we have

$$\frac{\partial L^{(0)}}{\partial \theta_0} \psi^{(0)} + L^{(0)} \frac{\partial \psi^{(0)}}{\partial \theta_0} = 0, \quad (150)$$

which implies

$$\left(\psi^{(0)}, \frac{\partial L^{(0)}}{\partial \theta_0} \psi^{(0)} \right) = 0. \quad (151)$$

Hence, Eq. (155) becomes

$$\begin{aligned} \frac{i}{\hbar} \frac{\partial \eta^{(0)}}{\partial (r^{1/2})} \left(\psi^{(0)}, \frac{\partial L^{(0)}}{\partial \omega^{(0)}} \psi^{(0)} \right) \frac{\partial \omega^{(0)}}{\partial \theta_0} \\ + \omega^{(1/2)} \eta^{(0)} \left(\psi^{(0)}, \frac{\partial L^{(0)}}{\partial \omega^{(0)}} \psi^{(0)} \right) = 0, \end{aligned} \quad (152)$$

from which we conclude

$$\omega^{(1/2)} = 0, \quad (153)$$

and

$$\frac{\partial \omega^{(0)}}{\partial \theta_0} = 0, \quad (160)$$

which determines θ_0 . Returning to Eq. (150), we have

$$L^{(0)} \hat{\psi}^{(1/2)} - \frac{i}{\hbar} \frac{\partial \eta^{(0)}}{\partial (r^{1/2})} \frac{\partial L^{(0)}}{\partial \theta_0} \psi^{(0)} = 0, \quad (161)$$

which becomes, using Eq. (156),

$$L^{(0)} \left(\hat{\phi}^{(1/2)} + \frac{i}{\epsilon q} \frac{\partial \eta^{(0)}}{\partial (\epsilon^{1/2} r)} \frac{\partial \psi^{(0)}}{\partial \theta_0} \right) = 0. \quad (162)$$

Thus,

$$\hat{\phi}^{(1/2)} = -\frac{i}{\epsilon q} \frac{\partial \eta^{(0)}}{\partial (\epsilon^{1/2} r)} \frac{\partial \psi^{(0)}}{\partial \theta_0} + \eta^{(1/2)} (\epsilon^{1/2} r) \psi^{(0)}. \quad (163)$$

Proceeding to next order in $\epsilon^{1/2}$, we find

$$L^{(0)} \hat{\phi}^{(1)} + L^{(1/2)} \frac{\partial \hat{\phi}^{(1/2)}}{\partial (\epsilon^{1/2} r)} + L^{(1)} \frac{\partial^2 \hat{\psi}^{(0)}}{\partial (\epsilon^{1/2} r)^2} + \frac{\partial L^{(0)}}{\partial \omega^{(0)}} \omega^{(1)} \hat{\phi}^{(0)} = 0. \quad (164)$$

From Eq. (147) it can be demonstrated that

$$\frac{\partial^2 L^{(0)}}{\partial \theta_0^2} = -2 \epsilon^2 q^{-2} L^{(1)} + \frac{\partial L^{(0)}}{\partial \omega^{(0)}} \frac{\partial^2 \omega^{(0)}}{\partial \theta_0^2}. \quad (165)$$

Using Eqs. (165), (151), and (163) in Eq. (164) leads to

$$\begin{aligned} \left(\psi^{(0)}, L^{(0)} \hat{\psi}^{(1)} \right) &= \frac{i}{\epsilon q} \eta^{(1/2)} \left(\psi^{(0)}, \frac{\partial L^{(0)}}{\partial \theta_0} \psi^{(0)} \right) \\ &+ \omega^{(1)} \left(\psi^{(0)}, \frac{\partial L^{(0)}}{\partial \omega^{(0)}} \psi^{(0)} \right) \eta^{(0)} - \frac{1}{\epsilon^2 q^{-2}} \frac{\partial^2 \eta^{(0)}}{\partial (\epsilon^{1/2} r)^2} \\ &\times \left[\left(\psi^{(0)}, \frac{\partial L^{(0)}}{\partial \theta_0} \frac{\partial \psi^{(0)}}{\partial \theta_0} \right) + \frac{1}{2} \left(\psi^{(0)}, \frac{\partial^2 L^{(0)}}{\partial \theta_0^2} \psi^{(0)} \right) \right. \\ &\left. - \frac{1}{2} \left(\psi^{(0)}, \frac{\partial L^{(0)}}{\partial \omega^{(0)}} \psi^{(0)} \right) \frac{\partial^2 \omega^{(0)}}{\partial \theta_0^2} \right] = 0. \end{aligned} \quad (166)$$

From Eq. (156) we find

$$\frac{1}{2} \left(\psi^{(0)}, \frac{\partial^2 L^{(0)}}{\partial \theta_0^2} \psi^{(0)} \right) + \left(\psi^{(0)}, \frac{\partial L^{(0)}}{\partial \theta_0} \frac{\partial \psi^{(0)}}{\partial \theta_0} \right) = 0, \quad (167)$$

so that Eq. (156) becomes

$$\frac{d^2 \eta^{(0)}}{d(\epsilon^{1/2} r)^2} + \omega^{(1)} \frac{2k^2 q^2}{(\partial^2 \omega^{(0)} / \partial \theta_0^2)} \eta^{(0)} = 0. \quad (168)$$

Defining

$$r = r_0 + \frac{z}{\epsilon^{1/2}}, \quad (169)$$

where

$$\left. \frac{\partial \omega^{(0)}}{\partial (\epsilon r)} \right|_{r=r_0} \equiv \frac{\partial \omega^{(0)}}{\partial (\epsilon r_0)} = 0 \quad (170)$$

determines r_0 , and using Eqs. (120) and (121), we finally obtain

$$\frac{d^2 \eta^{(0)}}{dz^2} + \left(s\omega - \frac{z^2}{2} \frac{\partial^2 \omega^{(0)}}{\partial (\epsilon r_0)^2} \right) \frac{2k^2 q^2}{(\partial^2 \omega^{(0)} / \partial \theta_0^2)} \eta^{(0)} = 0, \quad (171)$$

which is just the expected Weber equation form.

The Weber equation [Eq. (171)], with the boundary condition that $\eta^{(0)}$ vanishes as $z \rightarrow \pm \infty$, has a countably infinite number of solutions, the lowest of which is just

$$\eta^{(0)} = C \exp(-\alpha z^2/2), \quad (172)$$

where C and α are constants. Inserting Eq. (172) in Eq. (171), we see that

$$\begin{aligned}\omega &= \omega^{(0)}(r r_0) + \epsilon \delta \omega \\ &= \omega^{(0)}(r r_0) + \frac{\epsilon}{2^0 q} \left(\frac{\partial^2 \omega^{(0)}}{\partial t^2} - \frac{\partial^2 \omega^{(0)}}{\partial (r r_0)^2} \right)^{1/2},\end{aligned}\quad (173)$$

and that

$$\epsilon = r q \left(\frac{\partial^2 \omega^{(0)}}{\partial (t r_0)^2} \right)^{1/2} \left(\frac{\partial^2 \omega^{(0)}}{\partial r_0^2} \right)^{-1/2}.\quad (174)$$

These are the results that were obtained in Ref. 8 for ideal MHD modes, except now the eigenvalue is expressed in terms of $\omega^{(0)}$ instead of $[\omega^{(0)}]^2$.

IV. SOLUTION METHOD

In this section the method of solution for the lowest order one-dimensional integral equation along the magnetic field lines on some given flux surface will be presented. Since the purpose here is to compare the results of different mathematical formulations, rather than different physical assumptions, it is necessary at this point to make the same approximations as in Ref. 4, from which the two-dimensional results will be taken. Briefly, these are as follows: (1) the electrostatic approximation, appropriate for $\beta \equiv 8\pi n(T_e + T_i)/B^2 \ll m_e/m_i$, is made so that only ϕ , and not $A_{||}$, needs to be calculated; (2) the equilibrium magnetic surfaces are taken to be concentric and of circular cross-section, so that the simpler r, θ, ζ coordinate system can be used instead of the ψ, χ, ζ coordinate system; (3) the inverse

aspect ratio r/R is taken to be small, where R is the major radius of the magnetic axis and the magnetic field written as $\underline{B} = [B_0 \underline{e}_\theta + B_\theta^0(r) \underline{e}_\theta]/h(\theta)$ with $h(\theta) \equiv 1 + \varepsilon_0 \cos \theta$; and (4) we specialize to the case $\omega > \bar{\omega}_{di} > \bar{\omega}_{ti}$, where $\bar{\omega}_{di} \equiv \omega_{*i}(r_n/R)$ is the typical ion magnetic drift frequency, $\omega_{*i} \equiv ecT_i/(|e|R B_\theta^0 r_n)$, $r_n \equiv -(d \ln n/dr)^{-1}$, $\bar{\omega}_{ti} \equiv v_i/qR_i$ is the typical circulating ion transit frequency, $v_i \equiv (2T_i/m_i)^{1/2}$, and $q(r) \equiv rB_0/(RB_\theta^0)$ is the usual tokamak safety factor. The first three approximations were made in Sec. IIIB.

In the electrostatic limit, the equation to be solved is the lowest order quasineutrality equation, Eq. (147), which can be written as [dropping the zero superscript on $\hat{\phi}^{(0)}$],

$$L^{(0)} \hat{\phi}(\theta) = 0, \quad (175)$$

where $-\infty < \theta < \infty$. There is no periodicity requirement on $\hat{\phi}(\theta)$, and $L^{(0)}$ is a linear, complex, non-Hermitian, one-dimensional integral operator. Equation (175) is solved by a Ritz method, writing

$$\hat{\phi}(\theta) = \sum_{n=0}^{\infty} \hat{\phi}_n h_n(\theta), \quad (176)$$

where the basis functions h_n are

$$h_n(\theta) = H_n(\sigma^{1/2} \theta) \exp(-\sigma \theta^2/2) M_n^{-1/2}, \quad (177)$$

H_n is a Hermite polynomial, and $M_n \equiv (\pi/\sigma)^{1/2} 2^n n!$. Here σ is an adjustable parameter, which will normally be chosen to be real, and can be adjusted to maximize the rate of convergence of the series in Eq. (176). Of course, we require $\sigma > 0$, for convergence. Substituting Eq. (176) into Eq. (175) and operating with

$$\int_{-\infty}^{\infty} d\theta h_n(\theta)$$

yields the desired matrix equation

$$L_{nn}[\omega^{(0)}] \hat{\phi}_n = 0, \quad (178)$$

where

$$L_{nn}[\omega^{(0)}] \equiv \int_{-\infty}^{\infty} d\theta h_n(\theta) L^{(0)} h_n(\theta). \quad (179)$$

This is the procedure used in Ref. 4, with the important difference that here only a single expansion in one dimension is needed, while in Ref. 4 a double expansion was required. The method used for finding the eigenvalues of $\omega^{(0)}$ is the same as that described in detail in Ref. 4. Referring to Eq. (147), we separate $L^{(0)}$ and L_{nn} into terms for the different species involved:

$$L^{(0)} = L^{eC} + L^{eT} + L^i \equiv -\frac{T}{ne^2} (\tilde{n}^{eT} + \tilde{n}^{eC} - \tilde{n}^i) \quad (180)$$

for circulating electrons, trapped electrons and ions, respectively.

For circulating electrons the magnetic drift frequency will be neglected, corresponding to the approximation made in Ref. 4. The ω_2 integrations in Eq. (149) then give

$$L_a^b = (\omega + i\nu) \frac{qR}{v} [\psi(b) - \psi(a)] , \quad (181)$$

where $v_{||}(v) = v_{||} v [1 - \Lambda/h(v)]^{1/2}$, $\Lambda = \mu B_e / R$, $v_{||} = \text{sign}(v_{||})$ and

$$\psi(v) = \int_0^v d\theta [1 - \Lambda/h(\theta)]^{-1/2} . \quad (182)$$

For the circulating electron term in Eq. (147) it is convenient to Fourier transform $\hat{\psi}(v)$ in terms of the $\hat{\theta}$ variable, i.e.,

$$\begin{aligned} \psi(v) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\theta \exp(i\theta v) \psi(\theta) , \\ \hat{\psi}(k) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\hat{\theta} \exp[-ik\hat{\theta}] \hat{\psi}(\theta) \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\theta [1 - \Lambda/h(\theta)]^{-1/2} \exp\{-ik \int_0^\theta d\theta' [1 - \Lambda/h(\theta')]^{-1/2}\} \psi(\theta) . \end{aligned} \quad (183)$$

Using Eqs. (181) and (183), the ω_1 integrations in Eq. (147) may be performed to yield

$$I^{eC} = 1 - \frac{1}{n} \left\{ d^3 v F_{me} (\omega - \omega_{*e}^T) (2\pi)^{-1/2} \right. \\ \left. \times \int_{-\infty}^{\infty} d\kappa \hat{\phi}(\kappa) \exp [i\kappa \hat{\theta}(\theta)] [\omega + iv - \frac{qR_0}{v} \kappa]^{-1} \right\}, \quad (184)$$

where for circulating particles at fixed θ

$$\int_C d^3 v = \frac{1}{2} \left(\frac{2}{m_e} \right)^{3/2} \sum_{q||=\pm 1} \int_0^{\infty} dE E^{1/2} \int_0^{1-\epsilon_0} d\Lambda \{ h(\theta) [1-\Lambda/h(\theta)]^{1/2} \}^{-1},$$

and F_{me} is a Maxwellian distribution. The zero superscript on $\omega^{(0)}$ has been dropped. Also the form for $v(E, \Lambda)$ for circulating electrons is the same as that described in Ref. 4. For electrons, $J_0 \approx 1$.

For trapped electrons only the time-average part $\omega_d^{(0)}$ of the magnetic drift frequency ω_d is retained in Eq. (149), corresponding to the approximation made in Ref. 4. Since $\omega_d^{(0)}$ is independent of θ , I_a^b becomes

$$I_a^b = [\omega + iv - \omega_d^{(0)}] \frac{qR_0}{v} [\hat{\theta}(b) - \hat{\theta}(a)]. \quad (185)$$

For trapped particles $\hat{\phi}(\theta)$ must be decomposed as a Fourier sum instead of a Fourier integral, i.e.,

$$\begin{aligned}\hat{\phi}(\theta) &= \sum_{p=-\infty}^{\infty} \hat{\phi}_p \exp [i\pi p \hat{\theta}(\theta) / \hat{\theta}(\theta_r)] , \\ \hat{\phi}_p &= \frac{1}{2\hat{\theta}(\theta_r)} \int_{-\hat{\theta}(\theta_r)}^{\hat{\theta}(\theta_r)} d\theta \exp [-i\pi p \hat{\theta}(\theta) / \hat{\theta}(\theta_r)] \hat{\phi}(\theta) \\ &= \frac{1}{2\hat{\theta}(\theta_r)} \int_{-\theta_r}^{\theta_r} d\theta [1 - \Lambda/h(\theta)]^{-1/2} \\ &\quad \times \exp \left\{ -i\pi \frac{p}{\hat{\theta}(\theta_r)} \int_0^{\theta} d\theta' [1 - \Lambda/h(\theta')]^{-1/2} \right\} \hat{\phi}(\theta) ,\end{aligned}\quad (186)$$

where $\pm\theta_r = \arccos [(1-\Lambda)/\epsilon_0]$ are the trapped particle reflection or turning points. Using Eqs. (185) and (186), the θ_1 integrations in the trapped electron term in Eq. (147) may be performed to yield

$$\begin{aligned}L^{(T)} &= -\frac{1}{n} \int_T d^3v F_{me}(\omega - \omega_{*e}) \sum_p \hat{\phi}_p \exp [i\pi p \hat{\theta}(\theta) / \hat{\theta}(\theta_r)] \\ &\quad \times \left(\omega + i\nu - \omega_{de}^{(0)} - p \frac{q_{||} v}{qE} \frac{\pi}{\hat{\theta}(\theta_r)} \right)^{-1} .\end{aligned}\quad (187)$$

In Eq. (187), a term has been omitted which contributes only for modes with $\hat{\phi}(\theta)$ odd in θ ; here and in Sec. IV we will deal only with even modes. The forms for ν and $\omega_{de}^{(0)}$ are the same as those given in Ref. 4. For trapped particles at fixed θ

$$\int_T d^3v = \frac{\pi}{2} \left(\frac{2}{m_e} \right)^{3/2} \sum_{q_{||}=\pm 1} \int_0^{\omega} dE E^{1/2} \int_{1-\epsilon_0}^{h(\theta)} d\Lambda \{ h(\theta) [1 - \Lambda/h(\theta)]^{1/2} \}^{-1} .$$

For the circulating ions, a different procedure is necessary in the case considered where $\omega \sim \bar{\omega}_{di} \gg \bar{\omega}_{ti}$. With the approximations mentioned earlier, Eq. (138) for ω_d reduces to

$$\omega_d = \bar{\omega}_{di} (\hat{v}_\perp^2 + 2\hat{v}_\parallel^2) [\cos \theta + \frac{q'r}{q}(\theta - \theta_0) \sin \theta] \quad (188)$$

where $\hat{v}_\perp \equiv v_\perp/v_i$ and $\hat{v}_\parallel \equiv v_\parallel/v_i$. Integrating by parts twice in the circulating ion term in Eq. (147) and noticing that terms odd in $v_\parallel$ vanish on integration over velocity space yields

$$L^i = \frac{T_e}{T_i} - \frac{T_e}{T_i} \int d^3v F_{mi} (\omega - \omega_{*i}) J_0 \left\{ \frac{J_0}{\omega - \omega_d} - \frac{\hat{v}_\parallel^{2-2} \omega_{ti}}{[\omega - \omega_d]^3} \right. \\ \left. \times \left[\frac{\partial^2 J_0}{\partial \theta^2} + 2 \frac{\partial J_0}{\partial \theta} \frac{\partial}{\partial \theta} + J_0 \frac{\partial^2}{\partial \theta^2} - \frac{1}{[\omega - \omega_d]} \frac{\partial \omega_d}{\partial \theta} \left(\frac{\partial J_0}{\partial \theta} + J_0 \frac{\partial}{\partial \theta} \right) \right] \right\}. \quad (189)$$

In Eq. (189), the distinction between trapped and circulating ions is unimportant for $\omega \gg \bar{\omega}_{ti}$, so the integration has been extended over all of velocity space. For the ions

$$\int d^3v = 2\pi \int_{-\infty}^{\infty} dv_\perp v_\perp \int_{-\infty}^{\infty} dv_\parallel.$$

Also in Eq. (189)

$$J_0 = J_0 \{ k_\theta \rho_i \hat{v}_\perp [1 + (\frac{q'r}{q})^2 (\theta - \theta_0)^2]^{1/2} \} \quad (190)$$

where $k_\theta \equiv -eq(r)/r$ and $\rho_i \equiv v_i/\Omega_i$, with the approximations mentioned earlier.

For the sake of completeness, we also include an ion-ion collision term in the same approximate form as used in Ref. 4, i.e.,

$$L^{(i)} = i \frac{b_1}{1-b_1} \left\{ H(b_1-1) \frac{3(\pi+1)}{8^{1/2}} \left[\frac{T_e}{T_i} + \frac{\omega}{\nu} \left(1 - \frac{3\pi+2}{4\pi+4} b_1 \right) \right] \right. \\ \left. + b_1^{1/2} + H(1-b_1) \frac{7}{19} (1+0.2517b_1) \left[\frac{T_e}{T_i} + \frac{\omega}{\nu} \left(1 - \frac{3}{28} (1+5.4366b_1) b_1 \right) \right] b_1^2 \right\}, \quad (19)$$

where $b_1 = (k_B^2 r_i^2 / 2) [1 + (q^2 r / q)^2 (6 - \epsilon_i)^2]$,

$\epsilon_i = (\partial \ln T_i / \partial r) / (\partial \ln n / \partial r)$, and H is the Heaviside step function.

For purposes of comparison with the two-dimensional results of Ref. 4, it is useful to construct a pseudo-two-dimensional eigenfunction from (). For this purpose it is more convenient to employ the ballooning representation in Eqs. (10) and (12) than the ansatz, Eq. (13), used in preceding sections. With the ballooning expansion, θ would be replaced by $(\theta - 2\pi p)$ in the preceding equations. With the current approximations, from Eqs. (10) and (12), the total perturbed electrostatic potential is

$$\phi = \sum_{p=-\infty}^{\infty} \hat{\phi}(\theta - 2\pi p) \exp \{ i \ell [-q(r) \psi + 2\pi p q(r)] - i \omega t \}. \quad (192)$$

If we assume that $\hat{\phi}$ is independent of r , which is the case only if all of the equilibrium gradients are independent of r , then, by solving for $\hat{\phi}(\theta - 2\pi p)$ from Eq. (175), we can construct

a pseudo-two-dimensional eigenfunction. The real two-dimensional eigenfunction, including the variation in the equilibrium gradients, was discussed in Sec. III. In Eq. (192), we expand $q(r)$ around a reference mode-rational surface at r_s where $q(r_s) = m^2/k$ to obtain

$$\begin{aligned} \psi &= \tilde{\psi}(\theta, S) \exp(i\tilde{\omega} - im^2\theta - i\omega t) = \exp(i\tilde{\omega} - im^2\theta - i\omega t) \\ &\times \sum_p \tilde{\psi}_p(\theta - 2\pi p) \exp[-iS(\theta - 2\pi p)] , \end{aligned} \quad (193)$$

where $S \equiv q(r) - m^2/k$. Since the function $\tilde{\psi}(\theta, S)$ is calculated in the case where equilibrium gradients are constant, Eq. (193) gives the appropriate form for comparison with Ref. 4. When the radial variation of the equilibrium gradients is included, the envelope function $\psi^{(0)}(r)$ given, for example, by Eqs. (172) and (174) would multiply the dependence in Eq. (193). An example of this procedure is discussed in the next section.

V. NUMERICAL RESULTS AND COMPARISONS

For comparison purposes, we use the "basic case" parameters of Ref. 4. These are: $\epsilon_1 = 0.1$, $T_e/T_i = 2$, $q = 2.5$, $q'r/q = 1$, $\nu_1 = \nu_0 = 0$, $r_0/r = 1$, $m_i/m_e = 3672$, $\nu_0^* = \nu_0 \exp(3/2 \nu_0) = 0.03$, and $k_{\perp 1} = 0.63$. The one-dimensional code which implements the procedure described in Sec. IV contains terms corresponding to all of the terms in the two-dimensional code described in Ref. 4 except for the circulating ion (transit) resonance term. Accordingly, this term is suppressed for the two-dimensional code results presented in this section.

The one-dimensional code result for ω for the lowest even mode (the one with the least number of nodes) is $\omega/\omega_{ce} = 0.252 + 0.183i$. This was obtained retaining 80 terms in the sum in Eq. (176); this was numerically determined to be more than the number of terms needed for convergence in ω . The ν_0 value used in Eq. (177) was $\nu_0 = 5.62$. The corresponding eigenfunction $\tilde{\psi}(0)$ is shown in Fig. 1 for $-2 \leq \xi \leq 2$. The corresponding two-dimensional eigenvalue, when the equilibrium gradients are taken to be constant, is $\omega/\omega_{ce} = 0.322 + 0.175i$. The number of poloidal basis functions used was nine and the number of radial basis functions was nine, with a ν_0 value of $\nu_0 = 0.70 + 0.15i$. The real parts of ω differ by 27% and the imaginary parts by 5%. This constitutes reasonable agreement in the present context.

The eigenfunctions $\tilde{\psi}(\theta, S)$, defined in Eq. (193), are shown on the reference mode-rational surface at $S = 0$ in Fig. 2 for the one-dimensional code and in Fig. 3 for the two-dimensional

code. These figures show the variation of the perturbed electrostatic potential along the field line (where the magnetic field minima are at $\theta = 0$ and 2π), with the usual ballooning towards the outside of the torus. The "fine structure" in the one-dimensional eigenfunction is not reproduced by the two-dimensional code even when the number of poloidal basis functions is increased from 9 to 15, and is therefore presumably an artifact of the one-dimensional numerical methods.

The absolute magnitude $|\hat{\psi}(\cdot, S)|$ of the eigenfunction is shown in Fig. 4 for the one-dimensional code, and in Fig. 5 for the two-dimensional code, for $0 \leq S \leq \psi_q(r) - \psi^0 \leq 5$. The drop-off in $|\hat{\psi}|$ as S increases in Fig. 5 occurs because of the truncation in poloidal harmonics after the ones corresponding to the $S = +4$ ($m = m^0 + 4$) mode-rational surfaces. With an infinite number of poloidal harmonics, there would be no dropoff in $|\hat{\psi}|$ in the absence of variation in the equilibrium gradients. This is what is seen in Fig. 4 in the one-dimensional code results.

We next consider the question of the radial structure of these eigenmodes, including the radial variation of the equilibrium gradients. The appropriate toroidal calculation generalizing the lowest order one-dimensional calculation was discussed in Sec. IIIB for the special case of closely spaced turning points. The result requires the calculation of $\omega^{(n)}(\cdot, r, \theta_0)$ at several different r and θ_0 values in order to locate the extrema of $\omega^{(n)}$ in r and θ_0 , and to calculate the second derivatives of $\omega^{(n)}$ with respect to r and θ_0 at those

extrema required in Eqs. (173) and (174) for the eigenfrequency and radial envelope.

These results obtained with the one-dimensional "ballooning formalism" will be compared to two-dimensional results obtained with an extension¹² to the case of nonconstant equilibrium gradients of the calculation in Ref. 4, which assumes constant equilibrium gradients. The difference from the implementation described in Ref. 4 is that Eq. (31) of Ref. 4 is replaced by

$$\tilde{\psi}(\theta, S) = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \tilde{\phi}_{jn} g_j(\theta) h_n(S+j), \quad (194)$$

where $\tilde{\phi}(\theta, S)$ is defined in Eq. (193), h_n is defined in Eq. (177), and $g_j(\theta) = \exp(ij\theta)(2\pi)^{-1/2}$. Also $S+j = (\ell q - m')$ and $S+j = \ell q - m$, with the poloidal mode number $m = m' - j$. Thus for each poloidal harmonic labeled by either m or j , the corresponding radial basis functions h_n are centered on the corresponding $(q = m/\ell)$ mode-rational surface, instead of all being centered on the reference $(q = m^0/\ell)$ mode-rational surface as in Ref. 4. This change allows the extended two-dimensional code including equilibrium gradient variation to cover a range in r that is limited only by the number of poloidal harmonics that can be handled computationally. Unfortunately, this number is not very large (approximately 19). This then requires a rather artificial set of equilibrium profiles to be used if the radial width of the eigenfunction $\tilde{\psi}$ is to saturate, as poloidal harmonics are added, before this maximum

number is reached. For experimentally interesting values of the toroidal mode number, say $\ell \sim 10^2$, this requires that the gradients be localized around some radius r_0 , and that the gradients go to zero not too far from r_0 . This requirement also would tend to contribute to the validity of the assumption made in Sec. IIIB of closely spaced turning points.

The equilibrium profiles used here are

$$n(r) = \frac{n_0 - n_a}{2} [1 - \tanh(\frac{r - r_0}{\Delta_n})] + n_a, \quad (195)$$

$$T_j(r) = \frac{T_{0j} - T_{aj}}{2} [1 - \tanh(\frac{r - r_0}{\Delta_{Ti}})] + T_{aj}, \quad (196)$$

$$q(r) = q_0 + (q_a - q_0) (\frac{r}{a})^2. \quad (197)$$

The parameters used are: $\ell = 50$, $m^0 = 126$, limiter radius $a = 40$ cm, $R = 200$ cm, $q_0 = 1.25$, $q_a = 6.25$, $n_0 = 1.64 \cdot 10^{14}$ cm $^{-3}$, $T_{0e} = 2$ keV, $T_{0i} = 1$ keV, $\Delta_n/a = \Delta_{Te}/a = \Delta_{Ti}/a = 0.0556$, $n_a/n_0 = T_{ae}/T_{0e} = T_{ai}/T_{0i} = 0.8$, $m_i/m_e = 3672$, and $B_0 = 42.93$ kG. For these parameters, at $r_s/a = r_0/a = 0.5$, the local parameters are: $\epsilon_0 \equiv r_s/R = 0.1$, $q = 2.5$, $q'r/q = 1$, $\eta_i = \eta_e = 1$, $r_n/r = 1$, $T_e/T_i = 2$, $v_e^* \equiv v_{ei}/(\epsilon_0^{3/2} \omega_{te}) = 1$, and $k_{0i} = 0.63$ (or $b_{i0} \equiv k_{0i}^2/2 = 0.2$).

Using these profiles, the one-dimensional code was used to compute $\omega^{(n)}(r, \theta_0)$ for several different values of r around r_0 , with $\theta_0 = 0$. It was found that both ω_r and ω_θ , where $\omega^{(0)} = \omega_r + i\omega_\theta$, have maxima in r at r_0 . Also $\omega_\theta = 0$ turns out to be an extremum in ω_r for both ω_r and ω_θ , as is expected for the type of up-down symmetric equilibrium considered here. The results obtained are: $\omega^{(n)}(r_0, 0)/\omega_{*e}(r_0) = 0.192 + 0.084i$, with

$$\frac{1}{(r/a)^2} \left[\frac{\omega^{(n)}(r, 0)}{\omega_{*e}(r)} \right]_{r=r_0} = -167 - 185i, \quad (198)$$

and

$$\frac{1}{(r/a)^2} \left[\frac{\omega^{(n)}(r_0, \theta_0)}{\omega_{*e}(r_0)} \right]_{\theta_0=0} = 0.142 - 0.130i. \quad (199)$$

Then, using Eq. (173), $\omega/\omega_{*e}(r_0) = 0.194 + 0.070i$, so that the $1/\epsilon$ correction term is stabilizing, as in the ideal MHD case.⁸ Also, using Eq. (174), $\epsilon = 0.143 - 0.142i$ (with ϵ set to unity, since it is only a formal ordering parameter). The absolute magnitude of the corresponding eigenfunction, $|\hat{\phi}(\theta, S)|$, including the slow radial dependence given by $n^{(0)}$ in Eq. (172), is shown in Fig. 6. Here the solution for $\hat{\phi}(\theta)$ obtained at r_0 was used for all r .

The corresponding calculation has been done using the extended two-dimensional code including equilibrium gradient variation described earlier in this section. As the number of poloidal harmonics was increased, it was found that ϵ and the width of the

eigenfunction both saturated, with $\omega/\omega_{*e}(r_0) = 0.1865 + 0.077i$. Thus the real frequencies obtained with the two different computational approaches differ by 4%, and the growth rates by 10%. This again constitutes acceptable agreement.

The absolute magnitude $|\tilde{\phi}(\theta, S)|$ of the corresponding eigenfunction for the case with 17 poloidal harmonics is shown in Fig. 7. The full width at half maximum of $|\tilde{\phi}|$ in S at $\theta = 0$ is 6.2. In terms of r , this would correspond to a width of $6.2 \Delta r_s$, where $\Delta r_s \equiv 1/q'$ is the spacing of mode-rational surfaces. The corresponding width from the "ballooning formalism" one-dimensional calculation is 6.3, indicating good agreement.

Comparing Figs. 6 and 7, the maximum in S of $|\tilde{\phi}(\theta, S)|$ for the two-dimensional code is shifted to the right by about one mode-rational surface spacing, whereas the one-dimensional code result for $|\tilde{\phi}(\theta, S)|$ is required to have the maximum at $r = r_0$, or $S = 0$. This difference could be the result of the neglect of the symmetry-breaking third-derivative term in Eq. (120), and thus in Eq. (171). In any case, the difference is not large. Also there is some fine structure in Fig. 7 absent from Fig. 6 due to the different computational methods and to the truncation in poloidal harmonics in the two-dimensional calculation.

In conclusion, it can be seen from the foregoing comparisons that the one-dimensional, "ballooning formalism" calculations can adequately reproduce the results of the earlier two-dimensional calculations. The one-dimensional calculations can also be computationally more efficient in many situations.

VI. CONCLUSIONS

In this paper a general procedure has been presented for the application of the ballooning mode formalism (originally developed for ideal MHD studies) to the kinetic theory of low-frequency microinstabilities.¹³⁻¹⁶ A complete treatment of the lowest order one-dimensional problem including trapped-particle effects has been presented. In addition, the higher order radial structure problem in toroidal geometry has also been treated within the framework of this formalism for the case of closely spaced turning points. Numerical results from these calculations are found to be in reasonable agreement with corresponding results for eigenvalues and eigenfunctions from the more conventional two-dimensional calculations.^{4,12} As discussed in Sec. III, it is proposed that a WKB type procedure can in general be applied to determine the radial structure of the toroidal eigenmodes (i.e., even in the case of well-separated turning points). In the present paper, this procedure is illustrated in Sec. IIIA by considering a simplified sheared-slab geometry. The form of the resultant equations governing the radial structure tends to suggest that this basic approach should likewise be applicable to the toroidal problem. A more complete implementation of this analysis extending to the case of widely spaced turning points in toroidal geometry is currently under development.

On the basis of the results obtained in the present investigation, it can be concluded that the ballooning mode formalism leads to an efficient scheme for analyzing an important class of low-frequency microinstabilities in toroidal geometry. A systematic treatment of the influence of electromagnetic (finite- β) effects and of more complicated toroidal equilibria on these instabilities will likely necessitate the further application of this approach. The starting point for such a calculation is presented in Sec. II, where the derivation of the perturbed distribution function and the corresponding set of coupled integral eigenmode equations is given.

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REFERENCES

- ¹D. Dobrott, D. B. Nelson, J. M. Greene, A. H. Glasser, M. S. Chance, and E. A. Frieman, Phys. Rev. Lett. 39, 943 (1977).
- ²J. W. Connor, R. J. Hastie, and J. B. Taylor, Phys. Rev. Lett. 40, 396 (1978).
- ³A. H. Glasser, in Proceedings of the Finite-Beta Theory Workshop (Varenna, Italy, September 1977), ed. by B. Coppi and W. Sadowski (National Technical Information Service, Springfield, Virginia, 1979) p. 55.
- ⁴G. Rewoldt, W. M. Tang, and E. A. Frieman, Phys. Fluids 21, 1513 (1978).
- ⁵G. Rewoldt, W. M. Tang, and E. A. Frieman, Bull. Am. Phys. Soc. 23, 785 (1978).
- ⁶See, for instance, C. Mercier and H. Luc, Lectures in Plasma Physics: The Magnetohydrodynamic Approach to the Problem of Plasma Confinement in Closed Magnetic Configurations (Commission of the European Communities, Luxembourg, 1974).
- ⁷See, for instance, P. J. Catto, and K. T. Tsang, Phys. Fluids 20, 396 (1977); P. J. Catto, Plasma Phys. 20, 719 (1978).
- ⁸M. S. Chance, R. L. Dewar, E. A. Frieman, A. H. Glasser, J. M. Greene, R. C. Grimm, S. C. Jardin, J. L. Johnson, J. Manickam, M. Okabayashi, and A. M. M. Todd, in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 7th Int. Conf., Innsbruck, 1978) (IAEA, Vienna, 1979) I, 677.
- ⁹R. Dewar, private communication.

¹⁰G. B. Whitham, J. Fluid Mech. 22, 273 (1965).

¹¹J. Heading, An Introduction to Phase-Integral Methods (Methuen, London, 1962).

¹²L. Chen, C. Z. Cheng, E. A. Frieman, P. N. Guzdar, P. K. Kaw, W. W. Lee, W. M. Nevins, C. R. Oberman, H. Okuda, G. Rewolde, P. H. Rutherford, W. M. Tang, and R. B. White, in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 7th Int. Conf., Innsbruck, 1978) (IAEA, Vienna, 1979) I, 763.

¹³R. J. Hastie, K. W. Hesketh, and J. B. Taylor, Report CLM-P575, Culham Laboratory (1979).

¹⁴M. Bornatici, F. Englemann, A. Nocentini, F. Pegoraro, T. Schep, and T. Timors, in Plasma Physics and Controlled Nuclear Fusion Research (Proc. 7th Int. Conf., Innsbruck, 1978) (IAEA, Vienna, 1979) I, 519.

¹⁵Y. C. Lee and J. W. Van Dam, Report PPG-337, University of California at Los Angeles (1978); Y. C. Lee and J. W. Van Dam, in Proceedings of the Finite-Beta Theory Workshop (Varenna, Italy, September 1977) ed. by B. Coppi and W. Sadowski (National Technical Information Service, Springfield, Virginia, 1979) p. 93.

¹⁶D. Choi, W. Horton, and R. Estes, Report FRCR No. 184, University of Texas (1978).

Fig. 1. Real and imaginary parts of the eigenfunction $\hat{\phi}(\theta)$ from the one-dimensional code for the parameters given in the text. Here $\theta = 0, \pm 2\pi$ corresponds to the outside of the torus and $\theta = \pm\pi$ to the inside.

Fig. 2. Real and imaginary parts of $\tilde{\phi}(\theta, S=0)$ from the one-dimensional code. $S = 0$ corresponds to the reference mode-rational surface.

792027

Fig. 3. Real and imaginary parts of $\tilde{\phi}(\theta, S=0)$ from the two-dimensional code described in Ref. 4.

792029

Code

1. The function $f(s)$ is a constant function, $f(s) = 1$, for $s \in [0, 5]$. The area under the curve is shaded.

2-D Code

FIG. 5. Absolute magnitude of the eigenfunction for the two-dimensional code, for constant equilibrium gradient.

7-2028

Fig. 6. Absolute magnitude (6.13) of the eigenfunction from the one-dimensional code for nonequilibrium gradients, for the equilibrium profile and parameters given in the text.

792104
Fig. 7. Absolute magnitude $|\tilde{\phi}(\theta, S)|$ of the eigenfunction from the two-dimensional code for nonconstant equilibrium gradients.

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