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ELASTIC-PLASTIC CONSTITUTIVE MODELING
OF CONCRETE

by

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ABSTRACT

The need to understand concrete behavior under high temperatures in the nuclear industry has become rather acute. For this purpose, a constitutive model of concrete especially developed for this severe environment is indispensable. This report reviews the presently available constitutive models of concrete at standard-temperature conditions and considers their advantages and drawbacks. A rather simple but effective approach is selected to treat concrete behavior at high temperatures. Special emphasis is devoted to the modeling of concrete up to and including failure. The derived constitutive model is checked with biaxial and tri-axial benchmark experimental results. Very good agreement is obtained.

1. Introduction

The necessity of understanding the behavior of concrete structures in nuclear industry is increasing because of safety issues regarding the possible exposure of concrete to severe environments. Scenarios in which structural concrete could experience very high temperatures are being considered for both water reactors and fast reactors [1].

Because of this need, Marchertas [2] developed an explicit finite element code called DYNAPCON. Subsequently, Marchertas and Kulak [3] developed a computer code called TEMP-STRESS, which can be used for temperature calculation as well as for structural calculation. They analyzed the SANDIA crucible test [4] using this code and obtained reasonable cracking patterns.

However, only a rather simple constitutive model of concrete was available in these codes. This model assumed linear elastic stress-strain behavior, with different tensile and compression limits. A more accurate and sophisticated constitutive model is desired for the study of concrete under severe environments.

In fact, many models have already been proposed in the literature for the description of concrete behavior. The objective of this study is to seek the most applicable model for use in these codes. Considering the simplicity of the analysis, only the elastic-plastic type models are examined here. The results of four different models are compared in this report with biaxial experimental data, and the variations are discussed.

2. Survey of Constitutive Equations of Concrete

2.1 Basic Characteristics of Concrete

A typical uniaxial compression-tension behavior of concrete is shown in Fig. 1. As seen in the figure, concrete has different strength limits in tension and compression and also exhibits nonlinearity even under very low stress and strain values. The figure also shows that two different portions exist in the overall response, which are the strain-hardening and the strain-softening parts. An accurate analysis should take account of the above properties.

In some methods, the strain-hardening part is treated as nonlinear elasticity [5-8], while the others apply an elastic-plastic theory [9-13]. Nonlinear elastic models are relatively simple but they cannot represent strain path-dependent phenomena, especially the unloading behavior. On the other hand, the elastic-plastic models, which were

Fig. 1. Uniaxial Behavior of Concrete.

originally developed for metals, need to be changed into a form more suitable for concrete behavior.

The highest stress point is usually called the 'failure' point, although this does not always mean complete fracture in the case of a displacement-control test. There are many studies concerning the failure condition in the multiaxial stress state and quite a few expressions are proposed for it.

The strain-softening part can be treated in many different ways. In some investigations [10,11], it is considered as the effect of cracking or crushing, and constitutive equations for these phenomena are derived. Still in others, the strain-softening phenomenon is regarded as 'microcracking' and is considered as the continuum deformation by means of the plastic-fracturing theory [14]. The endochronic theory, the most recent approach, was also applied to treat concrete behavior [15].

It should be noted that because this study is the first step for the refinement of the constitutive equation in the code, we limited our concern within the strain-hardening part and tried to find out the most suitable model for our numerical calculation in that part.

2.2 Failure Criteria

Extensive experimental research was carried out with regard to the failure criterion and many analytical expressions were proposed by many investigators to represent the results. Although there are many rather simple models, e.g., maximum principal stress criterion or maximum shear stress criterion, they do not, in general, give a good prediction of concrete failure. On the other hand, as a result of numerous comparisons with test data it became clear that three stress invariants have influence on the failure. Therefore, almost all recent failure criteria proposed for concrete involve the three stress invariants in a rather similar way as follows,

$$f(I_1, J_2, J_3) = 0, \quad (1)$$

where

$$I_1 = \sigma_{ii},$$

$$J_2 = \frac{1}{2} S_{ij} S_{ij} ,$$

$$J_3 = \frac{1}{3} S_{ij} S_{jk} S_{ki} ,$$

and

$$S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} ,$$

where σ_{ij} and S_{ij} are the components of stress and deviatoric stress, respectively.

In many cases, the less complicated expressions are special cases of the more complicated expression. Thus, one should choose the method according to the accuracy and simplicity one desires in one's calculation.

Among these expressions, the four-parameter model by Hsieh, Ting and Chen [11], and the five-parameter model by William and Warnke [16] seem to be most up-to-date and also are the most general. Both of these models can be reduced to other simpler models (Drucker-Prager, von Mises, etc.).

2.3 Constitutive Equations Prior to Failure

In order to utilize uniaxial experimental data in multiaxial numerical analysis, appropriate constitutive relations must be developed. Many investigators are engaged in this work in an effort to find out a more realistic constitutive model.

Although there are a few other types of approaches, we can roughly divide almost all the proposed models into two categories; namely, (1) non-linear elastic models and (2) elastic-plastic models.

Investigators of the former category include Cedolin, Crutzen and Dei Poli [5], Ottosen [6], Darwin [7], and Gerstle [8]. Cedolin, et al., and Gerstle formed unified relationships between normal stress-normal strain and between deviatoric stress-deviatoric strain to construct constitutive equations based on these relationships. Gerstle compared these predictions with several triaxial experimental results and obtained good agreement. However, their equations can not generally represent path-dependent behaviors unless

other corrections are made. Ottosen and Derwin developed different theories based on experimental fitting.

Among models based on the classical elastic-plastic theory are those of Chen and Chen [9], Chen and Suzuki [10], and Hsieh, Ting and Chen [11]. They assumed a loading function which was of the same form as their failure surfaces. Furthermore, they used the associated flow rule. Argyris, et al. [12], and Willam and Warnke [16] also developed constitutive equations using their failure models. Buyukozturk and Tassoulas [13] developed a model in the compression region, using the Drucker-Prager type loading function and the nonassociated flow rule.

Bazant and co-workers [14,15,17] have also developed several kinds of constitutive models, namely, a plastic-fracturing theory, endochronic theory, and total strain theory. All of them can be applied to the strain-softening region as well as the hardening region. Their models involve many functions depending on the material uniaxial compressive strength. These functions are obtained by curve-fitting the data from many experiments.

3. Description of the Models Used in This Study

In this study, four different models are used. They are:

- (i) Hsieh-Ting-Chen four-parameter model [11]
- (ii) Willam-Warnke five-parameter model [16]
- (iii) von Mises model with a failure surface
- (iv) Two-parameter model with a failure surface

In models (i)-(iii), a single loading function is used, while two loading functions are used at the same time in model (iv). In the subsequent sections, the description of each model is given according to the order shown above. Because we need a rather different procedure for model (iv), its description is given in a different section.

3.1 Elastic-Plastic Equation Using Single Loading Function

3.1.1 General Description

The classical elastic-plastic theory consists of an initial yielding function, a subsequent loading function, and a plastic potential function. The subsequent loading function is determined according to a 'hardening rule'. In numerical calculation, the isotropic or the kinematic hardening rule is usually used. Recently, Azelesson and Samuelsson [18] proposed a mixed hardening rule which can represent the character of both rules, and Hsieh, Ting and Chen [11] used it in their constitutive model. In this study, we used the same mixed hardening law.

A loading function is a function of stress and hardening parameters:

$$f = f(\bar{\sigma}_{ij}, \tau) , \quad (2)$$

$$\bar{\sigma}_{ij} = \sigma_{ij} - \alpha_{ij} , \quad (3)$$

where σ_{ij} is the stress tensor, α_{ij} and τ are hardening parameters which change with the plastic deformation: α_{ij} represents the rigid movement of the loading surface (kinematic hardening part) and τ is a measure of the size of the loading surface (isotropic hardening part). The loading surface is used for the purpose of distinguishing plastic loading from elastic response as follows:

if $f = 0$ and $\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} > 0$, plastic loading occurs,

if $f < 0$ or $\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0$, elastic response occurs.

In the case of plastic loading, the following 'consistency condition' must be satisfied:

$$df = \frac{\partial f}{\partial \bar{\sigma}_{ij}} d\bar{\sigma}_{ij} + \frac{\partial f}{\partial \tau} d\tau = 0 . \quad (4)$$

In order to account for temperature-dependent plastic behavior, we must include this effect into the hardening parameters. For the sake of simplicity, it is assumed that the loading surface changes only its size, but does not move as the temperature changes. In other words, only τ changes as temperature varies and α_{ij} remains the same. Thus $d\tau$ can be expressed as

$$d\tau = \frac{\partial \tau}{\partial \bar{\epsilon}^p} d\bar{\epsilon}^p + \frac{\partial \tau}{\partial T} dT, \quad (5)$$

where $\bar{\epsilon}^p$ is an equivalent plastic strain and T is the temperature. Substituting Eqs. (3), (5) and the expression

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial \alpha_{ij}}, \quad (6)$$

into Eq. (4), we obtain

$$\frac{\partial f}{\partial \sigma_{ij}} (d\sigma_{ij} - d\alpha_{ij}) + \frac{\partial f}{\partial \tau} \left(\frac{\partial \tau}{\partial \bar{\epsilon}^p} d\bar{\epsilon}^p + \frac{\partial \tau}{\partial T} dT \right) = 0. \quad (7)$$

Moreover, the following equations are assumed for the hardening parameters.

$$d\alpha_{ij} = \frac{H}{\tau} (1 - M) d\bar{\epsilon}^p \bar{\sigma}_{ij}, \quad (8)$$

$$\frac{\partial \tau}{\partial \bar{\epsilon}^p} = MH, \quad (9)$$

$$d\bar{\epsilon}^p = \sqrt{d\epsilon_{ij}^p d\epsilon_{ij}^p}, \quad (10)$$

where H , M , ϵ_{ij}^p are the hardening rate, the mixed hardening coefficient, and the plastic strain, respectively.

The generalized Hooke's law, accounting for the change of the elastic moduli, is given as

$$d\sigma_{ij} = dC_{ijkl} \epsilon_{kl}^e + C_{ijkl} d\epsilon_{kl}^e, \quad (11a)$$

$$d\epsilon_{ij}^e = d\epsilon_{ij} - d\epsilon_{ij}^p - d\epsilon_{ij}^{th}, \quad (11b)$$

where C_{ijkl} , ϵ_{ij}^e , ϵ_{ij} , ϵ_{ij}^{th} represent the elastic stress-strain tensor, the elastic strain, the total strain, and the thermal strain, respectively.

Plastic strain increment is given as follows, using the plastic potential function, g :

$$d\epsilon_{ij}^p = d\lambda \frac{\partial g}{\partial \sigma_{ij}}, \quad (12)$$

where $d\lambda$ is a scalar coefficient to be determined. Substituting Eqs. (8)-(12) into (7), we can obtain

$$\begin{aligned} & \frac{\partial f}{\partial \sigma_{ij}} \left[dC_{ijkl} \epsilon_{kl}^e + C_{ijkl} \left(d\epsilon_{kl} - d\lambda \frac{\partial g}{\partial \sigma_{kl}} - d\epsilon_{kl}^{th} \right) \right. \\ & \left. - (1 - M) \frac{H}{\tau} d\lambda \frac{\partial g}{\partial \sigma_{mn}} \frac{\partial g}{\partial \sigma_{mn}} \bar{\sigma}_{ij} \right] + \frac{\partial f}{\partial \tau} \left(MH d\lambda \sqrt{\frac{\partial g}{\partial \sigma_{mn}} \frac{\partial g}{\partial \sigma_{mn}}} + \frac{\partial \tau}{\partial T} dT \right) = 0 \quad (13) \end{aligned}$$

Solving for $d\lambda$, we get

$$d\lambda = \frac{\frac{\partial f}{\partial \sigma_{ij}} \left[C_{ijkl} (d\epsilon_{kl} - d\epsilon_{kl}^{th}) + dC_{ijkl} \epsilon_{kl}^e \right] + \frac{\partial f}{\partial \tau} \frac{\partial \tau}{\partial T} dT}{\frac{\partial f}{\partial \sigma_{ij}} C_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} + \sqrt{\frac{\partial g}{\partial \sigma_{mn}} \frac{\partial g}{\partial \sigma_{mn}}} \left[\frac{H}{\tau} (1 - M) \bar{\sigma}_{ij} \frac{\partial f}{\partial \sigma_{ij}} - \frac{\partial f}{\partial \tau} MH \right]} \quad (14)$$

Once we have the value of $d\lambda$, we can calculate other quantities according to the above-mentioned equations. By back substitution of $d\lambda$ into Eqs. (12), (11b) and (11a), we calculate $d\sigma_{ij}$, which is what we seek.

These expressions are quite general, and we can account for the change of the elastic moduli. In this study, the elastic moduli are assumed to change only with temperature, T ; namely,

$$dC_{ijkl} = \frac{dC_{ijkl}}{dT} dT. \quad (15)$$

However, if it is desired and enough information is given, the degradation of the elastic moduli due to plastic deformation or microcracking can also be accounted for.

Now that we have the basic equations, we next choose a suitable loading function and a plastic potential function, and substitute their derivatives into Eq. (14). In many models, especially for metals, the identical function is used for the loading function and for the plastic potential function (the associated flow rule). But in certain models for concrete or soil, different formulations are used for the two functions (the nonassociated flow rule).

In this study, the associated flow rule is primarily used, but one nonassociated flow model is also adopted to control the volume change.

3.1.2 Loading Functions Used in This Study

(1) Four-Parameter Loading Function

Hsieh, Ting and Chen [11] proposed an elastic-plastic constitutive equation in which their four-parameter failure function is utilized as the loading function. Their original loading function was given as follows:

$$f = AJ_2/\tau + B\sqrt{J_2} + C\alpha_1 + DI_1 - \tau, \quad (16)$$

where α_1 represents the maximum principal stress and can be calculated as

$$\alpha_1 = I_1/3 + 2\sqrt{J_2/3} \cos \theta, \quad \theta = \frac{1}{3} \cos^{-1} \left(\frac{3}{2} \sqrt{\frac{J_3}{J_2^3}} \right). \quad (17)$$

Constants A, B, C, and D are parameters to be determined from simple test results. In this study, Eq. (16) is solved for τ , and the following loading function is used instead:

$$f = B\sqrt{J_2} + C\alpha_1 + DI_1 + \sqrt{(B\sqrt{J_2} + C\alpha_1 + DI_1)^2 + 4AJ_2} - 2\tau. \quad (18)$$

The reason for this change is purely one of convenience in the numerical calculation, as described later. The derivatives can be obtained according to the following formula:

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial f}{\partial J_2} \frac{\partial J_2}{\partial \sigma_{ij}} + \frac{\partial f}{\partial J_3} \frac{\partial J_3}{\partial \sigma_{ij}} . \quad (19)$$

Expressions for each derivative term are given in the Appendix.

As will be explained later, Eq. (16) or (18) appears to give too much volume expansion when f is used as the plastic potential as well as the loading function. For this reason, we employ the following plastic potential function to eliminate volume expansion:

$$g = B \sqrt{J_2} + C \alpha'_1 + \sqrt{(B \sqrt{J_2} + C \alpha'_1)^2 + 4A J_2} , \quad (20)$$

where $\alpha'_1 = \alpha_1 - I_1/3 + 2 \sqrt{J_2/3} \cos \theta$. This potential is derived by subtracting the hydrostatic part from Eq. (18) so that no volume change occurs.

(2) Five-Parameter Loading Function

William and Warnke [16] proposed a failure criterion involving five independent parameters as an extension of their previous three-parameter model. This criterion is used as a loading function in the following expression:

$$\begin{aligned} f &= \sqrt{5} \tau_m - r , \\ r &= \frac{2r_c (r_c^2 - r_t^2) \cos \theta + r_c (2r_t - r_c) \sqrt{4 (r_c^2 - r_t^2) \cos^2 \theta + 5r_t^2 - 4r_t r_c}}{4 (r_c^2 - r_t^2) \cos^2 \theta + (r_c - 2r_t)^2} , \\ r_t &= \sqrt{5} \tau \left[a_0 + a_1 \left(\frac{\sigma_m}{\tau} \right) + a_2 \left(\frac{\sigma_m}{\tau} \right)^2 \right] , \\ r_c &= \sqrt{5} \tau \left[b_0 + b_1 \left(\frac{\sigma_m}{\tau} \right) + b_2 \left(\frac{\sigma_m}{\tau} \right)^2 \right] , \end{aligned} \quad (21)$$

where τ_m , σ_m are stress invariants defined as

$$\tau_m = \sqrt{\frac{2}{5} J_2} , \quad (22)$$

$$\sigma_m = \frac{1}{3} I_1 , \quad (23)$$

while a_0 , a_1 , a_2 , b_0 , b_1 and b_2 are material constants which determine the shape of loading surface and are obtained from five individual failure data points.

The corresponding expressions for derivatives $\partial f / \partial \sigma_{ij}$, $\partial f / \partial \tau$ are given in the Appendix. It should be noted that Chen [19] derived similar formulas as those given in the Appendix; however, his expression pertaining to $\partial \theta / \partial \sigma_{ij}$ appears to be erroneous.

(3) von Mises Type Loading Function with Four- or Five-Parameter Failure Criteria

The von Mises type loading function may be the most common function used in metal plasticity. It involves only one stress invariant, J_2 , and in this study, the following form is used

$$f = \sqrt{3J_2} - \tau . \quad (24)$$

It is easy to obtain the differential value of this function. Thus,

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{1}{2\sqrt{3J_2}} S_{ij} , \quad (25)$$

$$\frac{\partial f}{\partial \tau} = -1 . \quad (26)$$

If we use only von Mises function, the same behavior is implied in both tensile and compressive regions, which is not true at all for concrete. Therefore, we must use the function in conjunction with another independent failure criterion which would cut-off elastic-plastic deformation

and produce different tensile and compressive responses. The concept of this approach is schematically shown in the biaxial principal stress space in Fig. 2.

Many investigators in the past have used the same function for a failure condition and a loading function for analytical convenience or because sufficient data were lacking to do otherwise. But this assumption is quite inadequate for concrete. Even in the biaxial case, we see that as the failure stress decreases according to the change of stress ratios, the deformation becomes more brittle; in other words, the nonlinear part of strain prior to failure decreases relatively. From this consideration, it becomes clear that independent formulations should be used for the two functions. This idea seems to be reasonable also from the physical viewpoint: the plastic deformation and failure, which is mainly due to tensile cracking, are rather independent phenomena from each other.

Fig. 2. Schematic Representation of the von Mises Loading Function with the Failure Surface.

3.2 Elastic-Plastic Constitutive Equation with Two Loading Functions

In the metallic materials, it is well-known that the volume change due to nonlinear deformation is so small that we can use the von Mises type function as a fairly good approximation. But, in materials such as concrete and soil, inelastic volume change is not so small in general. In a biaxial compression test of concrete, plastic compaction takes place first, followed by dilatation which is seen at stress level very close to the failure limit. The latter effect is considered as the result of microcracking and seems to be rather difficult to express by means of the elastic-plastic theory.

From the above consideration, it would be desirable to include plastic volume change, as well as shear deformation, for more refined analysis. For this purpose, Buyukozturk and Tessoulas [13], as well as Bazant

and Kim [14], used a nonassociated flow rule with Drucker-Prager type pressure-dependent loading function. However, their models require a loading function, a plastic potential, and, in general, a rather complicated numerical curve-fitting procedure.

In this study, an elastic-plastic model using two loading functions is examined. Here, both the plastic deviatoric strain and the volume change are controlled by different loading functions which involve shear stress intensity (J_2) and hydrostatic pressure (I_1), respectively. Mathematically, this can be expressed as,

$$d\epsilon_{ij}^p = \delta_{ij} d\epsilon_m^p + de_{ij}^p \quad (27)$$

$$de_{ij}^p = d\lambda_1 \left(\frac{\partial f_1}{\partial \sigma_{ij}} \right), \quad f_1 = \sqrt{3J_2} - \tau_1(\bar{e}^p, T), \quad (28)$$

$$d\epsilon_m^p = d\lambda_2 \left(\frac{\partial f_2}{\partial \sigma_{ij}} \right), \quad f_2 = -I_1/3 - \tau_2(\epsilon_m^p, T). \quad (29)$$

where $\epsilon_m^p = \frac{1}{3} \epsilon_{ii}^p$, e_{ij}^p represents plastic mean strain and deviatoric strain, respectively. The differentials $d\lambda_1$ and $d\lambda_2$ are coefficients to be determined from this analysis, $\bar{e}^p$ is defined as $\bar{e}^p = \int \sqrt{de_{ij}^p de_{ij}^p}$, τ_1 and τ_2 are the respective hardening parameters.

We must have two consistency conditions for each loading function instead of the one given in Eq. (4). These are:

$$df_1 = \frac{\partial f_1}{\partial \sigma_{ij}} (d\sigma_{ij} - d\alpha_{ij}^{(1)}) + \frac{\partial f_1}{\partial \tau_1} d\tau_1 = 0, \quad (30)$$

$$df_2 = \frac{\partial f_2}{\partial \sigma_{ij}} (d\sigma_{ij} - d\alpha_{ij}^{(2)}) + \frac{\partial f_2}{\partial \tau_2} d\tau_2 = 0. \quad (31)$$

Moreover, similar to Eq. (5), $d\tau_1$, $d\tau_2$ can be represented as follows:

$$d\tau_1 = \frac{\partial \tau_1}{\partial \bar{\epsilon}^p} d\bar{\epsilon}^p + \frac{\partial \tau_1}{\partial T} dT, \quad (32)$$

$$d\tau_2 = \frac{\partial \tau_2}{\partial \epsilon_m^p} d\epsilon_m^p + \frac{\partial \tau_2}{\partial T} dT. \quad (33)$$

Similarly, the hardening parameters can be represented as

$$d\alpha_{ij}^{(1)} = \frac{H_1}{\tau_1} (1 - M) d\bar{\epsilon}^p (\sigma_{ij} - \alpha_{ij}^{(1)}), \quad (34)$$

$$d\alpha_{ij}^{(2)} = \frac{H_2}{\tau_2} (1 - M) d\epsilon_m^p (\sigma_{ij} - \alpha_{ij}^{(2)}), \quad (35)$$

$$\frac{\partial \tau_1}{\partial \bar{\epsilon}^p} = MH_1, \quad d\bar{\epsilon}^p = d\lambda_1 \sqrt{\frac{\partial f_1}{\partial \sigma_{ij}} \frac{\partial f_1}{\partial \sigma_{ij}}}, \quad (36)$$

$$\frac{\partial \tau_2}{\partial \epsilon_m^p} = MH_2, \quad d\epsilon_m^p = \frac{1}{3} d\lambda_2 \frac{\partial f_2}{\partial \sigma_{ij}} \delta_{ij}, \quad (37)$$

where $H_1 = d(\sqrt{3}J_2)/d\bar{\epsilon}^p$ and $H_2 = -d\sigma_m/d\epsilon_m^p$ are hardening rates corresponding to f_1 and f_2 , respectively. The generalized Hooke's law in this case is

$$d\sigma_{ij} = dC_{ijkl} \epsilon_{kl}^e + C_{ijkl} \left(d\epsilon_{kl} - d\lambda_1 \frac{\partial f_1}{\partial \sigma_{kl}} - d\lambda_2 \frac{\partial f_2}{\partial \sigma_{kl}} - d\epsilon_{kl}^{th} \right). \quad (38)$$

Substituting Eqs. (32) to (38) into Eqs. (30) and (31), we obtain two simultaneous equations involving the unknowns $d\lambda_1$ and $d\lambda_2$:

$$\begin{aligned}
df_1 = & \frac{\partial f_1}{\partial \sigma_{ij}} \left[dC_{ijk1} \epsilon_{k1}^e + C_{ijk1} \left(d\epsilon_{k1} - d\lambda_1 \frac{\partial f_1}{\partial \sigma_{k1}} - d\lambda_2 \frac{\partial f_2}{\partial \sigma_{k1}} - d\epsilon_{k1}^{th} \right) \right. \\
& \left. - \frac{H_1}{\tau_1} (1 - M) d\lambda_1 \sqrt{\frac{\partial f_1}{\partial \sigma_{ij}} \frac{\partial f_1}{\partial \sigma_{ij}}} (\sigma_{ij} - \alpha_{ij}^{(1)}) \right] \\
& + \frac{\partial f_1}{\partial \tau_1} \left(MH_1 d\lambda_1 \sqrt{\frac{\partial f_1}{\partial \sigma_{ij}} \frac{\partial f_1}{\partial \sigma_{ij}}} + \frac{\partial \tau_1}{\partial T} dT \right) = 0 ,
\end{aligned} \tag{39}$$

$$\begin{aligned}
df_2 = & \frac{\partial f_2}{\partial \sigma_{ij}} \left[dC_{ijk1} \epsilon_{k1}^e + C_{ijk1} \left(d\epsilon_{k1} - d\lambda_1 \frac{\partial f_1}{\partial \sigma_{k1}} - d\lambda_2 \frac{\partial f_2}{\partial \sigma_{k1}} - d\epsilon_{k1}^{th} \right) \right. \\
& \left. - \frac{H_2}{3\tau_2} (1 - M) d\lambda_2 \frac{\partial f_2}{\partial \sigma_{ij}} (\sigma_{ij} - \alpha_{ij}^{(2)}) \right] \\
& + \frac{\partial f_2}{\partial \tau_2} \left(\frac{1}{3} MH_2 d\lambda_2 \frac{\partial f_2}{\partial \sigma_{ij}} + \frac{\partial \tau_2}{\partial T} dT \right) = 0 .
\end{aligned} \tag{40}$$

After some manipulation, we obtain the formulas for $d\lambda_1$ and $d\lambda_2$ in the following form:

$$d\lambda_1 = \frac{c_1 (a_2 - b_2) - a_{12} c_2}{(a_2 - b_2) (a_1 - b_1) - a_{12}} , \tag{41}$$

$$d\lambda_2 = \frac{c_2 (a_1 - b_1) - a_{12} c_1}{(a_2 - b_2) (a_1 - b_1) - a_{12}} , \tag{42}$$

where,

$$a_1 = \frac{\partial f_1}{\partial \sigma_{ij}} C_{ijk1} \frac{\partial f_1}{\partial \sigma_{k1}} ,$$

$$a_2 = \frac{\partial f_2}{\partial \sigma_{ij}} C_{ijk1} \frac{\partial f_2}{\partial \sigma_{k1}} ,$$

$$a_{12} = \frac{\partial f_1}{\partial \sigma_{ij}} C_{ijk1} \frac{\partial f_2}{\partial \sigma_{k1}} ,$$

$$b_1 = \left[\frac{\partial f_1}{\partial \tau_1} M H_1 + \frac{\partial f_1}{\partial \sigma_{ij}} (1 - M) \frac{H_1}{\tau_1} (\sigma_{ij} - \alpha_{ij}^{(1)}) \right] \sqrt{\frac{\partial f_1}{\partial \sigma_{k1}} \frac{\partial f_1}{\partial \sigma_{k1}}} ,$$

$$b_2 = \left[\frac{\partial f_2}{\partial \tau_2} M H_2 + \frac{\partial f_2}{\partial \sigma_{ij}} (1 - M) \frac{H_2}{\tau_2} (\sigma_{ij} - \alpha_{ij}^{(2)}) \right] \frac{1}{3} \frac{\partial f_2}{\partial \sigma_{k1}} \delta_{k1} ,$$

$$c_1 = \frac{\partial f_1}{\partial \sigma_{ij}} \left[C_{ijk1} (d\epsilon_{k1} - d\epsilon_{k1}^{th}) + dC_{ijk1} \epsilon_{k1}^e \right] + \frac{\partial f_1}{\partial \tau_1} \frac{\partial \tau_1}{\partial T} dT ,$$

$$c_2 = \frac{\partial f_2}{\partial \sigma_{ij}} \left[C_{ijk1} (d\epsilon_{k1} - d\epsilon_{k1}^{th}) + dC_{ijk1} \epsilon_{k1}^e \right] + \frac{\partial f_2}{\partial \tau_2} \frac{\partial \tau_2}{\partial T} dT .$$

When only one of the two loading functions is active, the above expressions simplify considerably. In other words,

$$\text{if } f_1 + \frac{\partial f_1}{\partial \sigma_{ij}} d\sigma_{ij} > 0 \quad \text{and} \quad f_2 + \frac{\partial f_2}{\partial \sigma_{ij}} d\sigma_{ij} < 0 ,$$

$$\text{then } d\lambda_1 = \frac{c_1}{a_1 - b_1} \quad \text{and} \quad d\lambda_2 = 0 ; \quad (43)$$

$$\text{if } f_1 + \frac{\partial f_1}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \quad \text{and} \quad f_2 + \frac{\partial f_2}{\partial \sigma_{ij}} d\sigma_{ij} > 0 ,$$

$$\text{then } d\lambda_1 = 0 \quad \text{and} \quad d\lambda_2 = \frac{c_2}{a_2 - b_2} . \quad (44)$$

The derivatives in those expressions can be obtained easily from Eqs. (28) and (29) as follows:

$$\frac{\partial f_1}{\partial \sigma_{ij}} = \frac{1}{2\sqrt{3}J_2} S_{ij} , \quad (45)$$

$$\frac{\partial f_1}{\partial \tau_1} = -1 , \quad (46)$$

$$\frac{\partial f_2}{\partial \sigma_{ij}} = -\frac{1}{3} \delta_{ij} \sigma_{ij} , \quad (47)$$

$$\frac{\partial f_2}{\partial \tau_2} = -1 . \quad (48)$$

Finally, the stress increment can be calculated by back substitution of Eqs. (41) to (44) into Eq. (38), using Eqs. (45) and (47).

The general forms of these expressions are rather complicated. However, if one can neglect the mixed hardening effect and/or the temperature variation, the forms become much simpler.

It is true that we can define two equivalent strains in other ways. But the plots of numerous experimental data show unique relationships

between $\sqrt{3}J_2$ and $\bar{\epsilon}^P$ or between I_1 and ϵ_m^P , except at the high pressure range. It may seem more natural to use $\bar{\epsilon}^P = \int \sqrt{d\epsilon_{ij}^P d\epsilon_{ij}^P}$ instead of $\bar{\epsilon}^P$. However, it is expected that the expression may become even more complicated because this would involve also the mean part of strain. These are the reasons why we choose the equivalent strains of these types.

Experimental data exhibit the coupling effect between shear and normal parts in the high compressive stress region. Our formulation can not simulate this effect. In order to do this, we would probably need to use the nonassociated flow rule. Nevertheless, in many practical cases we do not usually experience such high pressure, and within this restriction, our model is considered to be effective.

It should be noted that, even in this model, other failure criteria must be used as an envelope or a cut-off of elastic-plastic deformation. In this study, we utilize both the four-parameter and the five-parameter failure criteria as such an envelope.

4. Computer Implementation

In this section, the computer implementation of the above-mentioned models is described. Because a different approach is required in the single loading function method and in the two loading function method, the description of each method is given separately in the following two sections.

4.1 Implementation of the Single Loading Function Method

The computer code used in this study is the explicit dynamic finite element code TEMP-STRESS. In the case of implicit nonlinear analysis, we need to iterate in order to obtain an equilibrium state in which a given stress-strain relationship is also satisfied. In contrast, an explicit method does not require such an iteration, but only requires the change from strain to stress according to the given relationship between these two quantities.

The flow of calculation in the single loading function method is given in Fig. 3. In this figure, superscripts n and $n+1$ represent the time steps. The symbol r is used for the subdivision of strain increment into elastic and elastic-plastic parts as shown schematically in Fig. 4. This

Fig. 3. Flow of Calculation: Single Loading Function Method.

Fig. 4. Relation Between f and Stress Intensity.

figure shows the relation between the strain increment and the magnitude of the yielding function. The relation, in general, is not linear even in the elastic range because the yielding function may involve a stress value of arbitrary form. However, when the yielding function is linear with respect to stress, r represents the ratio of the elastic portion exactly.

In this study, the loading functions given by Eq. (18) and Eq. (24) are examples of this case. This is the reason for changing the form of the four-parameter loading function from Eq. (16) to Eq. (18).

In other cases, the relationship is no longer linear, and r does not give the correct ratio of the elastic portion. The loading function of Eq. (21) of this study is of this case. The correct ratio may be obtained by iterations; however, this is not done in this study. It is assumed that the errors are not significant because small time steps are usually used in the explicit method.

In a former type (linear type) of loading function, it is also easy to return the stress value to the loading surface after the stress value is updated [20]. This operation is carried out as follows. A loading function is assumed to be

$$f = f'(\sigma_{ij}) - k\tau, \quad (49)$$

where f' is a part of f depending on σ_{ij} and k is a constant. First, a correction ratio m , is obtained as follows:

$$m = \frac{f'(\sigma_{ij})}{k\tau} . \quad (50)$$

Then, the stress components are divided by m to satisfy the loading function; namely,

$$\sigma'_{ij} = \frac{\sigma_{ij}}{m} , \quad (51)$$

where σ'_{ij} represents the stress value after the correction. In our calculation, this 'returning' operation is used for models (i) and (iii), but it can not be used for the five-parameter loading function because of its nonlinear functional form.

4.2 Implementation of the Two Loading Function Method

The flow of calculation in the two loading function method is illustrated in Fig. 5. In this calculation, a more approximate method is used. The reason for this is that, in this case, reproducing the same sophisticated operation of the single loading function method would result in a considerably complicated procedure. Nevertheless, as long as the strain increment is sufficiently small, the result is considered to be accurate enough. The accuracy can be checked by plotting the τ_1 versus $\bar{\epsilon}^p$ and τ_2 versus ϵ_m^p curves obtained from numerical calculations and comparing them with the input curves.

5. Results of Computations

In this section, the results of computations based on the above-mentioned models are described. At first, the biaxial results are compared with the experimental data of Kupfer, et al. [21]. Secondly, the triaxial analytical results are compared with the experimental data of Linse and Aschl [22].

$$\Delta \sigma_{ij} = C_{ijk1} \Delta \epsilon_{k1}$$

$$\sigma_{ij}^{n+1} = \sigma_{ij}^n + \Delta \sigma_{ij}$$

$$\bar{\sigma}_{ij(1)}^{n+1} = \sigma_{ij}^{n+1} - \alpha_{ij(1)}^n$$

$$\bar{\sigma}_{ij(2)}^{n+1} = \sigma_{ij}^{n+1} - \alpha_{ij(2)}^n$$

Fig. 5. Flow of Calculation: Dual Loading Function Method.

5.1 Biaxial Study

5.1.1 Finite Element Model and Condition of Calculation

In the biaxial study, one plane stress element with one Gaussian integration point is used. The prescribed boundary condition and the applied external load are shown in Fig. 6.

Fig. 6. Finite Element Model Used in Biaxial Study.

Fig. 7. Plot of τ and $\bar{\epsilon}^P$ Relation Used in Models (i) and (ii).

5.1.2 Material Property Data

According to the experimental data of Kupfer, et al., material property data are determined as follows:

- (a) In models (i) and (iii), the elastic constants and a plot of τ and $\bar{\epsilon}^P$ relation, shown in Fig. 7, which are obtained from uniaxial compression test, are used. The following values are used for the constants in Eqs. (18) and (21):

$A = 2.01$	$B = 0.972$	$C = 9.14$	$D = 0.231$
$a_0 = 0.0535$	$a_1 = -0.508$	$a_2 = -0.0389$	
$b_0 = 0.0921$	$b_1 = -0.867$	$b_2 = -0.1425$	

It should be noted that the values of A, B, C and D are the same as those given in the original paper of the four-parameter model [11] and are consistent with the test results done by Kupfer et al, [21]. The values of a_0 , a_1 , a_2 , b_0 , b_1 and b_2 , which are used in the five-parameter model, are made consistent with the constants of the four-parameter model.

(b) In model (iii), the following constants are used:

$$\text{Young's Modulus } E = 325,000 \text{ kgf/cm}^2$$

$$\text{Poisson's Ratio } \nu = 0.2$$

In Fig. 8, a plot of τ and $\bar{\epsilon}^p$ ($\tau = \sqrt{3J_2}$) relations obtained from six experimental data are shown with the multilinear approximation used in the calculation. It is clear that the agreement of relations of all data is fairly good. Accordingly, the von Mises model is expected to give good predictions in all cases.

(c) In model (iv), the same elastic constants as in model (iii) and averaged $\sqrt{3J_2}$ versus $\bar{e}^p$ and σ_m versus $\bar{\epsilon}_m^p$ ($\sigma_m = I_1/3$, $\bar{\epsilon}_m^p = \bar{\epsilon}/3$) relations are used. In Fig. 9, the plot of $\sqrt{3J_2}$ and $\bar{e}^p$ relations from six experiments are shown together with the input data. The

Fig. 8. $\sqrt{3J_2}$ - $\bar{\epsilon}^p$ Relation.

Fig. 9. $\sqrt{3J_2}$ - $\bar{e}^p$ Relation.

Fig. 10. $I_1 - \epsilon_v^p$ Relation.

Fig. 11. Results of Uniaxial Compression.

agreement in this figure is as good as that of $\sqrt{3J_2}$ and $\bar{\epsilon}^p$. The plot of I_1 versus ϵ_v^p shown in Fig. 10 exhibits a little larger scattering than that of the previous case. The multilinear relations shown in the figures are used as input data.

5.1.3 Results of Calculations

Calculations are performed for five cases (i to v) with all four models. Figures 11 to 15 show the results for each case, compared with the experimental data. Figure 11 shows the result of the uniaxial compression condition. When the same loading and failure function is used with the Hsieh-Ting-Chen model and the Willam-Warnke model, they give stiffer response in compressive strain and softer response in tensile strain as compared with the experimental data. The other two models proposed in this study give better agreement with the experimental results. There appears to be a relatively small difference between the two models.

Figures 12 and 13 show the results of the biaxial compression cases. The same trend as before can be seen more clearly. In Fig. 13, better agreement with test data is achieved by the two-parameter model than by the von Mises model. Figure 14 shows the result for a compression-tension case. The deviation of tensile strain in models (i) and (ii) from experimental values is very large, while the other two models give fairly good agreement

Fig. 12. Results of Biaxial Compression.

Fig. 13. Results of Biaxial Loading in Compression.

Fig. 14. Results of Biaxial Loading in Compression and Tension.

Fig. 15. Results of Uniaxial Tension.

with the test data. Figure 15 shows the result of uniaxial tension. Similarly unacceptable deviation of the former two models can be seen here. On the other hand, the latter two models give good agreement with the test results, although a small nonlinearity which appeared in the experiment can not be represented by analytical models.

Also shown in Figs. 12 and 14 are the results obtained using the nonassociated flow rule described in the preceeding section. The use of the nonassociated flow rule improves the agreement with the test data to some extent, but shows deviation especially in the compression-tension stress condition as shown in Fig. 14.

5.2 Triaxial Study

In order to assess the applicability of mode? (iv) to triaxial stress condition, a comparison with experimental results of Linse and Aschl [22] is made for three different cases.

5.2.1 Finite Element Model and Condition of Calculation

For illustration purposes, an axisymmetric element with one Gaussian integration point is used. Because of the consistency between the internal force and the inertial force, the external force shown in Fig. 16 is applied; the boundary conditions are also indicated in this figure.

5.2.2 Material Property Data

According to the paper of Gerstle, et al. [23] Young's modulus and Poisson's ratio are assumed as follows:

Fig. 16. Finite Element Model Used in Triaxial Study.

$$E = 2000 \text{ N/mm}^2$$

$$\nu = 0.24$$

From the experimental data of Linse and Aschl [22], a plot of $\sqrt{3J_2}$ versus $\bar{\epsilon}^p$ and I_1 versus ϵ_v^p are obtained and displayed in Figs. 17 and 18, respectively. The scattering in these figures is somewhat larger than what appeared in the biaxial case. One reason for this wide scattering may be the statistical nature of the experimental data. The multilinear expressions indicated in Figs. 17 and 18 are used as input for the calculation.

Fig. 17. $\sqrt{3J_2} - \bar{\epsilon}^p$ Relation.

Fig. 18. $I_1 - \epsilon_v^p$ Relation.

5.2.3 Results of Calculation

The results of the calculation are shown in Figs. 19, 20 and 21 for different loading condition. The experimental data are also shown in these figures for comparison.

Figure 19 shows the results of case I ($\sigma_1:\sigma_2:\sigma_3 = 1:0.2:0.2$). Very good agreement of calculated and experimental results can be observed.

In Fig. 20, two sets of calculational results are shown together with the experimental data for case II ($\sigma_1:\sigma_2:\sigma_3 = 1:1:0.15$). One of

Fig. 19. Results of Triaxial Calculation (I).

Fig. 20. Results of Triaxial Calculation (II).

these analytical results pertain to model (iii), in which the τ versus $-\frac{p}{e}$ relation obtained from the same experimental data is used as input. The particularly good agreement of this model can be attributed to the use of the correct input. For this model, input was derived from the average distribution of the experimental results.

Figure 21 shows the results of case III (load history data). Fairly good agreement is obtained here between analytical and experimental values.

6. Discussion

In the biaxial calculations, it was found that the four- and five-parameter models using loading surfaces similar to the respective failure surfaces give large discrepancies especially in tension-

Fig. 21. Results of Triaxial Calculation (III)

tension and tension-compression regions. The same results would have been expected with any other model that uses the same shape for the loading and failure surface. Thus, the main observation of this investigation is that the loading surface should not be of the same shape as the failure surface.

In addition, the above models give poor results in the biaxial compression region, as well as the uniaxial compression state. In those cases, the tendency that tensile strain is too large and compressive strain is too small can be seen from the figures. The reason for this is clear. Both of these loading functions have a slope in the $\sqrt{3}J_2$ and I_1 plane [11, 19]. Therefore, the plastic volume expansion is inevitable as long as the associated flow rule is used. The use of the nonassociated flow rule improves the results to some extent, but not enough. This conclusion would also apply to any model which would include the hydrostatic pressure in the loading functions.

On the contrary, the von Mises model and the two-parameter model give fairly good agreement with experimental data. In both models, either the four- or the five-parameter failure models must be used as a cut-off of plastic deformation.

Some investigators have employed the von Mises loading function in the past. For example, Arnesen, et al. [24] used it as a loading function, as well as a failure function in the biaxial compression region. The simple tension cut-off criterion was used in tension-compression and tension-tension regions, and it resulted in a discontinuity at the uniaxial compression state. Moreover, the use of the von Mises function as a failure criterion can not account for the dependency of failure strength on hydrostatic pressure. Thus, the same strength results for the uniaxial and for the equally loaded biaxial compression tests.

The two-parameter model can control the plastic volume change as a function of mean stress. This can be considered as a refined model of the von Mises model in which no plastic volume change occurs. This model is very similar to the nonlinear elastic approaches using octahedral relationship [5,8], but these approaches do not account for path-dependent phenomena such as unloading. On the other hand, the two-parameter model can describe the unloading behavior, as shown in Fig. 22.

Fig. 22. Unloading Behavior in Uniaxial Compression.

The two-parameter model gives good agreement even in the triaxial stress condition. The agreement in some cases is as good as those of the nonlinear elastic model of Gerstle [8]. In some other cases, the present method gives better result than Gerstle's method.

One disadvantage of this method is that it can not represent the dilatancy effect which is observed just before the failure limit. It is expected that this disadvantage can be reduced by using the failure function as a loading function only just prior to the failure point.

The other disadvantage of this approach is that it can not take account of the so-called coupling effect. If need be, this discrepancy may be overcome by the use of the nonassociated flow rule.

7. Conclusions

For the description of stress-strain relation of the strain hardening part of concrete, four models are compared. As a result, the following points become clear.

- (1) If the four- or five-parameter failure function is used also as a loading function, biaxial deformation behavior can not be predicted with sufficient accuracy.
- (2) If von Mises loading function is used with one of the above failure functions, reasonable results can be obtained in biaxial tests in spite of the great simplicity of this model.

- (3) The two parameter method, using two loading functions at the same time, is found to be effective in the triaxial stress condition, as well as in the biaxial stress condition, except for the condition of very high hydrostatic pressure.

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APPENDIX

Loading Functions and Their Derivatives of the 4- and 5-Parameter Criteria

I. Loading Function and its Derivatives of the 4-Parameter Criterion

$$f = B\sqrt{J_2} + C\sigma_1 + DI_1 + \sqrt{(B\sqrt{J_2} + C\sigma_1 + DI_1)^2 + 4AJ_2} - 2\tau$$

$$\sigma_1 = I_1/3 + 2\sqrt{J_2/3} \cos \theta$$

$$\theta = \frac{1}{3} \cos^{-1} \left(\frac{3}{2} \sqrt{\frac{3}{J_2}} J_3 \right)$$

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial f}{\partial J_2} \frac{\partial J_2}{\partial \sigma_{ij}} + \frac{\partial f}{\partial J_3} \frac{\partial J_3}{\partial \sigma_{ij}}$$

$$K_1 = B\sqrt{J_2} + C\sigma_1 + DI_1$$

$$K_2 = \sqrt{K_1^2 + 4AJ_2}$$

$$\frac{\partial f}{\partial I_1} = \frac{\partial K_1}{\partial I_1} + \frac{K_1}{K_2} \frac{\partial K_1}{\partial I_1} = \left(\frac{C}{3} + D \right) \left(1 + \frac{K_1}{K_2} \right)$$

$$\frac{\partial f}{\partial J_2} = \frac{\partial K_1}{\partial J_2} + \frac{1}{2K_2} \left(2K_1 \frac{\partial K_1}{\partial J_2} + 4A \right) = K_3 + \frac{K_1 K_3 + 2A}{K_2}$$

where

$$K_3 = \frac{\partial K_1}{\partial J_2} = \frac{B}{2\sqrt{J_2}} + C \left(\frac{1}{\sqrt{3J_2}} \cos \theta - \frac{3}{2} \frac{\sin \theta}{\sin 3\theta} J_3/J_2^2 \right)$$

$$\frac{\partial f}{\partial J_3} = \frac{\partial K_1}{\partial J_3} + \frac{K_1}{K_2} \frac{\partial K_1}{\partial J_3} = C \frac{\sin \theta}{\sin 3\theta} \frac{1}{J_2} \left(1 + \frac{K_1}{K_2} \right)$$

At the points $\theta = 0, \frac{\pi}{3}$, we can not calculate $\mathcal{K}_1/\mathcal{J}_2$, $\mathcal{K}_1/\mathcal{J}_3$ so we go back to the original formula to obtain expressions in these cases. Thus,

(i) at $\theta = 0$, $\cos \theta = 1$:

$$\sigma_1 = I_1/3 + 2 \sqrt{J_2/3}$$

$$K_3 = \frac{\mathcal{K}_1}{\mathcal{J}_2} = \frac{B}{2\sqrt{J_2}} + C \frac{1}{\sqrt{3J_2}}$$

$$\frac{\mathcal{K}_1}{\mathcal{J}_3} = 0$$

(ii) at $\theta = \frac{\pi}{3}$, $\cos \theta = \frac{1}{2}$:

$$\sigma_1 = I_1/3 + \sqrt{J_2/3}$$

$$K_3 = \frac{\mathcal{K}_1}{\mathcal{J}_2} = \frac{B}{2\sqrt{J_2}} + C \frac{1}{2\sqrt{3J_2}}$$

$$\frac{\mathcal{K}_1}{\mathcal{J}_3} = 0$$

If we set the value of r_A equal to $\sin \theta / \sin 3\theta$ for $0 < \theta < \frac{\pi}{3}$ and equal to 0 for $\theta = 0$ or $\frac{\pi}{3}$, we can express K_3 and $\mathcal{K}_1/\mathcal{J}_3$ in a general form as follows:

$$K_3 = \frac{B}{2\sqrt{J_2}} + C \left(\frac{1}{\sqrt{3J_2}} \cos \theta - \frac{3}{2} r_A J_3/J_2^2 \right)$$

$$\frac{\partial I_1}{\partial J_3} = C r_A / J_2$$

In the numerical implementation, we need to take into account the round-off error, so r_A is determined in the following way,

$$r_A = 0 \quad \text{for } \theta < T \quad \text{or} \quad \left(\frac{\pi}{3} - \theta\right) < T$$

$$r_A = \frac{\sin \theta}{\sin 3\theta} \quad \text{otherwise}$$

In the present program, T is set to $\frac{1}{3} \times 10^{-2}$.

$$\frac{\partial I_1}{\partial \sigma_{ij}} = \delta_{ij}$$

$$\frac{\partial J_2}{\partial \sigma_{ij}} = S_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{mm}$$

$$J_3 = \begin{vmatrix} S_x & \tau_{xy} & 0 \\ \tau_{xy} & S_y & 0 \\ 0 & 0 & S_z \end{vmatrix}$$

$$\frac{\partial J_3}{\partial \sigma_x} = \frac{\partial J_3}{\partial S_x} \frac{\partial S_x}{\partial \sigma_x} + \frac{\partial J_3}{\partial S_y} \frac{\partial S_y}{\partial \sigma_x} + \frac{\partial J_3}{\partial S_z} \frac{\partial S_z}{\partial \sigma_x}$$

$$\frac{\partial J_3}{\partial \sigma_x} = \frac{2}{3} S_y S_z - \frac{1}{3} (S_x S_z + S_x S_y - \tau_{xy}^2)$$

Similarly,

$$\frac{\partial J_3}{\partial \sigma_y} = \frac{2}{3} S_z S_x - \frac{1}{3} (S_x S_y + S_y S_z - \tau_{xy}^2)$$

$$\frac{\partial J_3}{\partial \tau_{xy}} = -2 S_z \tau_{xy}$$

$$\frac{\partial J_3}{\partial \sigma_z} = \frac{2}{3} (S_x S_y - \tau_{xy}^2) - \frac{1}{3} S_z (S_x + S_y)$$

If we select the associated flow rule, the same formula is used for $\partial g / \partial \sigma_{ij}$.
In the case of the nonassociated flow rule, we set

$$\frac{\partial g}{\partial I_1} = 0.$$

II. Loading Function and its Derivatives of the 5-Parameter Criterion

$$f = \sqrt{5} \tau_m - r$$

$$r = \frac{2r_c (r_c^2 - r_t^2) \cos \theta + r_c (2r_t - r_c) \sqrt{4 (r_c^2 - r_t^2) \cos^2 \theta + 5 r_t^2 - 4 r_t r_c}}{4 (r_c^2 - r_t^2) \cos^2 \theta + (r_c - 2 r_t)^2}$$

$$r_t = \sqrt{5} \tau \left[a_0 + a_1 \left(\frac{\sigma_m}{\tau} \right) + a_2 \left(\frac{\sigma_m}{\tau} \right)^2 \right]$$

$$r_c = \sqrt{5} \tau \left[b_0 + b_1 \left(\frac{\sigma_m}{\tau} \right) + b_2 \left(\frac{\sigma_m}{\tau} \right)^2 \right]$$

$$r = \frac{S+T}{V} = \frac{U}{V}$$

$$S = 2r_c (r_c^2 - r_t^2) \cos \theta$$

$$T = r_c (2r_t - r_c) \sqrt{4 (r_c^2 - r_t^2) \cos^2 \theta + 5r_t^2 - 4r_t r_c}$$

$$V = 4 (r_c^2 - r_t^2) \cos^2 \theta + (r_c - 2r_t)^2$$

$$A. \quad \frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial \sigma_m} \frac{\partial \sigma_m}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \tau_m} \frac{\partial \tau_m}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial \sigma_{ij}}$$

$$1. \quad \frac{\partial f}{\partial \sigma_m} = - \frac{\partial r}{\partial \sigma_m} = - \frac{V \left(\frac{\partial S}{\partial \sigma_m} + \frac{\partial T}{\partial \sigma_m} \right) - \frac{\partial V}{\partial \sigma_m} (S + T)}{V^2}$$

$$\frac{\partial S}{\partial \sigma_m} = \frac{\partial S}{\partial r_c} \frac{\partial r_c}{\partial \sigma_m} + \frac{\partial S}{\partial r_t} \frac{\partial r_t}{\partial \sigma_m}$$

$$\frac{\partial S}{\partial r_c} = 2 \cos \theta (3r_c^2 - r_t^2), \quad \frac{\partial S}{\partial r_t} = -4 \cos \theta r_c r_t$$

$$\frac{\partial r_t}{\partial \sigma_m} = \sqrt{5} \tau \left(a_1 + \frac{2a_2 \sigma_m}{\tau} \right), \quad \frac{\partial r_c}{\partial \sigma_m} = \sqrt{5} \tau \left(b_1 + \frac{2b_2 \sigma_m}{\tau} \right)$$

$$\frac{\partial T}{\partial \sigma_m} = b_2 b_3 + a_2/a_3$$

$$a_2 = (2r_t r_c - r_c^2) (-8r_t \cos^2 \theta + 10r_t - 4r_c) \frac{\partial r_t}{\partial \sigma_m}$$

$$+ (8r_c \cos^2 \theta - 4r_t) \frac{\partial r_c}{\partial \sigma_m}$$

$$a_3 = 2 \sqrt{4 (r_c^2 - r_t^2) \cos^2 \theta + 5 r_t^2 - 4r_t r_c}$$

$$b_2 = 2r_c \frac{\partial r_t}{\partial \sigma_m} + 2(r_t - r_c) \frac{\partial r_c}{\partial \sigma_m}$$

$$b_3 = \frac{a_3}{2}$$

$$\frac{\partial V}{\partial \sigma_m} = \frac{\partial V}{\partial r_t} \frac{\partial r_t}{\partial \sigma_m} + \frac{\partial V}{\partial r_c} \frac{\partial r_c}{\partial \sigma_m}$$

$$\frac{\partial V}{\partial r_t} = 8r_t \sin^2 \theta - 4r_c$$

$$\frac{\partial V}{\partial r_c} = 8r_c \cos^2 \theta + 2r_c - 4r_t$$

$$1' \quad \frac{\partial \sigma_m}{\partial \sigma_{ij}} = \frac{1}{3} \delta_{ij}$$

$$2. \quad \frac{\partial f}{\partial \tau_m} = -\sqrt{5}$$

$$2' \quad \frac{\partial \tau_m}{\partial \sigma_{ij}} = \frac{\sqrt{5}}{\tau_m} \frac{\partial \tau_m}{\partial \sigma_{ij}} = \frac{\sqrt{5}}{\tau_m} S_{ij}$$

$$3. \quad \frac{\partial f}{\partial \theta} = -\frac{\partial r}{\partial \theta} = -\frac{\left(v \frac{\partial U}{\partial \theta} - U \frac{\partial V}{\partial \theta} \right)}{v^2}$$

$$\frac{\partial U}{\partial \theta} = 2r_c (r_t^2 - r_c^2) + \frac{a_4}{a_3} \sin \theta$$

$$\frac{\partial V}{\partial \theta} = 8(r_t^2 - r_c^2) \cos \theta \sin \theta$$

$$a_4 = 8r_c (2r_t - r_c) (r_t^2 - r_c^2) \cos \theta \sin \theta$$

$$\text{if } \theta = 0 :$$

$$r = r_t = \sqrt{5} \tau \left[a_0 + a_1 \left(\frac{\sigma_m}{\tau} \right) + a_2 \left(\frac{\sigma_m}{\tau} \right)^2 \right]$$

$$\frac{\partial f}{\partial \theta} = 0$$

$$\text{if } \theta = \frac{\pi}{3} :$$

$$r = r_c = \sqrt{5} \tau \left[b_0 + b_1 \left(\frac{\sigma_m}{\tau} \right) + b_2 \left(\frac{\sigma_m}{\tau} \right)^2 \right]$$

$$\frac{\partial f}{\partial \theta} = 0$$

$$3' \quad \frac{\partial \theta}{\partial \sigma_{ij}} = \frac{1}{\sin 3\theta} \left(\frac{3\sqrt{3}}{4} J_2^{-2.5} J_3 \frac{\partial J_2}{\partial \sigma_{ij}} - \frac{\sqrt{3}}{2} J_2^{-1.5} \frac{\partial J_3}{\partial \sigma_{ij}} \right)$$

$$B. \quad \frac{\partial f}{\partial \tau} = - \frac{\partial r}{\partial \tau} = - \frac{v \left(\frac{\partial S}{\partial \tau} + \frac{\partial T}{\partial \tau} \right) - u \frac{\partial v}{\partial \tau}}{v^2}$$

$$\frac{\partial S}{\partial \tau} = 2 \cos \theta \left[(3r_c^2 - r_t^2) \frac{\partial r_c}{\partial \tau} - 2r_t r_c \frac{\partial r_t}{\partial \tau} \right]$$

$$\frac{\partial T}{\partial \tau} = b_2' b_3 + \frac{a_2'}{a_3}$$

$$b_2' = 2r_c \frac{\partial r_t}{\partial \tau} + 2(r_t - r_c) \frac{\partial r_c}{\partial \tau}$$

$$a_2' = (2r_t r_c - r_c^2) \left[(-8r_t \cos^2 \theta + 10r_t - 4r_c) \frac{\partial r_t}{\partial \tau} \right]$$

$$+ (8r_c \cos^2 \theta - 4r_t) \frac{\partial r_c}{\partial \tau} \Bigg]$$

$$\frac{\partial V}{\partial \tau} = (8r_t \sin^2 \theta - 4r_t) \frac{\partial r_t}{\partial \tau} + (8r_c \cos^2 \theta + 2r_c - 4r_t) \frac{\partial r_c}{\partial \tau}$$

$$\frac{\partial r_t}{\partial \tau} = \sqrt{5} \left[a_0 - a_2 \left(\frac{\sigma_m}{\tau} \right)^2 \right]$$

$$\frac{\partial r_c}{\partial \tau} = \sqrt{5} \left[b_0 - b_2 \left(\frac{\sigma_m}{\tau} \right)^2 \right]$$

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