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THEORY OF ENERGETIC TRAPPED PARTICLE-INDUCED
RESISTIVE INTERCHANGE-BALLOONING MODES

By

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THEORY OF ENERGETIC TRAPPED PARTICLE-INDUCED RESISTIVE INTERCHANGE-BALLOONING MODES

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ABSTRACT

A theory describing the influence of energetic trapped particles on resistive interchange-ballooning modes in tokamaks is presented. It is shown that a population of hot particles trapped in the region of adverse curvature can resonantly interact with and destabilize the resistive interchange mode, which is stable in their absence because of favorable average curvature. The mode is different from the usual resistive interchange mode not only in its destabilization mechanism, but also in that it has a real component to its frequency comparable to the precessional drift frequency of the rapidly circulating energetic species. Corresponding growth rate and threshold conditions for this trapped-particle-driven instability are derived and finite banana width effects are shown to have a stabilizing effect on the mode. Finally, the ballooning/tearing dispersion relation is generalized to include hot particles, so that both the ideal and the resistive modes are derivable in the appropriate limits.

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I. INTRODUCTION AND MOTIVATION

The gross magnetohydrodynamic (MHD) stability of plasma systems containing a high-energy component has received considerable attention from the plasma physics community over the years. This is only appropriate since many of the experimental devices proposed for containing plasmas rely on an auxiliary energetic source for either heating or stability enhancement. The earliest of these investigations concerned themselves with hot particle compressional stabilization of plasma systems which would otherwise be magnetohydrodynamically unstable. Examples of such systems can be drawn from simple mirror machines,¹ field-reversed hot electron rings,² ion rings,³ astron configurations,⁴ and the Elmo Bumpy Torus.⁵ Harnessing hot particles in a stabilizing capacity has only recently been taken under advisement by the tokamak fusion community.

Connor *et al.*⁶ have found that in the nonadiabatic, nonresonant limit where the hot particles precess much faster than any other time scale of interest in the system, i.e., $\bar{\omega}_{dh} > |\omega|, \omega_{bh}, \omega_{th}$, where $\bar{\omega}_{dh}$ is the precessional drift frequency of the hot ions and ω_{bh} and ω_{th} are their bounce and transit frequencies, respectively, an isotropic population of energetic ions (equal trapped and circulating components) would have a significant stabilizing effect on localized ballooning modes. This is because the energetic ions, on the one hand, just move too fast on the time scale of the instability to be able to interact with it (i.e., by rapidly averaging the potential over many spatial oscillations, the hot particles effectively "see" a much weaker perturbation), while on the other, they dig a diamagnetic well in the equilibrium profile. More recently, Rosenbluth *et al.*⁷ inspired by these previous studies, examined energetic particle stabilization of high to moderate- n ballooning modes in the opposite adiabatic limit where the trapped hot particle bounce and precessional motion dominates mode growth, i.e., $\omega_{bh} \gtrsim \bar{\omega}_{dh} \gg |\omega|$. They were able to show that an anisotropic population of super-hot ($\sim$ MeV) particles, deeply trapped on the bad curvature side of the tokamak (so that curvature and diamagnetic drifts are in the same direction), compressionally stabilize ballooning modes and hence, provide direct stable access from the low- β first stability regime to the high- β second stability regime of confinement in tokamaks.

The first indication of a destabilizing influence of energetic particles

came in connection with high-power, near-perpendicular neutral beam injection experiments on the Poloidal Divertor Experiment (PDX) at Princeton. In these experiments, "fishbone oscillations," so-called because of their characteristic skeletal signature on the soft X-ray imaging system, were observed to rotate toroidally with a frequency comparable to the precessional drift frequency of the energetic trapped beam ions ($|\omega| \sim 100$ kHz).⁸ These fluctuations were theoretically explained as energetic trapped particle-induced ideal internal kink modes,⁹ while their high-frequency ($|\omega| \sim 500$ kHz), MHD-type precursors were identified as energetic trapped particle-induced ideal ballooning modes.^{10,11} The emergent physical picture then, is that while suprathermal particles can have a stabilizing influence on normal MHD modes when the mode frequency is strongly off-resonant with the hot particle precessional drift frequency ($|\omega| \neq \bar{\omega}_{dh}$), they become destabilizing when the wave structure begins to resonate with the precessional drift motion of the hot particle banana center ($|\omega| \simeq \bar{\omega}_{dh}$).

All the calculations listed above are subject to the ideal MHD constraint. More quantitatively, it will be shown later in this paper that they are restricted in their regime of validity to $\bar{\omega}_{dm} \gg S_M^{-1/3} \omega_A$, where $\bar{\omega}_{dm}$ is a measure of the bounce-averaged precessional drift frequency, S_M is the magnetic Reynolds number, and ω_A is the Alfvén frequency. However, it is not necessarily the case, especially as one goes to larger machines and higher magnetic field strengths, that this condition can be accommodated. For typical parameters, i.e., $S_M = 10^6$, $\omega_A = 10$ MHz, the condition is at best only marginally satisfied for PDX ($\bar{\omega}_{dm} \simeq 100$ kHz). Already for a device like the Joint European Torus (JET) where $\bar{\omega}_{dm} \simeq 1$ kHz, the condition breaks down. Thus, it becomes relevant to consider the resistive analogues of these energetic trapped particle-induced instabilities, i.e., frequency regimes wherein $\bar{\omega}_{dh} \lesssim S_M^{-1/3} \omega_A$. Recently, we reported our findings in this regard for the case of the fishbone instability.¹² This paper considers, among other things, the same question in regard to the high-frequency precursor to the fishbone.

A second motivation for this work is the universally observed degradation in confinement with increasing input power in all auxiliary heating experiments operating in the L-mode regime. A number of recent experiments¹³ have evinced a slow saturation of beta (i.e., the ratio of

kinetic to magnetic pressure) at values below those predicted by ideal ballooning mode theory. Several recent theoretical studies^{14,15} on resistive interchange-ballooning modes in tokamaks have concluded that, at least away from marginal stability, these latter are fairly innocuous and unlikely to be a leading cause of pressure saturation. Since among the various alternative heating schemes, electron cyclotron resonance heating (ECRH), heating in the ion cyclotron range of frequencies (ICRF), electron cyclotron heating (ECH), and perpendicular neutral beam injection give rise to a significant population of trapped energetic particles, we have been motivated to investigate their influences on resistive interchange-ballooning modes as a possible mechanism responsible for the experimentally-observed soft β -limit.

A final motivation for this work has been the conclusions drawn from the work of Weiland and Chen,¹⁰ and more recently, Van Dam and Rosenbluth.¹¹ These authors have invoked hot particle-induced ideal ballooning modes to explain the high-frequency MHD-type precursor oscillations which precede the fishbone fluctuations in perpendicular beam-heated PDX discharges. While their theory has been largely successful in describing the qualitative features of experimental observations, its applicability to distribution functions other than the singular type remains questionable. The reason for this can be seen from the dispersion relation, which symbolically, can be written as an energy balance¹⁰

$$-i \frac{\omega}{\omega_A} + \delta \hat{W}_c + \delta \hat{W}_k = 0 \quad (1)$$

Here, $\delta \hat{W}_c$ is related to the perturbed potential energy of the core plasma, $\delta \hat{W}_k$ is related to the perturbed energy content of the hot particles, and the first term is related to the energy sink inside the inertial layer. For a slowing-down distribution function [cf. Eq. (22)],

$$\delta \hat{W}_k \propto \langle \epsilon \beta_{ph} \rangle_t \frac{\omega}{|\bar{\omega}_{dm}|} \ln \left(1 - \frac{|\bar{\omega}_{dm}|}{\omega} \right) ,$$

where $\langle \epsilon \beta_{ph} \rangle_t$ is a measure of the trapped particle poloidal beta, $\epsilon = a/R$ is the inverse aspect ratio, a is the minor radius of the plasma, R is the plasma major radius, and $\bar{\omega}_{dm}$ is the hot particle precessional drift frequency evaluated at the beam injection energy. It is apparent that due to the unique

form of $\delta\hat{W}_k$, the threshold $(\epsilon\beta_{ph})_t$ is independent of $\delta\hat{W}_c$ and is obtained by balancing the first and last terms of the dispersion relation. The singular nature of $\delta\hat{W}_k$ allows one to satisfy the real part of the dispersion relation for any value of $\delta\hat{W}_c$, so that the latter plays no role in determining the instability criterion. For any distribution function other than the singular type, however, there will be a hot particle threshold for instability associated with positive-definite $\delta\hat{W}_c$ arising from a balance between the last two terms in the dispersion relation. For shear values of order unity, $\delta\hat{W}_c$ can be written as

$$\delta\hat{W}_c \propto (1 - \frac{\beta_c}{\beta_c^{cr}}) \quad , \quad (2)$$

where the first term is the large stabilizing contribution from bending field lines and the second term is the core pressure drive. It is clear that, at least away from marginal stability, the Weiland-Chen theory would predict a threshold $(\epsilon\beta_{ph})_t$ of order unity which, for all practical purposes, is too stringent a condition to render ballooning modes unstable. It is, however, reasonable to expect that if the frozen-in-flux condition were relaxed and if the resultant interchange-ballooning mode could be destabilized by hot particles, then an instability insensitive to the form of the distribution function could be triggered.

In this paper, we will show that energetic particles can indeed act as a destabilizing agent for resistive interchange-ballooning modes. In Sec. II, we will derive the governing set of eigenmode equations which describe energetic trapped particle-induced resistive interchange-ballooning modes, starting with the resistive MHD equations for the core plasma and adopting a gyrokinetic description for the hot particles. Our result will be the usual resistive ballooning mode eigenmode equations,¹⁴⁻¹⁷ generalized to include the effect of hot particle pressure. In Sec. III, we will obtain and analyze with the aid of Nyquist diagrammatic techniques the resulting dispersion relation. It will be shown there that the resistive interchange mode, which is stable in tokamaks because of favorable average curvature effects, can be rendered unstable by trapped energetic particles, yielding a threshold for instability that scales as $\epsilon\beta_{ph} \sim (\epsilon\beta_{pc})^2$ and that is qualitatively insensitive to the form of the distribution function. Here, β_p denotes the ratio of kinetic to poloidal magnetic pressure and the subscripts 'c' and 'h' denote the core and hot plasma, respectively. These results are generalized in Sec. IV to

incorporate hot particle finite banana width (FBW) effects. These effects have not been of great importance in the past. For drift waves, for example, which reside in a phase velocity regime intermediate between the electron and ion thermal speeds, i.e., $v_{te} \ll \omega/k_{\parallel} \ll v_{ti}$, trapped ion effects are typically unimportant as the ions do not have time to complete a bounce motion during an instability growth time, and trapped electrons, while important, have negligibly small banana widths for FBW effects to become relevant. In fact, ion Larmor radii are typically larger than electron banana widths. Furthermore, in previous ideal calculations of energetic trapped particle-induced instabilities,^{9,10} the effect of the hot particles inside the narrow inner layer was neglected relative to outside the layer, so that there was no scale length against which to compare their banana widths. In a recent work¹² where we examined the resistive analogue of the fishbone instability, we again legislated hot particle effects inside the resistive layer away relative to the outside by arguing that due to the radially global nature of the internal kink mode, hot particles had a significantly larger region to tap on the outer rather than the inner layer. It is only in this work that the mode is of such a highly localized nature that energetic particle effects need to be incorporated inside the resistive layer. Thus, this problem offers a rare glimpse at the relevance of FBW effects which, as one would intuitively expect from analogy with finite Larmor radius (FLR) effects, have a stabilizing effect on the mode. In Sec. V, we give a complete description of the mode for all frequencies. Thus, we provide a generalization of the usual dispersion relation of resistive interchange-ballooning (twisting)^{14,15} and high mode-number tearing modes¹⁴⁻¹⁶ to include energetic particle effects and show how the present theory matches on to that of Weiland and Chen¹⁰ in the ideal limit. Finally, we provide a summary and discussion of our conclusions in Sec. VI.

II. ANALYTICAL FORMULATION

In this section, we list the major assumptions and orderings and derive the governing set of coupled eigenmode equations describing resistive ballooning modes that include energetic particles. The plasma is taken to consist of two components: a warm core (denoted in the text by a subscript 'c')

which is treated using a resistive MHD description and a highly energetic component (denoted by subscript 'h') which is dealt with using gyrokinetic formalism. The core plasma is assumed to be in local thermodynamic equilibrium and weakly interacting with the high-energy component. It is thus a separate thermodynamic entity with its own density and temperature, hence pressure, and obeys an adiabatic equation of state. The energetic component, decoupled from the core, behaves in a non-MHD manner and is not in thermodynamic equilibrium. It is thus highly anisotropic and its pressure must be determined as a velocity moment of its distribution function which, in turn, is obtained from gyrokinetic formalism.

The ordering is carried out using two smallness parameters: the inverse aspect ratio ϵ and the inverse magnetic Reynolds number S_M^{-1} which represents the ratio of the resistive diffusion time to the ideal Alfvén time $S_M = \tau_R/\tau_A$. Since we wish to treat resistive ballooning modes, we consider the ideally stable β_c operating regime, i.e.,

$$\beta_c \sim O(\epsilon^2) < (\beta_c)_{cr}^{ideal} \sim O(\epsilon) ,$$

and formally order $\beta_h \sim O(\epsilon^3)$ which, in hindsight, is sufficiently large to demonstrate instability. It will be seen that this β ordering will make the Rosenbluth et al.⁷ stabilization scheme inaccessible from the point of view of this theory. We further assume very low density energetic particles, i.e., we take $n_h/n_c \sim O(\epsilon^3)$ which, taken together with $\beta_h/\beta_c \sim O(\epsilon)$ implies $T_c/T_h \sim O(\epsilon^2)$ and $\omega_c/\omega_{dh} \sim O(\epsilon)$.

The derivation, which parallels that of Glasser for the usual resistive ballooning mode,¹⁷ proceeds from the resistive MHD equations given below

$$\begin{aligned} \frac{d\rho_c}{dt} + \rho_c \nabla \cdot \mathbf{v}_c &= 0 , \\ \rho \frac{d\mathbf{v}}{dt} &= \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p_c - \nabla \cdot \mathbf{P}_h , \\ \frac{d}{dt}(p_c/\rho_c^{F_s}) &= 0 , \\ \nabla \cdot \mathbf{E} &= -\frac{1}{c} \frac{\partial B}{\partial t} , \\ \nabla \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{J} , \end{aligned}$$

$$\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} = \eta \cdot \mathbf{J}_c, \quad \left(\begin{array}{c} \eta \\ \mathbf{P}_h \end{array} \right) = \left(\begin{array}{c} \eta_{\perp} \\ p_{\perp h} \end{array} \right) (\mathbf{I} - \hat{\mathbf{e}}_{\parallel} \hat{\mathbf{e}}_{\parallel}) + \left(\begin{array}{c} \eta_{\parallel} \\ p_{\parallel h} \end{array} \right) \hat{\mathbf{e}}_{\parallel} \hat{\mathbf{e}}_{\parallel}.$$

These are, in order of appearance, the continuity equation, the equation of motion, the adiabatic equation of state for the core plasma (Γ_s is the adiabatic compressibility), Faraday's law, Ampere's law, and Ohm's law. Note the appearance of anisotropic pressure in the momentum equation and anisotropic resistivity in Ohm's law. Linearizing these equations assuming an $\exp(\gamma t)$ time dependence, we can obtain expressions for the perturbed core velocity and the total current

$$\delta \mathbf{v}_c = \left[\frac{\delta \mathbf{E} \times \mathbf{B}}{c} + \eta_{\perp} \left(\frac{\mathbf{J} \times \delta \mathbf{B}}{c} - \nabla \delta p_c \right) \right] / \left(\frac{B^2}{c^2} + \eta_{\perp} \rho \gamma \right) - \frac{\delta \mathbf{B} \cdot \nabla p_c + \mathbf{B} \cdot \nabla \delta p_c}{\rho B \gamma}, \quad (3)$$

$$\delta \mathbf{J} = \left[\left(\frac{\mathbf{J} \times \delta \mathbf{B}}{c} - \nabla \delta p_c - \nabla \cdot \delta \mathbf{P}_h \right) \times \mathbf{B} + \rho \gamma \delta \mathbf{E}_{\perp} \right] / \left(\frac{B^2}{c^2} + \eta_{\perp} \rho \gamma \right) + \frac{\delta E_{\parallel}}{r_{\parallel} B} \mathbf{B}. \quad (4)$$

In Eq. (3), the first term is the $\mathbf{E} \times \mathbf{B}$ drift, the second represents cross-field diffusion and the third is the total perturbed core parallel pressure gradient which drives the parallel flow and gives rise to sound waves. In Eq. (4), the first term is the diamagnetic current, the second is the polarization current (inertia term), while the last is the parallel Spitzer current which will come to represent field-line bending in the final set of equations. The resistive ballooning mode equations are obtained by using the separation between mode rational surfaces, i.e., Δr_s , as the maximal ordering parameter

$$\lambda = \frac{1}{n} \sim q' \Delta r_s \ll 1,$$

where n is the toroidal mode number, $q = r B_z / R B_p$ is the safety factor, $q' = dq/dr$, and B_z and B_p are the toroidal and poloidal magnetic fields, as well as invoking the two-spatial scale nature of the problem, i.e.,

$$\nabla_{\perp}(\text{perturbed quantities}) \sim O(\lambda^{-1})$$

and

$$\nabla_{\parallel}(\text{perturbed quantities}) \sim \nabla(\text{equilibrium quantities}) \sim O(1) \quad .$$

At this stage the maximal ordering is

$$\beta_h \sim \beta_c \sim \epsilon \sim \frac{\omega}{\omega_{A,S}} \sim O(1) \quad ,$$

$$S_M^{-1} \sim O(\lambda^2) \quad ,$$

where $\omega_S = (\Gamma_S \nu_c / \rho_c)^{1/2} / R$ is the core sound frequency. It is worth remarking that the use of the separation between mode rational surfaces as an ordering parameter is purely a matter of convenience at this stage. In fact, the theory of resistive ballooning modes does not hinge on the high- n assumption at all. We will have additional comments on this point later. Following Glasser, Greene, and Johnson,^{16,17} we adopt modified Hamada coordinates (V, θ, ν) as our working coordinate system. Here, V is the volume inside a given flux tube, θ is a poloidal-like angle and $\nu = \zeta - q\theta$ is an ignorable (assuming axisymmetry) toroidal-like angle (ζ is the true toroidal angle). The magnetic field in this representation is given by

$$\mathbf{B} = \Psi' \nabla \nu \times \nabla V \quad ,$$

where Ψ is the poloidal magnetic flux function. The orthonormal basis in this metric is

$$\Psi' \mathbf{B} \times \nabla \nu / B^2 \quad , \quad \Psi' \nabla V \times \mathbf{B} / B^2 \quad , \quad \mathbf{B} / B^2 \quad ,$$

while the Jacobian is $J^{-1} = -\Psi' / B^2$.

In terms of scalar and vector potentials $\delta\phi$ and $\delta\mathbf{A}$, our equations reduce in leading order to

$$\begin{aligned} \mathbf{B} \cdot \nabla \left[\frac{1}{\eta_{\parallel} B^2} (\mathbf{B} \cdot \nabla \delta\phi + \frac{\gamma}{c} \delta\mathbf{A} \cdot \mathbf{B}) \right] + c^2 \frac{\rho}{B^2} \gamma \nabla_{\perp}^2 \delta\phi \\ + c \frac{\boldsymbol{\kappa} \times \mathbf{B}}{B^2} \cdot \nabla_{\perp} (2\delta p_c + \delta p_{\parallel h} + \delta p_{\perp h}) = 0 \quad , \end{aligned} \quad (5)$$

$$\begin{aligned} \mathbf{B} \cdot \nabla \left\{ \frac{1}{B^2} [\mathbf{B} \cdot \nabla \delta p_c + \frac{P_c'}{\Psi'} \frac{\partial}{\partial \nu} (\delta \mathbf{A} \cdot \mathbf{B})] \right\} + \eta_{\perp} c^2 \frac{\rho}{B^2} \gamma \nabla_{\perp}^2 \delta p_c \\ + 2c\rho\gamma \frac{\boldsymbol{\kappa} \times \mathbf{B}}{B^2} \cdot \nabla_{\perp} \delta \phi - \rho\gamma \frac{B^2 + 4\pi\Gamma_S P_c}{B^2 \Gamma_S P_c} \left(\gamma \delta p_c + \frac{cP_c'}{\Psi'} \frac{\partial \delta \phi}{\partial \nu} \right) \\ - \rho\gamma \left\{ \gamma \delta p_{\perp h} - \frac{c}{B^2} [\mathbf{B} \times \nabla P_{\perp h} + (P_{\perp h} - F_{\parallel h}) \boldsymbol{\kappa} \times \mathbf{B}] \cdot \nabla \delta \phi \right\} = 0, \quad (6) \end{aligned}$$

$$\nabla_{\perp}^2 (\delta \mathbf{A} \cdot \mathbf{B}) = \frac{4\pi}{\eta_{\parallel} c} (\mathbf{B} \cdot \nabla \delta \phi + \frac{\gamma}{c} \delta \mathbf{A} \cdot \mathbf{B}), \quad (7)$$

where prime denotes $\partial/\partial V$.

Equation (6) is the quasineutrality or vorticity equation which is just the statement that the current is divergence-free. The first term, which is the divergence of the parallel current, is the so-called field-line bending term. The second term represents finite inertial effects associated with the divergence of the polarization current. The last term, which is the divergence of the diamagnetic current, represents the interaction of core and hot particle pressure with curvature ($\boldsymbol{\kappa} = \hat{\mathbf{e}}_{\parallel} \cdot \nabla \hat{\mathbf{e}}_{\parallel}$) which, in turn, acts to drive interchange-ballooning modes. Equation (7) is the adiabatic pressure law for the core plasma. The first term represents parallel compression, the second perpendicular compression due to cross-field resistive diffusion, the third perpendicular compression due to the $\mathbf{E} \times \mathbf{B}$ drift, the fourth $\mathbf{E} \times \mathbf{B}$ convection and compression of the core by itself and finally, the last term represents the $\mathbf{E} \times \mathbf{B}$ compression of the core by the hot particles. In deriving Eqs. (6) and (7), use has been made of the quasistatic condition for perturbed pressure balance

$$\delta B_{\parallel} = -4\pi \frac{\delta p_c + \delta p_{\perp h}}{B}.$$

Finally, Eq. (8) is the parallel Ampere's law, where the right-hand side is the parallel Spitzer current obtained from Ohm's law.

Turning our attention to hot particle dynamics, we start from the linearized gyrokinetic equation^{18,19} ignoring finite Larmor radius effects

$$\begin{aligned} \delta F_h = e_h \left(\delta \phi \frac{\partial}{\partial E} - \frac{\mu \delta B_{\parallel}}{c \Omega_h} \frac{\partial}{\partial \mu} \right) F_{0h} + \delta H_h, \\ [v_{\parallel} \hat{\mathbf{e}}_{\parallel} \cdot \nabla - i(\omega - \omega_{dh})] \delta H_h = i \frac{e_h}{m_h} Q_h \left(\kappa \phi - \frac{v_{\parallel}}{c} \delta A_{\parallel} + \frac{v_{\perp}^2}{2\Omega_h} \frac{\delta B_{\parallel}}{c} \right), \end{aligned}$$

where

$$\begin{aligned} Q_h &= (\omega \frac{\partial}{\partial E} + \hat{\omega}_{*h}) F_{0h} , \\ \hat{\omega}_{*h} &= -i \frac{\hat{e}_{\parallel} \times \nabla \ln F_{0h}}{\Omega_h} \cdot \nabla , \\ \omega_{dh} &= -i v_{dh} \cdot \nabla = -i \frac{\hat{e}_{\parallel} \times [(v_{\perp}^2/2) \nabla \ln B + v_{\parallel}^2 \kappa]}{\Omega_h} \cdot \nabla . \end{aligned}$$

Here, $\mu = v_{\perp}^2/2B$ is the magnetic moment, $E = v^2/2$ is the energy, F_{0h} and δF_h are the unperturbed and perturbed parts of the hot particle distribution function, $\Omega_h = e_h B/m_h c$ is the cyclotron frequency, and e_h and m_h are the hot particle charge and mass. Since the parallel wave number vanishes nowhere for circulating particles, their response is, to dominant order, adiabatic. Neglecting finite- β_h effects (which due to the ordering adopted for β_h/β_c will not enter our final set of eigenmode equations), the untrapped response is given by

$$\delta F_{h,u} = \frac{e_h}{m_h} [(\delta\phi - \delta\psi) \frac{\partial F_{0h}}{\partial E} - \frac{\hat{\omega}_{*h}}{\omega} F_{0h} \delta\psi] ,$$

where $\hat{e}_{\parallel} \cdot \nabla \delta\psi = i\omega \delta A_{\parallel}/c$. The trapped particles, on the other hand, have vanishing wave numbers at the turning points, so that they have both an adiabatic and a nonadiabatic response to the field perturbation. As we shall see, it is the latter contribution that gives rise to the instability. Thus,

$$\delta F_{h,t} = \frac{e_h}{m_h} [(\delta\phi - \delta\psi) \frac{\partial F_{0h}}{\partial E} - \frac{\hat{\omega}_{*h}}{\omega} F_{0h} \delta\psi + \frac{Q_h}{\omega} \overline{\delta\psi} - \frac{Q_h}{\omega - \bar{\omega}_{dh}} \overline{\delta\phi}] ,$$

where an overhead bar denotes bounce-averaging

$$\overline{(\dots)} = \frac{\oint (dl/v_{\parallel})(\dots)}{\oint (dl/v_{\parallel})} ,$$

and the closed line integral is between the turning points.

Two important assumptions have been made in deriving the expressions for the hot particle distribution function. First, the analysis has been carried out in the zero banana width limit. This restriction on the theory will be removed in Sec. IV. More importantly, our beta ordering, $\beta_h/\beta_c \sim O(\epsilon)$,

$\beta_e \sim O(\epsilon^2)$, has not only allowed us to ignore finite- β_h effects here, but also finite- β effects in the equilibrium pressure balance relation

$$\nabla \ln B = \kappa + O(\beta_{e,h})$$

This, in turn, makes it impossible for the ∇B -drift to compressionally stabilize and anchor any ballooning action. It is for this reason that the hot particle stabilization scheme of Rosenbluth *et al.*⁷ cannot be recovered from this theory.

Taking velocity moments of the perturbed distribution function, we obtain expressions for the hot particle pressure components

$$\begin{pmatrix} \delta p_{\parallel h} \\ \delta p_{\perp h} \end{pmatrix} = -\frac{e}{\gamma \Psi'} \frac{\partial \delta \psi}{\partial \nu} \begin{pmatrix} P_{\parallel h} \\ P_{\perp h} \end{pmatrix}_B + e_h \begin{pmatrix} \delta \phi - \delta \psi \\ \delta \phi - \delta \psi \end{pmatrix} + \sum_{\pm} e_h \int_{tr} d^3 \nu \left(\frac{v_{\parallel}^2}{v_{\perp}^2} \right) Q_h \left(\frac{\delta \psi}{\omega} - \frac{\delta \phi}{\omega - \bar{\omega}_{dh}} \right), \quad (8)$$

where the 'B' subscript means that the V -differentiation should be performed at fixed B , $\int_{tr}$ denotes an integration over the trapped particle population alone, and $\sum_{\pm}$ implies a summation over the two legs of the banana.

Taken together, Eqs. (5) through (8) form a set of four coupled partial differential equations which are analytically intractable to solve. Analytic progress is made possible by utilizing the two disparate parallel and perpendicular spatial scales, which come about due to the inhomogeneity of the magnetic field, to reduce the system of partial differential equations to a set of ordinary differential equations in the slow variable. The longer (parallel) length scale is the connection length which, in turn, reflects the toroidal periodicity, while the shorter length scale is related to the width of the resistive layer. The usual representation adopted is the well-known ballooning mode eikonal description for the perturbed quantities,^{17,20} where an appropriate linear superposition of aperiodic eigenfunctions simulates the physically periodic behavior:

$$\begin{pmatrix} \delta \phi \\ \delta \mathbf{A} \cdot \mathbf{B} \\ \delta \psi \end{pmatrix} (V, \chi, \nu) = \sum_{l=-\infty}^{\infty} \begin{pmatrix} \delta \hat{\phi} \\ \delta \hat{a}_{\parallel} \\ \delta \hat{\psi} \end{pmatrix} (V, \chi - 2\pi l) e^{-in[\nu + 2\pi l q]},$$

$$\delta p_{c,h}(V, \chi, \nu) = \frac{\partial}{\partial \nu} \sum_{l=-\infty}^{\infty} \delta \hat{p}_{c,h}(V, \chi - 2\pi l) e^{-in(\nu + 2\pi l q)}$$

Note that the pressure stream function is not quite the same as the real pressure because of the ν -derivative out in front. Here, χ is the (poloidal) ballooning coordinate which forms a Fourier conjugate pair with the distance away from a mode rational surface in (real) configuration space, i.e., $m - nq$. Henceforth, all discussions about length scales will refer to ballooning space, which by the definition just given, should be thought of as real space turned inside out.

The two-length scale nature of the equations arising from the secular-ity of $\nabla \nu$ makes feasible a systematic averaging procedure whereby, using $\lambda = (n^2/S_M)^{1/3} \ll 1$, the short equilibrium length scale (denoted by χ_0) is removed and averaged eigenmode equations defined on the longer resistive scale length (denoted by χ_{-1}) can be derived. Moreover, in looking for the least painful way to obtain the influence of energetic trapped particles on resistive interchange-ballooning modes, we are led to a subsidiary ordering in terms of the inverse aspect ratio $\epsilon \ll 1$ which formally sets the frequencies to be of order $\epsilon^{4/3}$ and the long resistive length scale to be of order $\epsilon^{-1/3}$. Then, taking $\beta_c \sim O(\epsilon^2)$ and $\beta_h \sim O(\epsilon^3)$, an optimal ordering which brings in hot particles only to leading order into the final set of eigenmode equations requires $\lambda \sim O(\epsilon^3)$. Normalizing all field quantities to the electrostatic potential and all frequencies to the Alfvén time as follows

$$\begin{aligned} \Lambda_{||} &= \lambda^{-1} \epsilon^{-1/3} (\gamma/c \Psi') \delta \hat{a}_{||} & \Phi &= \delta \hat{\phi} & \Pi_{c,h} &= -(\gamma \Psi' / c P_c') \delta \hat{p}_{c,h} \\ \Omega &= \lambda^{-1} \epsilon^{-4/3} \gamma \tau_A & \hat{\Omega}_{\perp,h} &= \epsilon^{-4/3} \hat{\omega}_{\perp,h} \tau_A & \hat{\Omega}_{dh} &= \lambda^{-1} \epsilon^{-4/3} \bar{\omega}_{dh} \tau_A \\ \xi &= \lambda \epsilon^{1/3} \chi_{-1} & \tau_A^2 &= \rho M / \Psi'^2 & \hat{P}_{||,\perp,h} &= P_{||,\perp,h} / P_c' \end{aligned} \quad (9)$$

we can write Eqs. (5)–(8) in the ballooning representation:

$$\epsilon^{1/3} \lambda \frac{\partial}{\partial \chi} \left(\frac{|\nabla \nu|^2}{B^2} \Lambda_{||} \right) + 4\pi \lambda^2 \epsilon^{8/3} \Omega^2 \frac{|\nabla \nu|^2}{MB^2} \Phi - \frac{4\pi P_c'}{\Psi'^4} \kappa_W (2\Pi_c + \Pi_{||h} + \Pi_{\perp h}) = 0, \quad (10)$$

$$\epsilon^{-8/3} M \frac{\partial}{\partial \chi} \left[\frac{1}{B^2} \left(\frac{\partial \Pi_c}{\partial \chi} + \lambda \epsilon^{1/3} \Lambda_{||} \right) \right] - \lambda \epsilon^{-4/3} \eta_{\perp} n^2 c^2 \tau_A \Omega \frac{|\nabla \nu|^2}{B^2} \Pi_c$$

$$\begin{aligned}
& -\lambda^2 \frac{2\Omega^2}{P_c'} \kappa_W \Phi - \lambda^2 \frac{\Omega^2}{B^2} (\Pi_{\perp h} - (\Psi' \frac{\partial \tilde{P}_{\perp h}}{\partial B} \Big|_V \frac{\mathbf{B} \times \nabla B}{B^2} \cdot \nabla \nu - \frac{\partial P_{\parallel h}}{\partial \ln B} \Big|_V \kappa_W) \Phi] \\
& + \lambda^2 \Omega^2 \frac{R^2 + 4\pi \Gamma_S P_c}{B^2 \Gamma_S P_c} (\Phi - \Pi_c) = 0 \quad , \quad (11)
\end{aligned}$$

$$\frac{\partial \Phi}{\partial \chi} + (\lambda \epsilon^{1/3} + \frac{\eta_{\parallel} c^2 n^2 \tau_A}{4\pi \Omega \epsilon} |\nabla \nu|^2) \Lambda_{\parallel} = 0 \quad , \quad (12)$$

$$\frac{\partial \psi}{\partial \chi} = -\lambda \epsilon^{1/3} \Lambda_{\parallel} \quad , \quad (13)$$

$$\begin{pmatrix} \Pi_{\parallel h} \\ \Pi_{\perp h} \end{pmatrix} = \begin{pmatrix} \tilde{P}_{\parallel h} \\ \tilde{P}_{\perp h} \end{pmatrix}_{B, P_c'} \psi + \sum_{\pm} \frac{\Psi'}{P_c'} \frac{e_h \epsilon^{4/3}}{n c \tau_A} \int_{tr} d^3 v \left(\frac{v_{\parallel}^2}{2} \right) \tilde{Q}_h \left(\frac{\omega}{\omega - \tilde{\omega}_{dh}} \Phi - \bar{\psi} \right) , \quad (14)$$

where

$$\begin{aligned}
\kappa_W &= \kappa_V - \chi q' \kappa_S \quad , \\
\kappa_V &= \Psi' \frac{\kappa \times \mathbf{B}}{B^2} \cdot (\nabla \nu - q' \chi \nabla V) = \pi \Psi' \frac{R B_c q'}{B^2} + \frac{\Psi''}{2\Psi'} + 2\pi \frac{P_c'}{B^2} \quad , \\
\kappa_S &= \Psi' \frac{\kappa \times \mathbf{B}}{B^2} \cdot \nabla V = -\frac{\partial}{\partial \chi} \left(\frac{\tilde{g}}{2B^2} \right) \quad , \\
\tilde{g} &= 2\pi \Psi' R B_c \quad , \\
\nabla \nu &= -\chi q' \nabla V + \nabla \zeta - q \nabla \chi \quad , \\
\tilde{Q}_h &= (\lambda i \Omega \frac{\partial}{\partial E} + \hat{\Omega} \cdot \mathbf{h}) F_{0h} \quad , \\
\hat{\omega}_{\perp h} &= n \frac{m_h c}{e_h \Psi'} \frac{\partial \ln F_{0h}}{\partial V} \quad , \\
\omega_{dh} &= -n \frac{m_h c}{e_h \Psi'} \kappa_W E \quad .
\end{aligned} \quad (15)$$

Here, κ_S is the geodesic (i.e., surface) component and κ_V the normal component of the field curvature. The averaging strategy discussed above entails an expansion of field quantities in powers of λ and ϵ , with the final eigenmode equations for the leading order *DC* quantities redeemed as solubility conditions on the higher order *AC* terms. Thus, for example, to leading order, Eq. (12) expresses the fact that the fluid perturbation endeavors to localize itself in a region where it can effectively decouple its motion from that of field lines in order to render communication between the regions

of good and bad curvature ineffective. Equation (11), to leading order, expresses the excitation of sound waves which propagate ballooning information along a parallel wavelength of the mode. This seeks to dampen the perturbation, but fails to do so on account of the fact that the mode has so localized itself that it manages to grow faster than the time it takes for the acoustic waves to carry the signal. Note that since we are in a frequency regime where

$$k_{\parallel} c_S \ll \omega \sim k_{\parallel} v_A \propto S_M^{-1/3} \omega_A \ll c_S / qR \quad ,$$

ion Landau damping effects are negligible. Carrying out the tedious algebra associated with the averaging scheme, we finally arrive at the following set of eigenmode equations

$$\Omega \frac{\partial \overline{\Phi^{(0)}}}{\partial \xi} + (\Omega + \epsilon^{-2} \xi^2) \overline{\langle \Lambda_{\parallel}^{(0)} \rangle} - \epsilon^{-2} H \xi \overline{\Pi_c^{(0)}} = 0 \quad , \quad (16)$$

$$\begin{aligned} \epsilon^2 \Omega \frac{\partial}{\partial \xi} \left(\frac{\partial \overline{\Phi^{(0)}}}{\partial \xi} + \overline{\langle \Lambda_{\parallel}^{(0)} \rangle} \right) + (E + F) \overline{\Pi_c^{(0)}} + H \xi \overline{\langle \Lambda_{\parallel}^{(0)} \rangle} \\ - \epsilon^2 (\Omega^2 \xi^2 - T_{\Omega}) \overline{\Phi^{(0)}} - \epsilon^2 N_{\Omega} \overline{\psi^{(0)}} = 0 \quad , \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{\partial}{\partial \xi} \left(\frac{\partial \overline{\Pi_c^{(0)}}}{\partial \xi} + \overline{\langle \Lambda_{\parallel}^{(0)} \rangle} \right) + \epsilon^2 \Omega^2 G (\overline{\Phi^{(0)}} - \overline{\Pi_c^{(0)}}) - \frac{M_{\perp}}{M} \Omega \xi^2 \overline{\Pi_c^{(0)}} \\ - \epsilon^2 \Omega^2 K (E \overline{\Phi^{(0)}} + F \overline{\Pi_c^{(0)}} + H \xi \overline{\langle \Lambda_{\parallel}^{(0)} \rangle}) = 0 \quad , \end{aligned} \quad (18)$$

where

$$\begin{aligned} \tau_R &= \frac{\langle B^2 / |\nabla V|^2 \rangle}{\pi c^2 \eta_{\parallel} q^2 \langle B^2 \rangle} \quad , \\ H &= \frac{4\pi}{q'} \frac{P_c'}{\Psi'^4} \tilde{g} \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle \left(\frac{\langle 1/|\nabla V|^2 \rangle}{\langle B^2 / |\nabla V|^2 \rangle} - \frac{1}{\langle B^2 \rangle} \right) \sim O(\beta_c) \quad , \\ \left(\frac{M}{M_{\perp}} \right) &= \frac{4\pi}{q'^2} \frac{P_c'}{\Psi'^4} \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle \left[\left(\frac{1}{\eta_{\parallel}} \right) \left\langle \frac{|\nabla V|^2}{B^2} \right\rangle \right. \\ &\quad \left. + \frac{\tilde{g}^2}{\Psi'^4} \left(\left\langle \frac{1}{B^2} \right\rangle - \frac{1}{\langle B^2 \rangle} \right) \right] \sim O\left(\frac{1}{\eta_{\parallel}} \right) \quad , \end{aligned}$$

$$\begin{aligned}
E &= \frac{4\pi}{q'^2} \frac{P_c'}{\Psi'^4} \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle \left(\frac{\Psi''}{\Psi'} + \tilde{g} q' \left\langle \frac{1}{B^2} \right\rangle \right) \sim O(\epsilon^2 \beta_{pc}^2) , \\
F &= \frac{16\pi^2}{q'^2} \frac{P_c'}{\Psi'^4} \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle \left\langle \frac{1}{B^2} \right\rangle \\
&\quad + \frac{\tilde{g}^2}{\Psi'^4} \left(\left\langle \frac{1}{B^2 |\nabla V|^2} \right\rangle - \frac{\langle 1/|\nabla V|^2 \rangle^2}{\langle B^2/|\nabla V|^2 \rangle} \right) \sim O(\epsilon^2 \beta_{pc}^2) , \\
G &= \frac{\langle B^2 \rangle}{M \Gamma_S P_c} \sim O(\beta_c^{-1}) , \\
K &= \frac{q'^2 \Psi'^4}{4\pi M P_c'^2} \frac{\langle B^2 \rangle}{\langle B^2/|\nabla V|^2 \rangle} \sim O(\epsilon^{-2} \beta_{pc}^{-2}) , \\
\left(\frac{T_{\hat{n}}}{N_{\hat{n}}} \right) &= \frac{8\pi}{q'^2 \Psi'^2} \frac{\omega_A}{c} \frac{e_h \epsilon^{-1/2}}{n} \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle \int_{tr} dE d\mu \\
&\quad \times E \tau_b (\kappa_V - q' \chi \kappa_S) \tilde{Q}_h \left(\frac{\frac{\omega}{\omega - \tilde{\omega}_{rh}}}{1} \right) \sim O(\beta_{ph}) .
\end{aligned}$$

In these equations, the superscript zero refers to the zeroth-order term in the λ -expansion and the angular brackets denote flux-surface averages

$$\langle \dots \rangle = \frac{\oint (dl/B) (\dots)}{\oint (dl/P)} .$$

To elucidate the physical significance of the various terms in Eq. (17), i.e., the averaged vorticity equation, the first term represents field line bending, the second and third represent coupling to sound waves, the fourth represents inertial effects (polarization term), while the last two terms represent hot particle effects. In the adiabatic pressure law, Eq. (18), the first term represents parallel compression, the next two terms represent the $\mathbf{E} \times \mathbf{B}$ convection of the core plasma (as can be seen by taking the limit of large G or $\Gamma_S \rightarrow 0$), the fourth term represents cross-field resistive diffusion, and the remaining terms represent perpendicular compression due to the $\mathbf{E} \times \mathbf{B}$ drift. Note how our judicious ordering of β has minimized algebraic pollution due to hot particles in the final set of equations. We remark in passing on a point that does not seem to have been widely appreciated. While the analytic theory of ideal ballooning modes²⁰ is restricted to high- n mode numbers (since n is the only large expansion parameter), the resistive theory, in utilizing $\lambda = (n^2/S_M)^{1/3}$ as the expansion parameter, does not

suffer from this handicap since S_M is typically large enough to ensure the smallness of λ even for low- n modes. The eikonal ballooning assumption, on the other hand, which again relied on large n in the ideal theory, still survives for low- n in the resistive theory because we are looking at highly localized modes for which $\chi \gg 1$.

The limit of $T_{\hat{n}}, N_{\hat{n}} \rightarrow 0$ recovers the usual resistive ballooning mode equations.^{14,15,17} These are well-known as Fourier transforms of the tearing layer equations in tokamaks as presented in the classic work of Glasser, Greene, and Johnson.¹⁶ We are primarily interested in the most virulent instability that can be promoted by energetic trapped particles in the resistive layer; hence, we focus attention on the deeply resistive regime, i.e., on short wavelength modes, $\xi \sim 1$, where any perturbation in the magnetic field is heavily dissipated by resistivity. As is clear from Ampere's law, Eq. (16), this is equivalent to an electrostatic assumption whereby we can neglect $\langle A_{\parallel}^{(0)} \rangle$ and hence, $\psi^{(0)}$ in the quasineutrality equation and the adiabatic pressure law. Equations (17) and (18) then become

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial \xi^2} + \left(\frac{T_{\hat{n}}}{\Omega} - \Omega \xi^2 \right) \Phi &= - \frac{\tilde{D}_{Re}}{\Omega} \Pi, \\ \frac{\partial^2 \Pi}{\partial \xi^2} - (\tilde{C} \Omega^2 + \Omega \xi^2) \Pi &= - \Omega^2 (\tilde{C} - \tilde{K} \tilde{D}_{Re}) \Phi, \end{aligned} \quad (19)$$

where we have simplified notation in an obvious way and normalized coefficients to order unity. In Eqs. (19), $\tilde{D}_{Re} = D_{Re}/\epsilon^2 = (E + F)/\epsilon^2$ is the usual resistive MHD quantity whose sign determines resistive stability,¹⁶ $\tilde{C} = (G + KF)\epsilon^2$, $\tilde{K} = \epsilon^2 K$, and we have specialized to $\eta_{\perp} = \eta_{\parallel}$ so that $M_{\perp} = M$ for simplicity. Before proceeding to obtain the dispersion relation, we digress to evaluate $T_{\hat{n}}$, which contains all hot particle information in Eqs. (19), for two special cases of interest.

First, we rewrite $T_{\hat{n}}$ more explicitly as

$$\begin{aligned} T_{\hat{n}} &= \frac{2^{7/2} m_h R_0 \pi^2}{q'^2 \Psi'} \epsilon^{-11/6} \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle \int_{tr} d\alpha \tilde{\tau}_h \bar{\kappa}_W^2 \\ &\times \int dE E^{3/2} \frac{\omega}{\omega - \tilde{\omega}_{dh} E} \left(\frac{\tilde{\omega}_{sh}}{\tilde{\omega}_{dh}} + \frac{\partial}{\partial \ln E} - \frac{\partial}{\partial \ln \alpha} \right) F_{0h} \sim \frac{\beta_{ph}}{\epsilon}, \end{aligned} \quad (20)$$

where $\alpha = \mu/E$ is the pitch angle, $\tilde{\omega}_{dh} = \tilde{\omega}_{dh}/E$ is the energy-independent

part of the precessional drift frequency,

$$\bar{\tau}_b = \oint d\chi_0 (1 - \alpha B)^{-1/2}$$

is proportional to the bounce time, and the energy integral over the resonant denominator should be interpreted in a principal value sense. In writing down Eq. (20), we have dropped a term proportional to $n_h/n_e \sim O(\epsilon^3)$ which is negligible in our ordering.

For realistic distribution functions, $T_{\bar{\Omega}}$ can only be evaluated numerically. However, in order to gain insight into the action of energetic trapped particles, it is instructive to evaluate $T_{\bar{\Omega}}$ for some model distribution functions. The most obvious candidate for beam-heated discharges is, of course, a slowing-down distribution function. Thus, the energetic beam is injected at an energy E_m , slows down due to frictional drag on core plasma electrons and finally, at some critical energy, changes its slowing-down character to interact primarily with the background ions and rapidly isotropizes due to pitch-angle scattering. It is the earlier phase that corresponds to our assumption of hot particle pressure anisotropy and the energy dependence in that phase obeys an inverse three-halves power law. Thus, the distribution function can well be modelled by

$$F_{0h}(V, E, \alpha) = \frac{P_h(V)}{\sqrt{2} \pi m_h B \bar{\tau}_{b0} E_m} E^{-3/2} \delta(\alpha - \alpha_0) \quad E \leq E_m, \quad (21)$$

where $P_h(V)$ is an arbitrary function of V with dimensions of pressure, $\bar{\tau}_{b0} = \bar{\tau}_b(\alpha = \alpha_0)$, α_0 is related to the injection angle, and $\delta(\dots)$ is the Dirac delta function. It is then straightforward to show that

$$T_{\bar{\Omega}} = -\frac{1}{q^2 \bar{s}^2} \frac{\bar{\omega}_{dA}}{\bar{\omega}_{dA}} \beta_{ph} I_{0S} \bar{\Omega} \ln(1 - \frac{1}{\bar{\Omega}}) \quad (22)$$

where

$$\begin{aligned} \frac{1}{q^2 \bar{s}^2} \beta_{ph} I_{0S}(\alpha_0, V) &= 8\pi \frac{R_0 P_h(V)}{q^2 \Psi' B} \epsilon^{-4/3} \bar{\kappa}_{W0}^2 \left\langle \frac{B^2}{|\nabla V|^2} \right\rangle \\ &\times [1 + \frac{\bar{\omega}_{dA}}{\bar{\omega}_{dA}} \frac{\partial}{\partial \ln \alpha_0} \ln(\bar{\tau}_{b0} \bar{\kappa}_{W0}^2)] \sim O(\beta_{ph} / \bar{s}^2) \quad , \end{aligned}$$

$$\begin{aligned}\bar{\kappa}_{W0} &= \bar{\kappa}_W(\alpha = \alpha_0) \quad , \\ \bar{\Omega} &= \omega/|\bar{\omega}_{dm}| \quad , \\ |\bar{\omega}_{dm}| &= \bar{\omega}_{dh}(E = E_m) \quad .\end{aligned}$$

A second case of interest arises in connection with energetic particle populations created by radio-frequency (RF) resonance heating. Motivated by the fact that these particles undergo something akin to a random walk process wherein they suffer a kick after every excursion close to the banana tips and that this kick may be in either direction, we model the energy dependence of this RF-generated distribution by a Maxwellian

$$F_{0h} = \frac{P_h(V)}{\sqrt{2\pi} m_h B E_m^{5/2}} e^{-E/E_m} \delta(\alpha - \alpha_0) \quad ,$$

where E_m denotes the thermal energy of the hot particles, $\alpha_0 = 1/B_t$, and B_t is the magnitude of the field at the banana tip where the hot particles are initially born. Using this distribution function in Eq. (20) we arrive at

$$T_{\bar{\Omega}} \simeq -\frac{1}{q^2 s^2} \frac{\bar{\omega}_{dh}}{\bar{\omega}_{dh}} \beta_{ph} I_{0M} \bar{\Omega} \left[\frac{1}{2} + \bar{\Omega} + \bar{\Omega}^{3/2} Z(\bar{\Omega}^{1/2}) \right] \quad , \quad (23)$$

where

$$\beta_{ph} I_{0M} = \frac{\beta_{ph} I_{0S}}{E_m^{5/2}} \quad ,$$

and $Z(\dots)$ is the plasma dispersion function²¹

$$Z(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv \frac{e^{-v^2}}{v - x} \quad .$$

Our choices for these two model distribution functions are motivated by the comments made in the introduction, namely, that stability needs to be investigated for regular as well as singular-type distribution functions.

III. DISPERSION RELATION

We are now prepared to derive the dispersion relation governing energetic trapped particle-driven resistive interchange modes. Noting that

the left-hand sides of Eqs. (19) are Hermite operators, we define a new independent variable $z = \Omega^{1/4} \xi$ to cast the equations in the form

$$\begin{aligned} \Phi_{zz} + \left(\frac{T_{\hat{\Omega}}}{\Omega^{3/2}} - z^2 \right) \Phi &= -\frac{\tilde{D}_{Rc}}{\Omega^{3/2}} \Pi, \\ \Pi_{zz} - (\tilde{C}\Omega^{3/2} + z^2) \Pi &= -\Omega^{3/2}(\tilde{C} - \tilde{K}\tilde{D}_{Rc}) \Phi. \end{aligned} \quad (24)$$

Expanding Φ and Π in an even basis of Hermite polynomials

$$\begin{pmatrix} \Phi(z) \\ \Pi(z) \end{pmatrix} = \sum_m \begin{pmatrix} \phi_{2m} \\ p_{2m} \end{pmatrix} H_{2m}(z) e^{-z^2},$$

where the Hermite polynomial obeys²²

$$\left[\frac{d^2}{dz^2} + (4m+1-z^2) \right] e^{-z^2} H_{2m}(z) = 0,$$

$$H_{2m}(z) = H_{2m}(-z),$$

and substituting into Eqs. (24), we get the following dispersion relation

$$-(4m+1)\tilde{C}\Omega^{3/2} - (4m+1)^2 + \tilde{D}_{Rc}(\tilde{C} - \tilde{K}\tilde{D}_{Rc}) + \left(\tilde{C} + \frac{4m+1}{\Omega^{3/2}} \right) T_{\hat{\Omega}} = 0. \quad (25)$$

The first term represents finite inertial effects, the second, coupling to sound waves, which in turn, includes a (de)stabilizing contribution due to (un)favorable average curvature and a stabilizing one due to perpendicular compression, while finally, the last term represents hot particle effects. Note from the behavior of the eigenfunctions that in the deeply resistive regime, $\xi \sim \Omega^{-1/4}$, which taken together with the electrostatic approximation $\xi \gg \epsilon\Omega^{1/2}$, yields the validity criterion for our deeply resistive approximation

$$\omega_R \gg |\bar{\omega}_{dm}|, \quad (26)$$

where as in Ref. 18, we have found it convenient to define a resistive interchange time scale given by $\omega_R = \lambda\omega_A$ (the reader will recall that $\lambda = (n^2/S_M)^{1/3}$ is our expansion parameter).

In the interest of readability, we will continue our analysis in the incompressible limit. Inclusion of compressibility is trivially performed and our final conclusions will not be hampered in any significant way by its absence.

For the remainder of this section then, we will examine the dispersion relation in the limit of $\tilde{C} \gg 1$ and $m = 0$, i.e.,

$$-\Omega^{3/2} + \tilde{D}_{Rc} + T_{\hat{n}} = 0 \quad (27)$$

The one-to-one correspondence between the terms in this resistive dispersion relation and the corresponding ideal one, i.e., Eq. (1), is worthy of the reader's attention at this point.

A. No Hot Particles ($T_{\hat{n}} = 0$)

In the absence of energetic particles and also in the adiabatic limit where the hot particles precess much faster than time scales of interest, the usual resistive interchange mode is driven by adverse curvature, i.e., $D_{Rc} > 0$:

$$\gamma = \left(\frac{n^2}{S_M}\right)^{1/3} D_{Rc}^{2/3} \quad (28)$$

Since tokamaks benefit from favorable average curvature, i.e., $D_{Rc} < 0$, this mode is suppressed and as a result, attention is shifted to the more benign resistive ballooning modes for which, except for very close marginal stability, the free energy source comes from the ideal region and the unstable mode has a tearing-like $S_M^{-3/5}$ growth rate scaling. What we shall show here is that the more dangerous resistive interchange mode can in fact be destabilized by a population of energetic particles trapped in a region of adverse curvature in spite of favorable D_{Rc} effects.

B. Fluid Limit ($|\omega| \gg |\tilde{\omega}_{dm}|$)

In the limit where the mode grows much faster than the time it takes for the hot particle banana center to precess once around the torus, $T_{\hat{n}}$ simplifies and we get a hot particle modified resistive interchange mode driven by both D_{Rc} and β_{ph} :

$$\gamma = \left(\frac{n^2}{S_M}\right)^{1/3} \left(\frac{\epsilon^2}{q^2 \tilde{s}^2} \frac{\tilde{\omega}_{-h}}{\tilde{\omega}_{dh}} \beta_{ph} I_0 S + D_{Rc}\right)^{2/3} \quad (29)$$

Already it is clear that for trapped particles undergoing unfavorable drift, i.e., hot species trapped on the outside so that $\tilde{\omega}_{-h}/\tilde{\omega}_{dh} > 0$, the resistive

interchange mode will be promoted even if the system has average favorable curvature, i.e., $D_{Re} < 0$, with a threshold for instability given by

$$(\beta_{ph} I_{0S})_{cr} = q^2 \hat{s}^2 \frac{\hat{\omega}_{dh}}{\hat{\omega}_{-h}} |D_{Re}|, \quad (30)$$

or $(\epsilon \beta_{ph})_{cr} \sim (\epsilon \beta_{pc})^2$ which is an order of magnitude lower than the corresponding threshold for the core plasma.

C. Resonant Destabilization ($|\omega| \sim |\bar{\omega}_{dm}|$)

The most interesting case occurs when the mode frequency just resonates with the precessional drift frequency of the hot particles. Consider, for the sake of illustration, the model slowing-down distribution function. The dispersion relation is given by

$$D(\bar{\Omega}) = \frac{\hat{\omega}_{-h}}{\hat{\omega}_{dh}} \frac{\beta_{ph} I_{0S}}{q^2 \hat{s}^2} \bar{\Omega} \ln(1 - \frac{1}{\bar{\Omega}}) + (\frac{|\bar{\omega}_{dm}|}{\varpi_R})^{3/2} e^{-3i\pi/4} \bar{\Omega}^{3/2} - \bar{D}_{Re} = 0, \quad (31)$$

where $\varpi_R = \epsilon^{4/3} \omega_R$, and noting Eq. (26), $|\bar{\omega}_{dm}|/\varpi_R \sim O(1)$. We will illustrate the stability properties of this dispersion relation using Nyquist diagrammatic techniques. In Fig. 1, we show a contour encircling the entire unstable half of the complex frequency plane. The number of times that the image path in the complex D plane encircles the origin equals the number of unstable roots of $D(\bar{\Omega}) = 0$. The real and imaginary parts of the dispersion relation, when $1 > \bar{\Omega} > 0$, are roughly given by

$$\text{Im}D = \frac{\hat{\omega}_{-h}}{\hat{\omega}_{dh}} \frac{\beta_{ph} I_{0S}}{q^2 \hat{s}^2} \pi \bar{\Omega}_r - \frac{1}{\sqrt{2}} (\frac{|\bar{\omega}_{dm}|}{\varpi_R})^{3/2} \bar{\Omega}_r^{3/2} = 0,$$

$$\text{Re}D = \frac{\hat{\omega}_{-h}}{\hat{\omega}_{dh}} \frac{\beta_{ph} I_{0S}}{q^2 \hat{s}^2} \bar{\Omega}_r \ln \left| \frac{1}{\bar{\Omega}_r} - 1 \right| - \frac{1}{\sqrt{2}} (\frac{|\bar{\omega}_{dm}|}{\varpi_R})^{3/2} \bar{\Omega}_r^{3/2} - \bar{D}_{Re} = 0,$$

where Re and Im denote the real and imaginary parts, respectively.

To touch base with previous results, first consider the case of no hot particles. Noting that for

$$\bar{\Omega}_r \rightarrow \pm\infty \Rightarrow D(\bar{\Omega}) \rightarrow (\frac{|\bar{\omega}_{dm}|}{\varpi_R})^{3/2} |\bar{\Omega}_r|^{3/2} e^{\mp 3i\pi/4}$$

and

$$\tilde{\Omega}_r \rightarrow 0 \Rightarrow \operatorname{Im} D \rightarrow 0 \text{ and } \operatorname{Re} D = -\tilde{D}_{Rc} ,$$

we can easily sketch the Nyquist plot in Fig. 2. Note that we get one unstable root depending on whether $\tilde{D}_{Rc}$ is positive or not. This is just the well-known resistive stability criterion of Glasser, Greene, and Johnson.¹⁶ The same Nyquist diagram would be obtained for the dispersion relation with hot particles included but in the fluid limit, i.e., case B.

Now, include a hot particle population and assume favorable average curvature, i.e., $\tilde{D}_{Rc} < 0$. This time, two conditions have to be met for resonant destabilization to occur: first, $\operatorname{Im} D$ has to change sign for some $\tilde{\Omega}_c$ such that $0 < \tilde{\Omega}_c < 1$ and second, $\operatorname{Re} D(\tilde{\Omega}_c) < 0$ in order for the origin to be encircled. These two conditions impose upper and lower bounds on the critical β_{ph} , respectively. Moreover, the Nyquist analysis reveals that there will be a hot particle-induced instability only for these latter trapped in the region of adverse curvature, so that they undergo unfavorable drifts, i.e., $\tilde{\omega}_h/\tilde{\omega}_{dh} > 0$ (cf. Figs. 3 and 4). Since trapped particles spend the greater fraction of their time near the banana tips, it follows that a sloshing (i.e., marginally trapped or trapped on the good curvature side) distribution of trapped particles will not foster instability. Finally, if the system has unfavorable average curvature, the Nyquist diagram manifests two unstable roots, one driven by the deeply trapped particles and the other by unfavorable average curvature.

More quantitatively, we can calculate threshold conditions near marginal stability, i.e., $\Omega = \Omega_{cr} + i0^+$:

$$\begin{aligned} (\beta_{ph} I_{0S})_{cr} &= \frac{q^2 \tilde{s}^2}{\sqrt{2} \pi} \frac{\tilde{\omega}_{dh}}{\tilde{\omega}_h} \left(\frac{|\tilde{\omega}_{dm}|}{\tilde{\omega}_R} \right)^{3/2} \tilde{\Omega}_{cr}^{1/2} , \\ \frac{1}{\sqrt{2} \pi} \left(\frac{|\tilde{\omega}_{dm}|}{\tilde{\omega}_R} \right)^{3/2} \tilde{\Omega}_{cr}^{3/2} \left| \pi \ln \left(\frac{1}{\tilde{\Omega}_{cr}} - 1 \right) - 1 \right| &= \tilde{D}_{Rc} . \end{aligned} \quad (32)$$

A rough estimation of the instability window alluded to above reveals that

$$1 \gtrsim \frac{\sqrt{2} \pi}{q^2 \tilde{s}^2} \left(\frac{\tilde{\omega}_R}{|\tilde{\omega}_{dm}|} \right)^{3/2} \frac{\tilde{\omega}_h}{\tilde{\omega}_{dh}} (\beta_{ph} I_{0S})_{cr} \gtrsim 0.2 .$$

We note two things: first, that the threshold is lower than the corresponding one for resistive ballooning modes, i.e.,

$$(\epsilon \beta_{ph})_{cr} \sim O(\epsilon^2) < (\epsilon \beta_{pc})_{cr} \sim O(1) , \quad (33)$$

and second, that the instability criterion depends only on the dissipation, i.e., $(\bar{\omega}_{dm}/\bar{\omega}_R)^{3/2}$, and retains independence from the source term, i.e., $\bar{D}_{Re}$. As advertised in the introduction, this is due to the singular nature of our chosen distribution function.

Turning our attention to the second model hot particle distribution function, namely, a Maxwellian, we remark that despite its very different analytic properties, it can be shown to manifest qualitatively similar Nyquist and stability properties as the slowing-down case just described, albeit at a higher cost of algebra. Hence, we will forgo presenting the analysis here. What is much more of interest, however, is that there is now a threshold associated with negative-definite $\bar{D}_{Re}$. In other words, for plasma systems such as tokamaks which possess favorable average curvature, the trapped hot particle population will have to overcome this stabilizing influence before it can render the resistive interchange mode unstable. To see this, consider again the dispersion relation near marginal stability:

$$\frac{\sqrt{2\pi}}{q^2 \bar{s}^2} \frac{\bar{\omega}_{sh}}{\bar{\omega}_{dh}} (\beta_{ph} I_{0M})_{cr} \bar{\Omega}_{cr} \exp(-\bar{\Omega}_{cr}) = \left(\frac{|\bar{\omega}_{dm}|}{\bar{\omega}_R} \right)^{3/2},$$

$$\frac{1}{q^2 \bar{s}^2} \frac{\bar{\omega}_{sh}}{\bar{\omega}_{dh}} (\beta_{ph} I_{0M})_{cr} \left\{ \frac{1}{2} + \bar{\Omega}_{cr} + \bar{\Omega}_{cr}^{3/2} [\text{Re} Z(\bar{\Omega}_{cr}^{1/2}) - \sqrt{\pi} \exp(-\bar{\Omega}_{cr})] \right\} = \bar{D}_{Re}.$$

By examining the limiting cases of weak (i.e., $\bar{D}_{Re} \ll 1$) and strong (i.e., $-\bar{D}_{Re} \gg 1$) sources, it is easy to show that instability will occur only if the hot particle drive exceeds the greater of either the source or the dissipation

$$\epsilon \beta_{ph} > \epsilon^2 \max(|\bar{D}_{Re}|, \left(\frac{|\bar{\omega}_{dm}|}{\bar{\omega}_R} \right)^{3/2}). \quad (34)$$

We see that unlike the ideal case,¹⁰ the resistive instability criterion, Eq. (34), is magnitude-wise insensitive to the form of the distribution function.

We conclude this section by calculating the growth rate just above marginal stability:

$$\gamma \simeq |\bar{\omega}_{dm}| \left| \frac{\beta_{ph} I_0}{(\beta_{ph} I_0)_{cr}} - 1 \right|. \quad (35)$$

Equation (35) has a universal form for all energetic trapped particle-induced instabilities.^{9,10,12}

IV. FINITE BANANA WIDTH (FBW) EFFECTS

All of the calculations carried out so far have been in the zero banana width limit, i.e., it has been assumed throughout that $k_{\perp} \rho_b \ll 1$, where $k_{\perp}$ is the inverse radial extent of the mode and ρ_b is the width of the hot particle banana orbits. However, the highly localized nature of the mode under consideration makes it relevant to consider situations where the finite banana width of the high-energy particles exceeds the extent of the mode. Since the bounce frequency of the energetic species is assumed to exceed greatly the mode growth rate, it can be expected that FBW averaging effects will mitigate the threshold results of the preceding section.

To estimate FBW effects, we take account of the variation of r with χ in the fast-varying eikonal

$$\nu = \zeta - q(r_0)\chi - \chi q'(r_0)\delta r \quad ,$$

where r_0 is the radial location of the symmetry flux surface and δr is the deviation of the banana orbit from it. The approximation $d\tau \simeq \pm R q d\chi / v_{\parallel}$, where $d\tau$ is the differential period of bounce motion, easily leads to

$$\dot{\chi} = \pm \sqrt{2} \omega_b (\cos \chi - \cos \chi_t)^{1/2} \quad ,$$

where $\omega_b = (\epsilon \mu B_0)^{1/2} / R_0 q$ is the bounce frequency, $\chi_t = \cos^{-1}(1 - E/\mu B_0)/\epsilon$ is the poloidal location of the turning point, and the overhead dot signifies a bounce time derivative. For deeply trapped particles, we find harmonic oscillation between turning points $\chi = \pm \chi_t$ at a frequency $\omega = \omega_b$:

$$\chi = \chi_t \cos \omega_b \tau \quad ,$$

and radial excursions given by

$$\delta r = r - r_0 = \mp \rho_b \chi_t \sin \omega_b \tau \quad ,$$

where $\rho_b = q_0 \rho_L / \epsilon^{1/2}$ is the banana width and ρ_L is the Larmor radius. The leading order corrected eikonal for deeply trapped particles then becomes

$$\nu = \zeta - q_0 \chi \pm q_0' \chi \rho_b \chi_t \sin \omega_b \tau \quad .$$

The net effect of the correction is to introduce Bessel functions modulating the field quantities when we bounce-average the gyrokinetic equation, i.e., when we transform from guiding center to banana center coordinates

$$\omega_b \oint d\tau \delta \hat{\phi} \exp[in(\zeta - q_0 \chi \pm q_0' \chi \rho_b \chi_t \sin \omega_b \tau)] = \delta \hat{\phi} J_0\left(\frac{\chi}{\chi_b}\right) \exp[in(\zeta - q_0 \chi)] ,$$

where $\chi_b = (nq_0' \rho_b \chi_t)^{-1}$. Putting this expression back into the vorticity equation and carrying out the various manipulations necessary to get the flux surface-averaged set of eigenmode equations, we note that all field quantities excluding those in the hot particle pressure term have phase at the exact particle position, whereas those in the latter have phase at the location of the banana center. Thus, another Bessel function appears when we transform back from banana-center coordinates to exact particle coordinates. Finally, the relevant eigenmode equation in the large- $\tilde{C}$, incompressible limit can be written as

$$\Phi_{\xi\xi} + \left(\frac{\tilde{D}_{Rc}}{\Omega} - \Omega \xi^2\right) \Phi = -\frac{T_{\tilde{\Omega}}}{\Omega} J_0^2\left(\frac{\xi}{\xi_b}\right) \Phi ,$$

where $\xi_b = (nq_0' \rho_b \theta_t)^{-1} \lambda \epsilon^{1/3}$. The limit $\xi/\xi_b \ll 1$, which corresponds to $k_{\perp} \rho_b \ll 1$, recovers the results of Sec. II. Changing variables to $z = (2\Omega)^{1/4} \xi$ and introducing $N = \tilde{D}_{Rc}/\sqrt{2}\Omega^{3/2}$ and $\Delta = -T_{\tilde{\Omega}} \xi_b/\pi 2^{1/4} \Omega^{5/4}$, we finally obtain

$$\Phi_{zz} + \left(N - \frac{z^2}{4}\right) \Phi = \frac{\pi \Delta}{z_b} J_0^2\left(\frac{z}{z_b}\right) \Phi . \quad (36)$$

Using well-known analytical techniques,²³ we can solve Eq. (36) via asymptotic matching in two regions (cf. Fig. 5): an outer region, $z \sim 1 \gg z_b$, where the Bessel function may be expanded to give

$$J_0^2\left(\frac{z}{z_b}\right) \simeq \frac{z_b}{\pi z} + O\left(\frac{z}{z_b}\right)^{3/2} ,$$

and an inner region, $1 \gg z \sim z_b$, where the full Bessel function is kept.

A. Region I ($1 \sim z \gg z_b$)

The eigenmode equation reduces to

$$\Phi_{zz} + \left(N - \frac{z^2}{4}\right) \Phi = \frac{\Delta}{z} \Phi . \quad (37)$$

Taking out the exponential dependence $\Phi = \Theta \exp(-z^2/4)$ reduces Eq. (37) to

$$\Theta_{zz} - z\Theta_z + \left(N - \frac{1}{2}\right)\Theta = \frac{\Delta}{z}\Theta ,$$

which, in turn, can be solved perturbatively. Two independent Frobenius series will be generated which, in the limit of large z , diverge exponentially as $z^{-N-1/2} \exp(z^2/4)$:

$$\lim_{z \rightarrow \infty} \Phi \sim \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4} - \frac{N}{2})} \left(1 + \frac{\Gamma(\frac{1}{4} - \frac{N}{2})}{\sqrt{2}\Gamma(\frac{3}{4} - \frac{N}{2})} \left\{ c + \frac{\Delta}{2} \times [\ln 2 - \gamma_E + 2 + cD(N)] \right\} \right) \left(\frac{2}{z^2} \right)^{N/2+1/4} e^{z^2/4} ,$$

where c is the ratio of the coefficients of the two independent series, $\gamma_E \simeq .577$ is the Euler-Mascheroni constant, and

$$D(N) = \sum_{l=0}^{\infty} \frac{\Gamma(l - \frac{N}{2} + \frac{3}{4})}{(l + \frac{1}{2})\Gamma(l - \frac{N}{2} + \frac{5}{4})} .$$

In order to ensure causality, the coefficient of the divergent term must vanish

$$1 + \frac{\Gamma(\frac{1}{4} - \frac{N}{2})}{\sqrt{2}\Gamma(\frac{3}{4} - \frac{N}{2})} \left\{ \frac{\Delta}{2} (\ln 2 - \gamma_E + 2) + c \left[1 + \frac{\Delta}{2} D(N) \right] \right\} = 0 . \quad (38)$$

The ratio c is obtained by matching the inner and outer solutions together. To obtain matching, we need the small- z behavior of Φ

$$\lim_{z \ll 1} \frac{d \ln \Phi}{dz} \sim c + \Delta \ln ez . \quad (39)$$

B. Region II ($1 \gg z \sim z_b$)

In this region, the eigenmode Eq. (36) reduces to

$$\Phi_{zz} = \frac{\pi \Delta}{z_b} J_0^2\left(\frac{z}{z_b}\right) \Phi .$$

Using a constant- Φ approximation,

$$\frac{d \ln \Phi}{dz} = \pi \Delta \int_0^{z/z_b} d\xi J_0^2(\xi) ,$$

which can be evaluated asymptotically to give

$$\lim_{z \gg z_b} \frac{d \ln \Phi}{dz} = \Delta \left(\ln \frac{z}{z_b} + \gamma_E + 3 \ln 2 \right) . \quad (40)$$

Matching Eqs. (39) and (40), we obtain the expression for c which, when substituted into Eq. (38), yields the dispersion relation

$$a e^{i5\pi/8} T_{\bar{\Omega}} \bar{\Omega}^{-5/4} \frac{\Gamma(\frac{1}{4} - e^{3i\pi/4} \frac{d_r}{\bar{\Omega}^{5/2}})}{\Gamma(\frac{3}{4} - e^{3i\pi/4} \frac{d_r}{\bar{\Omega}^{5/2}})} \left(b + i \frac{\pi}{2} - \ln \bar{\Omega} \right) = 1 , \quad (41)$$

where

$$\begin{aligned} a &= (2^{11/4} \pi n \theta_b q_0' \rho_b)^{-1} \lambda \left(\frac{|\bar{\omega}_{dm}|}{\omega_R} \right)^{-5/4} < 1 , \\ b &= 2 \gamma_E + \frac{1}{4} \ln [2^{13/4} \lambda \epsilon^{1/3} e^2 n \theta_b q_0' \rho_b \left(\frac{|\bar{\omega}_{dm}|}{\omega_R} \right)^{-5/4}] \sim 1 , \\ d_r &= \tilde{D}_{Re} \left(2 \frac{|\bar{\omega}_{dm}|}{\omega_R} \right)^{-3/2} . \end{aligned}$$

From Eq. (41), it can be shown that qualitatively the threshold for instability is raised by a factor $(\rho_b/\rho_R) \ln(\rho_b/\rho_R)$, where ρ_R is the width of the resistive mode.

V. CONNECTION TO IDEAL REGIME

The theory that has been expounded in the previous sections has been restricted in its regime of validity to the deeply resistive zone. As we showed in Sec. III, the validity criterion for the deeply resistive approximation is $|\bar{\omega}_{dm}| \ll \omega_R$. The ideal ballooning mode calculation¹⁰ is valid in exactly the opposite limit of $|\bar{\omega}_{dm}| \gg \omega_R$. One might therefore expect that the $|\bar{\omega}_{dm}| \sim \omega_R$ limit of the two theories must match together. In fact, however, the thresholds predicted by the two theories in this limit are far apart, as can be seen upon comparing Eq. (32) of this paper with Eq. (34a) of Ref. 10

$$\lim_{\bar{\omega}_{dm} \sim \omega_R} \frac{(\beta_{ph})_{cr}^{ideal}}{(\beta_{ph})_{cr}^{deeply\ resistive}} \sim S_M^{-1/3} \ll 1 .$$

This suggests the existence of an intermediate frequency regime between the two theories which acts to bridge them together. In this section, we will expand the deeply resistive dispersion relation to describe energetic trapped particle-induced resistive interchange-ballooning modes. In so doing, we generalize the results of Connor *et al.*¹⁴ and Antonsen and Drake¹⁵ to include hot particle effects. We will then derive a dispersion relation in the intermediate regime which will bridge the two theories together and show how these three dispersion relations match up to cover the entire frequency spectrum.

A. Energetic Trapped Particle-Induced Resistive Ballooning/Tearing Modes

To obtain the dispersion relation governing energetic trapped particle-induced resistive ballooning and tearing modes requires a three-region matching of the solutions to Eqs. (16) to (18), as illustrated in Fig. 6. Region I is the ideal region $\chi \sim 1$, where resistivity and inertia are negligible. Region II is the intermediate region $1 \gg \xi \sim \epsilon \Omega^{1/2}$ or $\lambda^{-1} \epsilon^{1/3} \gg \chi \sim \lambda^{-1} \epsilon^{2/3} \gg 1$ [recall that $\lambda \sim O(\epsilon^3)$], where field-line bending and parallel compressional effects dominate over other terms in Eqs. (16) and (18). Finally, region III is the deeply resistive (electrostatic) regime where hot particle effects come in. Matching the small- ξ values of Φ and Π of region III to the respective large- ξ quantities in region II, and the small- ξ results of the intermediate region to the large- ξ values of the ideal region and solving all these together, we obtain the dispersion relations:

$$\Delta'_B = \frac{2\epsilon^{5/3}}{\lambda \Omega^{1/4}} \frac{\left\{ (\tilde{C} \Omega^{3/2} - T_{\tilde{\Omega}} / \Omega^{3/2})^2 + 4[\tilde{D}_{Rc}(\tilde{C} - \tilde{F} \tilde{D}_{Rc}) + \tilde{C} T_{\tilde{\Omega}}] \right\}^{1/2}}{|\tilde{C}(1 - T_{\tilde{\Omega}} / \tilde{D}_{Rc}) - \tilde{K} \tilde{D}_{Rc}|(1 + T_{\tilde{\Omega}} / \Omega^{3/2} \epsilon^{1/2})(1 + T_{\tilde{\Omega}} / \Omega^{3/2} \sigma_-)} \\ \times \left\{ \frac{\Gamma[(1 + \sigma_-)/4]}{\Gamma[3 + \sigma_-)/4]} - \frac{\Gamma[(1 + \sigma_+)/4]}{\Gamma[3 + \sigma_+)/4]} \right\}^{-1}, \quad (42)$$

tearing parity

$$\Delta'_T = \frac{2\epsilon^{5/3}\Omega^{5/4}}{\lambda} \left\{ (\tilde{C}\Omega^{3/2} - T_{\tilde{n}}/\Omega^{3/2})^2 + 4[\tilde{D}_{Rc}(\tilde{C} - \tilde{K}\tilde{D}_{Rc}) + \tilde{C}T_{\tilde{n}}] \right\}^{-1/2} \\ \times \left\{ [\sigma_+ + (T_{\tilde{n}} + \tilde{D}_{Rc})/\Omega^{3/2}] \frac{\Gamma[(3 + \sigma_-)/4]}{\Gamma[(1 + \sigma_-)/4]} \right. \\ \left. - [\sigma_- + (T_{\tilde{n}} + \tilde{D}_{Rc})/\Omega^{3/2}] \frac{\Gamma[(3 + \sigma_+)/4]}{\Gamma[(1 + \sigma_+)/4]} \right\}, \quad (43)$$

where

$$2\sigma_{\pm} = \tilde{C}\Omega^{3/2} - T_{\tilde{n}}/\Omega^{3/2} \pm \left\{ (\tilde{C}\Omega^{3/2} - T_{\tilde{n}}/\Omega^{3/2})^2 + 4[\tilde{D}_{Rc}(\tilde{C} - \tilde{K}\tilde{D}_{Rc}) + \tilde{C}T_{\tilde{n}}] \right\}^{1/2},$$

and the factor $\lambda = (n^2/S_M)^{-1/3} \ll 1$ that occurs in these equations reflects the ratio of the resistive to the ideal spatial extents.

In Eqs. (42) and (43), the left-hand side is the usual Δ' quantity which reflects the ratio of the coefficients of the small to large ideal region solutions, and has been analytically evaluated without hot particles by Antonsen and Drake.¹⁵ An interesting point is that within our present beta ordering, hot particle effects do not enter Δ' , since their Mercier-like contribution is of higher order [i.e., $\sim O(\epsilon^3)$] than the core pressure terms [$\sim O(\epsilon^2)$].

Digressing momentarily to make contact and comment on previous calculations of the usual resistive ballooning mode, we look at the limit of Eqs. (42) and (43) in the absence of hot particle effects

$$\Delta'_B = \frac{2\epsilon^{5/3}}{\lambda\Omega^{1/4}} \frac{[\tilde{C}^2\Omega^3 + 4\tilde{D}_{Rc}(\tilde{C} - \tilde{K}\tilde{D}_{Rc})]^{1/2}}{(\tilde{C} - \tilde{K}\tilde{D}_{Rc})} \\ \times \left\{ \frac{\Gamma[(1 + \sigma_-)/4]}{\Gamma[(3 + \sigma_-)/4]} - \frac{\Gamma[(1 + \sigma_+)/4]}{\Gamma[(3 + \sigma_+)/4]} \right\}^{-1}, \quad (44)$$

$$\Delta'_T = \frac{2\epsilon^{5/3}\Omega^{5/4}}{\lambda} [\tilde{C}^2\Omega^3 + 4\tilde{D}_{Rc}(\tilde{C} - \tilde{K}\tilde{D}_{Rc})]^{-1/2} \\ \times \left\{ (\sigma_+ + \tilde{D}_{Rc}/\Omega^{3/2}) \frac{\Gamma[(3 + \sigma_-)/4]}{\Gamma[(1 + \sigma_-)/4]} - (\sigma_- + \tilde{D}_{Rc}/\Omega^{3/2}) \frac{\Gamma[(3 + \sigma_+)/4]}{\Gamma[(1 + \sigma_+)/4]} \right\}, \quad (45)$$

where

$$2\sigma_{\pm} = \tilde{C}\Omega^{3/2} \pm [\tilde{C}^2\Omega^3 + 4\tilde{D}_{Rc}(\tilde{C} - \tilde{K}\tilde{D}_{Rc})]^{1/2} .$$

Noting that the averaged resistive equations of Drake and Antonsen¹⁵ ignore the stabilizing influence of perpendicular compression, Eqs. (44) and (45) can be seen to recover their results as well as that of Connor *et al.*¹⁴ in the appropriate large- $\tilde{C}$ incompressible limit. Our Eq. (45), in fact, is identical to that derived by Glasser, Greene, and Johnson¹⁶ for tearing modes in a low- β , large aspect ratio tokamak. As is well-known,^{15,16} usual resistive ballooning (tearing) modes are driven unstable whenever the free-energy source in the ideal region, denoted by Δ'_B (Δ'_T) exceeds a threshold $(\Delta'_B)_{cr}$ [$(\Delta'_T)_{cr}$]. This critical value can be analytically evaluated near marginal stability in the limit of large average favorable curvature, i.e., $-\tilde{D}_{Rc} \gg 1$ to be

$$(\Delta'_B)_{cr} = \frac{\epsilon^{5/3}}{\lambda} \tilde{C}^{1/6} [2|\tilde{D}_{Rc}|(\tilde{C} + \tilde{K}|\tilde{D}_{Rc}|)]^{2/3} \sim S_M^{1/3} \gg 1 . \quad (46)$$

(Note an algebraic error in the corresponding calculation of Drake and Antonsen.¹⁵) This critical value is high enough that except for proximity to marginal stability, we are led to conclude that usual resistive ballooning modes are unlikely to be a cause of poor confinement in tokamaks. A similar conclusion follows for tearing modes.

Returning to Eqs. (42) and (43), we recognize that the dispersion relation of Sec. III is recovered in the limit of $\Delta' \ll 1$, since then,

$$-\sigma_{\pm} = 4m + 1 , \quad (47)$$

which after some manipulation, can be shown to be identical to Eq. (25). The large- Ω , small- T_0 , $\tilde{D}_{Rc}$ limit of Eq. (44), namely

$$\lim_{\Omega \gg 1} \Delta'_B = -\frac{1}{8\lambda} \frac{\Gamma(-1/4)}{\Gamma(5/4)} \Omega^{5/4} , \quad (48)$$

must match the small- Ω limit of some intermediate region which, in turn, acts to bridge the gap between the deeply resistive region and the Weiland-Chen theory.¹⁰

B. Intermediate Region

We proceed as before to the exclusion only of the subsidiary ϵ -ordering of Ω and ξ

$$\tilde{\Omega} = \gamma/\lambda\omega_A, \quad \vartheta = \lambda\chi.$$

Then, to leading order, the averaged resistive equations reduce to

$$\frac{\partial}{\partial \vartheta} \left(\frac{\vartheta^2}{1 + \vartheta^2/\tilde{\Omega}} \frac{\partial \Phi}{\partial \vartheta} \right) - \tilde{\Omega}^2 \vartheta^2 \Phi = 0,$$

whose solution is

$$\Phi = \tilde{\Omega}^{7/8} \vartheta^{-2} \frac{\partial}{\partial \vartheta} \left[\vartheta^{1/2} W_{\kappa, \mu}(\tilde{\Omega}^{1/2} \vartheta^2) \right],$$

where $W_{\kappa, \mu}(z)$ is Whittaker's function.²² The small- ϑ limit of Φ , given by

$$\lim_{|\vartheta| \ll 1} \Phi = \frac{4\sqrt{\pi} \tilde{\Omega}^{3/2}}{\Gamma[(\tilde{\Omega}^{3/2} - 1)/4]} \left\{ 1 - \frac{\tilde{\Omega}^{5/4}}{8|\vartheta|} \frac{\Gamma[(\tilde{\Omega}^{3/2} - 1)/4]}{\Gamma[(\tilde{\Omega}^{3/2} + 5)/4]} \right\}, \quad (49)$$

must match onto the ideal solution. The question as to which solution dominates as one approaches the ideal limit, i.e., $\chi \sim 1$, depends on the relative magnitude of the two coefficients

$$1 : \frac{|\tilde{\Omega}|^{5/4}}{\vartheta} \sim S_M^{3/5} \frac{|\bar{\omega}_{dm}|}{\omega_A}. \quad (50)$$

Thus, for example, if $|\bar{\omega}_{dm}|/\omega_A \gtrsim S_M^{-1/3} \gg S_M^{-3/5}$, the ϑ^{-1} term is the large solution and would dominate the constant Φ term. Then, matching is more easily performed in terms of the new variable $\psi = \vartheta\Phi$ since now, one can invoke a constant- ψ approximation. This is precisely what was done by Weiland and Chen.¹⁰ To see that we can indeed recover the Weiland-Chen theory in the limit of $|\omega| \gg \omega_R$, i.e., $|\tilde{\Omega}| \gg 1$, we make a large argument expansion of the gamma function to obtain

$$\lim_{\vartheta \ll 1 \ll |\tilde{\Omega}|} \Phi \propto 1 + \frac{1}{8i\tilde{\Omega}\chi}, \quad (51)$$

where $\tilde{\Omega}_r = \omega/\omega_A$, which is the same as the small- z limit of Eq. (20) in Ref. 10. Hence, the intermediate region dispersion relation in the notation of Weiland and Chen becomes

$$\delta\tilde{W}_c + \delta\tilde{W}_k = -\lambda\tilde{\Omega}^{-5/4} \frac{\Gamma[(\tilde{\Omega}^{3/2} + 5)/4]}{\Gamma[(\tilde{\Omega}^{3/2} - 1)/4]} \quad (52)$$

Given that the dispersion relation of Ref. 10, which can be recovered in the large- $\tilde{\Omega}$ limit of Eq. (52), i.e.,

$$\delta\tilde{W}_c + \delta\tilde{W}_k = i \frac{\omega}{\omega_A} \quad (53)$$

has precisely the same form as that governing fishbone instabilities⁹ (i.e., energetic trapped particle-induced ideal internal kink modes), it is not too surprising to find that the present dispersion relation is identical in form to that governing hot particle-induced resistive internal kink modes.¹² On the other hand, the dispersion relation in the deeply resistive limit, i.e., $|\tilde{\Omega}| \ll 1$, becomes

$$\delta\tilde{W}_c + \delta\tilde{W}_k = \lambda \left| \frac{\Gamma(5/4)}{\Gamma(-1/4)} \right| \left(\frac{|\tilde{\omega}_{dm}|}{\omega_R} \right)^{-5/4} e^{i5\pi/8} \tilde{\Omega}^{-5/4} \quad (54)$$

where $\tilde{\Omega} = \omega/|\tilde{\omega}_{dm}|$. Equation (54) has been analyzed exhaustively in Ref. 12. In particular, it is shown there that for a slowing-down distribution function, the threshold is given by

$$\langle \beta_{h,t} I_{0S} \rangle_{cr} = \frac{\sin 3\pi/8}{\pi} \left| \frac{\Gamma(5/4)}{\Gamma(-1/4)} \right| \left(\frac{|\tilde{\omega}_{dm}|}{\omega_R} \right)^{-5/4} \lambda \tilde{\Omega}_{cr}^{-5/4} \sim \lambda \left(\frac{|\tilde{\omega}_{dm}|}{\omega_R} \right)^{-5/4} \quad (55)$$

so that

$$\frac{(\beta_{ph})_{cr}^{ideal}}{(\beta_{ph})_{cr}^{resistive}} \sim \left(\frac{\omega_R}{|\tilde{\omega}_{dm}|} \right)^{5/4} \sim 1 \quad (56)$$

for $|\tilde{\omega}_{dm}| \sim \omega_R$. Note, moreover, that in this same limit, the constant term in Φ is the large solution, as is clear from Eqs. (49) and (50). Then, Δ' is given by

$$\lim_{\tilde{\Omega} \ll 1} \Delta' = -\frac{1}{8\lambda} \frac{\Gamma(-1/4)}{\Gamma(5/4)} \tilde{\Omega}^{5/4} \quad (57)$$

which precisely matches the large- $\tilde{\Omega}$ limit of the deeply resistive regime, i.e., Eq. (48).

VI. CONCLUSION

By adopting a maximal ordering scheme, we have systematically developed a general theory to investigate the influence of energetic particles on resistive interchange-ballooning modes. In the resistive limit where the hot particle precessional drift frequency is weaker than the characteristic inverse resistive interchange time scale (i.e., $\bar{\omega}_{dm} < \omega_R$), we have shown that a population of deeply trapped suprathermal particles, such as those that are created during auxiliary heating in the region of adverse curvature in tokamaks, can resonantly interact with and destabilize resistive interchange modes. It should be remarked that this trapped-particle-driven mode is a new and separate branch of the resistive interchange mode. In particular, it will assert itself even if the resistive MHD branch is stable, as is the case in tokamaks because of favorable average curvature. The destabilization mechanism is closely akin to the beam-plasma variety and as such, has the character of a negative energy/dissipation trapped particle precession mode which becomes positively dissipated at the resistive layer. The mode is triggered when the hot particle beta exceeds a threshold value given by

$$(\epsilon\beta_{ph})_{cr} \sim \epsilon^2 \max(|\tilde{D}_{Rc}|, (\frac{|\bar{\omega}_{dm}|}{\omega_R})^{3/2}) \sim O(\epsilon^2) .$$

Noting that unlike the ideal theory of energetic trapped particle-induced ballooning modes as propounded by Weiland and Chen,¹⁰ the threshold condition is magnitude-wise insensitive to the choice of the hot particle distribution function; we feel that the present theory provides a more adequate explanation for the high-frequency precursors to the so-called fishbone oscillations as observed on PDX. Needless to say, all the positive features of the Weiland-Chen theory, as enumerated in the concluding remarks to their paper, are preserved in the resistive theory.

The highly localized nature of the mode has necessitated an assessment of hot particle banana width (ρ_b) effects. We have shown that when the latter exceeds in extent the width of the resistive layer (ρ_R), the threshold for instability is raised by a factor $(\rho_b/\rho_R) \ln(\rho_b/\rho_R)$, a fact which has intuitive appeal since the fatter the hot particle banana, the less likely that it will "see" and thus interact with the localized resistive mode. Finally, the resistive interchange-ballooning dispersion relation¹⁴⁻¹⁶ was generalized to

include the influence of hot trapped particles and the connection between dispersion relations in various regions of ballooning space established.

The linear picture having thus emerged, it becomes relevant to enquire as to the implications of this mode for anomalous transport. Such a study necessitates a nonlinear calculation which is currently under way and will be reported in a future publication.

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FIGURE CAPTIONS

- Fig. 1 Contour encircling the upper half complex frequency plane, used in the Nyquist analysis.
- Fig. 2 Nyquist contour in the image D plane for the case of no hot particles.
- Fig. 3 Nyquist contour for the case of deeply trapped energetic particles and favorable average curvature ($D_{Re} < 0$).
- Fig. 4 Nyquist contour for the case of deeply trapped particles and unfavorable average curvature. :
- Fig. 5 Nyquist diagram for the case of sloshing energetic particles.
- Fig. 6 Separation of regions for the case of the finite banana width calculation.
- Fig. 7 Separation of regions for the energetic trapped particle-induced resistive interchange-ballooning mode calculation.

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Fig. 1

Fig. 2

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Fig. 3

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Fig. 4

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Fig. 5

Fig. 6

Fig. 7