MHD COMPUTATIONS FOR STELLARATORS

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ABSTRACT

Considerable progress has been made in the development of computational techniques for studying the magnetohydrodynamic equilibrium and stability properties of three-dimensional configurations. Several different approaches have evolved to the point where comparison of results determined with different techniques shows good agreement.

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I. INTRODUCTION

The good confinement observed in stellarators in the early 1980's has rekindled interest in this approach to magnetic confinement for fusion [1]. At the same time it has become recognized that tokamaks and other axisymmetric devices usually exhibit three-dimensional effects that affect their containment properties. Thus, it is becoming increasingly important to understand the nature and limitations of three-dimensional configurations. Since these systems possess no symmetries, the problem is quite difficult. The early work was mostly analytical, characterized by vacuum field mappings, expansion about force-free cylindrical models, or expansion about a nearly straight magnetic axis. Recent improvements in computer hardware have made it possible to complement and extend these analytical studies with numerical investigations. Codes have been developed using a variety of different techniques and computational algorithms. Although there is room for considerable refinement and more sophisticated algorithms, the accomplishments that have been made are impressive. The emphasis of this paper will be on the physical models that are employed, rather than on the numerical details of the implementation. The purpose is to show that numerical techniques have progressed to the point where results obtained with different schemes are in quantitative agreement and that three-dimensional calculations are giving useful physical information. This shows that the recent experimental progress with stellarators is being matched by the improvement in our theoretical techniques and understanding.

The general field is often separated into several categories: vacuum field considerations, magnetohydrodynamic (MHD) equilibrium and stability, transport, heating, and confinement of guiding-center plasmas and individual particles. I intend to discuss ways of treating the vacuum field and MHD

equilibrium and stability problems in this review. Much progress has been made in these areas, but even more remains to be done.

The basic fluid model is quite simple. It consists of equations for mass and momentum conservation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0 \quad , \quad (1)$$

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = \underline{J} \times \underline{B} - \nabla p \quad , \quad (2)$$

Ohm's law,

$$\underline{E} + \underline{v} \times \underline{B} = n \underline{j} \quad , \quad (3)$$

an equation of state,

$$\frac{d}{dt} (p \rho^{-\gamma}) = 0 \quad , \quad (4)$$

and Maxwell's equations,

$$\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E} \quad , \quad (5)$$

$$\underline{J} = \nabla \times \underline{B} \quad . \quad (6)$$

In this model

$$\rho \equiv (n_i m_i + n_e m_e) \quad , \quad (7)$$

$$\underline{v} \equiv (m_i \underline{v}_i + m_e \underline{v}_e) / (m_i + m_e) \quad . \quad (8)$$

Equations (1) and (2) are obtained by multiplying the Boltzmann-Vlasov equations for the individual species by m and m_e , respectively, integrating over velocity space, and adding them. Equation (3) results from subtracting them. The pressure p and resistivity η are introduced phenomenologically to account for interaction between the particles. The derivation can be partially justified by a short mean-free-path assumption [2] or by an expansion where the ratio of the gyration radius to the size of the system is small [3]. It describes plasma behavior on a space scale much longer than the Debye length so that $n_i \sim n_e$; the displacement current is ignored in Eq. (6) and the main constraint imposed by charge neutrality is that

$$\nabla \cdot \underline{J} = 0 \quad , \quad (9)$$

which follows from Eq. (6). Most of the simplifications and limiting assumptions are made in reducing Ohm's law to Eq. (3). Among other simplifications, we have ignored the effects of Hall currents, finite-particle gyration radius, and electron inertia. Even so, for hot plasmas the behavior is well represented by setting the left-hand side of this equation equal to zero, except in thin layers near magnetic surfaces on which the perturbations from equilibrium follow the magnetic field lines so that $\nabla \times (\underline{v} \times \underline{B})$ goes to zero. The phenomenological $\eta \underline{J}$ term is usually introduced to represent the physics that enables an electric field along $\underline{B}$ to exist. This allows breakage of the magnetic field lines [4] resulting in resistive instabilities and configurational change such as magnetic island formation. These resistive effects will be ignored in much of this review. Similarly, treating the

pressure tensor as a scalar provides another major simplification. In this case p is constant on a magnetic surface labeled by Ψ , such that $\underline{B} \cdot \nabla \Psi = 0$. The label Ψ is often taken to be the poloidal or toroidal flux inside the surface. For most studies of plasma behavior on an MHD time scale, it is adequate to ignore the details of thermal transport and use the simple equation of state given by Eq. (4) to close the system of equations.

The physical properties of the equilibrium solutions of these equations are quite complex, but a qualitative understanding of them is quite easily achieved [5]. A number of the properties of an equilibrium can be inferred from a consideration of a two-dimensional cross section of the toroidal system. If we follow a magnetic field line which passes through a given point on this cross section toroidally around the system and note its point of intersection each time it passes through the cross section, we can generate magnetic surfaces such as those in Fig. 1. The surfaces form chains of islands in regions near where the rotational transform

$$i = d\Psi/d\Phi \quad (10)$$

is rational, with Φ the toroidal flux and Ψ the poloidal flux inside a magnetic surface. The function i measures the average twist of a magnetic field line. We drop a factor of 2π that was employed in the older literature, but do not adopt the "iota-bar" symbol that has been used more recently for this function. In many cases, such as where a periodicity length can be introduced such that the transform over this length is small, the islands can be very thin. Then it is often convenient to treat the magnetic surfaces as a set of nested toroids. In this case it is useful to construct a set of coordinates in which the magnetic field lines appear straight. Indeed, Hamada

has shown that such a coordinate system can be constructed with a unit Jacobian, in which case the current lines are also straight. This makes the coordinates singular on the rational surfaces. This system has proven useful for analytic work, but the technique for construction is so difficult that it has received little or no use in computational application. The condition for the magnetic islands to be thin is quite stringent, but it can be satisfied if there exists a periodicity length such that the rotational transform over this period is small. Probably the single most important problem in stellarator research at the present time is the determination of the width and structure of these magnetic islands.

Although Hamada coordinates have not proven to be useful computationally, Boozer and his colleagues [6] have provided a technique for numerically constructing a coordinate system with straight magnetic field lines and a Jacobian proportional to $1/B^2$. In this case the field lines and their orthogonals, but not the current lines, are straight and the coordinates are always well behaved. They start by writing the magnetic field as

$$\underline{B} = \nabla\psi \times \nabla\theta_0 = \nabla\chi + B \nabla\psi \quad , \quad (11)$$

$$\theta = \theta_0 + \iota \phi \quad ,$$

$$\partial B / \partial \chi = (d\iota/d\psi) / B^2 \quad ,$$

where ψ is a poloidal flux, χ is a scalar potential for the magnetic field, θ is the poloidal angle, ϕ is the toroidal angle, and ι is the rotational transform. The coordinates $R = R(\psi, \theta, \chi)$, $Z = Z(\psi, \theta, \chi)$, the metric elements of the system, and their Fourier decomposition in θ and χ can be determined by

integrating along some magnetic field lines far enough to define the magnetic surfaces.

Quite a few groups are working in the area of MHD equilibrium and stability using different codes which do essentially the same thing. In most cases the codes are used to determine equilibrium properties. The typical output consists of contour plots of the flux surfaces at several toroidal cross sections, the rotational transform and average magnetic well $\Delta V'(\phi)/V'(\phi)$ (with V the volume inside a surface of toroidal flux ϕ and primes denoting derivatives with respect to the argument) as functions of ϕ , and the shift of the magnetic axis with increasing pressure as a function of the pressure or mesh size. I will not try to describe them all in detail, but will attempt to give enough material on a few typical models to make it clear what can be done. Work with vacuum configurations will be considered in the next section. One-dimensional models will be discussed in Sec. III, two-dimensional ones in Sec. IV, and fully three-dimensional work in Sec. V.

II. VACUUM-FIELD CONFIGURATIONS

Most vacuum-field calculations fall into one of two types of approach, which can be differentiated by the boundary conditions that are prescribed.

The most straightforward approach is where currents in external coils are prescribed and a Biot-Savart law is used to calculate the magnetic field at any point. Field-line trajectories can then be determined so as to construct magnetic surfaces and evaluate equilibrium properties such as the rotational transform, the shear $dt/d\phi$, the average magnetic well $\Delta V'/V'$, with V the volume inside a surface and the prime denoting a derivative with respect to the toroidal flux, etc. A good, well-documented code that does this by tracing field lines which are determined from filamentary coils has been

written at Princeton [7]. Others have been written elsewhere.

The other approach is a fixed-boundary one where the shape of a perfectly conducting wall defines an outer magnetic surface. This problem is frequently solved by introducing a scalar potential χ and solving the Laplace equation

$$\nabla^2 \chi = 0 \quad (12)$$

inside this surface, subject to the boundary condition $\mathbf{n} \cdot \nabla \chi = 0$.

It is often possible to introduce a Green's function and transform the problem into one of solving an integral equation. This has been done successfully for helically symmetric systems (i.e., in the infinite aspect ratio limit) [8]. In an (r, ζ, z) coordinate system with $\zeta = \theta - hz$, the Green's function is

$$G(r, \zeta, z; r', \zeta', z') = \frac{1}{(r^2 + r'^2 - 2rr' \cos[\zeta - \zeta' + h(z - z')]) + (z - z')^2}^{1/2}. \quad (13)$$

Merkel decomposed this into a series of products of modified Bessel functions and showed that its singularities can be summed up in closed form so that the series converges rapidly. The introduction of a discrete mesh reduces the problem to a straightforward one of matrix inversion.

Dommaschk [9] has developed a set of easily computable field harmonics which is especially useful for representing stellarator vacuum fields. The solution is obtained by representing the stream function χ as a superposition of these field harmonics and equating the Fourier coefficients in the boundary condition expressions for the different terms in $\exp im\theta$. Thus, in an R, ϕ, Z cylindrical coordinate system, one can write

$$\chi = \sum_{m,l} [a_{ml} D_{m,l}(Z,R) + b_{ml} N_{m,l-1}(Z,R)] \cos m\phi, \quad (14)$$

where $D_{m,l}$ and $N_{m,l-1}$ satisfy recursion relations beginning with

$$D_{0,0} = 1, \quad D_{0,1} = Z,$$

$$D_{0,2} = Z^2/2 + (1 - R^2 + 2 \ln R)/4,$$

...

$$D_{m,0} = (R^m + R^{-m})/2, \quad m = 1, 2, \dots,$$

$$D_{m,1} = Z(R^m + R^{-m})/2, \quad m = 1, 2, \dots,$$

$$D_{m,2} = Z^2(R^m + R^{-m})/4 + \{R^m[m(1-m)R^2 + (m+1)(m-2)] + R^{-m}[m(1+m)R^2 - (m-1)(m+2)]\}/[8m(m^2-1)], \quad m = 2, 3, \dots,$$

$$D_{1,2} = Z^2(R + 1/R)/4 - (R^3 + 4R \ln R - 4R + 3/R)/16,$$

...

$$N_{0,0} = \ln R, \quad N_{0,1} = Z \ln R,$$

...

$$N_{m,0} = (R^m - R^{-m})/2m, \quad m = 1, 2, \dots,$$

$$N_{m,1} = Z(R^m - R^{-m})/2m, \quad m = 1, 2, \dots,$$

...

Another approach for systems like Helic with strongly twisted magnetic axes has been given by Shestakov and Mirin [10]. They transform to a

nonorthogonal coordinate system in which the boundary looks circular, and Fourier decompose in the two angular coordinates and discretize in the radial one. This leads to a block-tridiagonal system of equations that can be solved by standard routines.

Merkel [11] has developed a scheme for determining the vacuum field that must be present to make $\mathbf{B} \cdot \mathbf{n} = 0$ on a prescribed surface $S(\underline{x})$, with $\mathbf{n}$ a vector normal to the surface, by converting Laplace's equation for the scalar potential $\chi(\underline{x})$ to the form

$$\chi(\underline{x}) + \frac{1}{2\pi} \int dS' \frac{\partial G(\underline{x}, \underline{x}')}{\partial n'} \chi(\underline{x}') = \frac{1}{2\pi} \int dS' G(\underline{x}, \underline{x}') \frac{\partial \chi(\underline{x}')}{\partial n'} \quad (15)$$

where $G(\underline{x}, \underline{x}') = 1/|\underline{x} - \underline{x}'|$ is the Green's function and $\underline{x}$ and $\underline{x}'$ are on the surface S . Decomposition in terms of two angle variables u and v transforms this integral equation to an infinite set of linear equations for the Fourier coefficients $\chi_{m,n}$. The singularity of the Green's function is treated by subtracting a function with the same singular behavior as $G(\underline{x}, \underline{x}')$ from G and adding its analytically calculated Fourier transform to the transformed integral equation. Once the potential χ is known on the surface, the magnetic field can be calculated at any point with the aid of Green's theorem. This approach should be useful not only for determining vacuum fields in three-dimensional systems, but also as a technique for calculating the vacuum field contributions in free-boundary equilibrium problems.

III. ONE-DIMENSIONAL MODELS

Toroidal curvature was neglected and the system was treated as a straight periodic system in the earliest work on conventional stellarators [12]. The helical field, the force-free current, the pressure, and the aspect ratio were

ordered small,

$$(B_\delta/B_0)^2 = \sigma a - \beta - \epsilon \approx 2\pi a/L \ll 1, \quad ha \sim 1,$$

with

$$B_\delta \equiv \frac{\delta}{h} I_\lambda(hr) \sin(2\theta - hz),$$

$$\sigma \equiv \underline{J} \cdot \underline{B}/B_0^2,$$

$$\beta \equiv 2p/B_0^2;$$

$I_\lambda(x)$ is a Bessel function, $h \equiv 2\pi/L_p$ measures the pitch of the helical field with L_p the periodicity length, and a is the lowest-order plasma radius. This "stellarator-expansion ordering," as it is commonly called, is similar to the well-known large-aspect-ratio tokamak expansion except that the pressure is an order larger. It was selected so that the contributions to the rotational transform from the helical fields and from the Ohmic heating current enter in the same order, that the transform over the length of the machine be finite, and that useful stability predictions be obtained. The evaluation of quantities which vary on the L_p scale length can be done analytically, so that they contribute to the physics only through averages.

Determination of equilibria in this model is trivial, and stability can be studied with the energy principle. The problem reduces to solving the equation

$$\frac{1}{r} \frac{d}{dr} r \frac{du}{dr} - \frac{\sigma}{r^2} u = 0, \quad (16)$$

$$\mu \equiv (rB_0/L)v \xi_r ,$$

$$\alpha \equiv m^2 + \frac{2\pi}{rv} \frac{d}{dr} \left(r^3 \frac{d\xi_\theta}{dr} \right) + \frac{2Lrp'V''}{v^2} ,$$

$$v \equiv 2\pi(\iota_\sigma + \iota_\delta - n/m) ,$$

with the boundary conditions that μ vanish at $r = 0$ and be continuous at the plasma-vacuum surface. Here μ is the radial component of the perturbed magnetic field, ξ is the displacement vector, ι_σ and ι_δ are the rotational transforms associated with the Ohmic heating current and the helical field, respectively, m and n are the mode numbers in θ and z associated with a periodicity length L , and V'' measures the average curvature of the magnetic field lines. As noted in Ref. 5, Newcomb has shown that the system is unstable for given values of m and n if a solution of this equation satisfying the boundary conditions at $r = 0$ vanishes inside the surface where $m\iota - n = 0$. Otherwise the system is stable. (He gives a prescription for investigating the behavior when there are two or more surfaces where $m\iota - n = 0$ with the same values of m and n .) If pressure gradients are nonzero at this surface, it is easy to construct a Mercier criterion for localized modes. Thus, numerical solution has been done mainly to investigate the weaker kink instabilities driven by the force-free Ohmic current crossed into the perturbed magnetic field where a local treatment is inadequate [13].

One can use the stellarator expansion which was described at the beginning of this section to reduce the full time-dependent model, Eqs. (1) through (6), to a one-dimensional problem by averaging over the helical period L_p analytically. The linearized model that is obtained with this procedure is

the same as that used in the original treatment of resistive instabilities. It is useful for discussions of tearing modes [13]. It can be pushed into the nonlinear regime [14] and has been employed for the investigation of the evolution of both ideal and resistive instabilities even though it neglects possibly short-wave-length couplings. Wakatani and his collaborators have made extensive use of this model for their studies of critical β 's in HELIOTRON E.

IV. TWO-DIMENSIONAL MODELS

Much progress has been made in the study of two-dimensional configurations and the results of these calculations have provided reliable insight into stellarator physics. Two distinctly different approaches have been used in this area. Studies of helically invariant configurations are needed because of interest in the Heliac device, a strongly twisted helical column with a large aspect ratio so that it can be approximated as a straight system. The ERATO code [15], which was developed to study ideal linear stability of tokamaks, has been modified for this purpose. We discuss this modification, the HERA code [16], in part A of this section. On the other hand, introduction of toroidicity into the stellarator expansion of the last section is also straightforward. It is useful for systems where the magnetic axis is nearly straight. The STEP code (stellarator expansion using PEST) [17,18] utilizes this model to modify the axisymmetric PEST code [19] and has been used at several laboratories for design and data interpretation studies. It and several other codes which use similar approximations are described in Part B. An equilibrium scheme where $R(\psi, \theta)$ and $Z(\psi, \theta)$ are found, rather than the usual approach determining $\psi(R, Z)$, has also been developed with the same assumptions as those made in the stellarator-expansion approximation. It is discussed in Part C.

A. Helical Invariant Configurations

For helically invariant configurations, it is convenient to introduce the scalar functions u , Ψ , and T , such that [12]

$$B_r = \partial\Psi/r\partial u \quad ,$$

$$B_\theta - hrB_z = -\partial\Psi/\partial r \quad ,$$

$$hrB_\theta + B_z = T \quad ,$$

and

$$u \equiv \theta - hz \quad .$$

Then the pressure p and stream function T are functions of Ψ only, which must satisfy

$$\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r}{(1+h^2 r^2)} \frac{\partial\Psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\Psi}{\partial u^2} = -p'(\Psi) - \frac{T(\Psi)T'(\Psi)}{1+h^2 r^2} + \frac{2hT(\Psi)}{(1+h^2 r^2)^2} \quad . \quad (17)$$

The surface label Ψ can be shown to be the difference between the flux-the-long-way and the flux-the-short-way around the magnetic axis measured over one helical period $2\pi/R$. Equation (17) reduces to the usual Grad-Shafranov equation for a straight system if h is made zero, or for an axisymmetric system in the limit where $h \rightarrow \infty$. In the HERA code [16], a (Ψ, χ, z) coordinate system is adopted with

$$dx = \frac{\rho d\theta}{J(\partial\psi/\partial\rho)} \quad , \quad (18)$$

where the Jacobian is

$$J = (1 + h^2 r^2) q / T \quad ,$$

the safety factor

$$q = \frac{T}{2\pi} \int_0^{2\pi} \frac{\rho d\theta}{(1+h^2 r^2)(\partial\psi/\partial\rho)} \quad ,$$

and ρ and θ are polar coordinates centered on the magnetic axis. Modification of the ERATO formalism to incorporate Eqs. (17) and (18) is straightforward. The code employs a Fourier decomposition in z where the different modes separate and uses a Galerkin procedure with two-dimensional hybrid elements. The vacuum can be treated by using the Green's function of Eq. (13), or by using hybrid elements in the vacuum region and keeping p and J zero.

B. Stellarator Expansion

When toroidal effects are introduced into the model of Sec. III using a coordinate system such that

$$d\mathbf{x}^2 = dr^2 + r^2 d\theta^2 + [1 + (r/R) \cos \theta]^2 dz^2 \quad ,$$

the equilibrium problem reduces to

$$\psi = \psi_0 - \frac{1}{B_0} \left(\int B_\theta dz \right) \cdot \nabla \psi_0 + \dots \quad , \quad (19)$$

$$\Psi_0 = \Psi_v + 2\pi RA \quad ,$$

$$\Psi_v \equiv \frac{2\pi R}{B_0} \langle B_{r\delta} \int B_{\theta\delta} dz \rangle \quad ,$$

$$\underline{B}_0 = \nabla z \times \nabla A \quad ,$$

$$\nabla^2 A = 4\pi\sigma B_0 = -2\pi^2 R p'(\Psi_0)\Omega + G(\Psi_0) \quad , \quad (20)$$

$$\sigma \equiv \underline{J} \cdot \underline{B} / B^2 \quad ,$$

$$\Omega \equiv \frac{\langle B_\delta^2 \rangle}{B_0^2} + \frac{2r}{R} \cos \theta \quad ,$$

with $\langle \rangle$ denoting averages over the helical period [17]. The Lagrangian associated with small perturbations from this equilibrium is [18]

$$L = 2(T - \delta W) = \int \rho \omega^2 \xi^2 d\tau - \int (Q^2 + \sigma \underline{B}_0 \times \underline{\xi}^* \cdot \underline{Q} + \underline{\xi} \cdot \nabla p \underline{\xi}^* \cdot \nabla \Omega) d\tau \quad , \quad (21)$$

with

$$\underline{\xi} = \nabla z \times \nabla \eta$$

and

$$\underline{Q} = \nabla z \times \nabla \left[B_0 \frac{\partial}{\partial z} + \nabla z \times \nabla \Psi_0 \cdot \nabla \right] \eta \quad .$$

The equilibrium formalism differs from that for tokamak modeling in three ways: (1) The flux function Ψ_0 contains, in addition to the usual stream

function A which is associated with toroidal current, a contribution $\nabla_{\parallel}$ from the average field-line twist associated with the helical vacuum field. (2) The magnetic-field-line curvature contains a contribution from the helical field as well as the usual $(R + r \cos \theta)^2$ coming from the toroidal field. (3) The large-aspect-ratio assumption reduces the usual Δ^* operator in the Grad-Shafranov equation to ∇^2 and leads to incorporation of only the leading-order term in r/R into Eq. (20). The stability formalism is similar, but it is simplified by the condition that $\nabla \cdot \xi_{\perp} = 0$ to lowest order, which eliminates the magnetoacoustic waves and reduces the problem to a functional of only one dependent variable η . Thus it is easy to modify the PEST equilibrium and stability codes to incorporate these changes. The PEST equilibrium code uses a fast-Fourier transform technique to invert the Grad-Shafranov equation. In the stability problem, Fourier decomposition is made in θ and ϕ (or z), and finite-element cent functions are used for the radial coordinate. Since this STEP (Stellarator Expansion using PEST) code is presently the only operational free-boundary stability program, it has been used by several groups. It is also possible to get information about magnetic surface structure by using the equilibrium code to determine the part of the magnetic field associated with the stream function A , add it to the externally prescribed vacuum field, and use a magnetic field line following code to construct puncture plots [20]. A typical application is the study of the magnetic surfaces of WENDELSTEIN VII-A given in Fig. 2. This equilibrium model has also been implemented as the RSTEQ code [21] into an analogous tokamak code [22] at Oak Ridge.

Clearly, initial value codes based on this expansion can also be developed. Strauss and Monticello [23] developed one of the early ones. They obtained the averaged, reduced, nonlinear equations of motion

$$\frac{\partial A}{\partial t} = (B_0 \frac{\partial}{\partial z} - \nabla z \times \nabla \Psi_0 \cdot \nabla) u + \eta \nabla^2 A, \quad (22)$$

$$\rho \left(\frac{\partial}{\partial t} - \nabla z \times \nabla u \cdot \nabla \right) \nabla^2 u = (B_0 \frac{\partial}{\partial z} - \nabla z \times \nabla \Psi_0 \cdot \nabla) \nabla^2 A + \nabla z \times \nabla \Omega \cdot \nabla p, \quad (23)$$

$$\left(\frac{\partial}{\partial t} - \nabla z \times \nabla u \cdot \nabla \right) p = 0, \quad ,$$

with A , Ψ_0 , and Ω defined in conjunction with Eqs. (19) and (20) and the magnetic field stream function A , the fluid velocity u , and the pressure p the variables being advanced. To determine equilibrium configurations they add a viscosity term $\nu \nabla^4 u$ to Eq. (23) and restrict consideration to the two-dimensional problem with $\partial/\partial r = 0$. They introduced these modifications into the tokamak code HIB [24], which uses Fourier decomposition in θ and z and finite differencing in r . To study stability properties, the Oak Ridge group has implemented them in a modified version of the RST code [25] using a similar decomposition of the dependent variables. In this work they point out that it is not necessary to make the large-aspect-ratio approximation to get to a two-dimensional limit.

Kovrizhnykh and Shchepetov [26] have also shown that an averaging method similar to that of the stellarator expansion can be used to reduce the MHD equations to a two-dimensional form. In their application they have expanded about a set of concentric lowest-order magnetic surfaces. Since the Shafranov shift (the outward shift of the inner magnetic surfaces with respect to the outer ones due to the currents along the field lines that must be set up to keep $\mathbf{j}$ divergence free) is large in most cases of interest, they have had to carry many higher-order terms in the expansion. Thus, this attractive scheme has had little useful numerical application.

An interesting modification of the stellarator expansion for determining equilibria was given recently by Hender and Carreras [27]. They carry through the usual analysis but work in the coordinate system of Eq. (11) for the vacuum-field configuration. They break all the functions into components independent of ϕ and varying with ϕ in this system and introduce the usual orderings of the parameters. They recover the stellarator Grad-Shafranov equation from the ϕ -independent part of the equation in the form

$$\begin{aligned} & \frac{1}{\rho_v} \frac{\partial}{\partial \rho_v} \left(\rho_v \langle g^{\rho\rho} \rangle \frac{\partial \Psi}{\partial \rho_v} + \langle g^{\rho\theta} \rangle \frac{\partial \Psi}{\partial \theta_v} \right) + \frac{1}{\rho_v} \frac{\partial}{\partial \theta_v} \left(\langle g^{\rho\theta} \rangle \frac{\partial \Psi}{\partial \rho_v} + \langle g^{\theta\theta} \rangle \frac{1}{\rho_v} \frac{\partial \Psi}{\partial \theta_v} \right) \\ &= - \frac{F_v}{B_0} \langle \frac{1}{J_v} \rangle \frac{dp}{d\Psi} - F \frac{dF}{d\Psi} - \frac{B_0 F}{F_v} \left[\frac{1}{\rho_v} \frac{\partial}{\partial \rho_v} \left(\rho_v^2 \iota_v \langle g^{\rho\rho} \rangle \right) + \frac{1}{\rho_v} \frac{\partial}{\partial \theta_v} \left(\rho_v \iota_v \langle g^{\rho\theta} \rangle \right) \right]. \end{aligned} \quad (24)$$

Here ρ_v and θ_v form a cylindrical coordinate system centered on the magnetic axis in the vacuum configuration, $\langle g^{\rho\rho} \rangle$, $\langle g^{\rho\theta} \rangle$, and $\langle g^{\theta\theta} \rangle$ are the averaged metric elements, J_v is the Jacobian, ι_v the vacuum field rotational transform, $p(\Psi)$ and $F(\Psi)$ are the usual functions for the pressure and toroidal fields with F_v the vacuum field value, and Ψ is a poloidal flux function. The exciting feature of this work is that the toroidally varying components of the field can be expressed in terms of $\tilde{B}_\phi$, which must satisfy

$$\begin{aligned} & \frac{1}{\rho_v} \frac{\partial}{\partial \rho_v} \left[\rho_v \left(\langle \frac{g^{\rho\rho}}{J_v} \rangle \frac{\partial \tilde{B}_\phi}{\partial \rho_v} + \frac{1}{\rho_v} \langle \frac{g^{\rho\theta}}{J_v} \rangle \frac{\partial \tilde{B}_\phi}{\partial \theta_v} \right) + \frac{1}{\rho_v} \frac{\partial}{\partial \theta_v} \left(\langle \frac{g^{\rho\theta}}{J_v} \rangle \frac{\partial \tilde{B}_\phi}{\partial \rho_v} + \frac{1}{\rho_v} \langle \frac{g^{\theta\theta}}{J_v} \rangle \frac{\partial \tilde{B}_\phi}{\partial \theta_v} \right) \right] \\ &+ \frac{B_0}{F_v} \frac{\partial^2 \tilde{B}_\phi}{\partial \phi_v^2} = - \frac{1}{\rho_v} \frac{\partial}{\partial \rho_v} \left[\rho_v \left(\langle \frac{g^{\rho\rho}}{J_v} \rangle \tilde{p}_\rho + \langle \frac{g^{\rho\theta}}{J_v} \rangle \tilde{p}_\theta + \frac{\partial}{\partial \phi_v} \tilde{b}_\rho \right) \right] \\ &- \frac{1}{\rho_v} \frac{\partial}{\partial \theta_v} \left(\langle \frac{g^{\rho\theta}}{J_v} \rangle \tilde{p}_\rho + \langle \frac{g^{\theta\theta}}{J_v} \rangle \tilde{p}_\theta + \frac{\partial}{\partial \phi_v} \tilde{b}_\theta \right), \end{aligned} \quad (25)$$

where

$$\tilde{p}_\rho = \frac{(1/J_v)(\partial \langle p \rangle / \partial \rho_v)}{\langle B_\phi / J_v \rangle} ,$$

$$\tilde{p}_\theta = \frac{(1/J_v)\rho_v(\partial \langle p \rangle / \partial \theta_v)}{\langle B_\phi / J_v \rangle}$$

and

$$\tilde{b}_\rho = \left(\frac{\tilde{\rho}^{\rho\rho}}{J_v} \right) \langle B_\rho \rangle + \left(\frac{\tilde{\rho}^{\rho\theta}}{J_v} \right) \langle B_\theta \rangle ,$$

$$\tilde{b}_\theta = \left(\frac{\tilde{\rho}^{\rho\theta}}{J_v} \right) \langle B_\rho \rangle + \left(\frac{\tilde{\rho}^{\theta\theta}}{J_v} \right) \langle B_\theta \rangle ,$$

$$\langle B_\rho \rangle = \frac{B_o}{F_v \alpha_v} + \langle g^{\rho\rho} \rangle \frac{\partial \Psi}{\partial \rho_v} + \langle g^{\rho\theta} \rangle \frac{1}{\rho_v} \frac{\partial \Psi}{\partial \theta_v} + \frac{B_o \rho_v F}{F_v} \langle g^{\rho\theta} \rangle ,$$

$$\langle B_\theta \rangle = - \frac{B_o}{F_v \alpha_v} \langle g^{\rho\rho} \rangle \frac{\partial \Psi}{\partial \rho_v} + \langle g^{\rho\theta} \rangle \frac{1}{\rho_v} \frac{\partial \Psi}{\partial \theta_v} + \frac{B_o \rho_v F}{F_v} \langle g^{\rho\rho} \rangle ;$$

$$\alpha_v \equiv 1 + \left(\frac{B_o^2 \rho_v^2}{F_v^2} \right) \langle g^{\rho\rho} \rangle .$$

Everything in this equation is known once Eq. (24) has been solved. Thus, it is possible to determine the fields associated with the helical Pfirsch-Schlüter current to an order higher in the expansion than had been done previously. They employed an energy-minimization technique in the NAV code [21,27] to solve these equations, introducing a fictitious force and using Fourier decomposition in θ_v and ϕ_v to reduce the toroidally varying equation to a large set of ordinary differential equations. Working in the Boozer coordinate system provides several other advantages. Since the averaging that

leads to a two-dimensional problem does not have to be done over the toroidal angle at fixed r and θ , the resulting equations are much more robust for systems with a nonplanar axis, such as the Helic, than in the usual stellarator expansion. Similarly, the entire coordinate system space is used to represent the plasma region so that a larger useful volume is available [21].

C. Inverse-Solving Techniques

Significant progress has been made at the Kurchatov Laboratory in formulating and solving the stellarator equilibrium problem by making the specific variable approximation [28].

$$r = R - a_v (1 + \delta) \cos (\theta_v + \lambda) \quad ,$$

$$z = a_v (1 + \delta) \sin (\theta_v + \lambda) \quad ,$$

where $a_v(1 + \delta)$, $(\theta_v + \lambda)$, ϕ form a vacuum field flux coordinate system with straight magnetic field lines; a_v and θ_v are independent of ϕ with δ and λ associated with the potential

$$\phi = \frac{RB_0 c}{m} I_2 \left(\frac{ma_v}{R} \right) \sin(l\theta_v - m\zeta)$$

through

$$\delta = \frac{Rc}{ma_v} I_2' \left(\frac{ma_v}{R} \right) \cos(l\theta_v - m\zeta) \quad ,$$

$$\lambda = - \frac{R^2 c l}{m^2 a_v^2} I_2 \left(\frac{ma_v}{R} \right) \sin(l\theta_v - m\zeta) \quad .$$

Formally introducing the two-dimensional coordinates

$$X = R - a_v \cos \theta_v, \quad Y = a_v \sin \theta_v,$$

reduces the determination of the desired functions $X(a, \theta)$, $Y(a, \theta)$ to solving the equation

$$\frac{\partial \alpha_{22}^0}{\partial \psi} - \frac{\partial \alpha_{12}^0}{\partial \theta} = -4\pi^2 (\sqrt{g})^0 p'(\psi) - \frac{FF'(\psi)}{\alpha_{33}^0} - \frac{FD}{R^2} \frac{d(a_v^2)}{a_v da_v}. \quad (26)$$

Here $\alpha_{ik}^0 \equiv (g_{ik}/\sqrt{g})^0$ are the z -independent parts of the metric tensor elements in the flux coordinates a, θ, ζ . In the stellarator approximation, they are

$$\alpha_{12}^0 = (X' \dot{X} + Y' \dot{Y})/RD,$$

$$\alpha_{22}^0 = (\dot{X}^2 + \dot{Y}^2)/RD,$$

$$\alpha_{33}^0 = [X + Rh_1(a_v)]/D,$$

$$(\sqrt{g})^0 = [X - Rh_2(a_v)]/D,$$

$$D = \dot{X}Y' - X'\dot{Y},$$

where $X' = \partial X(a, \theta)/\partial a$, $\dot{X} \equiv \partial X(a, \theta)/\partial \theta$, and h_1 , h_2 , and the vacuum field rotational transform are determined by the external helical field as

$$h_1 = \frac{\epsilon^2}{2} \left[\left(\frac{l^2 + 1}{a_v^2} + \frac{m^2}{R^2} \right) I_z'^2 \left(\frac{ma_v}{R} \right) - \left(\frac{3l^2}{a_v^2} + \frac{m^2}{R^2} \right) \frac{R}{ma_v} I_z' \left(\frac{ma_v}{R} \right) I_z \left(\frac{ma_v}{R} \right) \right]$$

$$\begin{aligned}
& + \left(\frac{l^2}{a_v^2} + \frac{m^2}{R^2} \right)^2 \frac{R^2}{m^2} I_l^2 \left(\frac{ma_v}{R} \right)] , \\
h_2 = & \frac{\epsilon^2}{2} \left[\left(\frac{l^2+1}{a_v^2} \right) I_l'^2 \left(\frac{ma_v}{R} \right) - \left(\frac{3l^2}{a_v^2} + \frac{m^2}{R^2} \right) \frac{R}{ma_v} I_l' \left(\frac{ma_v}{R} \right) I_l \left(\frac{ma_v}{R} \right) \right. \\
& \left. + \left(\frac{l^2}{a_v^2} + \frac{m^2}{R^2} \right) \frac{l^2 R^2}{m^2 a_v^2} I_l^2 \left(\frac{ma_v}{R} \right) \right] , \\
v_v = & \frac{l \epsilon^2 R^4}{4 m^3 a_v^3} \frac{d}{da_v} \left[\frac{1}{a_v} \frac{d}{da_v} I_l^2 \left(\frac{ma_v}{R} \right) \right] .
\end{aligned}$$

Again, this formalism has been incorporated into an existing tokamak code. Stability studies for localized modes using Mercier and ballooning mode equations have been done with these equilibria.

V. THREE-DIMENSIONAL TREATMENTS

Much progress has been made towards addressing the three-dimensional problem. Several codes have been developed and are in active use. Again, the codes can be classified as Eulerian where the flux function ψ is determined on a fixed computational grid or Lagrangian where the inverse problem of finding $R(\psi, \theta, \phi)$ in flux coordinates is solved.

A. Fixed-Grid Formulations

The Chodura-Schlüter code [29] was one of the first successful implementations. In the program, the plasma is contained in a straight or toroidal domain of fixed rectangular cross section. The pressure, density, and field are prescribed initially at appropriate points on this computational mesh and the energy

$$W = \int [B^2/2 + p/(\gamma-1)] d\tau$$

minimized, subject to the constraints

$$\partial \underline{B} / \partial t = \nabla \times (\underline{v} \times \underline{B}) \quad ,$$

$$\partial p / \partial t = -\nabla \cdot \rho \underline{v} \quad ,$$

with $\underline{v} = 0$ at the conducting boundary and $p \rho^{-\gamma} = 1$. Thus

$$\frac{dW}{dt} = - \int \underline{E} \cdot \underline{v} d\tau \quad (27)$$

with

$$\underline{E}(\underline{r}, t) = (\nabla \times \underline{B}) \times \underline{B} - \nabla p \quad .$$

Different routes to the energy minimum are prescribed, such as

$$\underline{v} = \alpha \underline{E}(\underline{r}, t)$$

with α a positive number or operator representing friction. The minimization is achieved by discretizing Eq. (27) in both space and time and choosing a short enough time step to ensure numerical stability. Flux lines and surfaces are then obtained by integrating the magnetic field lines from the values at the grid points with a standard line-following code and marking the points of intersection with the different cross-sectional planes. Since the finite size of the mesh prevents minimization to the exact equilibrium, it is usually

necessary to do the calculation for several mesh sizes and then extrapolate to infinitely fine mesh.

These three-dimensional minimization techniques should lead to stable equilibria. If, however, an unstable mode of the final state had not been excited by the choice of $\mathbf{y}(\mathbf{r}, t)$ used in the course of the iteration, it would be necessary to continue the calculation by disturbing this state either in an ordered fashion by a Fourier mode or more or less stochastically. If the state were stable, the energy would rise with the disturbance and then return to its previous value because of the numerical viscosity. If the system were unstable, the energy would decrease to a lower value. Frequently a symmetry is imposed in the equilibrium which could also be relaxed by an unstable mode. The ability to investigate these problems has been built into the code.

A major advantage of this program is that the magnetic field lines need not form a single set of nested toroids. Indeed, the code is appropriate for treatment of systems with separatrices or island structure as in Fig. 1. Although, in principle, the initial magnetic topology with $p(\psi)$ and $i(\psi)$ specified should be conserved in the energy minimization, the finite viscosity allows the field lines to break and reconnect. Thus, the final state can have more physical significance than might have been expected. The code suffers, however, from several serious limitations. As written, the computational mesh fills a cylindrically symmetric region. Because of this and the condition that it has a rectangular cross section, only a small part of the mesh provides useful information in many applications. A more serious objection is that the equilibrium is determined by the initial prescription of the plasma parameters so that one has little control of the shapes of the pressure distribution or that of the rotational transform (or equivalently, the net current on each magnetic surface). Furthermore, since the viscosity and

friction associated with the finite mesh and the choice of $\mathbf{F}(\mathbf{r}, t)$ leads to loss of flux conservation, the final equilibrium is partially determined by the computational program. Nevertheless, results obtained with this program have proven useful.

An obvious improvement of this program would be to work in the vacuum-field flux-surface coordinate system of Eq. (11). This has been done at Oak Ridge in the equilibrium code NEAR [30]. The coordinates of the vacuum field surfaces, the metric elements, and their components in a Fourier decomposition in θ and ϕ are obtained by the techniques of Ref. 6 and the potential-energy minimization of Eq. (27) is effected using the conjugate gradient iteration scheme of Chodura and Schlüter [29]. Several useful refinements are introduced. Since the coordinate system is described in terms of a Fourier decomposition in θ and ϕ , the dependent variables are also Fourier analyzed. The dependent variables are chosen to be v^0 , (B^0/D_v) , p , etc., where the incorporation of the Jacobian D_v maintains the condition that $\mathbf{v} \cdot \mathbf{B} = 0$ throughout the calculation. They are represented in finite-difference form with v^0 and (B^0/D_v) discretized on a uniformly spaced radial mesh whose first point is the coordinate axis and whose last point is the wall. The remaining variables are on an intermediate mesh whose points lie halfway between. This Fourier representation requires that convolutions of Fourier series be calculated numerically. These are performed using simple trigonometric formulas for the products of sines and cosines. More recently, an additional outer iteration loop has been added to the code to permit the calculation of equilibria with zero net current [31]. Each time an equilibrium is obtained, the net current on each surface is calculated, a Poisson-type equation for the magnetic field associated with this current is solved; it is subtracted from the existing field, and a new flux-conserving equilibrium is computed.

Clearly, three-dimensional initial-value codes can also be developed. One application of such codes is for the evaluation of equilibrium configurations. This can be accomplished by introducing viscosity into the equation to advance the velocity to damp the kinetic energy and resistivity into Ohm's law to make the average parallel current vanish. However, a straightforward application requires a large amount of computing time because dissipative coefficients must be quite small for the final Ohmic configuration to be equivalent to an equilibrium state. An even larger obstacle is the implementation of the condition $\underline{B} \cdot \nabla p = 0$, which requires that the pressure conform to magnetic island shapes and be constant in stochastic regions. Park and his collaborators [32] have developed a method to circumvent these difficulties. They start with the usual MHD equations. To circumvent the relaxation problem, they note that dropping the convection term $\rho \underline{v} \cdot \nabla \underline{v}$ does not change either the equilibrium nor the linear stability properties, but makes the final Ohmic steady state depend only on the product of μ and η . Thus, they assume a large resistivity and a small viscosity so that the elimination of the kinetic energy occurs because of resistive Alfvén modes rather than the usually employed viscous damping. The pressure constraint is implemented by introducing an artificial velocity $\underline{v}_a$ along the magnetic field lines. Then the usual equation to advance the pressure is replaced by the system

$$\partial p / \partial t = \underline{B} \cdot \nabla \underline{v}_a, \quad (28)$$

$$\partial \underline{v}_a / \partial t = \underline{B} \cdot \nabla p.$$

The equilibrium state of this system can easily be found by an energy reduction method. The solution proceeds by alternating the solution of Eq. (28) with a fixed magnetic field and the relaxation process. At present, these methods have been implemented in the HIBS code [32], using an axisymmetric computational domain with a circular cross section.

The FAR code [33] is an incompressible initial-value code where the poloidal flux ψ , the toroidal flux χ , the poloidal velocity stream function ϕ , the toroidal velocity stream function γ , the electrostatic potential Φ , and the pressure p are advanced by the system of equations

$$\frac{\partial \psi}{\partial \tau} = -\frac{\partial \Phi}{\partial \zeta} - v^\theta B^\theta + v^\theta B^\rho + \eta j_\zeta, \quad ,$$

$$\frac{\partial \chi}{\partial \tau} = -\frac{1}{\rho} \frac{\partial \Phi}{\partial \theta} - v^\zeta B^\rho + v^\rho B^\zeta + \eta j_\theta, \quad ,$$

$$\frac{\partial \Phi}{\partial \rho} = -v^\theta B^\zeta + v^\zeta B^\theta + \eta j_\rho, \quad ,$$

$$\begin{aligned} \frac{\partial u^\theta}{\partial \tau} = & S^2 \left[\frac{\partial}{\partial \zeta} (v^\theta u^\zeta - v^\zeta u^\theta) + \frac{\partial}{\partial \rho} (v^\theta u^\rho - v^\rho u^\theta) - \frac{\beta_0}{2\epsilon^2} \left(\frac{\partial R^2}{\partial \zeta} \frac{\partial p}{\partial \rho} - \frac{\partial R^2}{\partial \rho} \frac{\partial p}{\partial \zeta} \right) \right. \\ & \left. + \frac{\partial}{\partial \zeta} (j^\theta B^\zeta - j^\zeta B^\theta) + \frac{\partial}{\partial \rho} (j^\theta B^\rho - j^\rho B^\theta) \right], \quad , \end{aligned}$$

$$\begin{aligned} \frac{\partial u^\zeta}{\partial \tau} = & S^2 \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} [\rho (v^\zeta u^\rho - v^\rho u^\zeta)] + \frac{1}{\rho} \frac{\partial}{\partial \theta} (v^\zeta u^\theta - v^\theta u^\zeta) \right. \\ & \left. - \frac{\beta_0}{2\epsilon^2} \left(\frac{\partial R^2}{\partial \rho} \frac{1}{\rho} \frac{\partial p}{\partial \theta} - \frac{1}{\rho} \frac{\partial R^2}{\partial \theta} \frac{\partial p}{\partial \rho} \right) \right. \\ & \left. + \frac{1}{\rho} \frac{\partial}{\partial \rho} [\rho (j^\zeta B^\rho - j^\rho B^\zeta)] + \frac{1}{\rho} \frac{\partial}{\partial \theta} (j^\zeta B^\theta - j^\theta B^\zeta) \right], \quad , \end{aligned}$$

$$\frac{\partial p}{\partial \tau} = -v^\rho \frac{\partial p}{\partial \rho} - v^\theta \frac{1}{\rho} \frac{\partial p}{\partial \theta} - v^\zeta \frac{\partial p}{\partial \zeta}, \quad (29)$$

with the magnetic field, fluid velocity, and vorticity given by

$$\underline{B} = \nabla\theta \times \nabla\chi + \nabla\zeta \times \nabla\psi \quad ,$$

$$\underline{v} = \frac{R^2}{R_0} (\nabla\theta \times \nabla\lambda + \nabla\zeta \times \nabla\phi) \quad ,$$

$$\underline{u} = \nabla \times \underline{v} \quad ,$$

where R and R_0 describe an equilibrium mass density $\rho_m = \rho_{m0}(R_0/R)^2$, η measures the resistivity, S is a ratio of the resistive to Alfvén time, β_0 is the equilibrium pressure at the axis, and ϵ measures the aspect ratio. The main use for this code is to investigate the stability properties of equilibria that have been obtained by other methods when the constraint of periodicity over the short helical length is relaxed. The calculation is usually made in a two-dimensional flux coordinate system with a perfectly conducting wall at the plasma surface, which was previously determined with a stellarator-expansion equilibrium solver. Both the equilibrium and dynamic quantities are Fourier analyzed in θ and ζ . A finite number of radial grid points is used for the calculation, with convergence to infinite mesh necessary for meaningful results. Although most of the present work has been in the linear regime, the program may be useful for investigation of the nonlinear evolution of stellarator configurations.

Several other three-dimensional initial-value codes have been developed which could be used to investigate behavior near an equilibrium configuration. A particular problem that has been addressed is the reconnection of magnetic field lines due to tearing modes that occur on a time

scale which is much longer than the one associated with the propagation of Alfvén waves around the torus. Useful results can be obtained by developing implicit algorithms for using time steps much longer than those tied to MHD behavior. Such a code, with the terms in the equations that drive compressional waves treated implicitly, has been constructed by Aydemir and Barnes [34]. Recently, Harned and Schnack [35] developed a semi-implicit scheme in which time step limitations are not imposed by numerical stability considerations associated with either compressible or shear Alfvén waves. They accomplish this with a predictor-corrector algorithm for advancing in time. The corrector velocity advance is given by

$$\begin{aligned} \underline{v}^{n+1} &= \frac{(\Delta t)^2}{\rho^*} \{ \underline{v} \times [\underline{v} \times (\underline{v}^{n+1} \times \underline{C})] \} \times \underline{C} \\ &= \underline{v}^n + \frac{(\Delta t)}{\rho^*} \underline{F}(\rho, \underline{v}, \underline{B}, p) - \frac{(\Delta t)^2}{\rho^*} \{ \underline{v} \times [\underline{v} \times (\underline{v}^n \times \underline{C})] \} \times \underline{C}, \end{aligned} \quad (30)$$

with ρ^* and $\underline{F}$ the predictor-advanced density and force associated with momentum balance. The algorithm is numerically stable if each component of $\underline{C}$ is larger than the analogous component of $\underline{B}$ and if the components of the semi-explicit terms containing $C_i C_j$ are replaced by $C_i C_j \delta_{ij}$ with δ_{ij} the usual delta function. Although these codes have been used mainly to investigate the evolution of plasmas in two-dimensional systems like tokamaks and reversed field pinches, they could be applied easily to stellarator studies.

A straightforward iteration method for determining stellarator equilibria which does not use energy minimization was proposed at the beginning of the fusion program [36]. The basic formulation is simple. Given a magnetic field $\underline{B}_k(\underline{r})$, and thus $p_k(\psi)$, one uses the equilibrium condition to determine $\underline{J}_{k+1}$. Then the field $\underline{B}_{k+1}$ associated with this plasma current is determined and the

iteration is continued until convergence is achieved. Because of the large amount of computation involved with this approach, it has only recently been implemented.

Kisslinger and Wobig [37] have written a code in which they express the current as $j_{n+1} = p'(\psi) \nabla \psi \times \nabla \psi$ and calculate the stream function v from

$$\nabla_{\perp} v = 1, \quad (31)$$

with the constant of integration chosen to make the current lines $v = \text{constant}$ poloidally closed curves in order to obtain a net-current-free stellarator. They avoid problems with magnetic islands by calculating j_{n+1} only on well-behaved magnetic surfaces. They replace the continuum of these plasma currents with a finite set of current filaments and then use the Biot-Savart law to determine the new field B_{n+1} at points on the B_n magnetic surfaces. These field lines are integrated to obtain the new surfaces and the procedure is continued.

Boozer [38] recently proposed another implementation of this iteration approach which is also being pursued [39]. The calculation is performed in the coordinate system of Eq. (11) using either the vacuum field or the field B_k . Integration along some magnetic field lines determines the coordinates, metric elements, the function $1/B_k^2$, and their Fourier decompositions. From this J_{k+1} can be obtained easily. At this point it becomes manifestly obvious that the Pfirsch-Schlüter current becomes infinite at a surface where the rotational transform is rational, $t = n/m$, unless $p'(\psi)$ or the m, n 'th Fourier coefficient of $\langle 1/B^2 \rangle$ vanishes there. Thus, the prescription of $p(\psi)$ can be adjusted in the course of the calculation to keep the problem well posed. The field $B_{k+1} = B_{\text{vacuum}} + b$, with b the part associated with the current in the

plasma, is then calculated. This is accomplished by writing

$$\underline{b} = \underline{h} + \nabla f \quad (32)$$

with

$$\underline{h} = \left(\int_{\psi}^{\psi_{\text{boundary}}} \frac{4\pi g(\psi)}{B_o^2} \frac{d\psi}{d\psi} \nabla \phi \right. \\ \left. + \left[\frac{4\pi g(\psi)}{B_o^2} \frac{d\psi}{d\psi} \int' \frac{\delta_{nm}}{(n-m)} \sin(n\phi - m\theta) \right] \nabla \psi \right) .$$

Here $g(\psi)$ is the current which provides the toroidal field and δ_{nm} is the m, n 'th Fourier coefficient of $\langle 1/B^2 \rangle$. The function h is chosen to vanish outside the plasma and have its curl equal to the plasma current inside. A Poisson equation for f is imposed to make $\underline{b}$ have zero divergence. The principal advantage of this approach is that it provides a clear indication of the physics that is taking place at each step of the iteration so that one can adjust the parameters of the system as one proceeds. Thus, it should be a useful tool for discussions of configurations where magnetic islands and stochastic regions are important. Since this approach is not based on a variational formulation, one has no assurance that it will converge to an interesting result. However, one can visualize it as a way of proceeding in small steps from a known configuration, such as inserting a low-pressure plasma into a vacuum system, converging to a solution, and then increasing the pressure.

Shestakov [40] has proposed a similar iteration using the implicit Fourier expansion method of Ref. 10. He determines the new magnetic field $\underline{B}_{k+1}$ by introducing a vector potential

$$\underline{b} = \nabla \times \underline{A}_k$$

and solving the three scalar Poisson equations

$$\nabla^2 A_{k,i} = -\frac{4\pi}{c} J_{k,i} \quad (33)$$

for its Cartesian components.

B. Inverse-Solution Techniques

The approach of defining a set of flux coordinates and varying the positions of the mesh points to satisfy the equilibrium constraints provides a powerful technique for determining three-dimensional configurations. It suffers from the need for assuming the existence of nested toroidal surfaces, but efforts are being made to relax this constraint. The first successful implementation of this approach was the BETA code, developed at New York University [41]. A good discussion of the existing codes that use this approach was given by Schwenn [42]. He notes that all of them introduce coordinates s, u, v , such that

$$\underline{B} = \nabla s \times \nabla \psi, \quad \psi = -\frac{dF_t(s)}{ds} u + \frac{dF_p(s)}{ds} v + \lambda(s, u, v), \quad (34)$$

where s labels the magnetic surface, $F_t(s)$ and $F_p(s)$ are toroidal and poloidal flux functions, respectively, and the stream function $\lambda(s, u, v)$ is a periodic function of the poloidal and toroidal coordinates u and v . By introducing

constraints relating the pressure and density, imposing conservation conditions on the total mass and magnetic fluxes inside surfaces of constant s , using modified cylindrical variables such as R_0 (the major radius), $r(s,u,v)$, $z(s,u,v)$, and relating the toroidal variable v to the angle ϕ in the cylindrical system, $v = \phi$, the energy integral can be put into the form

$$W = \frac{1}{2} \int \frac{1}{D} \left[\left(\frac{2\pi R_0}{N} \right)^2 \left(1 + \frac{r}{R_0} \right)^2 + \left(\frac{\partial r}{\partial v} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2 \right] \left(\frac{\partial \psi}{\partial u} \right)^2 + \left[\left(\frac{\partial r}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial u} \right)^2 \right] \left(\frac{\partial \psi}{\partial v} \right)^2 - 2 \left(\frac{\partial r}{\partial u} \frac{\partial r}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} \right) \frac{\partial \psi}{\partial u} \frac{\partial \psi}{\partial v} ds du dv . \quad (35)$$

The determination of equilibrium configurations then consists of minimizing this functional. In some codes it is done by using the conjugate gradient approach employed by Chodura and Schlüter in their treatment of Eq. (27). In others, it is accomplished by introducing artificial acceleration and damping terms such that, if ξ represents the dependent variable and $L(\xi) = 0$ at equilibrium, the minimization equation takes the form

$$\gamma \frac{d^2 \xi}{dt^2} + \delta \frac{d\xi}{dt} = -L(\xi) . \quad (36)$$

The major differences between codes are in the choices of the coordinates, such as how v and ϕ are related, and in the selection of minimization techniques, such as how δ/γ is chosen.

The best known and most extensively employed code of this type is BETA [41,43]. The New York University group set $v = \phi$ and replace r and z by the functions $r_0(v)$ and $z_0(v)$ for the position of the magnetic axis and a single radius function $R(u,v,s)$, chosen to keep the $u = \text{constant}$ lines straight and to vary between 0 and 1 at the axis and on the plasma boundary $r_1(u,v)$,

$z_1(u,v)$. The dependent variables are then r_0 , z_0 , R , and λ . They introduce an accelerated steepest decent method, adjusting the ratio δ/γ in Eq. (36) to stay close to the dominant eigenvalue in the calculation at any given time. The principal difficulty in the scheme occurs at the magnetic axis $s = 0$, which is a singularity in the computational coordinate system. This problem is resolved by locally modifying the Jacobian. Although the system of equations can properly represent only systems with nested magnetic surfaces, it can give some indication of the nature and severity of magnetic islands. The inability of the representation to incorporate an island structure manifests itself in an ill-defined Jacobian, so that recognition of where this quantity becomes small is useful. As has been noted in connection with previously described three-dimensional formulations, the difficulties are associated with large Pfirsch-Schlüter currents on the rational surfaces. Calculations of these currents can provide significant understanding of the physics.

The results obtained with this model are strongly dependent on the size of the mesh, so that it is always necessary to carry through the calculations for several mesh sizes and then extrapolate to an infinitely fine mesh. The results that are typically presented are the shapes of the magnetic surfaces or current distributions for the finest mesh used, the extrapolated position of the magnetic axis, and the extrapolated value of the Mercier criterion for localized stability [5] on a set of magnetic surfaces. That this can be evaluated accurately enough to be useful is quite impressive.

As in other methods of determining three-dimensional equilibria, one can not be sure that the minimization algorithm has excited the most unstable mode of the system so that the obtained equilibrium is really the lowest-energy state. Of equal interest are the stability properties with respect to

perturbations with low mode numbers when periodicity constraints associated with the equilibrium determination are relaxed. To study these questions the energy integral of Eq. (35) is again minimized, subject to a constraint

$$\int \xi \cdot \xi_0 \, d\tau = \text{constant} \quad ,$$

where ξ_0 is a prescribed test function corresponding to a perturbation around the equilibrium. Although the final value of the energy after this minimization is always larger than that found in the equilibrium minimization for a given grid size, it frequently extrapolates to a lower value in the limit of an infinitely fine mesh, in which case instability of the equilibrium has been demonstrated. The calculation is complicated by two factors. First, the test function ξ_0 must contain a significant part of an unstable perturbation for the instability to manifest itself, especially because the use of a finite mesh makes it difficult to represent an unstable mode. The use of physical intuition or the results of other modeling can partially answer this problem by providing guidance in the choice of ξ_0 . Second, the subtraction of the energies for the perturbed and unperturbed systems, together with a further loss of significance in the extrapolation to zero mesh size, leads to a loss of significant figures, typically six for problems of physical interest. Nevertheless, as will be noted later, the stability predictions have been shown [19] to agree well with those obtained by other techniques.

Most of the other Lagrangian formalisms that have been developed employ Fourier decomposition in u and v in order to reduce the energy minimization problem to one of solving a large set of ordinary differential equations in the surface parameter s . They differ primarily in the choice of the dependent

variables. The use of these spectral techniques is motivated by the geometric considerations that the magnetic surfaces can be approximated well by only a few terms in the expansion of $R(u,v,s)$ and $Z(u,v,s)$, provided that the parameter λ is properly chosen.

Hirshman and his colleagues provided one of the early implementations of the spectral approach [44]. They adopt a pseudo-polar representation for R and Z , set $v = \phi$, use the condition that there be no net radial current to determine the spectral components of λ , and obtain from the equilibrium condition a set of coupled second-order ordinary differential equations for $R_{m,n}$ and $Z_{m,n}$. They expand the poloidal flux as a Taylor series in R and Z near the magnetic axis to obtain equations that resolve the singularity in the coordinate system. They too use Eq. (36), adjusting their dissipation to fit the dominant eigenvalue. More recently, they have developed techniques for optimizing the choice of the poloidal coordinates by properly choosing λ [45]. This has enable them to replace their earlier pseudo-spectral representation by an augmented Fourier representation concurrently with minimizing the potential energy [46].

A group at GA Technologies has a code [47] which is quite similar to the earlier Oak Ridge one [44]. One of the main advantages of their work is the use of an adaptive integration technique to obtain much better radial resolution than can be obtained with a standard gridded code. A novel feature of their treatment is an examination of the form of the highest-order derivatives in the equilibrium equations, which correspond to d^2/ds^2 . They show that the equations have the form

$$(B \cdot \nabla)^2 \frac{\partial^2}{\partial s^2} R_{m,n} = \text{lower-order terms,}$$

which is very singular on each surface where the magnetic field lines close. Since the truncated Fourier representation is not complete enough to express the functions that vanish under the $\mathbf{B} \cdot \nabla$ operator when $\lambda \neq 0$, the singularities do not affect the integrations that are actually performed in finding equilibria. This analysis, like the study of Boozer [37], should provide guidance to the development of useful codes as well as for the choice of the physical input parameters for obtaining useful solutions.

Schwenn's spectral code FIT [42] has much in common with Hirshman's latest code [46]. The dependent variables used are r and z with $v = \phi$. The stream function λ can be set equal to zero so that u is a genuine Lagrangian variable, or it can be treated as a dependent variable. Both approaches have been implemented. A conjugate gradient method for minimization, using a quadratic expansion of the energy with respect to all the Fourier coefficients of r and z , is employed. The variation with respect to the doubly periodic function λ provides a linear, elliptic equation on each magnetic surface which is solved directly. The set of equations that must be solved is large, but owing to its symmetric block triangular structure it can be solved. Good agreement between the results of this code with those obtained from BETA [43] and MOMCON [44] have been reported [48].

Bhattacharjee and his colleagues have also developed a spectral code [49]. They have two ways of choosing their representation of the dependent variables, either solving for $R(s,u,v)$, $Z(s,u,v)$ and $\phi(s,u,v)$ or for $R(s,u,\phi)$, $Z(s,u,\phi)$ and $v(s,u,\phi)$. These choices serve primarily to change the way the boundary conditions are expressed and provide a means for affecting the convergence properties of the numerical scheme for a given calculation. In the first approach, the inverse-mapping program, they typically set $v = \phi$ and use the equations from an energy variation to adjust u so that the magnetic

field lines are straight. In the latter, the mixed-mapping approach, they fix u and vary v by defining $v = \phi - \lambda(s, u, \phi)/i(s)$ and use the Fourier coefficients of λ as dependent variables. They solve the Euler-Lagrange equations obtained by varying $R_{m,n}$, $Z_{m,n}$ and either $\phi_{m,n}$ or $\lambda_{m,n}$ directly by a method of collocation at Gaussian quadrature points, expanding their dependent variables in a set of basis functions employing Hermite cubic B-splines.

Another approach [50] that may lead to useful computational techniques is to work with two coordinate systems, a "natural" one based on the contravariant representation of the vector $\underline{B}$ (where the magnetic field lines are straight) and a "conatural" one using the vector $\underline{B} \times \nabla s$ (where the current lines are straight). Here, the function s is constant on a magnetic surface. The "natural" system is characterized by the poloidal and toroidal flux functions $\psi(s)$ and $\phi(s)$; the "conatural" one uses the toroidal current $J(s)$ and external poloidal current $F(s)$. The conditions that $\underline{B}$ be divergence-free and that $\underline{B} \cdot \nabla s = 0$ and $\nabla \times \underline{B} \cdot \nabla s = 0$ provide relations between the metric elements that relate these two systems to a reference system a, θ, ζ . The magnetic field can be decomposed into the form

$$\underline{B} = \nabla \psi \times \underline{b}_F + F \underline{b}_\psi$$

where

$$\underline{b}_F = \frac{\partial \underline{r} / \partial v_F}{(\partial \underline{r} / \partial v_F) \cdot (\partial \underline{r} / \partial v_\psi)},$$

$$\underline{b}_\psi = \frac{\partial \underline{r} / \partial v_\psi}{(\partial \underline{r} / \partial v_F) \cdot (\partial \underline{r} / \partial v_\psi)},$$

with $\partial \underline{r} / \partial v_i$ the contravariant basis vector associated with the toroidal coordinate in the two respective coordinate systems. Then the equilibrium condition can be put into the form

$$\begin{aligned} \nabla \cdot \underline{b}_F^2 \nabla \psi = & - \frac{dP}{da} - F \frac{dF}{da} \underline{b}_\psi \cdot \nabla \psi + F \underline{b}_F \cdot \nabla \times \underline{b}_\psi \\ & - \underline{b}_F \times \underline{b}_\psi \cdot \nabla F + \underline{b}_F \times (\nabla \times \underline{b}_F) \cdot \nabla \psi - F \underline{b}_\psi \cdot \nabla v_\psi \end{aligned} \quad (37)$$

with

$$\left(\frac{d\phi}{da} \frac{\partial}{\partial v_\psi} - \frac{d\psi}{da} \frac{\partial}{\partial v_\phi} \right) v_\psi = \frac{dP}{da} \left[\frac{dV}{da} - \sqrt{g_\psi} \right] .$$

The equilibrium relations have a form similar to that which has been solved in two-dimensional cases. The new feature is the need to work with the two coordinate systems simultaneously. Neither set of reference vectors can be specified at the outset, but they must be reconciled with the solution $\psi(\underline{r})$ at each step of the computation. The formalism clearly shows the problem with three-dimensional configurations, that the m, n 'th Fourier component of v_ψ blows up at a surface where $m\psi' + n\phi' = 0$ unless $(\sqrt{g})_{mn}$ or $p'(\psi)$ equals zero there.

VI. COMMENTS

Much progress has been made in both the analysis and understanding of three-dimensional magnetohydrodynamic equilibria and in the development and application of computational techniques for determining them. The three primary areas of concern are the magnetic topology or surface structure of the configuration, equilibrium properties, and stability considerations. The

first of these is clearly the most serious. It has been said [51] that static equilibria obtained with the ideal MHD equations would have to possess either highly pathological pressure profiles or large singular currents. However, it can be shown that good surfaces can exist to all orders in an expansion where some parameter, such as the rotational transform per helical period, is small [5] and reasonable confinement in stellarators has been observed experimentally [1]. Thus, steady-state configurations which differ little from what can be calculated must exist in many configurations, and it is clear that the equilibrium concept that is being developed has some validity. The use of finite computational grids to avoid the singularities or the use of truncated Fourier series to weaken them provides reasonable expedients for investigating the problem. Even better, the use of formalisms where the plasma behavior near rational surfaces is made evident as the calculation proceeds [38,39] makes it possible to adjust the plasma parameters so as to avoid difficulties. Evaluation of the equilibrium properties such as rotational transform or net current on a magnetic surface, average field line curvature, etc., is straightforward provided the topology of the magnetic surfaces is sufficiently well behaved. Turning to the question of stability, it is well recognized that three-dimensional configurations obtained by minimizing an energy integral should be stable. However, it must be remembered that the chosen minimization path does not always lead to the lowest energy state, so that stability studies are often needed. Of even more interest, the calculated equilibria usually have some built-in symmetries, such as periodicity over some helical wave length, whose destruction by some longer-length perturbation could lower the potential energy even further. Since the change in energy associated with such a displacement is small, these considerations are difficult. The problem is quite important, however, since

it is similar to that in tokamak research where beta limits obtained from ideal MHD stability considerations agree well with experimental observation.

The results obtained with these different methods show remarkable agreement. A comparison of the equilibrium flux surfaces computed with the Chodura-Schlüter code [29] and the Oak Ridge NEAR code [30] for the $\ell = 2$, $M = 12$ torsatron configuration ATF-1 with $\beta(0) = 5\%$ is given in Fig. 3. Evaluation of the equilibrium shift and magnetic well profile with the two calculations shows remarkable agreement. A comparison of the results from the NAV code [27] with those from the NEAR code [30] for an ATF-1 model with $\beta(0) = 5\%$ also shows good agreement (Fig. 4), indicating that the stellarator expansion provides an adequate treatment of this device. A similar comparison has been made [21] for a four field period Helicac with an aspect ratio of eight and a rotational transform profile that varies from $\iota = 0.71$ at the magnetic axis to $\iota = 0.62$ at the plasma edge (Fig. 5). It too shows good agreement even though the magnetic axis is highly nonplanar.

It is surprising that the stability studies also show quantitative agreement. Comparisons of the growth rates for an $m = 2$, $n = 1$ internal mode in a straight $\ell = 2$, $M = 5$ stellarator with pitch $h_a = 0.66$, as calculated with the HERA [15] and BETA [42] codes, showed good agreement [52]. More striking results were obtained when calculations with the STEP code [18] were included in the comparison [53], Fig. 6. The differences can be attributed to the restrictions on how the pressure function and plasma boundary are specified in the different codes. This agreement between the PEST and BETA codes holds when toroidal curvature is introduced (Fig. 7) even though the eigenfunction shows considerable structure.

These MHD codes have already contributed to the analysis and interpretation of results obtained from the WENDELSTEIN VII-A stellarator at

Garching and from HELIOTRON E at Kyoto. They are essential tools for the design of the next generation of stellarators, including the Garching WENDELSTEIN VII-AS device, the Oak Ridge ATF-1 torsatron, and the Madrid JT-II heliac. They are being incorporated into detailed physical studies, such as the treatments of ballooning mode stability that have appeared recently [54,55].

Useful two and three-dimensional codes have been developed, but much remains to be done. They have already contributed much and will do so increasingly in the future. They will refine our understanding of the MHD aspects of stellarator confinement.

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FIGURE CAPTIONS

- FIG. 1. Typical magnetic surface cross sections in stellarators. (Reprinted with permission of Academic Press from Ref. 5.)
- FIG. 2. Magnetic surfaces in the $\ell = 2$, $M = 5$ WENDELSTEIN VII-A stellarator with $\nu \approx 0.5$ as constructed with the STEP code [17]. (a) $\beta(0) = 1\%$. (b) $\beta(0) = 2\%$. (Reprinted with permission of IAEA from Ref. 20.)
- FIG. 3. Comparison of the equilibrium flux surfaces computed with the NEAR [30] and Chodura-Schlüter [27] codes for the $\ell = 2$, $M = 12$ torsatron ATF-1 with $\beta(0) = 5\%$. (Reprinted with permission of Academic Press from Ref. 30.)
- FIG. 4. Comparison of the equilibrium flux surfaces computed with the NEAR code [30] and the stellarator-expansion code NAV [27] for the $\ell = 2$, $M = 12$ torsatron ATF-1 with $\beta(0) = 5\%$. The vacuum field flux surfaces are given for comparison. (Reprinted with permission of the American Institute of Physics from Ref. 27.)
- FIG. 5. Comparison of the equilibrium flux surfaces computed with the NEAR code [30] and the NAV code [27] for an $M = 4$, $A = 8$, $\nu(0) = 0.71$, $\nu(a) = 0.62$, $\beta(0) = 3.5\%$ heliac. (Reprinted with permission of Academic Press from Ref. 21.)

FIG. 6. Growth rates for an $m = 2$, $n = 1$ instability in a straight $l = 2$, $M = 5$ helical system with pitch $h_a = 0.66$ and $\langle \beta \rangle = 1.5\%$ and 3.0% as functions of the rotational transform at the axis. The solid curve is from STEP, the broken curve from HERA, and the dashed curve from BETA. (Reprinted with permission of Academic Press Press from Ref. 53.)

FIG. 7. Maximum growth rate as a function of ι for the configuration of Fig. 6, but with toroidal curvature introduced such that the ratio of the periodicity length to the toroidal circumference is $L/2\pi R$. The circles are from STEP, and the squares from BETA. (Reprinted with permission of Academic Press from Ref. 53.)

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Fig. 1

Fig. 2

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Fig. 3

Fig. 4

Fig. 5

Fig. 6

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Fig. 7

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