TITLE: A MONTE CARLO LEARNING/BIASING EXPERIMENT WITH INTELLIGENT
RANDOM NUMBERS

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A MONTE CARLO LEARNING/BIASING EXPERIMENT WITH INTELLIGENT RANDOM NUMBERS

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ABSTRACT

A Monte Carlo learning and biasing technique is described that does its learning and biasing in the random number space rather than the physical phase-space. The technique is probably applicable to all linear Monte Carlo problems, but no proof is provided here. Instead, the technique is illustrated with a simple Monte Carlo transport problem. Problems encountered, problems solved, and speculations about future progress are discussed.

INTRODUCTION

Biassing of Monte Carlo transport calculations has typically been accomplished by altering the random walk taken for the same sequence of random numbers. For instance, the analog sampling for the distance to collision, s , in a medium with constant macroscopic total cross section, σ , is

$$s = -\frac{1}{\sigma} \ln \xi \quad \text{or} \quad \xi = e^{-\sigma s} \quad (1)$$

where ξ is a random number uniformly distributed on $(0,1]$. This samples the analog density

$$p(s) = \sigma e^{-\sigma s} \quad (2)$$

Sometimes it is advantageous to modify the effective cross section and sample the density

$$p'(s') = \sigma' e^{-\sigma' s'} \quad (3)$$

instead of Eq. (2). In this case, the same random number ξ used in Eq. (1) will result in a different distance to collision

$$s' = -\frac{1}{\sigma'} \ln \xi \quad (4)$$

with a weight multiplication

$$w_m = p(s')/p(s) = \frac{\sigma e^{-\sigma s'}}{\sigma' e^{-\sigma' s'}} \quad (5)$$

so that the expected weight colliding at s' is the same as in the analog case. That is, if the particle weight before sampling is w , the expected weight colliding in ds' about s' is

$$wp(s')ds' \quad (6)$$

in the analog case, and

$$w_m (wp'(s')ds') = wp(s')ds' \quad (7)$$

in the biased case as desired. If $\sigma' < \sigma$, then for a given ξ , $s' > s$, and if $\sigma' > \sigma$, then for a given ξ , $s' < s$. This is essentially the exponential transform technique.

There is another way to accomplish the exponential transform technique; that is, there is another way to sample the density in Eq. (3). Note that ξ is sampled from a density

$$\begin{aligned} f(\xi) &= 1 & 0 < \xi \leq 1 \\ &= 0 & \text{otherwise} \end{aligned} \quad (8)$$

and Eq. (1) gives $s(\xi)$. Thus, the density $p(s)$ can be expressed in terms of the density $f(\xi)$ by using the change of variable theorem, that is

$$p(s) = f(\xi[s]) \left| \frac{d\xi(s)}{ds} \right| = 1 \cdot |- \sigma e^{-\sigma s}| = \sigma e^{-\sigma s} \quad (9)$$

which, of course, had to come out the same as Eq. (2) or Eq. (1) would not sample the density $p(s)$.

Suppose that ξ is not sampled from Eq. (6) but rather from a density $\hat{f}(\xi)$; in this case, s will have the density

$$\hat{p}(s) = \hat{f}(\xi) \left| \frac{d\xi}{ds} \right| = \hat{f}(\xi) \sigma e^{-\sigma s} \quad (10)$$

The density $p'(s)$ (see Eq. (3)) can then be sampled if $\hat{f}(\xi)$ is chosen so that $p(s) = p'(s)$, namely, using Eq. (1)

$$\begin{aligned}\hat{f}(\xi) &= \frac{\sigma' e^{-\sigma' s}}{\sigma e^{-\sigma s}} = \frac{\sigma'}{\sigma \xi} \exp \left[-\sigma' \left(\frac{-1}{\sigma} \ln \xi \right) \right] = \frac{\sigma'}{\sigma \xi} \exp \left[\ln(\xi)^{\sigma'/\sigma} \right] \\ &= \frac{\sigma'}{\sigma} \frac{\xi^{\sigma'/\sigma}}{\xi} = \frac{\sigma'}{\sigma} \xi^{\left(\frac{\sigma'}{\sigma} - 1 \right)}.\end{aligned}\quad (11)$$

As a check, compute

$$\int_0^1 \hat{f}(\xi) d\xi = \frac{\sigma'}{\sigma} \int_0^1 \xi^{\left(\frac{\sigma'}{\sigma} - 1 \right)} d\xi = \left[\frac{\sigma'}{\sigma} \frac{\xi^{\sigma'/\sigma}}{\sigma'/\sigma} \right]_0^1 = 1 \quad (12)$$

as desired. To reiterate, the distance to collision, s , may be sampled from $p'(s)$ by sampling the density $\hat{f}(\xi)$ in Eq. (11) for ξ and finding s from Eq. (1).

There is, of course, a weight correction for having sampled the random number ξ from a non-uniform density. If a given random number is made y times as likely to occur as it should, then the sample must be weighted by y^{-1} . In particular, the weight multiplier here is

$$\begin{aligned}w_m &= \frac{f(\xi)}{\hat{f}(\xi)} = \frac{1}{\frac{\sigma'}{\sigma} \xi^{\left(\frac{\sigma'}{\sigma} - 1 \right)}} = \frac{\sigma}{\sigma'} \xi^{\left(\frac{\sigma - \sigma'}{\sigma} \right)} = \frac{\sigma}{\sigma'} (e^{-\sigma s})^{\frac{\sigma - \sigma'}{\sigma}} \\ &= \frac{\sigma}{\sigma'} e^{-(\sigma - \sigma')s} = \frac{\sigma e^{-\sigma s}}{\sigma' e^{-\sigma' s}}\end{aligned}\quad (13)$$

the same as Eq. (5).

The preceding example demonstrates that a Monte Carlo calculation can either be biased by using uniformly distributed random numbers, Eq. (8), in a non-analog formula, (e.g. Eq. (4) versus Eq. (1)), or by using biased random numbers, (e.g. Eq. (11) versus Eq. (8)), in the analog formula, (e.g. Eq. (1)). Section III explains why one might wish to bias the random numbers rather than bias the selection formulas. For now, simply note that no matter how the random numbers are biased, a given ξ always corresponds to the same s through Eq. (1).

STANDARD BIASING VERSUS RANDOM NUMBER BIASING

The example in the Introduction only considered the sampling of a single event, but the biased random number technique can be applied throughout the entire Monte Carlo calculation. An interesting way to compare the biased random number technique to more conventional biasing techniques is to consider what is really calculated by analog Monte Carlo. Each history is associated with a random number sequence $\vec{r}_1$ and a history score $T(\vec{r}_1)$ that depends on the random walk specified by $\vec{r}_1$. The random number sequence $\vec{r}_1$ is selected from a uniform density $f(\vec{r})$ of random number sequences. The expected score is then

$$E = \int T(\vec{r})f(\vec{r})d\vec{r} . \quad (14)$$

When the calculation is biased $T(\vec{r})$ is normally altered to some $T'(\vec{r})$, but $f(\vec{r})$ is left unaltered. Note that T and f appear in Eq. (14) in exactly the same way, so that it makes just as much sense to alter f to some f' and leave T unaltered. Although it will not be discussed here, it is also possible to alter both T and f . That is, it is possible to use biased random numbers together with standard variance reduction techniques.

LEARNING IN THE RANDOM NUMBER SPACE

Before discussing learning in the random number space, it should be pointed out that many people, especially in the past decade, have invented techniques that allow machines to estimate appropriate biasing parameters based on the machine's experience with the random walks that were sampled. The interested reader should examine Ref. 2-11. The salient difference here is that the learning and biasing are done in the random number space.

At first, learning and biasing in the random number space may seem like a strange idea until one realizes that there is a real advantage to working in the random number space. In Ref. 2-11, the user had to supply information about what to learn. Typically, the user states a set of parameters he would like optimized and the computer will estimate optimal values for these parameters. Unfortunately, it is difficult to write a computer program general enough to allow the user to optimize on all sets of parameters that a user can dream up. In my own work, for example, it was necessary to optimize on space-energy dependent parameters for one class of problems,⁶ and space-angle dependent parameters for a second class of problems.⁷ Furthermore, a special biasing technique had to be developed for the second class of problems. Learning and biasing in the random number space tends to ameliorate these problems.

Before discussing why these problems are ameliorated, consider the very simple Monte Carlo problem illustrated in Fig. 1. The problem consists of a point mono-energetic source impinging normally on a very thin slab with macroscopic cross section σ . The slab thickness, T , is a small fraction of a free path. The tally here is the fraction of particles having at least one collision in the slab. (The problem can be solved analytically by noting that $e^{-\sigma T}$ is the

fraction that traverse the slab without collision and hence $1-e^{-\sigma T}$ is the fraction that collide.) How might learning and biasing in the random number space be used to improve the Monte Carlo calculation?

Suppose that an analog Monte Carlo calculation is done and each time a particle enters its first collision, the random number used to select the distance to collision is recorded; nothing is recorded if the particle does not collide. Eq. (1) is used to determine the distance, s , to collision; if $s \leq T$, then a collision occurs; and if $s > T$, then no collision occurs. If enough particles are run in the analog calculation, then upon inspection (after the analog calculation is finished) of the "successful" (i.e., recorded) random numbers, one finds a roughly uniform distribution of successful random numbers in the interval $[e^{-\sigma T}, 1]$.^{*} That is, the successful random numbers are distributed very differently compared with the standard random numbers that are uniformly distributed on $(0,1]$. The standard and successful random number densities are compared in Fig. 2.

For the moment, assume that an infinite number of particles have been run so that there is no statistical uncertainty in the distribution of successful random numbers. That is, the successful random numbers are known to be uniformly distributed on $[e^{-\sigma T}, 1]$. (Section VII discusses the problem of inferring the successful random number distribution from a finite number of successful random numbers.) Biasing in the random number space consists of selecting the ξ in Eq. (1) from the successful random number distribution rather than from the standard random number distribution.

Here it is useful to introduce a convention to avoid confusion between random numbers selected from the successful random number distribution (also called "biased random numbers") and standard random numbers selected directly from a uniform random number generator on $(0,1]$. Let η be associated with uniform random numbers on $(0,1]$ and let ξ be associated with (possibly) biased random numbers.

Returning to the thin slab problem, note that a biased random number ξ_i can be sampled from a uniform distribution on $[e^{-\sigma T}, 1]$ by setting

$$\xi_i = e^{-\sigma T} + (1-e^{-\sigma T})\eta_i . \quad (15)$$

There is a weight correction for drawing ξ_i from a biased density. As usual, this weight correction is the ratio of the true density for selecting an unbiased random number on $(0,1]$, i.e., 1, to the sampled density $(1-e^{-\sigma T})^{-1}$ or

$$w_m = 1-e^{-\sigma T} . \quad (16)$$

^{*} $1-e^{-\sigma T}$ is obtained by solving Eq. (1) for ξ when $s = T$.
 1 is obtained by solving Eq. (1) for ξ when $s = 0$.

The biased Monte Carlo calculation proceeds as follows. Select n_i from the random number generator and use Eq. (15) to compute biased random number ξ_i . Use ξ_i in Eq. (1) to compute the distance, s_i , to collision and multiply the particle weight by w_m from Eq. (16). Because $e^{-\sigma T} \leq \xi_i \leq 1$, then $0 \leq s_i \leq T$ and every source particle will collide in the slab with weight $w_m = 1 - e^{-\sigma T}$. This is thus a zero variance solution with mean $1 - e^{-\sigma T}$, the same as the analytic solution.

In practice, one cannot achieve a zero variance solution because there will always be some error in the inferred distribution of successful random numbers because of statistical uncertainty associated with the finite number of successful random numbers used to infer the distribution. The only reason the example above was a zero variance solution was that there was assumed to be no error in the distribution of successful random numbers, because it was based on an infinite number of samples. Section VII discusses the problem of inferring the distribution from a finite number of samples.

Recall that I asserted earlier that there were advantages to doing the learning and biasing in the random number space. Note that the analogous biasing technique in standard Monte Carlo biasing is the forced collision¹² technique wherein the particle is deterministically split into a collided weight $1 - e^{-\sigma T}$ and an uncollided weight $e^{-\sigma T}$. The collision site, s , for the collided weight is selected from an exponential distribution where $0 \leq s \leq T$. Thus both the forced collision technique and the random number biasing result in each particle having a first collision exponentially distributed between 0 and T with weight $1 - e^{-\sigma T}$. They are equivalent techniques in the same sense as the exponential transform techniques, described in Section I, are equivalent.

What are the advantages to working in the random number space? Note that in the random number space, no parameters had to be specified, whereas if the the learning were done in the physical phase-space, there would typically be a set of parameters to learn. Furthermore, learning these parameters may or may not allow (with perfect learning) zero variance solutions. If the user does not supply an appropriate set of parameters to learn, it may be impossible to generate a forced collision density for a subsequent calculation. Most Monte Carlo codes have a myriad of special variance reduction techniques that tend to make the codes large and cluttered with special cases. Biasing in the random number space requires only an analog Monte Carlo code with a call to a subroutine that supplies the biased random numbers. The "special cases" appear only as data to the Monte Carlo code rather than requiring any special coding. In the example above the forced collision technique was duplicated simply by sampling the appropriate random number distribution inferred from the data on successful random numbers. The important problem thus becomes: how to infer appropriate random number distributions from the data. This problem is considered in Section VII.

DESCRIPTION OF THE TEST PROBLEM

Although I have been unable to see any significant limitations to learning and biasing in the random number space, the ideas have only been tested on the simple test problem described here. Thus the ideas will be discussed in reference to the test problem and the wild speculations on how general and useful the ideas might be will be left to Section XI.

The test problem (see Fig. 3) consists of a point isotropic mono-energetic source on one surface of a 5-cm thick slab. There is a vacuum on either side of the slab and the mean free path is 1 cm. At each collision there is a 50% absorption probability and a 50% scattering probability. The scattering is isotropic with no energy change. The only important phase-space coordinates in this problem are the distance (x) into the slab and the cosine (μ) of the particle's direction with respect to the x direction. The tally of interest is a simple surface crossing tally on the slab surface opposite the source; unit weight particles reaching 5 cm score 1 and the particles not reaching 5 cm score 0. Figure 4 shows a flow diagram for the analog Monte Carlo calculation.

LEARNING ONLY IN THE RANDOM NUMBER SPACE - A BIASING FAILURE

This section discusses an early approach to learning in the random number space that was rejected in favor of the approach described in Section VI. The uninterested reader may skip ahead to Section VI because nothing there depends on the discussion here. However, this section will help the reader understand why it is helpful to consider the particle's position in the physical phase-space and the random number space rather than just the position in the random number space as considered here.

The distribution of successful random numbers is essentially an importance function in the random number space. In the example in Section III, the importance function was

$$\begin{aligned} I(\eta) &= 0 & 0 < \eta < e^{-\sigma T} \\ I(\eta) &= 1 & e^{-\sigma T} \leq \eta \leq 1 \end{aligned} \quad (17)$$

and the learning process estimated this importance function. For more complicated problems where many random numbers are thrown, the importance function can be denoted by $I(\vec{\xi})$, where $\vec{\xi}$ is a sequence of random numbers associated with a history. The importance function is the history score associated with a unit weight particle and the random number sequence $\vec{\xi}$. A perfect (zero variance) bias results when the probability of selecting random number sequence $\vec{\xi}$ is equal to the importance, $I(\vec{\xi})$, divided by the average importance, $\langle I \rangle$. That is,

$$p'(\vec{\xi}) = I(\vec{\xi})/\langle I \rangle \text{ is the sampled density for } \vec{\xi},$$

$$p(\vec{\xi}) = 1 \text{ is the analog density for } \vec{\xi},$$

$$w_m = p(\vec{\xi})/p'(\vec{\xi}) \text{ is the weight correction for sampling } p' \text{ versus } p, \text{ and}$$

hence the history score for $\vec{\xi}$ is $w_m I(\vec{\xi}) = \langle I \rangle$; i.e., every $\vec{\xi}$ sampled results in the same history score.

In the test problem (Fig. 3) here each $\vec{\xi}$ was either successful and scored 1, or was unsuccessful and did not score. It seemed plausible to me that if $\vec{\xi}$ scored, then it was likely that random sequences close to $\vec{\xi}$ would also score. For instance, consider random number sequences $\vec{\xi} = (\xi_1, \xi_2)$ that result in a collisionless penetration of the slab. If $\mu > 0$ and $s > 5\mu^{-1}$, then the particle will have a collisionless penetration of the slab. Relating μ, s to ξ_1, ξ_2

$$\mu = -1 + 2\xi_1$$

$$s = -\ln(\xi_2)$$

one sees that $\mu > 0$ implies $\xi_1 > 1/2$ and $s > 5\mu^{-1}$ implies

$$-\ln(\xi_2) > 5/\mu = 5/(-1 + 2\xi_1)$$

$$\xi_2 < \exp[-5/(-1 + 2\xi_1)]$$

Figure 5 shows a graph of all (ξ_1, ξ_2) pairs that penetrate the slab without collision. Note that for any pair (ξ_1, ξ_2) that penetrates, there is a neighborhood of points that will also penetrate.

I attempted to bias the test problem in the random number space in the following manner. First, an analog calculation was done and the random number sequences $\vec{n} = (n_{i1}, n_{i2}, \dots)$ were recorded, where n_{ij} is the j^{th} random number of the i^{th} successful particle. Next, a biased calculation was run and the first random number ξ_1 was selected from the learned distribution¹ of the n_{i1} . The second random number, ξ_2 , was then selected from a distribution formed from a subset of the n_{i2} for which n_{i1} was "close" to the ξ_1 actually sampled. This is necessary because the k^{th} random number in a successful random number sequence is correlated with the previous $k-1$ random numbers. (As an example, consider Fig. 5 again. If ξ_1 were equal to .8, then $0 < \xi_2 \leq .00024$ if the particle is to penetrate the slab without collision, whereas, if ξ_1 were equal to .98, then $0 < \xi_2 \leq .00547$ if the particle is to penetrate the slab without collision.) The third random number ξ_3 was then selected from a distribution formed from a subset of the n_{i3} for which n_{i1} was close to the ξ_1 actually sampled and n_{i2} was close to the ξ_2 actually sampled, and so forth.

¹For the moment, it is not important how the distribution was formed.

I tried two ways of defining close; both gave problems. The first way was to select (when forming the distribution from which to sample ξ_{j+1}) a subset of the η_{ij+1} such that

$$|\eta_{im} - \xi_m| < c \text{ for } m = 1, j.$$

This worked well for small j , but the number of points in the subset dwindled to zero as j increased and was therefore unacceptable. The second way was to take the l closest η_i to ξ . That is, compute

$$c_i = \sum_{m=1}^j (\eta_{im} - \xi_m)^2$$

and form the distribution (for selecting ξ_{j+1}) from the η_i with the smallest c_i . Unfortunately, I observed that as j became large, the distributions for the $j+1, j+2, \dots$ all tended to become identical. In other words, the correlation between distributions for successive random numbers disappeared. I rejected this technique because I knew that the successful random numbers were correlated.

A little investigation reveals what, perhaps, should have been obvious. If one looks at points in the physical phase-space, one finds that there are some points very close in the physical phase-space, but very far apart in the random number space. This is because there are many different random paths that arrive at a given point in the physical phase-space. Figure 6 illustrates this. Remember that the only important coordinates are x and μ , and if one looks at particles entering a collision, then x is the only important coordinate because the scattering is isotropic and the after-collision coordinates do not depend on the incoming μ .

Figure 7 shows two particles with close random number sequences whose tracks are nonetheless diverging in the physical phase-space. The distribution of random numbers (after entering the fourth collision) that will penetrate the slab is much different for particles 1 and 2. However, the distribution of random numbers (after entering a collision at x_0) is identical. Stated another way, for a particle arriving at a particular point in the physical phase-space, the same random number sequences (after arriving) will penetrate the slab, regardless of how the particle got to that point. The next section describes a method based on this fact.

LEARNING IN THE PHYSICAL SPACE TOGETHER WITH THE RANDOM NUMBER SPACE

After an attempt to ignore the physical space entirely and to only consider the random number space failed (see Section V), a successful method based on

both the physical space and the random number space was developed. This successful method learns the appropriate distribution of random numbers that should be used from any point in the physical phase-space to find the next point in the phase-space. This is in contrast to the unsuccessful technique described in Section V where the method attempted to learn the appropriate distribution of random numbers all the way from source to tally.

Fortunately, it is not necessary to explicitly know what random number sequences will cause a source particle to tally. It is enough to know what distribution of random numbers should be used from any point in the phase-space to find the next point in the phase-space, because the importance of a particle at some point $\vec{P}$ in phase-space is independent of the random walk taken to $\vec{P}$. In terms of the random number sequences: if two successful random number sequences, $\xi = (\xi_1, \xi_2, \dots)$ and $r = (r_1, r_2, \dots)$, arrive at some location $\vec{P}$ after having used j and k random numbers respectively then $(\xi_1, \xi_2, \dots, \xi_j, r_{k+1}, r_{k+2}, \dots)$ and $(r_1, r_2, \dots, r_k, \xi_{j+1}, \xi_{j+2}, \dots)$ will also be successful random number sequences.

Figure 8 takes this idea one step further. Three successful random walks are depicted along with the random number sequences corresponding to those walks. Each segment is labeled with the random numbers that took the particle from one point in the physical phase-space to its next point in the physical phase-space. For example, ξ_{23} is that portion of random number sequence ξ_2 that takes the particle from phase-space point B to phase-space point C. The hatched random walk was inferred from the three observed successful random walks and likewise the random number sequence $i = (\xi_{11}, \xi_{23}, \xi_{32})$ was inferred from the three observed successful random number sequences. How a particle arrives at a particular point in the phase-space is not important to the selection of the next point in phase space. For example, ξ_{23} takes the particle from B to C regardless of whether the particle got to B by ξ_{11} or (ξ_{21}, ξ_{22}) . Thus rather than learn the distribution of random numbers (on successful tracks) from source to tally, the computer need only learn the distribution of a subset of random numbers (on successful tracks) that takes the particle from one phase-space point to the next.

Although the distribution of successful random number sequences departing a phase-space point $\vec{P}$ does not depend on the random number sequence used to get to $\vec{P}$, the j^{th} random number used after departing $\vec{P}$ will, in general, depend on the first $j-1$ random numbers used after departing $\vec{P}$. For example, in the simple penetration example of Fig. 3, not only will successful random walks have a preference for $\mu > 0$ (forward direction), but there will be a preference for s (distance to collision) that depends on μ . When $\mu > 0$

(i.e., $\xi > 1/2$), then successful random walks will preferentially have large s (i.e., small ξ') and when $\mu < 0$, they will preferentially have small s (i.e., large ξ'). The next few paragraphs illustrate how this dependence is incorporated in the biasing technique.

Before doing any biasing, some information had to be obtained and so an analog Monte Carlo calculation was done and information about the successful (i.e. penetrating) particles was recorded. In particular, recall that three choices are made at each collision site:

- (a) absorption vs scattering,
- (b) scattering angle (μ), and
- (c) distance to next collision.

(If choice 1 results in an absorption, then the other choices need not be made; note that for successful particles, all three choices are made because a successful particle is always scattered upon collision.) Upon entering collision, the only important phase-space coordinate is x because the new μ will be selected isotropically regardless of the incoming μ . Thus, for each collision along successful track, four things are recorded:

- 1. x , the position;
- 2. η_1 , the random number used to determine absorption versus scattering;
- 3. η_2 , the random number used to determine scattering angle (μ); and
- 4. η_3 , the random number used to determine distance to collision (s).

The biased random number distributions are formed by using collections of these recorded data points that are in the same "neighborhood;" that is, they are close in a sense described shortly.

There are many ways to define what points are close to each other and I have chosen a definition that is simple, though admittedly far from optimal. (However, better definitions are under consideration.) In particular, two points y and z are close in D dimensions if the Euclidean distance

$$l = \left(\sum_{i=1}^D (y_i - z_i)^2 \right)^{1/2}$$

is small. The dimension, D , will depend on which random number is being chosen. Recall that the distribution used to select the j^{th} random number may depend on the $j-1$ random numbers previously selected. Let ξ_1 , ξ_2 , and ξ_3 be three biased random numbers corresponding to the three choices described in the preceding paragraph. Suppose (in the biased game) that a collision occurs at x' , ξ_1 is chosen from a distribution inferred from a subset of η_1 's that have x close to x' . ξ_2 is then chosen from a distribution inferred from a subset of η_2 's that have (x, η_1) close to (x', ξ_1) . ξ_3 is then chosen from a distribution inferred from a subset of η_3 's that have (x, η_1, η_2) close to (x', ξ_1, ξ_2) . The next paragraph gives an example using real data.

Table I shows the recorded collision data from a 20,000 particle analog calculation of the test problem. Note that there were 85 collisions along successful tracks and these have been ordered by x coordinate. Tables II-IV give sets of the 9 closest data points to the indicated data point in 1, 2, and 3 dimensions. Now consider the biased calculation using the data from Tables II-IV. Suppose that a particle enters a collision $x' = 3.12$. The 51st collision has x coordinate closest to x' (item 1). Item 2 indicates the 9 collisions with x coordinates closest to the 51st x coordinate. Item 3 shows 9 random numbers thrown after a collision near x' that lay along successful tracks. A distribution is formed from these 9 random numbers and ξ_1 is then sampled from this distribution. Suppose ξ_1 is sampled as $\xi_1 = .19$, then $(3.12, .19)$ is closest to the 52nd collision data (see Table III). Item 4 indicates the 9 data points closest, in (x, ξ_1) , to the 52nd collision data point, and item 5 shows the 9 random numbers thrown after a collision near (x', ξ_1) that lay along successful tracks. A distribution is formed from these 9 random numbers and ξ_2 is then sampled from this distribution. Suppose ξ_2 is sampled as $\xi_2 = .96$, then $(3.12, .19, .96)$ is closest to the 50th collision data point (see Table IV). Item 6 indicates the 9 closest data points, in (x, η_1, η_2) , to the 50th collision data; item 7 shows the 9 random numbers thrown after a collision near (x', ξ_1, ξ_2) that lay along successful tracks. A distribution is formed from these 9 random numbers and ξ_3 is then sampled from this distribution. $(x', \xi_1, \xi_2, \xi_3)$ determines the next collision site and the process starts again (provided the particle does not exit the slab).

INFERRING APPROXIMATE DISTRIBUTIONS FOR SUCCESSFUL RANDOM NUMBERS

Thus far I have avoided describing how approximate distributions for biased random numbers are generated. There are two reasons for this. First, the ideas presented previously did not depend on how the distributions were formed. Second, I really do not know a good way to form the distributions and many readers may well know (or might quickly develop) a better way. However, even the crude way presented here is sufficient to demonstrate a factor of 30 decrease in the history variance by using biased random number distributions. Table I shows the data in its raw form. I wanted to form distributions such that (1) no random walks are precluded by sampling the biased distributions, and (2) as the number of data points approaches infinity, the inferred successful random number distributions approach the true distribution of successful random numbers.

The first condition is desirable because the inferred biased distributions are never perfect and thus it is impossible to be sure that any given random walk is a zero importance walk. If a walk does not have zero importance and the walk is precluded, then the tally will be wrong. The second condition ensures that as more and more is learned, the biasing will become closer and closer to a zero variance biasing.

Recall that every decision along successful random walks has been recorded. In fact, Tables II-IV list the decisions made under similar (i.e., close as defined on page 24) circumstances. In the tables, each collision site is associated with its 9 closest (including itself) neighboring collisions. The nine closest depend on what decision is being made because points that are closest in one coordinate are not necessarily closest in two coordinates and so forth. The inferred distributions of successful random numbers will approach the true distributions of successful random numbers if (1) the number of data points in the neighborhoods approaches infinity, and (2) the neighborhoods shrink to

zero. There are undoubtedly better solutions, but these two conditions led me to choose the following (based on C recorded collisions in the entire problem): for each recorded collision, the random number distribution shall be inferred from the $\sqrt{C}$ closest recorded collisions. For example, in Table I, $C = 85$ and this implies use of $\sqrt{85} \approx 9$ closest points to infer the successful random number distribution.

Perhaps the best way to explain how the distributions were inferred from the $\sqrt{C}$ random numbers is through an example. Consider forming a distribution based on the nine random numbers indicated as item 3 in Table II. These nine random numbers, and the tally associated with each (in this example the tally is always 1), are plotted in Figure 9a. The number of density intervals 7 (Fig. 9b) is initially calculated from

$$\begin{aligned} S &= [C^{1/2}] \\ T &= [S^{1/2}] \\ I &= [(S-1)/(T-1)] = \# \text{ intervals} \end{aligned} \tag{18}$$

where $[]$ indicates the greatest integer function. T points are associated with each interval, with the endpoints associated with both bordering intervals. In the example of Fig. 9b, $C = 85$, $S = 9$, $T = 3$, and $I = 4$. The unnormalized density function in the j^{th} interval (ξ_{j1}, ξ_{jT}) is calculated as:

$$p_j(\xi) = \frac{\frac{1}{2}s_{j1} + \sum_{m=2}^{T-1} s_{jm} + \frac{1}{2}s_{jT}}{\xi_{jT} - \xi_{j1}} \tag{19}$$

(s_{jm} = m^{th} score in the j^{th} random number interval)

where the $\frac{1}{2}$ factors are needed because the endpoints are associated with two intervals and thus are counted twice. Without the $\frac{1}{2}$, the endpoints would have twice as much input into the inferred successful random number distribution as the interior points.

The successful random number density pictured in Fig. 9b is unacceptable because there are random number intervals with density zero and this will preclude some random walks. This particular decision (scattering versus absorption) has the true successful random number density indicated in Fig. 9c. Note that the true density is non-zero on (0,.5) while the inferred density is non-zero only on (.028,.430). How can one make sure that these outer edges are not missed? One way is to add some more intervals and geometrically decrease the densities (in the added intervals) as the added intervals get further from the original intervals.

The added intervals were chosen to have the same length as the interval that they are placed next to and half the density. For the example, the lowest interval has length $.064 - .028 = .036$ so that a new interval from $(.028-.036,.028)$ is added with density $55.56/2 = 27.78$. The lower endpoint must terminate at zero so the interval is adjusted to $(0,.028)$ with density 27.78. The highest interval has length $.430-.337 = .093$ and density 21.51 so the intervals added on the high side are:

<u>Interval</u>	<u>Density</u>	<u>Area</u>
(.430,.523)	10.76	1
(.523,.616)	5.38	.5
(.616,.709)	2.69	.25
(.709,.802)	1.34	.125
(.802,.895)	0.67	.0625
(.895,.988)	0.34	.03125
(.988,1.00)	0.17	.022

The density function is now normalized to have unit integral and the result is shown in Table V.

This density is described in Table V and pictured in Fig. 10. Note from Fig. 10 that the inferred density is a poor approximation to the true density. This is because the density was inferred from only nine data points in order to illustrate the technique.

Figure 10 also shows that the added intervals are a non-trivial part of the density function for this example. However, as the number of data points increases, the fraction of the area under the added intervals will shrink because the lengths of the added intervals are tied to the length of the adjacent (data) interval. The lengths of the data intervals will shrink because there will be [see Eq. (18)] ever more intervals as the number of data points increases. This is illustrated in Fig. 11.

In the example above, the number of intervals and the number of data points (see Fig. 9b) are such that there are exactly three points (two endpoints and one interior point) in each interval. If instead of nine points there are ten points, something special has to be done. Figure 12 shows what is done.

The intervals are divided up as before (ignoring the incomplete interval at the right) and their densities are calculated from Eq. (19). The density of the incomplete interval is calculated by taking as many points to the left of the incomplete interval as necessary to complete it. The density in the incomplete interval is equated to the density of the completed interval calculated from Eq. (19).

RESULTS FOR THE TEST PROBLEM

The methods described in Sections VI and VII were applied to the test problem of Fig. 3. An analog calculation with 500,000 particles produced a mean 1.824×10^{-3} , a standard deviation 6.03×10^{-5} , and 1751 collisions (along successful tracks) in 8.2 seconds.*¹ A biased calculation with 40,000

*1/oray - 1 times.

particles was then done using the data from the 1751 collisions. The biased calculation produced a mean 1.836×10^{-3} and a standard deviation 3.71×10^{-5} in 537 seconds.

Two things are apparent from these results. First, the standard deviation has been improved by the learning process. Whereas the analog calculation needed 500,000 particles for a standard deviation of 6.03×10^{-5} , the biased calculation needed only 40,000 particles for a standard deviation of 3.71×10^{-5} . Second, a biased particle history (on average) consumes over 800 times more computer time than an analog particle. Thus for this particular test problem, the analog calculation is still far better than the biased calculation. However, the factor of 800 can probably be reduced drastically, as speculated in the next section. If so, learning and biasing in the random number space may yet be beneficial.

SPECULATIONS AND FUTURE INVESTIGATIONS

The principal problem confronting the random number learning and biasing techniques described here is the large amount of time required to process a biased particle's history. The sampling itself does not require very much more time in the biased calculation versus the analog calculation. Most of the time is required simply to find what biased distribution to sample. Recall that for each decision the biased distribution is selected by finding which of the previous data points (from the analog run) is closest to the current point in the biased calculation. Only after the closest point is found can the appropriate distribution be sampled. This closest point is currently being found in an extremely clumsy and inefficient manner. I firmly believe that an efficient method can be developed; I am investigating the problem and believe two methods show promise. One method can find the closest point extremely rapidly but currently requires too much memory. The other method is slower, but is not limited by memory. These methods are immature and will not be discussed here.

Once the time problem is solved, it should be possible to learn and bias in the same calculation. That is, one starts with an analog calculation that becomes self-biasing based on what random numbers have been successful. I see no fundamental reason why a Monte Carlo calculation cannot become ever better as the program learns more and more about what random numbers produce "good" random walks. However, there are some questions that need to be answered:

1. How fast can adaptive Monte Carlo converge?
2. How would a confidence estimate be obtained?
3. What stability problems might occur when the sampling is continuously adjusted based on the computer's experience?

These three questions all occur because the standard central limit theorem used in particle transport¹ does not apply because the samples are neither independent nor identically distributed.

CONCLUSION

This work has demonstrated that it is possible both to bias in the random number space and learn the appropriate bias. The advantage of learning and biasing in the random number space are twofold. First, an analog Monte Carlo code will suffice for all calculations because biasing in the random number space uses exactly the same pathways in the code, only with different probabilities. Second, no special variance reduction techniques are required because the biasing occurs naturally based on what was successful in the past.

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TABLE I
DATA FROM THE 85 COLLISIONS

	J	X	η_1	η_2	η_3
1	1	8.298112E-03	0.12141926383	0.82822618397	0.22198421509
2	2	8.455787E-03	0.48463433169	0.91535987174	0.02532443844
3	3	8.078988E-03	0.20055489737	0.91634192126	0.00137592571
4	4	1.932981E-01	0.01945335525	0.91439578790	0.00276366270
5	5	2.090215E-01	0.00804427473	0.99489448825	0.00836601064
6	6	2.209894E-01	0.04255007496	0.86748518373	0.01947137479
7	7	2.998572E-01	0.14627270453	0.99060178578	0.00256199010
8	8	2.742986E-01	0.30636797218	0.99629634477	0.00828870178
9	9	2.761407E-01	0.05445539478	0.49821892065	0.63066202452
10	10	3.052687E-01	0.25729546537	0.68667578304	0.18984064415
11	11	3.250284E-01	0.00331249579	0.99374109824	0.01918478601
12	12	3.860367E-01	0.13602005001	0.96403613806	0.94136349534
13	13	3.827179E-01	0.03055628925	0.92705535286	0.00161877200
14	14	4.121163E-01	0.48595825180	0.73209769446	0.00185727903
15	15	4.733246E-01	0.40941981131	0.92507684645	0.00637694526
16	16	4.992219E-01	0.32727411080	0.98899078468	0.02349838657
17	17	5.473197E-01	0.7456633169	0.93469536025	0.00174272803
18	18	6.023828E-01	0.30628837793	0.21036391955	0.85360524410
19	19	6.239383E-01	0.20845912250	0.37167585331	0.85607684232
20	20	6.494517E-01	0.07929203133	0.92250813259	0.02092550245
21	21	6.721778E-01	0.46025498789	0.93344560378	0.00527850908
22	22	6.898380E-01	0.46209556919	0.83885119503	0.11817976691
23	23	1.017127E+00	0.43248902925	0.92332838443	0.06578047004
24	24	1.181947E+00	0.05168054565	0.88235933600	0.39193629467
25	25	1.336232E+00	0.0183636874	0.83476899885	0.06988192920
26	26	1.801089E+00	0.36194714237	0.67012909848	0.66542384398
27	27	1.839687E+00	0.38199319623	0.78299683578	0.50414037110
28	28	1.852145E+00	0.02810487718	0.90452359577	0.43933529837
29	29	1.748276E+00	0.00875124394	0.62342038143	0.35726427293
30	30	1.802322E+00	0.33589809812	0.67843891051	0.18852983752
31	31	1.809283E+00	0.21563266569	0.78944689335	0.64998609115
32	32	1.999940E+00	0.46009258509	0.99543156074	0.34142002581
33	33	2.002344E+00	0.07017098681	0.91887825983	0.50086817875
34	34	2.032824E+00	0.02218878306	0.71314422117	0.63635932225
35	35	2.055888E+00	0.32338925894	0.94689013502	0.06558880198
36	36	2.068074E+00	0.43834530010	0.71786787778	0.05498183311
37	37	2.076934E+00	0.04376748492	0.78930155741	0.83838287751
38	38	2.171880E+00	0.22227128324	0.78213976774	0.09076532091
39	39	2.226173E+00	0.21181814155	0.94648903254	0.00915188142
40	40	2.317880E+00	0.36188386388	0.07660961926	0.74303308817
41	41	2.387761E+00	0.09890032518	0.92286429485	0.02578246765
42	42	2.437104E+00	0.21676580891	0.83118711481	0.39348378145
43	43	2.581717E+00	0.03966376797	0.81413521871	0.60103310985
44	44	2.596110E+00	0.37578574375	0.83461040390	0.28886439068
45	45	2.687035E+00	0.25013642459	0.83689403918	0.00048281878
46	46	2.799369E+00	0.25031278301	0.83848643521	0.07188284154
47	47	2.966167E+00	0.02780378956	0.93383575942	0.04118350141
48	48	3.054911E+00	0.03444580954	0.83893213421	0.02284900022
49	49	3.061783E+00	0.13813011315	0.96838550911	0.05061809626
50	50	3.118888E+00	0.16871534311	0.94003620270	0.11892787249
51	51	3.117838E+00	0.06360328521	0.32672066297	0.39436196203
52	52	3.142333E+00	0.18829878828	0.68867701100	0.18253091062
53	53	3.146381E+00	0.37144752300	0.81543289183	0.00444013586
54	54	3.162223E+00	0.43041008433	0.89945726188	0.04798733963
55	55	3.321633E+00	0.33716364826	0.34588835460	0.85960974967
56	56	3.328397E+00	0.10309787475	0.67468354844	0.00204193042
57	57	3.331275E+00	0.32117552472	0.91576439346	0.04789138197
58	58	3.481300E+00	0.12973112685	0.99828648389	0.99092748921
59	59	3.518684E+00	0.48107846945	0.81687205028	0.09643770195
60	60	3.529856E+00	0.43393431907	0.00390922380	0.98980684284
61	61	3.618988E+00	0.24900894284	0.79354259481	0.30884434288
62	62	3.786013E+00	0.17234991886	0.71700583406	0.42227992283
63	63	3.784190E+00	0.37135227590	0.84128663837	0.12422804011
64	64	3.898102E+00	0.08853459889	0.21625931740	0.27486952897

TABLE I (continued)
DATA FROM THE 85 COLLISIONS

65	4.11824E+00	0.04613831408	0.96532555507	0.50671466853
66	4.129371E+00	0.18783547182	0.51285299867	0.07991014453
67	4.130169E+00	0.45825199707	0.21441884867	0.40718630829
68	4.134424E+00	0.53267018713	0.98076459529	0.11319231223
69	4.107458E+00	0.34566136270	0.40311130273	0.82156011413
70	4.229178E+00	0.10772103808	0.80713788930	0.48060265019
71	4.306765E+00	0.30591005409	0.26859446685	0.68909212138
72	4.488185E+00	0.43549891768	0.87288179003	0.26782195395
73	4.490886E+00	0.17446046508	0.83556716684	0.32593762248
74	4.493514E+00	0.46371720693	0.89962558746	0.06371552269
75	4.494727E+00	0.04399576986	0.48048799092	0.96791293148
76	4.579265E+00	0.10588071506	0.93116725979	0.13213678759
77	4.699697E+00	0.31846767003	0.66843981593	0.28780727254
78	4.733806E+00	0.36970297050	0.82367569426	0.08241328621
79	4.749613E+00	0.00269376859	0.96211830225	0.00901891297
80	4.770908E+00	0.44899140758	0.81917114725	0.80006727279
81	4.809647E+00	0.43828365659	0.42901756278	0.58593634323
82	4.846200E+00	0.43767206009	0.57600290231	0.09106332789
83	4.813296E+00	0.12634675518	0.90742748976	0.59106800074
84	4.943754E+00	0.19398793726	0.29878741973	0.78482773082
85	4.988783E+00	0.41834528487	0.91495182275	0.83013886962

POINTS CLOSE IN X (ONE DIMENSION)

J	X	η_1	η_2	η_3	
590	THE CLOSEST TO J=	51	3.117838E+00		① CLOSEST X TO X'=3.12
591	51 3.117838E+00	0.06360325521	0.32672056297	0.39436196203	
592	50 3.115898E+00	0.16871534311	0.94003620270	0.11892767249	
593	52 3.142333E+00	0.18829878828	0.68867701100	0.18253091062	
594	53 3.146381E+00	0.37144752300	0.81543289183	0.00444013586	
595	54 3.163223E+00	0.43041008433	0.89945726188	0.04798733963	
596	49 3.084763E+00	0.13813011315	0.96838550911	0.05061809626	
597	48 3.054911E+00	0.03444560994	0.83893213421	0.02284900022	
598	47 2.986167E+00	0.02780378956	0.93363575942	0.04118350141	
599	55 3.321633E+00	0.33716364526	0.34556835460	0.55960974967	
600	THE CLOSEST TO J=	52	3.142333E+00		③ PREVIOUSLY SUCCESSFUL RANDOM NUMBER IN THIS NEIGHBORHOOD
601	52 3.142333E+00	0.18829878828	0.68867701100	0.18253091062	
602	53 3.146381E+00	0.37144752300	0.81543289183	0.00444013586	
603	54 3.163223E+00	0.43041008433	0.89945726188	0.04798733963	
604	51 3.117838E+00	0.06360325521	0.32672056297	0.39436196203	
605	50 3.115898E+00	0.16871534311	0.94003620270	0.11892767249	
606	49 3.084763E+00	0.13813011315	0.96838550911	0.05061809626	
607	48 3.054911E+00	0.03444560994	0.83893213421	0.02284900022	
608	47 2.986167E+00	0.02780378956	0.93363575942	0.04118350141	
609	55 3.321633E+00	0.33716364526	0.34556835460	0.55960974967	

9 COLLISIONS
WITH X CLOSEST
TO

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TABLE III
POINTS CLOSE IN (X, η_1) (TWO DIMENSIONS)

A MONTE CARLO
LEARNING/BIASING EXPERIMENT
WITH INTELLIGENT
RANDOM NUMBERS

J	X	η_1	η_2	η_3	
1444	THE CLOSEST TO J=	51	3.117838E+00		④ 9 DATA POINTS CLOSEST TO (3.12, 0.19)
1445	51 3.117838E+00	0.06360325521	0.32672056297	0.39436196203	
1446	48 3.054911E+00	0.03444560994	0.83893213421	0.02284900022	
1447	49 3.084763E+00	0.13813011315	0.96838550911	0.05061809626	
1448	50 3.115898E+00	0.16871534311	0.94003620270	0.11892767249	
1449	52 3.142333E+00	0.18829878828	0.68867701100	0.18253091062	
1450	47 2.986167E+00	0.02780378956	0.93363575942	0.04118350141	
1451	58 3.328297E+00	0.10309787875	0.87468354844	0.00204193042	
1452	53 3.146381E+00	0.37144752300	0.81543289183	0.00444013586	
1453	57 3.331275E+00	0.32117552472	0.91576439346	0.04789138197	
1454	THE CLOSEST TO J=	52	3.142333E+00		⑤ PREVIOUSLY SUCCESSFUL RANDOM NUMBERS IN THIS NEIGHBORHOOD
1455	52 3.142333E+00	0.18829878828	0.68867701100	0.18253091062	
1456	50 3.115898E+00	0.16871534311	0.94003620270	0.11892767249	
1457	49 3.084763E+00	0.13813011315	0.96838550911	0.05061809626	
1458	51 3.117838E+00	0.06360325521	0.32672056297	0.39436196203	
1459	48 3.054911E+00	0.03444560994	0.83893213421	0.02284900022	
1460	53 3.146381E+00	0.37144752300	0.81543289183	0.00444013586	
1461	56 3.328297E+00	0.10309787875	0.87468354844	0.00204193042	
1462	57 3.331275E+00	0.32117552472	0.91576439346	0.04789138197	
1463	55 3.321633E+00	0.33716364526	0.34556835460	0.55960974967	
1464	THE CLOSEST TO J=	53	3.146381E+00		
1465	53 3.146381E+00	0.37144752300	0.81543289183	0.00444013586	
1466	54 3.163223E+00	0.43041008433	0.89945726188	0.04798733963	
1467	55 3.321633E+00	0.33716364526	0.34556835460	0.55960974967	
1468	52 3.142333E+00	0.18829878828	0.68867701100	0.18253091062	
1469	57 3.331275E+00	0.32117552472	0.91576439346	0.04789138197	
1470	50 3.115898E+00	0.16871534311	0.94003620270	0.11892767249	
1471	49 3.084763E+00	0.13813011315	0.96838550911	0.05061809626	
1472	51 3.117838E+00	0.06360325521	0.32672056297	0.39436196203	
1473	56 3.328297E+00	0.10309787875	0.87468354844	0.00204193042	

TABLE IV
POINTS CLOSE IN (X, η_1, η_2) (THREE DIMENSIONS)

	J	X	η_1	η_2	η_3
2278	THE CLOSEST TO J= 49 3.064763E+00				
2279	49	3.064763E+00	0.13813011315	0.96838550911	0.08061809626
2280	50	3.115898E+00	0.18871534311	0.94003620270	0.11892767249
2281	47	2.966167E+00	0.02780378956	0.93363575942	0.04118350141
2282	48	3.054911E+00	0.03444560994	0.83893213421	0.02284900022
2283	53	3.146381E+00	0.37144752300	0.81543289183	0.00444013586
2284	46	2.795369E+00	0.25031275301	0.93845643521	0.07183284184
2285	52	3.142333E+00	0.18829878828	0.88867701100	0.18253091062
2286	54	3.163223E+00	0.43041008433	0.89945726188	0.04798733963
2287	57	3.331275E+00	0.32117852472	0.91576439346	0.04789138197
2288	THE CLOSEST TO J= 50 3.115898E+00				
2289	50	3.115898E+00	0.18871534311	0.94003620270	0.11892767249
2290	49	3.064763E+00	0.13813011315	0.96838550911	0.08061809626
2291	48	3.054911E+00	0.03444560994	0.83893213421	0.02284900022
2292	47	2.966167E+00	0.02780378956	0.93363575942	0.04118350141
2293	53	3.146381E+00	0.37144752300	0.81543289183	0.00444013586
2294	52	3.142333E+00	0.18829878828	0.88867701100	0.18253091062
2295	57	3.331275E+00	0.32117852472	0.91576439346	0.04789138197
2296	54	3.163223E+00	0.43041008433	0.89945726188	0.04798733963
2297	46	2.795369E+00	0.25031275301	0.93845643521	0.07183284184
2298	THE CLOSEST TO J= 51 3.117656E+00				
2299	51	3.117656E+00	0.06360325521	0.32672056297	0.39476196203
2300	55	3.321633E+00	0.33718364526	0.34558835460	0.5590974987
2301	52	3.142333E+00	0.18829878828	0.88867701100	0.18253091062
2302	56	3.328297E+00	0.10309787875	0.67468354844	0.00204193042
2303	48	3.054911E+00	0.03444560994	0.83893213421	0.02284900022
2304	43	2.881717E+00	0.03966376797	0.51413521671	0.80103310885
2305	53	3.146381E+00	0.37144752300	0.81543289183	0.00444013586
2306	50	3.115898E+00	0.18871534311	0.94003620270	0.11892767249
2307	47	2.966167E+00	0.02780378956	0.93363575942	0.04118350141

⑥ 9 DATA POINTS CLOSEST TO (3.12, 0.19, 0.96)

⑦ PREVIOUSLY SUCCESSFUL RANDOM NUMBERS IN THIS NEIGHBORHOOD

TABLE V
INFERRED DENSITY

Interval	Unnormalized Density	Area	Normalized Density	Length Interval
(0,.028)	27.78	0.78	2.5842	.028
(.028,.064)	55.56	2.00	5.1684	.036
(.064,.169)	19.05	2.00	1.7721	.105
(.169,.337)	11.90	2.00	1.1070	.168
(.337,.430)	21.51	2.00	2.0009	.093
(.430,.523)	10.76	1.00	1.0005	.093
(.523,.616)	5.38	.50	.5002	.093
(.616,.709)	2.69	.25	.2501	.093
(.709,.802)	1.34	.125	.1251	.093
(.802,.895)	0.67	.0625	.0625	.093
(.895,.988)	0.34	.03125	.0313	.093
(.988,1.00)	0.17	.0002	.00002	.012

Figure 1.

Figure 2. Standard and successful random number densities.

Figure 3. The test problem.

Figure 4. Flow chart for analog calculation.

Figure 5. (ϵ_1, ϵ_2) pairs that penetrate slab without collision.

Figure 6. Four successful random walks entering collisions at x_0 despite radically different random walks to x_0 .

Figure 7. Two successful random walks with close random number sequences but diverging random walks and one random walk with a very different random number sequence but entering collision very close to collision 4 on track #1.

Figure 8. Three empirically observed successful random walks (1-3) and one inferred successful random walk (hatched).

Figure 9a Successful Random Numbers and Scores

Figure 9b Random Number Distribution Intervals and Unnormalized Densities

Figure 9c True Successful Random Number Density

Figure 10.

Figure 11.

Figure 12.