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**Application and Experimental Qualification of the
Equivalent Solid Plate Method to the Analysis of
Perforated Plates with Square Penetration Patterns**

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October 1979

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ABSTRACT

The analysis of perforated plates containing circular penetrations arranged in a square pattern is greatly complicated by the orthotropy and circumferential variation of the equivalent solid plate material properties. These complications make the application of the equivalent solid plate concept for this type of penetration pattern to problems involving axisymmetric geometries and loadings very difficult. Unfortunately, the alternatives to the equivalent solid plate concept in a large number of practical situations require explicit three-dimensional modeling of each penetration and therefore is both costly and time consuming if not impossible due to problem size.

Approximate techniques are developed in this paper which facilitate the application of the equivalent solid plate properties for the square penetration pattern perforated plate geometry to axisymmetric analysis. The problem is discussed within the context of the finite element method and appropriate material property relationships are derived which facilitate axisymmetric analysis. These techniques and relationships are also used in the formulation of a thin plate analysis. Finally, after a summary of the experimental methods, comparisons with experimental results are given which substantiate the assumptions made permitting the axisymmetric analysis.

Nomenclature

A	Radius of plate to center of bolt hole ring, in. [mm]
a	Outer radius of perforated region in thin plate model, in. [mm]
a_{ij}	Entries of equivalent solid plate material stiffness matrix, psi [MPa]
$B_1, B_2, B_3, C_1, C_2, C_3$	Constants of integration in thin plate model
b	Outer radius of solid plate region in thin plate model, in. [mm]
C_{ijkl}	Entries of axisymmetric equivalent solid plate material stiffness matrix, psi [MPa]
$[C_0], [C_4], [S_4]$	Material stiffness matrices for the harmonic ASAAS analysis, psi [MPa]
D	Isotropic plate rigidity in.-lb [N-mm]
D_{ij}	Equivalent solid plate rigidity in.-lb [N-mm]
$[D]$	Differential operator defined in Equation (7)
E	Young's Modulus, psi [MPa]
G	Shear Modulus, psi [MPa]
h	Ligament width, in. [mm]
$K_r, K_\theta, K_{r\theta}, K_x, K_y, K_{xy}$	Stress multipliers
M_r, M_θ	Plate bending moments, in.-lb/in. [N-mm/mm]
N	Number of holes in bolt ring

P	Pitch of penetration pattern, in. [mm]
Q	Shearing force, lbs/in. [N/mm]
q	Applied pressure, psi [MPa]
$R_o, R^*, R_p, R_H, R_1, R_2$	Various plate radii defined in Figure 3, in. [mm]
r, θ	In-plane cylindrical coordinates
S	Surface length, in. [mm]
t, t_1, t_2	Plate thicknesses, in. [mm]
u, v, w, w_1, w_2	Displacements in the r, θ, z coordinate system, in. [mm]
V	Volume, in. ³ [mm ³]
x, y	In-plane rectangular coordinates
z	Out-of-plane coordinate
μ	Ligament efficiency (h/P)
ν	Poisson's ratio
ρ_1, ρ_2	Radii of curvature, in. [mm]
Π	Potential energy functional, in.-lbs [mm-N]
σ_i, ϵ_i	Stresses and strains in rectangular coordinates ($i = x, y, z, xy, yz, zx$)

σ_ϕ	Azimuthal stress on penetration boundary, psi [MPa]
ϕ	Azimuthal angle
Φ	Slope of plate

Notation: $\bar{\sigma}_i$ or $\bar{\epsilon}_i$ imply equivalent solid plate quantity

Superscript, *, implies equivalent solid plate property

Subscript, p, implies pitch direction

Summation is implied on repeated i, j, k or l subscripts

Brackets, [ ], imply matrix quantities and braces, { }, imply vector quantities

Introduction

In the design stress analysis of power generating equipment, the use of closed form solutions to approximations of the actual structure is giving way to the use of the more generally applicable finite element structural analysis computer programs. However, in the analysis of perforated plate structures, the present generation of computers and finite element programs make explicit modeling of each penetration impractical especially for steam-generator tubesheet geometries.

For situations involving a large number of periodically placed penetrations such as tubesheet and flow-distributor plate geometries, it is common practice to replace the perforated material with a solid homogeneous material whose mechanical properties have been adjusted to reflect the presence of the holes. This adjustment is done in such a way that the average displacements in the perforated plate are identical to the average displacements in the equivalent solid plate. Once the analysis of the equivalent solid plate is completed, the resulting equivalent solid plate stresses are converted to actual perforated plate stresses through application of appropriate stress multipliers.

The equivalent solid plate procedure simplifies the structural analysis as explicit modeling of each penetration is not necessary. For plates with the holes arranged in a triangular pattern, the equivalent solid plate mechanical properties are isotropic in the plane of the plate and are independent of ligament orientation. Unfortunately, such is not the case for plates with the holes arranged in a square penetration pattern.

As shown by Slot [1]* and Slot and O'Donnell [2], the effective elastic constants for the equivalent solid plate are orthotropic for the square penetration pattern. Because of the symmetry present in the square pattern, this orthotropy is described by six independent elastic constants rather than the usual (for isotropy) two. It is this orthotropy that causes the difficulty in analysis since it can cause a geometrically axisymmetric body to deform non-axisymmetrically even if it is loaded axisymmetrically.

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* Numbers in brackets refer to list of references at end of text.

Methods of analysis for orthotropic bodies are discussed in [3] and closed form solutions are given in [3], and [4] for thin plates that deform axisymmetrically. For plates which cannot be assumed to deform axisymmetrically, three-dimensional solution capabilities are the only recourse.

From the standpoint of application to axisymmetric finite element programs, the orthotropy present in the square penetration pattern equivalent solid plate properties requires additional approximations. Since perforated plates are found in many structures which are nominally axisymmetric there is a clear motivation for the development of methods which permit the analyst to take advantage of the axisymmetric nature of the geometry.

The work presented in this paper is directed at developing equivalent solid plate methods for use in axisymmetric finite element programs for the analysis of thick plates having a square array of penetrations. The material properties appropriate for axisymmetric finite element programs are derived from the orthotropic equivalent solid plate properties given in [1]. The reasonableness of this approach is investigated by comparing the finite element stress and deflection results obtained both by complete orthotropic and the approximate axisymmetric material representations with the experimental results for the laterally loaded perforated plates tested by Bynum & Lemcoe [5] at Southwest Research Institute (SWRI). The approximate axisymmetric material representations are also used to formulate a thin plate theory from which appropriate stress and deflection results are obtained for the SwRI test geometries. The thin plate results provide additional evaluation and demonstrate useful application of the methods developed in this paper.

The comparisons of the results obtained from the test and the analyses were favorable for both displacements and stresses. These comparisons support the underlying assumptions made in developing the methods presented in this paper at least as they were applied to the test situations.

Equivalent Solid Plate Properties For Square Penetration Patterns

The coordinate system under consideration is given in Figure 1. As shown in [1] and [2], the material properties turn out to be a function of ligament efficiency ($\mu = \frac{h}{p}$), elastic constants of the base material (E and ν), and ligament orientation. As given in [2], for the square penetration patterns, the constitutive equations are

$$\begin{Bmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\epsilon}_z \\ \bar{\gamma}_{xy} \\ \bar{\gamma}_{yz} \\ \bar{\gamma}_{zx} \end{Bmatrix} = \begin{pmatrix} \frac{1}{E_p^*} & -\frac{\nu_p^*}{E_p^*} & -\frac{\nu}{E_z^*} & 0 & 0 & 0 \\ -\frac{\nu_p^*}{E_p^*} & \frac{1}{E_p^*} & -\frac{\nu}{E_z^*} & 0 & 0 & 0 \\ -\frac{\nu}{E_z^*} & -\frac{\nu}{E_z^*} & \frac{1}{E_z^*} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_p^*} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_z^*} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_z^*} \end{pmatrix} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_z \\ \bar{\tau}_{xy} \\ \bar{\tau}_{yz} \\ \bar{\tau}_{zx} \end{Bmatrix} \quad (1)$$

It is important to note that in the plane of the plate (x-y plane), there are three independent constants, E_p^* , ν_p^* and G_p^* ("p" is the pitch direction) with the out-of-plane direction (z) adding three more, E_z^* , ν , and G_z^* . For this situation, the anisotropy complication is reflected in the fact that $G_p^* \neq E_p^*/2(1+\nu_p^*)$. For completeness, thermal strains should be added to equation (1). However, since the coefficient of thermal expansion for the equivalent solid material is identical to that of the actual material, no specific treatment is considered necessary for the purposes of this paper.

Transforming the properties of equation (1) into cylindrical coordinates gives

$$\begin{Bmatrix} \bar{\epsilon}_r \\ \bar{\epsilon}_\theta \\ \bar{\epsilon}_z \\ \bar{\gamma}_{r\theta} \\ \bar{\gamma}_{\theta z} \\ \bar{\gamma}_{zr} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_r \\ \bar{\sigma}_\theta \\ \bar{\sigma}_z \\ \bar{\tau}_{r\theta} \\ \bar{\tau}_{\theta z} \\ \bar{\tau}_{zr} \end{Bmatrix} \quad (2)$$

where

$$a_{11} = a_{22} = \frac{1}{E_p^*} + \frac{1}{8} \left(\frac{1}{G_p^*} - \frac{2(1+\nu_p^*)}{E_p^*} \right) [1 - \cos 4\theta]$$

$$a_{21} = a_{12} = -\frac{\nu_p^*}{E_p^*} + \frac{1}{8} \left(\frac{2(1+\nu_p^*)}{E_p^*} - \frac{1}{G_p^*} \right) [1 - \cos 4\theta]$$

$$a_{31} = a_{13} = a_{23} = a_{32} = -\frac{\nu_z^*}{E_z^*}$$

$$a_{41} = a_{14} = -a_{24} = -a_{42} = -\frac{1}{4} \left(\frac{2(1+\nu_p^*)}{E_p^*} - \frac{1}{G_p^*} \right) \sin 4\theta$$

$$a_{33} = \frac{1}{E_z^*}$$

$$a_{44} = \frac{1}{G_p^*} + \frac{1}{2} \left(\frac{2(1+\nu_p^*)}{E_p^*} - \frac{1}{G_p^*} \right) [1 - \cos 4\theta]$$

$$a_{55} = a_{66} = \frac{1}{G_z^*}$$

The a_{ij} entries of the cylindrical coordinate system constitutive matrix are functions of θ which is a direct result of the shear modulus G_p^* being independent of E_p^* and ν_p^* (i.e., $G_p^* \neq E_p^*/2(1+\nu_p^*)$). The effect of the θ variation in the material property relations can be measured by the term $[1 - 2G_p^*(1+\nu_p^*)/E_p^*]$ since this factor is the coefficient of the terms that are a function of angle in the constitutive equation. From Figure 2 it can be seen that as the ligament efficiency gets small, the degree of θ -dependence becomes significant. For this reason, consideration of the θ -dependence of the material properties for perforated plates with a square array of holes is important especially for smaller ligament efficiencies.

Orienting the x-y coordinate system as in Figure 1, the variation with θ is periodic with the period being $\frac{\pi}{2}$ radians. To use equation (2) in axisymmetric finite element programs, it is clear that the program must allow for periodic variation of the material properties. A finite element code that does permit angular variation of material properties is ASAAS [6]. Unfortunately, the capacity of the program limits the degree of geometric complexity to less than 250 nodes thus compromising its general applicability.

Lacking axisymmetric finite programs which do allow for periodic variation in material properties as well as reasonably large nodal capacity, the recourse is to three-dimensional capabilities. These three-dimensional methods can be used in two ways. The first method is to model each of the holes explicitly. This method eliminates the usefulness of the equivalent solid plate method. The second method is to use the equivalent solid plate properties of equation (1) in a three-dimensional program. This paper investigates one- and two-dimensional approximate alternatives to these approaches.

Approaches involving one-dimensional (thin plate) or two-dimensional (axisymmetric thick plate) analysis of perforated plates containing a square array of holes by necessity require approximation to eliminate the periodic material property variation. That this can be a reasonable assumption depends on the loading and support conditions of the plate structure. Given that the period of the constitutive relations is $\frac{\pi}{2}$ and that most perforated regions are attached to relatively stiff stiffening rings and bolting ring flanges, it appears excellent results should be possible using some sort of θ -averaged values for the constitutive relations.

Continuing with this assumption, i.e., the deformation of the plate is nominally axisymmetric, equivalent solid material properties can be obtained which are theoretically consistent with the normal axisymmetric finite element formulations. The consequence of this premise is investigated in the next section through application of the Potential Energy Theorem.

Finite Element Requirements

Since the material properties are functions of θ in the plane of the plate, a rigorous elastic analysis of the perforated region is difficult at best. However, it is expected that in a predominate number of applications, reasonable accuracy is possible using an approximate axisymmetric analysis. As mentioned, this expectation is based on the assumption that the stiffening ring will force axisymmetric behavior at the boundary of the plate in the presence of axisymmetric loading functions and hence the displacements of the perforated region will behave as nominally axisymmetric. However, even though the displacements may not depend strongly on θ , the stresses do. Given the θ -independent displacements and hence strains, the equivalent solid plate stresses may be calculated as a function of θ through equation (2).

Finite element capabilities are usually derived through the Principle of Virtual Work or the Minimum Potential Energy Theorem. Mathematically, the Minimum Potential Energy Theorem states that for equilibrium, the functional

$$\Pi = \frac{1}{2} \int_V \epsilon_{ij} E_{ijkl} \epsilon_{kl} dV - \oint_S T_i u_i dS \quad (3)$$

must be stationary, or the variation of Π set to zero as

$$\delta \Pi = 0. \quad (4)$$

To obtain displacement formulations for finite element work, strain-displacement equations along with assumed displacement shapes are used in conjunction with equation (3). In an axisymmetric formulation, the displacements and hence the strains are functions only of the radius (r) and thickness (z) of the plate. Mathematically, this implies that v is zero and $\frac{\partial}{\partial \theta}$ is a null operator. Hence, only four strains are non-zero and they are

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \epsilon_z \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{Bmatrix} u \\ w \end{Bmatrix} \quad (5)$$

From this consideration, it can be seen that the only quantities that are a function of θ in the kernel of the volume integral of equation (3) are the entries of E_{ijkl} . Hence, θ can be integrated out immediately and using the notation

$$C_{ijkl} = \frac{1}{2\pi} \int_0^{2\pi} E_{ijkl} d\theta \quad (6)$$

the finite element stiffness formulation may be stated in a fashion consistent with the minimum Potential Energy Theorem

$$\Pi = \frac{1}{2} \langle u \rangle^T 2\pi \int [D]^T [C] [D] r dr dz \langle u \rangle - \oint_S \langle T \rangle^T \langle u \rangle dS \quad (7)$$

Here, $[D]$ is the product of the differential operator of equation (5) and the finite element displacement shape functions while $[C]$ is defined as in equation (6).

Computing the entries of equation (6) requires the inverse of equation (2). With the help of Reference [8], the E_{ijkl} matrix can be written as

$$\begin{Bmatrix} \bar{\sigma}_r \\ \bar{\sigma}_\theta \\ \bar{\sigma}_z \\ \bar{\tau}_{r\theta} \\ \bar{\tau}_{\theta z} \\ \bar{\tau}_{zr} \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & 0 & 0 \\ E_{21} & E_{22} & E_{23} & E_{24} & 0 & 0 \\ E_{31} & E_{32} & E_{33} & 0 & 0 & 0 \\ E_{41} & E_{42} & 0 & E_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{66} \end{bmatrix} \begin{Bmatrix} \bar{e}_r \\ \bar{e}_\theta \\ \bar{e}_z \\ \bar{\gamma}_{r\theta} \\ \bar{\gamma}_{\theta z} \\ \bar{\gamma}_{zr} \end{Bmatrix} \quad (8)$$

where

$$E_{11} = E_{22} = \frac{E_p^*}{f(1+v_p^*)} \left[1 - \frac{E_p^*}{E_z^*} v^2 \right] + \frac{1}{2} \left[G_p^* - \frac{E_p^*}{2(1+v_p^*)} \right] (1 - \cos 4\theta)$$

$$E_{12} = E_{21} = \frac{E_p^*}{f(1+v_p^*)} \left[v_p^* + v^2 \frac{E_p^*}{E_z^*} \right] - \frac{1}{2} \left[G_p^* - \frac{E_p^*}{2(1+v_p^*)} \right] (1 - \cos 4\theta)$$

$$E_{13} = E_{31} = E_{32} = E_{23} = \frac{E_p^* v}{f}$$

$$E_{41} = E_{14} = -E_{24} = -E_{42} = \frac{1}{2} \left[G_p^* - \frac{E_p^*}{2(1+v_p^*)} \right] \sin 4\theta$$

$$E_{33} = \frac{E_z^* (1-v_p^*)}{f}$$

$$E_{44} = G_p^* - \frac{1}{2} \left[G_p^* - \frac{E_p^*}{2(1+v_p^*)} \right] (1 - \cos 4\theta)$$

$$E_{55} = E_{66} = G_z^*$$

$$f = 1 - v_p^* - 2 \frac{E_p^*}{E_z^*} v^2$$

The fourth and fifth rows and columns of E_{ijkl} may be collapsed out since $\bar{\gamma}_{r\theta}$ and $\bar{\gamma}_{\theta z}$ are zero for this case. $[C]$ then becomes a four by four matrix whose entries are obtained by integration of E_{ijkl} with θ and this process yields

$$\begin{Bmatrix} \bar{\sigma}_r \\ \bar{\sigma}_\theta \\ \bar{\sigma}_z \\ \bar{\tau}_{zr} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & C_{22} & C_{23} & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_r \\ \bar{\epsilon}_\theta \\ \bar{\epsilon}_z \\ \bar{\gamma}_{zr} \end{Bmatrix} \quad (9)$$

where

$$C_{11} = C_{22} = \frac{E^* P}{f(1+\nu^*_p)} \left[1 - \frac{E^* P}{E^*_z} \nu^2 \right] + \bar{E}$$

$$C_{21} = C_{12} = \frac{E^* P}{f(1+\nu^*_p)} \left[\nu^*_p + \nu^2 \frac{E^* P}{E^*_z} \right] - \bar{E}$$

$$C_{13} = C_{23} = C_{31} = C_{32} = \frac{E^* \nu}{f}$$

$$C_{33} = \frac{E^*_z (1-\nu^*_p)}{f}$$

$$C_{44} = E_{66} = G^*_z$$

$$\bar{E} = \frac{1}{2} \left[G^*_p - \frac{E^* P}{2(1+\nu^*_p)} \right]$$

Note that for the plane stress values from Reference [1], the ratio of $\bar{E}$ to C_{11} is 0.356 for a ligament efficiency of 20%. Hence the correction term, $\bar{E}$, is a significant contributor to the constitutive relationships.

Application To Finite Element Programs

The above axisymmetric interpretation of the equivalent solid plate properties for square penetration patterns are now applied to the perforated plates tested at SwRI by Bynum and Lemcoe [5]. Dimensions and properties of the three perforated plate configurations tested at SwRI are given in Table I. Figure 3 shows the geometric detail and loading associated with each configuration.

To the extent that Bynum and Lemcoe [5] detail their results, two finite element programs are used to predict the same results thus permitting useful comparison and evaluation of the numerical techniques presented here. The two programs used are ASAAS [6] and BESTRAN [9].

ASAAS [6] is a finite element program which provides elastic analysis capability for axisymmetric structures subjected to non-axisymmetric loadings given material properties which may vary circumferentially and exhibit general anisotropy. The program uses quadrilateral elements made up of four constant-strain triangular elements.

BESTRAN [9] is a finite element structural analysis system which includes a nine-node isoparametric quadratic element for two-dimensional axisymmetric analyses. In this mode of operation, BESTRAN is capable of orthotropic elastic substructured axisymmetric analysis and it is to this program that the circumferentially-averaged orthotropic perforated plate properties are input.

The models used for both programs consisted of 175 elements as shown in Figure 4. However, since the ASAAS element has four nodes per element, there were 216 nodes in the ASAAS model. BESTRAN incorporates the nine-node isoparametric element which gives 781 nodes for the same number of elements.

TABLE I
MATERIAL PROPERTIES AND DIMENSIONS
OF PERFORATED PLATES
TESTED AT SwRI

	Configuration I	Configuration II	Configuration III
h/P	0.35	0.35	0.20
t/P	3.90	3.10	3.10
t	4.872	3.875	3.875
E^*/E_P	0.4753	0.4753	0.3135
ν^*_P	0.2024	0.2024	0.1244
G^*/G_P	0.2797	0.2797	0.1001
E^*_Z/E	0.6682	0.6682	0.4974
G^*_Z/G	0.5000	0.5000	0.3221

$$R_O = 15 \text{ in. (381 mm)}$$

$$A = 14 \text{ in. (356 mm)}$$

$$R_P = 12 \text{ in. (305 mm)}$$

$$R^* = 11.75 \text{ in. (298 mm)}^*$$

$$2R_H = 13/16 \text{ in. (20.6 mm)}$$

$$q = 1000 \text{ psi (6.895 MPa)}$$

$$R_1 = 13.594 \text{ in. (345 mm)}$$

$$R_2 = 14.407 \text{ in. (366 mm)}$$

$$E = 10.5 \times 10^6 \text{ psi (72400 MPa)}$$

$$\nu = 0.326$$

Number of bolt holes is 48

* Currently there are several different methods of calculating R^* , none of which have a rigorous theoretical basis. The given value was taken from O'Donnell [7] to provide a consistent basis for comparison. This value can be obtained from the equation $R^* = R + 1/4(P-h)$ given in the ASME Code (where R is the radial distance to the center of the outermost hole). The value given was used for all configurations for convenience.

The perforated plate properties were obtained from Slot [1] for the moderately thick plates analyzed. Generalized plane strain properties were used since the plate thickness to pitch ratio are 3.1 and 3.9 for the configurations tested by SwRI.

Specific material input to BESTRAN is obtained by computing the [C] matrix of Equation (9) and inverting it to arrive at the material definition which is

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \epsilon_z \\ \gamma_{r\theta} \\ \gamma_{\theta z} \\ \gamma_{zr} \end{Bmatrix} = \begin{bmatrix} 1/E_r & -\nu_{r\theta}/E_\theta & -\nu_{rz}/E_z & 0 & 0 & 0 \\ & 1/E_\theta & -\nu_{\theta z}/E_z & 0 & 0 & 0 \\ & & 1/E_z & 0 & 0 & 0 \\ \text{[SYMMETRIC]} & & & 1/G_{r\theta} & 0 & 0 \\ & & & & 1/G_{\theta z} & 0 \\ & & & & & 1/G_{rz} \end{bmatrix} \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{r\theta} \\ \tau_{\theta z} \\ \tau_{zr} \end{Bmatrix} \quad (10)$$

The entries of the matrix in Equation (10) are then obtained by comparison with the corresponding entries in the $[C]^{-1}$ matrix. The explicit values used in this analysis are given in Table II.

It is pointed out that most finite element programs define the inputs for the material properties in terms of the flexibility form of the constitutive equations such as Equation (10). The programs immediately invert this equation for use in the stiffness formulation of Equations (3) and (7). For the type of orthotropy of Equation (2) (i.e., with a_{14} and a_{24} entries) the inverse of the θ -average of [A] is not equal to the θ -average of [E] of Equation (8) hence requiring the procedure outlined in the previous paragraph.

As shown in Figure 3, the perforated region extends to $r = R^*$ with the stiffening and/or bolting ring attached. The bolting ring is seen to be made up of a perforated ring representing the bolt holes sandwiched between two isotropic rings. For the bolt hole ring the material properties of Equation (10) are given by

$$E_{\theta} \ll E$$

$$E_r = E_z = E[1 - (NR_H^2)/(R_2^2 - R_1^2)]$$

$$\nu_{rz} = 0.3$$

$$\nu_{r\theta} = \nu_{\theta z} = 0.0$$

(11)

The term $NR_H^2/(R_2^2 - R_1^2)$ is the area ratio of the total area removed by the bolt holes to the area of the ring. The shear moduli $G_{r\theta}$ and $G_{\theta z}$ are zero and $G_{rz}/G = E_r/E$. For the plates tested by SwRI, E_r/E is 0.6518.

The plate is assumed to be continuously simply supported at a radius equal to the inside edge of the bolt holes. This condition is considered a reasonable approximation to the spherical washer seating arrangement used in the SwRI tests. A uniform pressure was distributed over the region $r \leq R_p$ as shown in Figure 3.

The circumferential variation in the material matrix of Equation (8) is required by ASAAS. Equation (8) can be expanded to the form

$$\{\sigma\} = [C_0] \{\epsilon\} + [C_4] \{\epsilon\} \cos 4\theta + [S_4] \{\epsilon\} \sin 4\theta \quad (12)$$

where the matrices $[C_0]$, $[C_4]$ and $[S_4]$ are available from Equation (8) and explicit values used for the SwRI plates are listed in Table II.

TABLE II

PERFORATED PLATE MATERIAL PROPERTY
DEFINITIONS FOR FINITE ELEMENT PROGRAMS

A. BESTRAN Input

	h/P = 0.20		h/P = 0.35	
	[10 ³ KSI]	[10 ³ MPa]	[10 ³ KSI]	[10 ³ MPa]
E _r	2.489	(17.16)	4.219	(29.09)
E _θ	2.489	(17.16)	4.219	(29.09)
E _z	5.223	(36.01)	7.016	(48.38)
ν _{rθ}	0.3380	(dimensionless)	0.3257	(dimensionless)
ν _{rz}	0.3260	(dimensionless)	0.3260	(dimensionless)
ν _{θz}	0.3260	(dimensionless)	0.3260	(dimensionless)
C _{rθ}	0.6238	(4.301)	1.444	(9.956)
G _{θz}	1.275	(8.791)	1.980	(13.65)
G _{rz}	1.275	(8.791)	1.980	(13.65)

B. ASAAS InputNon-Zero Entries for the [C₀] Matrix*

I J	h/P = 0.20		h/P = 0.35	
	[10 ³ KSI]	[10 ³ MPa]	[10 ³ KSI]	[10 ³ MPa]
1 1	3.149	(21.71)	5.452	(37.59)
1 2	1.289	(8.888)	2.269	(15.64)
1 3	1.447	(9.977)	2.517	(17.35)
2 2	3.149	(21.71)	5.452	(37.59)
2 3	1.447	(9.977)	2.517	(17.35)
3 3	6.166	(42.51)	8.657	(59.69)
4 4	6.238	(43.01)	1.444	(9.956)
5 5	1.275	(8.791)	1.980	(13.65)
6 6	1.275	(8.791)	1.980	(13.65)

Non-Zero Entries for the [C₄] Matrix*

1 1	0.5337	(3.680)	0.4839	(3.336)
1 2	-0.5337	(-3.680)	-0.4839	(-3.336)
2 2	0.5337	(3.680)	0.4839	(3.336)
4 4	-0.5337	(-3.680)	-0.4839	(-3.336)

Non-Zero Entries for the [S₄] Matrix*

1 4	-0.5337	(-3.680)	-0.4839	(-3.336)
2 4	0.5337	(3.680)	0.4839	(3.336)

* Matrices are symmetric.

Bynum and Lemcoe [5] measured the local peak stress on the periphery of the penetrations along the pitch and diagonal directions of the penetration pattern. Special finite element post-processing is required to extract these quantities from the finite element solutions. First, the stresses are obtained from the strains using Equation (2) noting that although the strains may be circumferentially invariant in the BESTRAN case, the stresses will vary with θ . These stresses are the equivalent solid plate stresses which must then be multiplied by the appropriate stress multipliers to get the peak azimuthal stress σ_ϕ from the equation

$$\sigma_\phi = K_x \bar{\sigma}_x + K_y \bar{\sigma}_y + K_{xy} \bar{\tau}_{xy} \quad (13)$$

The multipliers K_x , K_y and K_{xy} are functions of the azimuthal angle ϕ and are given in Slot [1]. Further discussion of these multipliers is given after equation (35) of the next section. For the SwRI plates, the stress multipliers used are given in Table III.

TABLE III

STRESS MULTIPLIERS FOR THE PEAK STRESS CALCULATIONS

	Configuration III h/P = 0.20		Configurations I and II h/P = 0.35	
	$\phi = 0^\circ$	$\phi = 45^\circ$	$\phi = 0^\circ$	$\phi = 45^\circ$
K_x	-0.023	1.396	-0.3065	1.2505
K_y	5.773	1.396	3.935	1.2505
K_{xy}	0.0	-14.51	0.0	-9.5225
	$\theta = \phi = 0^\circ$	$\theta = \phi = 45^\circ$	$\theta = \phi = 0^\circ$	$\theta = \phi = 45^\circ$
K_r	-0.0230	-5.859	-0.3065	-3.511
K_θ	5.773	8.651	3.935	6.012
$K_{r\theta}$	0.0	-2.792	0.0	-2.501

Application To Thin Plate Theory

The approximate axisymmetric analysis formulations derived above are now applied to classical thin plate theory. A complete discussion of the development of the theory for isotropic plates is given by Harvey [10] and Timoshenko [11].

For thin circular plates, the bending moments are defined as

$$\begin{Bmatrix} M_r \\ M_\theta \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \bar{\sigma}_r \\ \bar{\sigma}_\theta \end{Bmatrix} z dz \quad (14)$$

Assuming small deflections, i.e., $\Phi = -dw/dr$, the radii of curvature of the plate are

$$\begin{aligned} \frac{1}{\rho_1} &= \frac{d\Phi}{dr} \\ \frac{1}{\rho_2} &= \frac{\Phi}{r} \end{aligned} \quad (15)$$

Recalling from Kirchhoff plate theory that

$$\begin{aligned} \bar{\epsilon}_r &= z \frac{d\Phi}{dr} \\ \bar{\epsilon}_\theta &= z \frac{\Phi}{r} \\ \bar{\sigma}_z &= 0 \text{ or a constant,} \end{aligned} \quad (16)$$

results in the moment-displacement relations for an isotropic plate

$$\begin{aligned} M_r &= -D \left(\frac{dw^2}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right) = D \left(\frac{d\Phi}{dr} + \nu \frac{\Phi}{r} \right) \\ M_\theta &= -D \left(t \frac{dw}{dr} + \nu \frac{dw^2}{dr^2} \right) = D \left(\frac{\Phi}{r} + \nu \frac{d\Phi}{dr} \right) \end{aligned} \quad (17)$$

where the plate rigidity is

$$D = \frac{Et^3}{12(1-\nu^2)}$$

For the equivalent solid plate constituting relations of Equation (9) these equations are

$$\begin{aligned} M_r &= \frac{-t^3}{12} \left\{ \left[C_{11} - \frac{C_{13} C_{31}}{C_{33}} \right] \frac{d^2 w}{dr^2} + \frac{1}{r} \left[C_{12} - \frac{C_{13} C_{32}}{C_{33}} \right] \frac{dw}{dr} \right\} \\ M_\theta &= -\frac{t^3}{12} \left\{ \left[C_{21} - \frac{C_{23} C_{31}}{C_{33}} \right] \frac{d^2 w}{dr^2} + \frac{1}{r} \left[C_{22} - \frac{C_{23} C_{32}}{C_{33}} \right] \frac{dw}{dr} \right\} \end{aligned} \quad (18)$$

Defining the equivalent solid plate rigidity as

$$D_{ij} = \frac{t^3}{12} \left[C_{ij} - \frac{C_{i3} C_{3j}}{C_{33}} \right] \quad (19)$$

permits the moments M_r and M_θ to be written as

$$\begin{aligned} M_r &= -D_{11} \frac{d^2 w}{dr^2} - D_{12} \frac{1}{r} \frac{dw}{dr} = D_{11} \frac{d\phi}{dr} + D_{12} \frac{\phi}{r} \\ M_\theta &= -D_{21} \frac{d^2 w}{dr^2} - D_{22} \frac{1}{r} \frac{dw}{dr} = D_{21} \frac{d\phi}{dr} + D_{22} \frac{\phi}{r} \end{aligned} \quad (20)$$

The equilibrium equation for circular plates

$$M_r + r \frac{dM_r}{dr} - M_\theta + Qr = 0 \quad (21)$$

yields the differential equation for isotropic circular plates

$$\frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \frac{\phi}{r^2} = - \frac{qr}{2D} \quad (22)$$

This equation can be integrated to give

$$\phi = \frac{-qr^3}{16D} + \frac{C_1 r}{2} + \frac{C_2}{r} \quad (23)$$

and

$$w = \frac{qr^4}{64D} - \frac{C_1 r^2}{4} - C_2 \ln r + C_3 \quad (24)$$

where C's are constants of integration.

The differential equation for an anisotropic circular plate is

$$D_{11} \frac{d^3 w}{dr^3} + \frac{1}{r} \left[D_{11} + D_{12} - D_{21} \right] \frac{d^2 w}{dr^2} - D_{22} \frac{1}{r^2} \frac{dw}{dr} = Q \quad (25)$$

Since the plate rigidities are symmetric and $D_{11} = D_{22}$, the governing differential equation becomes

$$\frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} = \frac{Q}{D_{11}} \quad (26)$$

which, upon integration, gives

$$\phi = - \frac{qr^3}{16D_{11}} + \frac{C_1 r}{2} + \frac{C_2}{r} \quad (27)$$

and

$$w = \frac{qr^4}{64D_{11}} - \frac{C_1 r^2}{4} - C_2 \ln r + C_3 \quad (28)$$

where C's are constants of integration.

The thin plate model is now developed using equations (23), (24), (27) and (28). As shown in Figure 4 the model consists of an equivalent solid plate region of thickness t_1 extending from $r = 0$ to $r = a$ and an isotropic plate region of thickness t_2 extending from $r = a$ to $r = b$. This composite plate is simply supported at its outer edge, $r = b$. For the SwRI plates b is taken to be equal to R_1 , the radius of the inside edge of the bolt holes as on the finite element analysis. For dimensions and properties of the SwRI plates see Table I. Equations (23), (24), (27) and (28) can now be written as

$$\begin{aligned}\phi_1 &= \frac{-qr^3}{16D_{11}} + \frac{C_1 r}{2} + \frac{C_2}{r} \\ w_1 &= \frac{qr^4}{64D_{11}} - \frac{C_1 r^2}{4} - C_2 \ln r + C_3 \\ \phi_2 &= \frac{-qr^3}{16D_2} + \frac{B_1 r}{2} + \frac{B_2}{r} \\ w_2 &= \frac{qr^4}{64D_2} - \frac{B_1 r^2}{4} - B_2 \ln r + B_3\end{aligned}\tag{29}$$

where the subscript 1 denotes the equivalent solid plate region and subscript 2 denotes the isotropic plate region. The boundary conditions for a simply supported circular plate are

$$\begin{aligned}\text{@ } r = 0 \quad \phi_1 &= 0 \\ \text{@ } r = a \quad \phi_1 &= \phi_2; w_1 = w_2; M_{r1} = M_{r2} \\ \text{@ } r = b \quad M_{r2} &= 0; w_2 = 0\end{aligned}\tag{30}$$

Using equations (29) and the boundary conditions (30) the system can be reduced to 6 equations in 6 unknowns. These can be solved for the 6 constants of integration to yield

$$\begin{aligned}
B_1 &= \frac{qa^2}{8D_2} - \frac{qD''A'}{8a^2D_2} \\
B_2 &= \frac{qD'A'}{16D_2} \\
B_3 &= \frac{a(a^2b^2 - \frac{b^4}{2})}{32D_2} + \frac{qA'}{32D_2} \left(2D' \ln b - \frac{D''b^2}{a^2} \right) \\
C_1 &= \frac{qa^2}{8D_{11}} - \frac{qA'}{4a^2} \\
C_2 &= 0 \\
C_3 &= \frac{qa^4}{64D_{11}} + \frac{q}{64D_2} \left(2a^2b^2 - b^4 - a^2 + 2A'D'(1-2\ln a + 2\ln b) - 2A'D'' \frac{b^2}{a^2} \right)
\end{aligned} \tag{31}$$

where

$$A' = \frac{h^2 (3 + \nu) - a^2 (1 + \nu)}{\frac{-D''}{a^2} (1 + \nu) - \frac{D'}{b^2} (1 - \nu)}$$

$$D' = D_{11} + D_{12} - D_2 (1 + \nu)$$

$$D'' = D_{11} + D_{12} + D_2 (1 - \nu)$$

Substituting these 6 constants into equations (29) gives a closed form solution to the thin plate model. Equation (8) with $\sigma_z = 0$ permits the calculation of $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, and $\bar{\tau}_{r\theta}$ as a function of θ . Specifically, for the equivalent solid plate region,

$$\begin{aligned}
 \bar{\sigma}_{r1} &= \frac{t_1}{2} \left[\frac{C_1}{2} - \frac{3qr^2}{16D_{11}} \right] \left[E_{11} - \frac{E_{31}^2}{E_{33}} \right] + \frac{t_1}{2} \left[\frac{C_1}{2} - \frac{qr^2}{16D_{11}} \right] \left[E_{12} - \frac{E_{31}^2}{E_{33}} \right] \\
 \bar{\sigma}_{\theta 1} &= \frac{t_1}{2} \left[\frac{C_1}{2} - \frac{3qr^2}{16D_{11}} \right] \left[E_{12} - \frac{E_{31}^2}{E_{33}} \right] + \frac{t_1}{2} \left[\frac{C_1}{2} - \frac{qr^2}{16D_{11}} \right] \left[E_{11} - \frac{E_{31}^2}{E_{33}} \right] \\
 \bar{\tau}_{r\theta 1} &= \frac{t_1}{2} \left[\frac{C_1}{2} - \frac{3qr^2}{16D_{11}} \right] E_{41} + \frac{t_1}{2} \left[\frac{C_1}{2} - \frac{qr^2}{16D_{11}} \right] E_{42}
 \end{aligned} \tag{32}$$

Although thin plate theory requires that $\bar{\gamma}_{zr}$ be zero, it is well known that the shear stress varies parabolically with z such that the average satisfies gross equilibrium of the transversely loaded plate. This error is a consequence of Kirchhoff plate theory and is present in this theoretical development as well as in isotropic thin plate theory. Equation (8) also points out the presence of the planar shear stress, $\bar{\tau}_{r\theta}$, which for isotropic theory is identically zero. Here, $\bar{\tau}_{r\theta}$ is only zero on a θ -averaged basis.

Another inconsistency in this analysis involves the interaction moment at the juncture of the perforated region and the support ring. This moment can be computed from the bending stress, $\bar{\sigma}_{r1}$ at $r = a$, only on a θ -averaged basis. That is, M_{r1}^* as calculated by

$$M_{r1}^* = - \frac{t_1^2}{6} \left[\bar{\sigma}_{r1} \right]_{r=a} \tag{33}$$

is equal to M_{r1} as calculated by equation (17) only when averaged over θ from 0 to 2π . M_{r1}^* varies periodically (period of $\pi/2$) about the average value M_{r1} . M_{r1}^* is equal to M_{r1} at θ equal to $\pi/8, 3\pi/8, 5\pi/8, \dots$ etc. For Configurations I and II, M_{r1}^* varies as much as $\pm 26.7\%$, about the average value M_{r1} , while for Configuration III M_{r1}^* varies as much as $\pm 81.6\%$. All other things being held constant the difference in the variations can be directly attributed to the difference in ligament efficiencies. Since these variations also hold for the stresses, they give an indication of the inaccuracies that would be encountered if the average values were used in the stress calculations. It should be recognized that if G_p^* was equal to $E_p^*/2(1+\nu_p^*)$ then the stresses would be independent of θ in this analysis and M_{r1}^* would equal M_{r1} for all values of θ .

Next the stresses $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, and $\bar{\tau}_{r\theta}$ of equation (32) are then transformed to the x-y coordinate system (see Figure 1) by the equations

$$\begin{aligned}\bar{\sigma}_x &= \frac{\bar{\sigma}_r + \bar{\sigma}_\theta}{2} + \frac{\bar{\sigma}_r - \bar{\sigma}_\theta}{2} \cos 2\theta + \bar{\tau}_{r\theta} \sin 2\theta \\ \bar{\sigma}_y &= \frac{\bar{\sigma}_r + \bar{\sigma}_\theta}{2} - \frac{\bar{\sigma}_r - \bar{\sigma}_\theta}{2} \cos 2\theta - \bar{\tau}_{r\theta} \sin 2\theta \\ \bar{\tau}_{xy} &= \frac{\bar{\sigma}_\theta - \bar{\sigma}_r}{2} \sin 2\theta + \bar{\tau}_{r\theta} \cos 2\theta\end{aligned}\tag{34}$$

These stresses are the equivalent solid plate stresses which must then be multiplied by the appropriate stress multipliers to get the peak azimuthal stress σ_ϕ from equation (13) which is rewritten for convenience as

$$\sigma_\phi = K_x \bar{\sigma}_x + K_y \bar{\sigma}_y + K_{xy} \bar{\tau}_{xy}\tag{35}$$

where K_x , K_y , and K_{xy} , taken directly from Slot [1], are given in Table III.

For use with the axisymmetric finite element results and with stresses derived from circular thin plate theory, it is often convenient to write equation (35) in terms of r- θ -z coordinates. Hence, the transformations of equation (34) can be used with equation (35) to give

$$\sigma_\phi = K_r \bar{\sigma}_r + K_\theta \bar{\sigma}_\theta + K_{r\theta} \bar{\tau}_{r\theta}\tag{36}$$

where

$$K_r = K_x \cos^2 \theta + K_y \sin^2 \theta + K_{xy} \sin \theta \cos \theta$$

$$K_\theta = K_x \sin^2 \theta + K_y \cos^2 \theta - K_{xy} \sin \theta \cos \theta$$

$$K_{r\theta} = (\cos^2 \theta - \sin^2 \theta)K_{xy} - 2\sin \theta \cos \theta (K_x + K_y)$$

The K_r , K_θ and $K_{r\theta}$ multipliers are then functions of both θ and ϕ and are tabulated in Table III for the configuration and orientations of interest.

Experimental Qualification

Douglas Bynum, Jr. and M. M. Lemcoe [5] conducted experimental tests, at Southwest Research Institute (SwRI), on moderately-thick circular aluminum plates having a square pattern of transverse perforations. The purpose of these tests was to provide experimental stress and deflection data to help in the development and modification of analytical theories. Uniform lateral loads were applied to one face of the simply-supported test specimens.

A simple-support boundary condition was obtained by oversizing the bolt-holes and using spherical washers with the bolts which attach the plate to the fixture. A thin aluminum diaphragm, which acted as a membrane, was placed between the fixture and the specimens to seal the perforations. A groove was provided on the fixture near the plate edges for an "O"-ring seal. With this simple and effective design, hydraulic pressure was applied to the diaphragm to provide the uniform lateral loading of the specimens.

Deflection measurements were taken with mechanical displacement type gages. A birefringent coating was applied to the test specimens, and relative retardation measurements were taken. With normal and oblique incidence meters, enabling the computation of the maximum and minimum principal stresses. A limited number of electric resistance, foil type, strain gages were used to provide a means of comparison with the results from the birefringent coating analysis.

Bynum and Lemcoe report that the stresses from the strain gage results are accurate with $\pm 5\%$ and the stresses obtained with the birefringent coating technique agree with the strain gage results within $\pm 6\%$.

Comparison of Results

The experimental data obtained from the SwRI tests are compared with the results of three analysis: (1) The BESTRAN finite element solutions using the circumferentially averaged equivalent solid plate properties; (2) The thin-plate theoretical solution using the circumferentially averaged equivalent solid plate properties and solid stiffening ring model; and (3) The ASAAS finite element solution which includes the periodic variation of the equivalent solid plate properties explicitly. Given that the plates tested qualify as moderately-thick to thin, good agreement is expected between the theoretical thin plate solution and the BESTRAN results. The ASAAS results allow assessment of the error introduced by the assumption of axisymmetric deformation made in the BESTRAN and thin-plate analyses.

The displacement results from the various analyses are compared with the SwRI test results in Figures 5 and 6. Figure 5 shows the transverse displacement comparisons for $\theta = 0$ and 90° while Figure 6 is for $\theta = 45^\circ$. Excellent agreement is shown between ASAAS, BESTRAN, thin-plate theory and the experimental values of Bynum and Temcoe [5] for all three configurations. Displacements at $\theta = 0$ and $\theta = 45^\circ$ are also seen to be in close agreement, thus supporting the circumferentially invariant displacement assumption.

It is expected that the more flexible the stiffening ring and its structural connection become, the less realistic becomes the assumption of circumferentially invariant displacements. For these problems the continuous simple support boundary condition constitutes an infinitely stiff structural connection. The validity of the assumption of the circumferentially invariant displacements is demonstrated in the Figure 5 comparison of the transverse displacements as computed by BESTRAN and the thin plate theory with the ASAAS results for $\theta = 45^\circ$. Figures 4 and 5 show negligible circumferential variation of the transverse displacements as computed by ASAAS. Therefore, the conditions are appropriate for the axisymmetric analysis and the displacement results support the assumptions made to permit the analysis.

As shown in Figure 5, the experimental results indicate a zero displacement at the center of the bolt hole ($r/A = 1.0$) which is in conflict with the assumption made in all of the analyses that the displacement would be zero at the inside of the bolt hole radius. This assumption is important because for a thin plate the maximum transverse displacement is proportional to the plate radius raised to the fourth power as seen in equation (28). The maximum displacements from the analyses are 3.22% lower than the experimental value for Configuration III. Presuming the A^4 dependence of the displacement, this error would change to 8.02% if the fixed point would be moved to the center of the bolt hole. This implies that the support is somewhere between $r/A = 1.0$ and the inside of the bolt hole ring. Regardless, the results of the present analyses are only 3.22% lower than the experimental values and well within the experimental error of the test. Therefore, further refinement of the boundary conditions is not warranted.

Bynum and Lemcoe [5] measured the local peak stress on the periphery of the penetrations along the pitch and diagonal directions of the penetration pattern. Special finite element post-processing is required to extract these quantities from the finite element solutions. Additional calculations are also required to obtain the appropriate thin-plate stress results. Equations (34), (35) and (36) were used for these additional calculations and the comparisons are all made on the basis of elastic peak stress σ_ϕ .

The comparisons for the stresses along the pitch ($\theta = 0^\circ$) direction are shown in Figure 7 with Figure 8 giving the stresses for the diagonal ($\theta = 45^\circ$) direction. These comparisons show that along the pitch direction ($\theta = 0^\circ$) the stresses predicted by ASAAS are within about 5% of those computed using the BESTRAN procedure with the greatest difference being near $r/R^* = 0.50$. The stress at $r/R^* = 0.0$ computed along the diagonal direction ($\theta = 45^\circ$) using BESTRAN is within 4% of the ASAAS value. However, there is as much as a 27% difference at $r/R^* = 0.50$. This large difference is attributed to the circumferential variation in properties which should be of greatest influence at $r/R^* = 0.50$. For these problems though, the stress results from BESTRAN do conservatively bound the theoretically more correct orthotropic analysis results of ASAAS in regions where the differences are large. The stress results from BESTRAN and the thin-plate theory agree well with the thin-plate theory being 2.5% higher than the BESTRAN value for Configuration I, 3.0% higher for Configuration II and 3.4% higher for Configuration III at the center of the plate.

Comparison of the calculated results with the experimental results shows that the data are within 10% at $r/R^* = 0.0$ for both $\theta = 0$ and $\theta = 45^\circ$. The experimental data do tend to support the circumferentially averaged axisymmetric analyses more conclusively than the ASAAS orthotropic analysis. This is particularly evident in Figure 8 in the range $0.30 \leq r/R^* \leq 0.60$. There is not however a consistent trend regarding the lower or upper bounding of the experimental data. The experimental results are obtained by strain gages placed on the periphery of the holes and are compared with the calculated values referenced to the equivalent solid plate stresses calculated for the center of the hole. For the equivalent solid plate method to be valid, the change in stress between any two adjacent holes must be small. From the thin plate theory, the slope of the stresses at the center of the plate is zero while it can produce a maximum hole-to-hole variation of 6% at $r = R^*$. In addition to this, the rapid variation with azimuthal angle of the stress on the rim of the hole makes precise placement of the strain gages very important when comparing with calculations. For Configuration III, a 5° misplacement of the gage could result in a 30% change in calculated stress. Hence, experimental/calculational differences are most likely attributable to knowledge of the precise azimuthal location of the gages.

The calculated maximum stress values that are shown in Figure 7 @ $r/R^* = 0.0$ and obtained by using the equivalent solid plate method as applied with the axisymmetric interpretation of the material properties are considered to be well within the range of the experimental error. Hence it is concluded that the axisymmetric equivalent solid plate method provides sufficient accuracy for this class of perforated plate problem.

The results of Figures 5, 6, 7 and 8 further suggest that the thicker the plate and the higher the ligament efficiency, the greater the expected accuracy from the approximate axisymmetric method. For example, Configuration I was either thicker or had a higher ligament efficiency than either Configurations II or III. As observed particularly from Figures 7 and 8 there is better agreement between the results for Configuration I than for the other two configurations.

Further studies are required regarding the general applicability of these procedures to practical engineering structures. In particular, the effect of the stiffening ring and stiffness of the structural connection are important considerations in assessing expected accuracy of the displacements and stresses derived from an equivalent solid plate analysis. The Bynum and Lemcoe tests do not help in this assessment because of the nature of the test fixture.

Conclusions

The analysis of perforated plates containing penetrations arranged in a square pattern is greatly complicated by the anisotropy of the equivalent solid plate material properties. The development of an approximate treatment of the equivalent solid plate anisotropy consistent with axisymmetric finite element programs was presented in this paper. This approximation was then used in both finite element analysis and a thin plate theoretical analysis of the three perforated plate configurations tested by Bynum and Lemcoe [5]. These plates were also analyzed using a finite element program which can explicitly treat the anisotropy.

Satisfactory agreement was shown to exist between the experimental and calculated results for the displacements and stresses. The comparisons between the experimental and numerical results included in this report have shown that when geometry and boundary conditions warrant, an adequate stress analysis is possible through the use of approximate axisymmetric analysis methods based on material properties that are averaged in the circumferential direction.

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FIGURE 1
PERFORATED PLATE GEOMETRY AND PARAMETERS

FIGURE 2
MEASURE OF ANGULAR DEPENDENCE
OF EQUIVALENT SOLID PLATE MATERIAL
PROPERTIES FOR SQUARE PENETRATION
PATTERNED PERFORATED PLATES

FIGURE 3

SWRI PRESSURE VESSEL RESEARCH COUNCIL PERFORATED PLATE TESTS

FIGURE 4
MODEL CONFIGURATIONS

FIGURE 5. COMPARISON OF TRANSVERSE DISPLACEMENTS FOR $\theta=0^\circ$ AND 90°

FIGURE 6. COMPARISON OF TRANSVERSE DISPLACEMENTS FOR $\theta = 45^\circ$

FIGURE 7. COMPARISON OF PEAK STRESS ON THE PENETRATION BOUNDARY FOR $\theta = \phi = 0^\circ$

FIGURE 8. COMPARISON OF PEAK STRESS ON THE PENETRATION BOUNDARY FOR $\theta = \phi = 45^\circ$