RESISTIVE INTERCHANGE INSTABILITY CRITERION IN
ANISOTROPIC TOROIDAL SYSTEMS

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ABSTRACT

Ideal (Mercier) and resistive criteria for localized magnetohydrodynamic instabilities are derived for general toroidal systems with anisotropic pressure.

MASTER

The ideal (Mercier) and resistive criteria for localized modes have been useful for predicting stability behavior of stellarators and tokamaks with scalar pressure.¹ Since they can be evaluated easily on each magnetic surface, these criteria are often included as part of a numerical equilibrium calculation. They provide a measure of the energy that is available locally to drive an instability and thus can even provide insight into the nonlinear evolution of the modes. Indeed, it has recently been shown² that the resistive criterion governs the nonlinear evolution of magnetic islands. Our purpose here is to generalize these criteria and not to evaluate the wave properties of the magnetohydrodynamic modes associated with them. The ideal criterion has been generalized to include anisotropic pressure in axisymmetric systems.³ Here, we derive this and the resistive criterion for arbitrary toroidal systems. Since the plasma anisotropy created by auxiliary heating in the present and future generations of tokamaks and stellarators can modify the equilibrium properties of the devices, this generalization seems useful.

In the Appendix we follow the derivation of a set of averaged equations that was given by Johnson and Greene.⁴ Since these equations are similar to those considered by Glasser *et al.*,¹ we can then identify with that work to obtain the criteria:

Ideal Interchange,

$$D_I \equiv E + H - \frac{1}{4} < 0 \quad \text{for stability;}$$

Resistive Interchange,

$$D_R \equiv E + H^2 = D_I + (H - \frac{1}{2})^2 < 0 \quad \text{for stability.}$$

Here

$$E \equiv \frac{\langle B^2/\sigma |\nabla V|^2 \rangle}{\Lambda^2} \left[\frac{\nabla V \cdot \nabla (p_{\perp} + B^2/2)}{\tau \sigma B^2 |\nabla V|^2} \left(\sigma \frac{\partial p_{\perp}}{\partial V} \Big|_B + \tau \frac{\partial p_{\parallel}}{\partial V} \Big|_B \right) \right. \\ \left. + \left\langle \frac{\sigma (\underline{J} \cdot \underline{B})^2}{B^2 |\nabla V|^2} \right\rangle - \frac{(\langle \underline{J} \cdot \underline{B} / |\nabla V|^2 \rangle)^2}{\langle B^2/\sigma |\nabla V|^2 \rangle} + \phi \psi' \left\langle \frac{\sigma \underline{J} \cdot \underline{B}}{B^2} \frac{\partial}{\partial \zeta} \frac{\nabla V \cdot \nabla u}{|\nabla V|^2} \right\rangle \right] ,$$

$$H \equiv \frac{\langle B^2/\sigma |\nabla V|^2 \rangle}{\Lambda} \left[\frac{\langle \underline{J} \cdot \underline{B} / |\nabla V|^2 \rangle}{\langle B^2/\sigma |\nabla V|^2 \rangle} - \frac{\langle \underline{J} \cdot \underline{B} \rangle}{\langle B^2/\sigma \rangle} \right] ,$$

$$\Lambda \equiv \psi' \chi'' - \chi' \psi'' ,$$

$$\phi \psi' \frac{\partial}{\partial \zeta} \frac{\nabla V \cdot \nabla u}{|\nabla V|^2} = \Lambda - \frac{\underline{B} \times \nabla V \cdot [\nabla \times (\underline{B} \times \nabla V)]}{|\nabla V|^4} ,$$

$$\langle A \rangle \equiv \frac{\oint A d\ell/B}{\oint d\ell/B} , \quad \sigma \equiv 1 - \frac{\partial p_{\perp}}{B \partial B} \Big|_V , \quad \tau \equiv 1 + \frac{\partial p_{\parallel}}{B \partial B} \Big|_V .$$

The parameter Λ measures the local shear, with primes representing derivatives with respect to V .

Since all of the functions that enter these relations are determined in an equilibrium calculation, these expressions can be evaluated easily. A (V, θ, ζ) set of Hamada coordinates with a unit Jacobian was used for the calculation, with V the volume inside the relevant magnetic surface. Use of the function $u = m\theta - n\zeta$, such that $\underline{B} = \nabla V \times \nabla u$ on the surface of interest rather than θ , simplifies the work.

In the isotropic limit, the pressure functions, $P_{\perp}(V, B)$ and $P_{\parallel}(V, B)$,

become surface functions and the criteria reduce to their well-known scalar pressure forms.¹ The first term in E is clearly associated with the normal component of the magnetic field line curvature. The remaining terms in E and H can be identified with the geodesic curvature through its relation to the surface variation of the parallel current. As before, the major difference between the ideal and resistive criteria is the factor of $1/4$ introduced into the ideal form by the shear. The modification associated with the different way that H enters is due to the identification of v^2_B with ξ rather than ω_B with ξ in Ohm's law, resulting in a different weighting over the magnetic field lines.

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APPENDIX

We consider a toroidal equilibrium with anisotropic pressure such that

$$\underline{J} \times \underline{B} = \nabla \cdot \underline{P} \quad , \quad (A1)$$

$$\underline{P} = p_{\perp} \underline{\underline{1}} + (p_{\parallel} - p_{\perp}) \frac{\underline{B}\underline{B}}{B^2} \quad . \quad (A2)$$

It is convenient to define⁵

$$\underline{K} \equiv \nabla \times \sigma \underline{B} \quad ,$$

$$\sigma \equiv 1 + \frac{(p_{\perp} - p_{\parallel})}{B^2} \quad ,$$

$$\tau \equiv 1 + \frac{\partial p_{\perp}}{B \partial B} \Big|_V \quad .$$

Then

$$\underline{K} \times \underline{B} = \frac{\partial p_{\parallel}}{\partial V} \Big|_B \nabla V, \quad (A3)$$

$$\frac{\partial p_{\parallel}}{B \partial B} \Big|_V = 1 - \sigma \quad . \quad (A4)$$

The $\underline{B}$ and $\underline{K}$ lines lie on magnetic surfaces, so that much of the formalism that has been developed for isotropic systems can be applied. The plasma current is given by

$$\begin{aligned} \underline{J} = & \frac{B^2}{|\nabla V|^2} \frac{(\sigma - \tau)}{(\sigma \left. \frac{\partial p_{\perp}}{\partial V} \right|_B + \tau \left. \frac{\partial p_{\parallel}}{\partial V} \right|_B)} \underline{B} \cdot \nabla \left(\frac{\sigma \underline{J} \cdot \underline{B}}{B^2} \right) \nabla V \\ & + \left(\frac{\nabla V \cdot \nabla p_{\perp}}{\sigma B^2 |\nabla V|^2} + \frac{(\sigma - 1) \nabla V \cdot \nabla B^2}{2 \sigma B^2 |\nabla V|^2} \right) \underline{B} \times \nabla V + \left(\frac{\underline{J} \cdot \underline{B}}{B^2} \right) \underline{B} \end{aligned} \quad (A5)$$

with

$$\underline{B} \cdot \nabla \left(\frac{\sigma \underline{J} \cdot \underline{B}}{B^2} \right) = \frac{B \times \nabla V \cdot \nabla B^2}{2 \sigma B^4} \left(\sigma \left. \frac{\partial p_{\perp}}{\partial V} \right|_B + \tau \left. \frac{\partial p_{\parallel}}{\partial V} \right|_B \right) \quad (A6)$$

We adopt a Hamada V, θ, ζ coordinate system with unit Jacobian. V is the volume inside a magnetic surface, and θ and ζ are angular coordinates that increase by unity in going once around the torus the short and long ways, respectively. We write

$$\underline{B} = \nabla V \times [\psi'(V) \nabla \theta - \chi'(V) \nabla \zeta] \quad (A7)$$

with $\psi'(V)$ and $\chi(V)$ the magnetic fluxes the long and short ways. Primes denote derivatives with respect to V . Since we are concerned with modes localized near a surface V_0 , on which the safety factor $\psi'(V_0)/\chi'(V_0) = m/n$ is rational, we transform to an x, u, ζ coordinate system where

$$x \equiv V - V_0,$$

$$u \equiv m\theta - n\zeta,$$

and write

$$\Lambda \equiv \psi'(V_0) \chi''(V_0) - \chi'(V_0) \psi''(V_0) ,$$

$$\Phi \equiv \psi'(V_0)/\pi = \chi'(V_0)/n .$$

Then

$$\underline{B} = \Phi \nabla V \times \nabla u \left[1 + \psi''(V_0)/\psi'(V_0) \right] + \nabla V \times \nabla \zeta \frac{\Delta x}{\psi'} + \dots \quad (A8)$$

We use the usual linearized MHD equations for the stability analysis,

$$\alpha q^2 \underline{\xi} = \underline{j} \times \underline{B} + \underline{j} \times \underline{b} - \nabla \cdot \hat{\underline{p}} , \quad (A9)$$

$$\underline{b} - \frac{n}{q} \nabla^2 \underline{b} = \nabla \times (\underline{\xi} \times \underline{B}) , \quad (A10)$$

with the modes varying with time as $\exp(qt)$ and the resistivity η a constant. Lower-case symbols, $\underline{\xi}$, $\underline{j}$, $\underline{b}$, $\hat{\underline{p}}_{\perp}$, and $\hat{\underline{p}}_{\parallel}$ denote perturbed quantities.

The determination of the quantities D_I and D_R in the final averaged equations is not affected by the choice of the equations of state for $p_{\parallel}$ and $p_{\perp}$. For simplicity we adopt a model in which the distribution functions are merely convected with the field lines. Then,

$$\hat{\underline{p}}_{\parallel} = (1 - \sigma) \frac{\underline{b} \cdot \underline{B}}{B^2} - \frac{\partial p_{\parallel}}{\partial V} \bigg|_B \frac{\underline{\xi} \cdot \nabla V}{|\nabla V|^2} , \quad (A11)$$

$$\hat{\underline{p}}_{\perp} = (\tau - 1) \frac{\underline{b} \cdot \underline{B}}{B^2} - \frac{\partial p_{\perp}}{\partial V} \bigg|_B \frac{\underline{\xi} \cdot \nabla V}{|\nabla V|^2} . \quad (A12)$$

This is the appropriate generalization to anisotropic equilibria of the pressure convection model of scalar pressure MHD (adiabatic index $\gamma = 0$) in which no sound waves propagate. As noted previously^{6,7} this permits new instabilities. However, it does not prevent us from identifying the anisotropic D_R and D_I .

It is convenient to decompose the perturbations as

$$\underline{\xi} = \xi_V \frac{\nabla V}{|\nabla V|^2} + \xi_S \frac{\mathbf{B} \times \nabla V}{B^2} + \xi_B \frac{\mathbf{B}}{B^2} , \quad (A13)$$

$$\underline{b} = b_V \frac{\nabla V}{|\nabla V|^2} + b_S \frac{\mathbf{B} \times \nabla V}{B^2} + b_B \frac{\mathbf{B}}{B^2} , \quad (A14)$$

and to introduce the orderings

$$\eta \sim \epsilon^8, \quad q \sim \epsilon^2, \quad \frac{\partial}{\partial x} \sim \epsilon^{-3}, \quad \frac{\partial}{\partial u} \sim \epsilon^{-1}, \quad \frac{\partial}{\partial \zeta} \sim \epsilon^0 ,$$

$$\xi_S \sim \epsilon^0, \quad \xi_V \sim b_S \sim b_B \sim \epsilon^2, \quad b_V \sim \epsilon^4 .$$

This rather artificial choice of ordering was introduced in Ref. 4 because it simplified the algebra. Similar results follow from the usual $\partial/\partial x \sim \eta^{1/3}$ ordering.¹ The calculation can be made in the same way as in Appendix B of Ref. 4. In lowest order, the condition that the perturbed field be divergence-free requires that

$$\frac{\partial b_V^{(4)}}{\partial x} + \frac{1}{\phi} \frac{\partial b_S^{(2)}}{\partial u} = 0 . \quad (A15)$$

The ∇V component of the induction equation, Eq. (A10), leads to

$$\begin{aligned}
 \psi' \frac{\partial \xi_V^{(2)}}{\partial \zeta} &= 0, \\
 \psi' \frac{\partial \xi_V^{(3)}}{\partial \zeta} &= 0, \\
 b_V^{(4)} - \frac{n}{q} |\nabla V|^2 \frac{\partial^2 b_V^{(4)}}{\partial x^2} &= \psi' \frac{\partial \xi_V^{(4)}}{\partial \zeta} + \frac{\Delta x}{\phi} \frac{\partial \xi_V^{(2)}}{\partial u}.
 \end{aligned} \tag{A16}$$

The component of this equation along $\underline{B}$ gives

$$\begin{aligned}
 \frac{\partial \xi_V^{(2)}}{\partial x} + \frac{1}{\phi} \frac{\partial \xi_S^{(0)}}{\partial u} &= 0, \\
 \frac{\partial \xi_V^{(3)}}{\partial x} + \frac{1}{\phi} \frac{\partial \xi_S^{(1)}}{\partial u} + \frac{\underline{B} \times \nabla V \cdot \nabla \zeta}{B^2} \frac{\partial \xi_S^{(0)}}{\partial \zeta} &= 0, \\
 \frac{\partial \xi_V^{(4)}}{\partial x} + \frac{\nabla V \cdot \nabla u}{|\nabla V|^2} \frac{\partial \xi_V^{(2)}}{\partial u} + \frac{1}{\phi} \frac{\partial \xi_S^{(2)}}{\partial u} + \frac{\underline{B} \times \nabla V \cdot \nabla \zeta}{B^2} \frac{\partial \xi_S^{(1)}}{\partial \zeta} &= 0.
 \end{aligned} \tag{A17}$$

The $\underline{B} \times \nabla V$ component of the induction equation is not needed because of the relation imposed by the $\nabla \cdot \underline{b} = 0$ constraint. In Ref. 4, we used the condition that $\hat{p} \sim \epsilon^2$ to make $\nabla \cdot \underline{\xi} \sim \epsilon^2$. We thus tied $\partial \xi_B / \partial \zeta$ to $\xi_{\perp}$ and kept $\nabla \cdot \underline{\xi}^{(2)}$ as an independent variable. Here, our incompressibility assumption in the equation of state removes this constraint. For this reason we only have to go one order lower in this equation than before.

We now turn to the momentum equation. The ∇V component requires that

$$\frac{\partial}{\partial x} (\hat{p}_{\perp}^{(2)} + b_B^{(2)}) = 0,$$

$$\frac{\partial}{\partial x} (\hat{p}_\perp^{(3)} + b_B^{(3)}) = 0 ,$$

$$\frac{\partial}{\partial x} (\hat{p}_\perp^{(4)} + b_B^{(4)}) + \frac{\nabla V \cdot \nabla u}{|\nabla V|^2} \frac{\partial}{\partial u} (\hat{p}_\perp^{(2)} + b_B^{(2)}) = 0 . \quad (A18)$$

The $\underline{B} \times \nabla V$ component gives

$$\frac{\partial}{\partial u} (\hat{p}_\perp^{(2)} + b_B^{(2)}) = 0 ,$$

$$\begin{aligned} & - \frac{B^2}{\phi} \frac{\partial}{\partial u} (\hat{p}_\perp^{(3)} + b_B^{(3)}) - \underline{B} \times \nabla V \cdot \nabla \zeta \frac{\partial}{\partial \zeta} (\hat{p}_\perp^{(2)} + b_B^{(2)}) \\ & + B^2 \psi' \frac{\partial}{\partial \zeta} \left(\frac{\sigma |\nabla V|^2}{B^2} b_s^{(2)} \right) + \frac{\tau B^2 \psi' \partial / \partial \zeta (\sigma \underline{J} \cdot \underline{B} / B^2)}{(\sigma \partial \hat{p}_\perp / \partial V|_B + \tau \partial \hat{p}_\parallel / \partial V|_B)} (2b_B^{(2)} + \hat{p}_\perp^{(2)} - \hat{p}_\parallel^{(2)}) = 0 , \end{aligned}$$

$$\begin{aligned} & - \frac{B^2}{\phi} \frac{\partial}{\partial u} (\hat{p}_\perp^{(4)} + b_B^{(4)}) - \underline{B} \times \nabla V \cdot \nabla \zeta \frac{\partial}{\partial \zeta} (\hat{p}_\perp^{(3)} + b_B^{(3)}) \\ & + B^2 \psi' \frac{\partial}{\partial \zeta} \left(\frac{\sigma |\nabla V|^2}{B^2} b_s^{(3)} \right) + \frac{\tau B^2 \psi' \partial / \partial \zeta (\sigma \underline{J} \cdot \underline{B} / B^2)}{(\sigma \partial \hat{p}_\perp / \partial V|_B + \tau \partial \hat{p}_\parallel / \partial V|_B)} (2b_B^{(3)} + \hat{p}_\perp^{(3)} - \hat{p}_\parallel^{(3)}) = 0 . \end{aligned} \quad (A19)$$

It will turn out that, to the order we go, these two sets of equations contain the same information. Thus, it is expedient to use the usual annihilation operator, $\nabla \cdot \underline{B} / B^2 \times$, on the equation to get

$$\begin{aligned} & - \psi' \frac{\partial}{\partial \zeta} \left(\frac{\sigma |\nabla V|^2}{B^2} \frac{\partial b_s^{(2)}}{\partial x} \right) \\ & + \frac{[\psi' \partial / \partial \zeta (\sigma \underline{J} \cdot \underline{B} / B^2)]}{(\sigma \partial \hat{p}_\perp / \partial V|_B + \tau \partial \hat{p}_\parallel / \partial V|_B)} \left[(\sigma - \tau) \frac{\partial b_B^{(2)}}{\partial x} + \sigma \frac{\partial \hat{p}_\perp^{(2)}}{\partial x} + \tau \frac{\partial \hat{p}_\parallel^{(2)}}{\partial x} \right] = 0 , \end{aligned}$$

$$\begin{aligned}
 & - \psi' \frac{\partial}{\partial \zeta} \left(\frac{\sigma |\nabla V|^2}{B^2} \frac{\partial b_s^{(3)}}{\partial x} \right) \\
 & + \frac{\psi' \partial / \partial \zeta (\sigma \underline{J} \cdot \underline{B} / B^2)}{(\sigma \nabla p_{\perp} / \partial V|_B + \tau \partial p_{\parallel} / \partial V|_B)} \left[(\sigma - \tau) \frac{\partial b_B^{(3)}}{\partial x} + \sigma \frac{\partial \hat{p}_{\perp}^{(3)}}{\partial x} + \tau \frac{\partial \hat{p}_{\parallel}^{(3)}}{\partial x} \right] = 0, \\
 & - \sigma q^2 \frac{|\nabla V|^2}{B^2} \frac{\partial \xi_s^{(0)}}{\partial x} = - \psi' \frac{\partial}{\partial \zeta} \left(\frac{\sigma |\nabla V|^2}{B^2} \frac{\partial b_s^{(4)}}{\partial x} + \frac{\sigma \nabla V \cdot \nabla u}{B^2} \frac{\partial b_s^{(2)}}{\partial u} \right) \\
 & - \frac{\sigma |\nabla V|^2 \Delta x}{\phi B^2} \frac{\partial^2 b_s^{(2)}}{\partial x \partial u} \\
 & + \frac{[\psi' \partial / \partial \zeta (\sigma \underline{J} \cdot \underline{B} / B^2)]}{(\sigma \partial p_{\perp} / \partial V|_B + \tau \partial p_{\parallel} / \partial V|_B)} \left\{ [(\sigma - \tau) \frac{\partial b_B^{(4)}}{\partial x} + \sigma \frac{\partial \hat{p}_{\perp}^{(4)}}{\partial x} + \tau \frac{\partial \hat{p}_{\parallel}^{(4)}}{\partial x}] \right. \\
 & + \frac{\nabla V \cdot \nabla u}{|\nabla V|^2} \left\{ (\sigma - \tau) \frac{\partial b_B^{(2)}}{\partial u} + \sigma \frac{\partial \hat{p}_{\perp}^{(2)}}{\partial u} + \tau \frac{\partial \hat{p}_{\parallel}^{(2)}}{\partial u} \right\} \\
 & + \frac{\nabla V \cdot \nabla p_{\perp}}{\phi \sigma B^2 |\nabla V|^2} \left(\frac{\partial b_B^{(2)}}{\partial u} - \frac{\partial \hat{p}_{\parallel}^{(2)}}{\partial u} \right) \\
 & - \frac{\nabla V \cdot \nabla B^2}{2 \phi \sigma B^2 |\nabla V|^2} \left[(\sigma - \tau) \frac{\partial b_B^{(2)}}{\partial u} + \sigma \frac{\partial \hat{p}_{\perp}^{(2)}}{\partial u} + \tau \frac{\partial \hat{p}_{\parallel}^{(2)}}{\partial u} \right]. \quad (A20)
 \end{aligned}$$

The component along $\underline{B}$ is not needed in the calculation, again because of the incompressibility constraint.

It is convenient to decompose all of the variables into functions which are constant on a field line and those which vary along the line but with zero average value. Thus,

$$\xi_V = \bar{\xi}_V(v,u) + \tilde{\xi}_V(x,u,z) ,$$

$$\oint \tilde{\xi}_V \frac{dl}{B} = 0 ,$$

etc. Then we see that

$$\tilde{\xi}_V^{(2)} = 0 \quad (A21)$$

from Eq. (A16) and, from Eq. (A17), that

$$\begin{aligned} \tilde{\xi}_S^{(0)} &= 0 , \\ \frac{\partial \tilde{\xi}_S^{(0)}}{\partial u} &= - \oint \frac{\partial \tilde{\xi}_V^{(2)}}{\partial x} . \end{aligned} \quad (A22)$$

The perpendicular components of the momentum equation, Eqs. (A18) and (A19), give

$$\hat{p}_L^{(2)} + b_B^{(2)} = 0 . \quad (A23)$$

Combining this with the equations of state, Eqs. (A11) and (A12), gives

$$\hat{p}_L^{(2)} = - b_B^{(2)} = - \frac{1}{\tau} \frac{\partial p_\perp}{\partial V} \Big|_B \tilde{\xi}_V^{(2)} ,$$

$$\hat{p}_{\parallel}^{(2)} = \left(\frac{(\sigma - \tau)}{\tau} \frac{\partial p_{\perp}}{\partial V} \Big|_B - \frac{\partial p_{\parallel}}{\partial V} \Big|_B \right) \bar{\epsilon}_V^{(2)} \quad (A24)$$

Inserting these and Eq. (A15) into Eq. (A20) and integrating with respect to τ , we obtain

$$\frac{\partial^2 b_V^{(4)}}{\partial x^2} = \left[\frac{J \cdot B}{|\nabla V|^2} - \frac{B^2 \oint J \cdot B / |\nabla V|^2 d\ell / B}{\sigma |\nabla V|^2 \oint B^2 / \sigma |\nabla V|^2 d\ell / B} \right] \frac{1}{\phi} \frac{\partial \bar{\epsilon}_V^{(2)}}{\partial x \partial u} \quad (A25)$$

$$+ \frac{B^2 \oint d\ell / B}{\sigma |\nabla V|^2 \oint B^2 / \sigma |\nabla V|^2 d\ell / B} \frac{\partial^2 b_V^{(4)}}{\partial x^2} \quad (A26)$$

When we insert these into the higher order equations, we find that the resulting conditions on the next order independent variables are the same as those on the lowest order ones. Thus, we can set

$$\zeta_V^{(3)} = \epsilon_s^{(1)} = b_V^{(5)} = b_s^{(3)} = \hat{p}_{\perp}^{(3)} = \hat{p}_{\parallel}^{(3)} = 0 \quad (A27)$$

We are now ready to treat the higher order equations. Averaging Eq. (A16) over τ gives

$$\bar{b}_V^{(4)} \oint \frac{d\ell}{B} - \frac{n}{q} \oint |\nabla V|^2 \frac{\partial^2 b_V^{(4)}}{\partial x^2} \frac{d\ell}{B} = \frac{\Lambda x}{\phi} \oint \frac{d\ell}{B} \frac{\partial \bar{\epsilon}_V^{(2)}}{\partial u} \quad (A28)$$

Insertion of Eq. (A26) then provides a relation

$$\begin{aligned} \bar{b}_V^{(4)} - \frac{n}{q} \frac{\oint B^2 / \sigma d\ell / B}{\oint B^2 / \sigma |\nabla V|^2 d\ell / B} \frac{\partial^2 b_V^{(4)}}{\partial x^2} &= \frac{\Lambda x}{\phi} \frac{\partial \bar{\epsilon}_V^{(2)}}{\partial u} \\ + \frac{n}{\phi q} \left[\frac{\oint J \cdot B d\ell / B}{\oint d\ell / B} - \frac{\oint J \cdot B / |\nabla V|^2 d\ell / B}{\oint B^2 / \sigma |\nabla V|^2 d\ell / B} \frac{\oint B^2 / \sigma d\ell / B}{\oint d\ell / B} \right] &\frac{\partial^2 \bar{\epsilon}_V^{(2)}}{\partial x \partial u} \end{aligned} \quad (A28)$$

between $b_V(4)$ and $\xi_V(2)$. The perpendicular components of the momentum equation, together with the equations of state, give

$$(\sigma - \tau) b_B^{(4)} - \tau \hat{p}_{\parallel}^{(4)} - \sigma \hat{p}_{\perp}^{(4)} = \left(\sigma \frac{\partial p_{\perp}}{\partial V} \Big|_B + \tau \frac{\partial p_{\parallel}}{\partial V} \Big|_B \right) \xi_V^{(4)}. \quad (A29)$$

We substitute this into Eq. (A20) and use Eq. (A16) to remove the $\partial/\partial \zeta (\dots \partial^2 \xi_V^{(4)}/\partial x \partial u)$ term. We also multiply Eq. (A26) by $\sigma \underline{J} \cdot \underline{B} B^2$ and integrate with respect to ζ to obtain an expression for $\oint [\sigma (\underline{J} \cdot \underline{B}/B^2) \partial^2 b_V^{(4)}/\partial x \partial u] d\ell/B$. Use of this expression and Eq. (A20) leads to

$$\begin{aligned} & \phi \rho q^2 \oint \frac{|\nabla V|^2}{B^2} \frac{d\ell}{B} \frac{\partial^2 \xi_V^{(2)}}{\partial x^2} + \frac{n}{\phi q} \left[\oint \frac{\sigma (\underline{J} \cdot \underline{B})^2}{B^2} \frac{d\ell}{B} - \frac{(\oint \underline{J} \cdot \underline{B} d\ell/B)^2}{\oint B^2/\sigma d\ell/B} \right] \frac{\partial^4 \xi_V^{(2)}}{\partial x^2 \partial u^2} \\ &= \left[\oint \frac{\nabla V \cdot \nabla (p_{\perp} + B^2/2)}{\tau \sigma B^2 |\nabla V|^2} \left(\sigma \frac{\partial p_{\perp}}{\partial V} \Big|_B + \tau \frac{\partial p_{\parallel}}{\partial V} \Big|_B \right) \frac{d\ell}{B} + \phi \psi' \oint \frac{\sigma \underline{J} \cdot \underline{B}}{B^2} \frac{\partial}{\partial \zeta} \frac{\nabla V \cdot \nabla u}{|\nabla V|^2} \frac{d\ell}{B} \right. \\ &+ \left. \left(\oint \frac{\sigma (\underline{J} \cdot \underline{B})^2}{B^2 |\nabla V|^2} \frac{d\ell}{B} - \frac{(\oint \underline{J} \cdot \underline{B}/|\nabla V|^2 d\ell/B)^2}{\oint B^2/\sigma |\nabla V|^2 d\ell/B} \right) \right] \frac{1}{\phi} \frac{\partial^2 \xi_V^{(2)}}{\partial u^2} - \frac{q}{n} \Lambda^2 x^2 \frac{(\oint d\ell/B)^2}{\oint B^2/\sigma d\ell/B} \frac{\partial^2 \xi_V^{(2)}}{\partial u^2} \\ &+ \oint \frac{d\ell}{B} \left(\frac{\oint \underline{J} \cdot \underline{B}/|\nabla V|^2 d\ell/B}{\oint B^2/\sigma |\nabla V|^2 d\ell/B} - \frac{\oint \underline{J} \cdot \underline{B} d\ell/B}{\oint B^2/\sigma d\ell/B} \right) \frac{\partial^2 b_V^{(4)}}{\partial x \partial u} + \frac{q}{n} \Lambda x \frac{(\oint d\ell/B)^2}{\oint B^2/\sigma d\ell/B} \frac{\partial b_V^{(4)}}{\partial u}. \end{aligned} \quad (A30)$$

It is useful to use a WKB-type approximation for variations along u , $m\partial/\partial u = i\alpha$, define the parameters,

$$C \equiv \left\langle \frac{B^2}{\sigma} \right\rangle \left\langle \frac{B^2}{\sigma |\nabla V|^2} \right\rangle,$$

$$M \equiv \psi^2 \left\langle \frac{B^2}{\sigma |\nabla V|^2} \right\rangle \left\langle \frac{|\nabla V|^2}{B^2} \right\rangle ,$$

$$K \equiv \frac{(\langle B^2/\sigma |\nabla V|^2 \rangle)^2}{\Lambda^2 \langle B^2/\sigma \rangle} \left[\left\langle \frac{\sigma (\underline{J} \cdot \underline{B})^2}{B^2} \right\rangle - \frac{(\langle \underline{J} \cdot \underline{B} \rangle)^2}{\langle B^2/\sigma \rangle} \right] ,$$

$$H \equiv \frac{\langle B^2/\sigma |\nabla V|^2 \rangle}{\Lambda} \left[\frac{\langle \underline{J} \cdot \underline{B} / |\nabla V|^2 \rangle}{\langle B^2/\sigma |\nabla V|^2 \rangle} - \frac{\langle \underline{J} \cdot \underline{B} \rangle}{\langle B^2/\sigma \rangle} \right] ,$$

and

$$E \equiv \frac{\langle B^2/\sigma |\nabla V|^2 \rangle}{\Lambda^2} \left[\left\langle \frac{\nabla V \cdot \nabla (p_\perp + B^2/2)}{\tau \sigma B^2 |\nabla V|^2} \right\rangle \left(\sigma \frac{\partial p_\perp}{\partial V} \Big|_B + \tau \frac{\partial p_\parallel}{\partial V} \Big|_B \right) \right.$$

$$\left. + \psi^2 \left\langle \frac{\sigma \underline{J} \cdot \underline{B}}{B^2} \frac{\partial}{\partial \zeta} \frac{\nabla V \cdot \nabla u}{|\nabla V|^2} \right\rangle + \left\langle \frac{\sigma (\underline{J} \cdot \underline{B})^2}{B^2 |\nabla V|^2} \right\rangle - \frac{(\langle \underline{J} \cdot \underline{B} / |\nabla V|^2 \rangle)^2}{\langle B^2/\sigma |\nabla V|^2 \rangle} \right] ,$$

with $\langle A \rangle = \oint A d\ell / B / \oint d\ell / B$, introduce the scale lengths,

$$X_0 \equiv \left(\frac{\rho M \eta^2 C^2}{\alpha^2 \Lambda^2} \right)^{1/6} ,$$

$$Q_0 \equiv \left(\frac{\alpha^2 \Lambda^2 \eta C}{\rho M} \right)^{1/3} ,$$

and set

$$x = X X_0 ,$$

$$q = Q Q_0 ,$$

$$\bar{\xi}_V^{(2)} = \Xi ,$$

$$\bar{b}_V^{(4)} = \frac{i\alpha\Lambda X_0}{\psi'} \Psi .$$

Then Eqs. (A28) and (A30) become

$$\frac{d^2\Psi}{dX^2} - H \frac{d\Xi}{dX} = Q (\Psi - X\Xi) , \quad (A31)$$

$$\Xi\Xi + H \frac{d\Psi}{dX} = - (Q^2 - \frac{K}{Q}) \frac{d^2\Xi}{dX^2} - QX (\Psi - X\Xi) . \quad (A32)$$

These are almost the same as Eqs. (40) and (41) of Ref. 4 and Eqs. (9) and (10) of reference 1 in the limit where the adiabatic constant $\gamma = 0$ so that $G = \infty$. The major difference is the introduction of a new term, $(-K/Q) d^2\Xi/dX^2$, in Eq. (A33). This same modification of the equations would have been obtained in the earlier work with isotropic pressure if the $\gamma = 0$ assumption had been made in the beginning. Because of this, ξ_B does not enter the equations, except in the inertial term in the $\underline{E}$ projection of the momentum equation. In the general $\gamma \neq 0$ case it was convenient to recognize that the divergence of $\underline{\xi}$ had to be second order because of the perturbed pressure balance so that $\nabla \cdot \underline{\xi}^{(2)}$ could be used as an independent variable. We then had to use the parallel component of the momentum equation to eliminate the fourth order perturbed pressure in terms of $\xi_B^{(0)}$ and then use information from one order higher in the component of the induction equation. Here we can use the ∇V component of the induction equation to eliminate $\xi_V^{(4)}$ and avoid some of this work. This results in the separation of the parameter M in the earlier work into the two parameters M and K of the present formalism. This

modification of the equations was found previously in a study of ballooning modes.⁶ The only other difference in these equations from before is the introduction of the parameters σ and τ associated with the pressure anisotropy.

A minor problem should be mentioned. In the $\gamma = 0$ anisotropic case, the parallel component of the momentum equation was automatically satisfied because of the conditions imposed by the other equations. Here this does not seem to occur. It is possible to choose ξ_B to eliminate the difficulty, but a better solution should be available.

We can ignore the K term because it is a consequence of the choice of equations of state. Since our purpose is to determine the generalization of D_R and not to examine the properties of modes where sound wave propagation can not provide stabilization, we do this and recover the starting point of Glasser's analysis.¹ Then the ideal condition, $D_I < 0$ for stability, follows from the assumption that $\varepsilon \sim X^S$ near the rational surface; as X goes to ∞ , $s = -1/2 \pm (1/4 - E - H)^{1/2}$. The resistive criterion, $D_R < 0$ for stability, follows from an ordering near marginal stability, $\nu \sim 1$, $\varepsilon \sim \delta^{-1/6}$, $\chi \sim \delta^{1/6}$, $Q \sim \delta^{2/3}$, $E \sim H \sim 1$, $D_R \sim \delta$.

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