Tearing-Mode Stability of a Forming Spheromak Plasma

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Abstract

The results of numerical calculations of Δ' for a class of equilibria typical of those encountered during the early formation stage of the S1 Spheromak are presented. The equilibrium plasma is assumed to be cylindrically symmetric and pressureless. It encloses a current carrying perfect conductor (flux core) and is surrounded by a vacuum with zero longitudinal field. Stability boundaries in the space formed by the equilibrium parameters are mapped. The plasma is tearing mode stable provided B_z/B_θ at the flux core is below a certain critical value which depends on the equilibrium parameters. For typical equilibria, this critical value is 0.65.

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A method for producing a spheromak plasma in which both poloidal and toroidal currents are inductively transferred from a flux core to a surrounding plasma sleeve has been proposed [1] and successfully tested experimentally [2]. This formation method operates on a relatively slow time scale so as to limit peak power requirements and improve the energy efficiency of the formation process. The success of the method requires that the plasma progress through a succession of meta-equilibria, each of which must be stable to instabilities that grow in times shorter than the formation timescale τ_F . In particular, to form the plasma on the timescale

$$\tau_t < \tau_F \ll \tau_R,$$

where $\tau_t = \tau_R^{3/5} \tau_A^{2/5}$ is the characteristic time for the tearing mode instability [3], $\tau_R = \mu_0 a^2 / \eta$ is the resistive diffusion time, and $\tau_A = (\mu_0 \rho)^{1/2} a / B_0$ is the Alfvén transit time, a sequence of equilibria stable to the tearing mode must exist. If such a sequence does not exist, the inductive formation of the spheromak must proceed on the faster time scale $\tau_F < \tau_t \ll \tau_R$.

In this letter, we assess the tearing mode stability of a class of MHD equilibria encountered during the inductive formation of the S1 Spheromak. During the early stages of this formation process, the plasma is well represented by the following circular cylindrical system (Fig. 1): a perfectly conducting flux core of radius a and periodicity length $2\pi R$ is surrounded by a pressureless plasma that fills the region $a < r < b$. A vacuum region extends from $r = b$ out to a conducting wall at $r = c$. No current flows outside of the plasma ($r > b$) although the flux core does in general carry a current. The current $\vec{J}$ within the plasma satisfies $\vec{J} = \nabla \times \vec{B}$ and

$\tilde{B}(r) = \alpha(r) \hat{B}(r)$. The function $\alpha(r)$ is chosen to be

$$\alpha(r) = \alpha_0 \left\{ 1 - [(r-a)/(b-a)]^j \right\}^i,$$

where α_0 , i , and j are "free" parameters. With this choice of $\alpha(r)$ the equilibrium current, longitudinal field B_z , and safety factor $q = rB_z/RB_\theta$ vanish at the plasma edge. We consider only equilibria whose safety factors decrease monotonically between $r = a$ and $r = b$.

We determine the tearing mode stability of these equilibria by using a procedure similar to the one described in Refs. [4] and [5]. The equilibrium equation in the plasma is

$$d^2 B_z / dr^2 + [1/r - 1/\alpha \, d\alpha/dr] \, dB_z / dr + \alpha^2 B_z = 0.$$

This is solved by expanding B_z in power series about the regular singular point $r = b$. The radius of convergence of these series extend beyond $r = a$ to the origin. One of the two linearly independent solutions is rejected because it does not satisfy the boundary condition $B_z(b) = 0$. The retained series converges rapidly for $a/(b-a) \gtrsim 0.5$ and is typically truncated after sixty terms. To assess the stability of this equilibrium we subject it to a perturbed radial magnetic field,

$$\tilde{B}_r(\vec{r}) = i\psi(r) \exp[i(kz - m\theta)],$$

where $k = (n/R)$ and m and n are integers, and analyze the stability of that mode. A necessary condition for tearing instability of a particular mode is that the mode rational surface r_s lie in the plasma; i.e., that $q(r_s) = (m/n)$

for $a < r_s < b$. For the monotonically decreasing q profiles considered here, an equivalent necessary condition for instability is that $q_s \equiv q(a) \geq (m/n)$.

Everywhere except in the vicinity of such a mode rational surface, the function ψ obeys the "infinite conductivity equation"

$$\frac{d}{dr} \left(H \frac{d\psi}{dr} \right) - \psi \left[\frac{q}{F^2} + \frac{1}{F} \frac{d}{dr} \left(H \frac{dF}{dr} \right) \right] = 0 ,$$

where

$$F = \mathbf{k} \cdot \mathbf{B} = k B_z - (m/r) B_\theta ,$$

$$H = r^3 / (k^2 r^2 + m^2) ,$$

$$q = \frac{(m^2 - 1) r^2}{k^2 r^2 + m^2} + \frac{k^2 r^2 F}{k^2 r^2 + m^2} \left[r F + \frac{2(kr B_z + m B_\theta)}{(k^2 r^2 + m^2)} \right] .$$

Since the perturbed radial field at the flux core must vanish, one boundary condition is $\psi(a) = 0$. The boundary condition at the plasma-vacuum interface is deduced from continuity of pressure and continuity of tangential electric field across the interface. For a pressureless plasma with $B_z(b) \neq 0$ these conditions imply that the perturbed fields satisfy $\tilde{B}_\theta(b^-) = \tilde{B}_\theta(b^+)$ and $\tilde{B}_r(b^-) = \tilde{B}_r(b^+)$. The vacuum fields are expressed in terms of modified Bessel functions and $\tilde{B}_\theta$ is eliminated using the Euler-Lagrange equations [7] yielding the boundary condition at $r = b$

$$\begin{aligned} \psi' + \psi \left(\frac{(2k^2 b^2 + 2m^2)}{m^2 b} + \frac{kb}{m} \frac{B_z'}{B_\theta} + \frac{k^2 b^2}{m^2} \frac{B_\theta'}{B_\theta} \right) \\ = \frac{(k^2 b^2 + m^2)}{kb^2} [\mathcal{K}_m'(kc) I_m(kb) - I_m'(kc) \mathcal{K}_m(kb)] . \end{aligned}$$

To determine tearing mode stability, we solve this equation for ψ in four distinct regions and match solutions; the matching coefficients determine tearing stability. First the equation is solved for $\psi_-(r)$ and $\psi_+(r)$ in the "outer regions" $a < r < r_g$ and $r_g < r < b$ using an accurate extrapolative differential equation solver [6]. If ψ vanishes only at the boundaries of these "outer regions," the perturbation is a stable ideal MHD mode [7] and we proceed with the resistive MHD analysis. In the "intermediate regions," $r = r_g \pm \epsilon$, we expand the functions F , H , and g in Taylor series about $r = r_g$ and find that $\psi_{\pm} \sim A_{\pm}\phi_L + B_{\pm}\phi_S$ where ϕ_L and ϕ_S are of the form

$$\phi_L(x) = 1 + a_1 x \ln|x| + a_2 x^2 \ln|x| + a_3 x^2,$$

$$\phi_S = x + b_1 x^2,$$

where

$$x = r - r_g,$$

and the a_i and b_i depend on the functions F , H , and g . The constants A_- and B_- are determined by matching the "outer" solution on the left side to the "intermediate" solution on the left side; A_+ and B_+ are found in the same fashion on the right side. In practice, this matching is accomplished by incrementally reducing the value of ϵ until the ratio B/A changes by one part in 10^{-4} when ϵ is halved. The value of $\Delta' \equiv (\psi'_+ - \psi'_-)/\psi = B_+/A_+ - B_-/A_-$ is then computed, the condition for tearing mode stability being that $\Delta' \leq 0$. The values of Δ' calculated by following this procedure typically change by one

part in 10^{-4} when the number of terms in the power series for B_z are doubled or when the accuracy of the differential equation solver is increased by a factor of ten.

Newcomb [7] has shown that if a linear pinch is stable to ideal MHD perturbations for $m = 1$, $-\infty < n < \infty$, then it is stable to all ideal perturbations with $m > 1$. We find that this comparison theorem is also valid for tearing mode stability. Consequently, we restrict our subsequent discussion to modes with $m = 1$.

We consider three different values of the current shaping parameters i and j :

Peaked

$$\alpha = \alpha_c \left[1 - \left[\frac{(r-a)}{(b-a)} \right] \right]^2 ,$$

Rounded

$$\alpha = \alpha_0 \left[1 - \left[\frac{(r-a)}{(b-a)} \right] \right] ,$$

Broad

$$\alpha = \alpha_0 \left[1 - \left[\frac{(r-a)}{(b-a)} \right]^3 \right] .$$

In Fig. 2, representative marginally stable current and "q" profiles are plotted for the three cases, "peaked," "rounded," and "broad." $b/a = 2.0$ in these graphs.

The effect on stability of the geometrical and current shaping parameters is summarized in the nine curves that constitute Fig. 3. The longitudinal mode number n , the rotational transform q , and the normalized length of the cylinder $2\pi R/a$ appear in the calculation of Δ' only in the combinations $ka = na/R$ and nq . The curve $\Delta' = 0$ (the boundary between regions of stability and instability) is therefore plotted in Fig. 3 as a function of nq_a and ka with the geometrical parameter b/a and the two current shaping parameters i and j

held fixed. The radius of the external conductor c/a is effectively infinite in these diagrams.

A specific equilibrium (b/a , c/a , R/a , α_0 , i and j all specified) corresponds in Fig. 3 to a line that passes through the origin with a slope equal to the ratio of poloidal field to toroidal field, $B_\theta(r=a)/B_z(r=a)$. Mode rational surfaces with $m = 1$ and $k = n/R$ correspond to discrete points along that line. Points for which $nq_a < 1$ are always stable since there is no corresponding mode rational surface. Points along the line for which $nq_a > 1$ may or may not be stable, depending on the slope of the line and the details of the stability boundary. In every case plotted in Fig. 3 there exist equilibria with sufficiently small ratio of toroidal field to poloidal field that all modes are stable. Listed in Table 1 are the critical values of this ratio above which tearing mode instability occurs. Also included in Table 1 are these critical values expressed in terms of the ratio of magnetic field pitch to the thickness of the current layer, $aB_z(r=a)/(b-a)B_\theta(r=a)$.

The results in Table 1 are in qualitative agreement with analytical results for the tearing mode in slab geometry. Furth, Killeen, and Rosenbluth [3] also found a critical value of pitch to thickness of the current layer above which instability occurs. For their "smooth" current profile, $j = 1/\cosh^2(y/a)$, this critical value is 1 while for their "broad" profile,

$$j = \begin{cases} 1 & |y/a| < 1 \\ 0 & |y/a| > 1 \end{cases},$$

the critical value is 1.6. We note that the critical values of pitch to current layer thickness we have calculated for forming spheromak equilibria are, like the results of Ref. [3], of $O(1)$. Further, just as with the analytical results in slab geometry, forming spheromak equilibria with broader

current profiles have larger critical coefficients of pitch to current layer thickness than equilibria with more narrow profiles. For the same value of $aB_z(a)/(b-a)B_\theta(a)$, a broad current profile has weaker current gradients which leads to an effectively larger shear length than a more narrow profile (Fig. 2a,b). Since it is the effective shear length of the current layer and not the plasma width (b-a) that most directly affects tearing mode stability, broad profiles are more stable.

One of the equilibria (rounded, $b/a = 2.5$) has an anomalously low critical value of toroidal field to poloidal field due to a narrow region of the plasma with unfavorable stability properties (the spike near $nq_a = 1$ in Fig. 3c). In a real physical system with discrete mode rational surfaces this narrow region is of little consequence since the plasma can "straddle" this unstable region by rearranging itself slightly. A more realistic critical value of B_z/B_θ for this equilibrium is therefore 0.46.

The position of the outer conductor c/b , has only a slight effect on tearing mode stability (Fig. 4). If the conductor is placed at the plasma edge it can sometimes stabilize an otherwise weakly unstable mode. However, for $c/b > 1.1$ shell stabilization is ineffective.

These theoretical results have immediate relevance to the inductive formation method for producing a spheromak plasma of Refs. [1] and [2]. Since there exists a critical value of pitch to plasma width below which the plasma is stable to the tearing mode, it should be possible, by operating below this critical value, to form the plasma on a time scale that is fast only compared to the resistive time of the plasma. If the ratio of pitch to plasma width exceeds this critical value, it may still be possible to burst through this region by forming on a time scale fast compared to the tearing mode growth rate.

<u>Current</u>	<u>b/a</u>	<u>B_z/B_θ</u>	<u>$aB_z/(b-a)B_\theta$</u>
Peaked	1.5	0.32	0.64
Peaked	2.0	0.42	0.42
Peaked	2.5	0.39	0.26
Rounded	1.5	0.65	1.3
Rounded	2.0	0.65	0.65
Rounded	2.5	0.31	0.21
Broad	1.5	1.5	3.0
Broad	2.0	2.2	2.2
Broad	2.5	2.5	1.7

Table 1.

Critical values of the ratio of toroidal field to poloidal field for nine equilibrium configurations.

Acknowledgments

The authors have benefited from discussions with Drs. H. P. Furth and M. Yamada.

This work was supported by the United States Department of Energy Contract No. DE-AC02-CH03073.

REFERENCES

- [1] JARDIN, S., PARK, W., Phys. Fluids 24 (1981) 679.
- [2] YAMADA, M., FURTH, H., HSU, W., JANOS, A., JARDIN, S., OKABAYASHI, M., SINNIS, J., STIX, T., YAMAZAKI, K., Phys. Rev. Lett. 46 (1981) 188.
- [3] FURTH, H., KILLEEN, J., ROSENBLUTH, M., Phys. Fluids 6 (1963) 459.
- [4] FURTH, H., RUTHERFORD, P. SELBERG, H., Phys. Fluids 16 (1973) 1054.
- [5] CHANCE, M., FURTH, H., GLASSER, A., SELBERG, H., Princeton Plasma Physics Lab. Rep. PPPL-1817 (1981).
- [6] BORIS, J., WINSOR, N., Princeton Plasma Physics Lab. 652 (1970).
- [7] NEWCOMB, W., Ann. Phys. 10 (1960) 232.

Figure Captions

Fig. 1. Class of equilibria analyzed for tearing mode stability.

Fig. 2. Comparison of the equilibrium longitudinal current (a), azimuthal current (b), and safety factor q (c) for three different values of the "current-shaping" parameters i and j . To facilitate comparison of the profiles, the ordinates have been normalized.

Fig. 3. Tearing mode stability diagrams. The shaded regions are unstable.

Fig. 4. Effect of position of the external conductor on Δ' .

(a) peaked current profile, $b/a = 2.0$, $k_a = 3.0$.

(b) rounded current profile, $b/a = 2.0$, $k_a = 2.0$.

(c) broad current profile, $b/a = 2.0$, $k_a = 1.2$.

Fig. 1

Fig. 2

Fig. 3

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Fig. 1