MAGNETIC MONOPOLES AND DIPOLES IN QUANTUM MECHANICS*

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Magnetic monopole charges present a peculiar challenge to quantum mechanical theories. Even the simplest problem of one moving electric charge interacting with one externally fixed magnetic charge has the feature that the angular momentum in the crossed electric and magnetic fields is independent of the distance between the two charges, and that makes both for singular behavior at zero distance and for a kind of lack of separability at infinite distance. Formally, the trouble shows up as the nonexistence of a vector potential to describe the Lorentz force. That is no obstacle to classical mechanics because the equations of motion can be obtained from Newton's laws, but with no vector potential there is no Lagrangian or Hamiltonian, and the usual ways of quantizing the theory do not work.

Dirac¹ first addressed the quantum theoretical problem in 1931. He showed that for quantized values of the electric and magnetic charges e and g which obey

$$eg/c = n\hbar/2, \quad (1)$$

n being the integers, he could introduce a singular vector potential that

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served as the basis for a quantum mechanical theory with particles that can be interpreted as magnetic monopoles. Since Dirac's original work there have been many developments of the theory and some formulations have removed or substantially hidden the singular strings,^{2,3} but the main physical obstacles have remained and we still have no satisfactory theory of particles interacting with fields and no practical ability to deal with several electrons interacting with several monopoles. In terms of its impact upon experiment, the theory can really make only two claims. Eq. (1) tells us what value of g to seek, and the singularity at the origin results in spin flips at high energy that may imply baryon number changes under the assumptions of grand unified theories.⁴

In this paper, we first consider the force on and the energy of a "di-monopole", the limiting case of a dipole made from two monopoles at zero separation and finite magnetic moment, interacting with an externally fixed magnetic field whose source is an electric current. Our original motivation was to see how the experimental signature of a di-monopole differs in quantum theory from the signature of an ordinary magnetic dipole whose magnetic moment arises from a circulating electric current or from some form factor which behaves like a circulating current.⁵ In fact we found that we do not know how to deal with di-monopoles near electric currents in quantum mechanics, and that the obstacle is again the lack of an appropriate classical Hamiltonian from which to begin. We also found that the same ideas applied to the problem of a single monopole coexisting with an external electric current lead to interesting contradictions which illuminate the nonexistence of the Hamiltonian for the monopole without a flux string in a novel way that is physically different from and in some respects more general than the impossibility of defining a vector potential. This is illustrated by a simple

"toy model" for the nonrelativistic motion of a monopole around an electric current.

In the simplest example of the Dirac procedure, a spinless monopole charge g is fixed at the origin and we wish to describe the nonrelativistic motion of a spinless electron nearby. Classically, the equation of motion is determined by the Lorentz force.

$$m\dot{\mathbf{v}} = (e/c) \mathbf{v} \times \mathbf{B} = (eg/c) \mathbf{v} \times \mathbf{r}/r^3 \quad (2)$$

To do quantum mechanics we want a Hamiltonian

$$H = (1/2m) [\mathbf{p} - (e/c)\mathbf{A}]^2, \quad (3)$$

but there is no vector potential $\mathbf{A}$ whose curl equals the monopole field $\mathbf{B}$, even if the origin where $\nabla \cdot \mathbf{B} \neq 0$ is excluded³ from the region where we require $\mathbf{B} = \nabla \times \mathbf{A}$. Dirac solved that problem for quantized values of the charges by introducing his now famous singular vector potential

$$A_\phi = g(1 - \cos\theta)/r\sin\theta; \quad A_r = A_\theta = 0. \quad (4)$$

Dirac's vector potential does not really represent a pure monopole field. Its curl contains a delta function singularity along the negative z axis which physically represents a string of magnetic flux $\Phi = 4\pi g$ running from minus infinity to the origin. Dirac's monopole can be regarded as the end of a current-carrying solenoid, semi-infinite in length and infinitesimal in diameter. However, for quantized values of eg/c , Dirac showed that the flux string or solenoid has no physical reality in the sense that it can be moved

from the negative z axis to any other line connecting infinity with the origin by a gauge transformation.

Dirac's trick does not apply to the di-monopole, even if we are prepared to go the limiting case through quantized values of g . A vector potential can be had by connecting the monopole-antimonopole pair with a flux string or solenoid, but then what we actually have is an ordinary dipole whose source is circulating currents.

However, the difficulty is more general than that. Imagine a di-monopole placed at the origin in a stationary external magnetic field B , and oriented as shown in Fig. 1. The x component of force on the di-monopole is given by

$$\begin{aligned} F_x &= \lim\{g[B_x(0,y,0) - B_x(0,-y,0)]\} \\ &= \mu \partial B_x / \partial y , \end{aligned} \tag{5}$$

where the limit means $y \rightarrow 0$, $g \rightarrow \infty$ with finite $2gy = \mu$. Alternatively, we could try to get the force from a Hamiltonian by introducing the potential energy suggested by classical mechanics,

$$V = - \mu B_y . \tag{6}$$

Then the force would be given by

$$F_x = - \partial V / \partial x = \mu \partial B_y / \partial x . \tag{7}$$

Expression (7) for the force F_x is not the same as the correct expression (5)

unless $(\nabla \times \mathbf{B})_z = 0$, and a Hamiltonian based on this V will be inconsistent with the correct equation of motion unless the current density vanishes at the origin.

This paradox results from an ambiguity in the definition of the energy of a di-monopole in the magnetic field generated by an electric current. The potential energy of a monopole in a magnetic field cannot be defined uniquely in a region where electric currents are present, and the energy needed to move a monopole-antimonopole pair from one point to another depends upon the path chosen for the motion. That path dependence does not vanish in the limiting case of a di-monopole. Suppose a monopole-antimonopole pair are moved from $-x$ to $+x$. If they move along the paths shown in Fig. 2a the work done agrees with Eq. (5), but if they move along the paths shown in Fig. 2b, the work done agrees with Eq. (7). The difference between the two results is just the work done in carrying one monopole around the loop defined by the four corners, as shown in Fig. 3. If the current density j_z at the loop is finite, this energy difference remains finite in the di-monopole limit.

The paradox can be illuminated in one way by examining the energy in the magnetic fields. The total magnetic field is the sum of an external magnetic field $\mathbf{B}_{\text{ext}}$ and the dipole field $\mathbf{B}_{\text{dip}}$. The interaction energy between the two fields is given by the integral of the cross term in the energy density $(\mathbf{B}_{\text{ext}} \cdot \mathbf{B}_{\text{dip}})/8\pi$. First suppose the dipole field is an ordinary one due to a current loop. If the external field is produced by currents, this can be integrated by parts to obtain the familiar $\mathbf{j}_{\text{dip}} \cdot \mathbf{A}_{\text{ext}}$ interaction. However, if the external field is produced by magnetic charges, then $\mathbf{B}_{\text{ext}}$ is irrotational and the interaction energy vanishes.

Next, suppose the dipole field is due to a static di-monopole. Then a magnetostatic potential M can be defined so that $B_{\text{dip}} = \nabla M$. Now if all fields vanish sufficiently rapidly at infinity, the interaction energy can be integrated by parts to give an energy density proportional to $M \nabla \cdot \mathbf{B}_{\text{ext}}$. If $\mathbf{B}_{\text{ext}}$ is produced by magnetic charges, this is just the magnetostatic energy of the di-monopole in the field of all the other magnetic charges. But this magnetic interaction energy vanishes when the external field is produced by electric currents instead of by magnetic charges. This is another manifestation of the impossibility of defining a Hamiltonian for a di-monopole interacting with electric currents in a consistent way.

A simple model involving only a monopole rather than a di-monopole illustrates all the physical principles involved when magnetic sources move in a solenoidal magnetic field whose source is an electric current. Here we can easily see how the troubles of the Hamiltonian theory arise from simple physical considerations which do not involve formalistic problems such as the existence or nonexistence of a vector potential. Consider a magnetic charge which moves freely on a circle of radius r in the xy plane. A long resistanceless wire along the z axis connects two capacitor plates far from the circle where the monopole moves. The dynamical variables are the charge q on the capacitor, the angle coordinate θ of the monopole on its circle, and their time derivatives. The current $\dot{q}$ in the wire is the source of a magnetic field in the θ direction

$$B_{\theta}(t) = 2\dot{q}(t)/rc \quad (8)$$

and the moving magnetic charge is in turn the source of an electric force on the current. These interactions are described by the Lagrangian

$$\mathcal{L} = (1/2)I\dot{\theta}^2 + (1/2)L\dot{q}^2 - (1/2C)q^2 - (2g/c)q\dot{\theta}, \quad (9)$$

where $\mathcal{L}$ is the self inductance of the wire, C is the capacitance of the two plates, and $I=mr^2$. The corresponding Hamiltonian

$$H = \left(\frac{1}{2L}\right) p_q^2 + \frac{1}{2} L\omega^2 q^2 + \frac{1}{2I} p_\theta^2 + \frac{4g}{Ic} qp_\theta \quad (10)$$

equals the conserved total energy

$$E = \frac{1}{2} L\dot{q}^2 + \frac{1}{2C} q^2 + \frac{1}{2} I\dot{\theta}^2. \quad (11)$$

The solutions of the equations of motion are

$$q(t) = q_0 + \frac{i_0}{\omega} \sin\omega(t-t_0)$$

$$\dot{\theta}(t) = -\frac{c}{2gC} q_0 + \frac{2g}{\omega Ic} i_0 \sin\omega(t-t_0), \quad (13)$$

where

$$\omega^2 = \frac{1}{L} \left(\frac{1}{C} + \frac{4g^2}{I} \right), \quad (14)$$

and q_0 and i_0 are constants.

In general, energy flows between the magnetic charge and the electrical system with frequency ω .

We can use these results to treat the motion of a monopole in an externally fixed magnetic field produced by a stationary current. That case is represented in our model by the limit $L \rightarrow \infty$, $\omega \rightarrow 0$, while $L\omega^2$ remains finite as do q_0 and i_0 . C may be finite or infinite. For any finite time t , the current and magnetic field approach the constant values

$$\lim\{\dot{q}(t)\} = i_0$$

$$\lim\{B_\theta(t)\} = 2i_0/rc, \quad (15)$$

but the energy of the electrical system

$$E_q = \frac{1}{2} L \dot{q}^2 + \frac{1}{2C} q^2 \quad (16)$$

approaches

$$\lim\{E_q\} = \frac{1}{2} L i_0^2 + \frac{1}{2} (L\omega^2) q_0^2 + (L\omega^2) q_0 i_0 (t-t_0) \quad (17)$$

for finite values of $(t-t_0)$.

The first term on the rhs of Eq. (17) is an infinite constant amount of energy. The last term represents a finite amount of energy which is exchanged between the magnetic charge and the fixed external magnetic field in which the magnetic charge moves. This term compensates the kinetic energy of the magnetic charge to conserve the total energy of the interacting system.

Because of the last term in Eq. (17), it is impossible to obtain a Hamiltonian function of θ and p_θ by subtracting (17) from the full Hamiltonian in the external field limit. When the variables of the electrical system are included in the dynamical description, the Hamiltonian theory works correctly. However, the Hamiltonian does not separate into an electrical part and a magnetic charge part, even in the limit where the inertia of the electrical system becomes infinite and the magnetic field becomes fixed, because the energy of the electrical system is then infinite and a finite amount of energy passes between the magnetic charge and the current as the magnetic charge moves.

All these results for the external field case are physically correct. However, they cannot be expressed in terms of a Hamiltonian with the interactions of the monopole described by a local interaction involving only the dynamical variables of the monopole and the electromagnetic fields or potentials at the position of the monopole. Newton's laws using only the field and monopole variables describe the motion correctly, but there is no Hamiltonian description without including the interaction with the current source, and that interaction must be nonlocal. Therefore the ordinary ways of doing quantum mechanics do not work. The interaction terms in the Lagrangian (9) and the Hamiltonian (10) between the monopole and the electrical system involve the product of the monopole velocity or canonical momentum and the charge q . The charge q is a variable which is far away from the monopole, and the interaction is manifestly nonlocal. Since the magnetic field at the position of the monopole is proportional to the current and therefore to the variable $\dot{q}$, the variable q can be expressed as a time integral of the magnetic field or vector potential at the position of the monopole. This gives an interaction which is local in space but nonlocal in time and is equally unsatisfactory.

One could perhaps evade the paradox by attaching a flux string to the monopole so that the string winds around the current and $\theta=0$ is not physically the same as $\theta=2\pi$. That is what happens if the Lagrangian $\mathcal{L}$ of Eq. (9) is replaced by

$$\mathcal{L} = + \frac{d}{dt} (q\theta) \approx \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} L \dot{q}^2 - \frac{1}{2C} q^2 + \frac{2q}{c} \dot{q}\theta . \quad (18)$$

The equations of motion are unchanged and the interaction term now has the form of the product of the monopole co-ordinate θ and the current variable

which is proportional to the field at the position of the monopole. However, the variable θ is not a legitimate operator because it is not single valued. It carries additional information beyond the actual position of the monopole; namely a kind of winding number describing the number of times the monopole has moved around the current. That represents the motion of a monopole carrying a string, with the string winding and unwinding around the current as the monopole moves back and forth.

This example suggests that the introduction of a singular string may not be harmless even when the Dirac quantization condition is obeyed.

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References

1. Dirac, P. A. M., Proc. Roy. Soc. (London) A133, 60 (1931).
2. Lipkin, H. J., W. I. Weisberger and M. Peshkin, Ann. Phys. 53, 203 (1969).
3. Wu, T. T. & C. N. Yang, Phys. Rev. D12, 3845 (1975); and Nucl. Phys. B107, 365 (1976).
4. Callan, Jr., G. C., In "Magnetic Monopoles" eds. R. A. Carrigan, Jr. and W. P. Trower, Plenum Press, New York and London (1982) p. 97, and references therein.
5. Jackson, J. D., "The Nature of Intrinsic Dipole Moments", CERN Report 77-17, theory division (1977).

Figure Captions

- Fig. 1 The force on a di-monopole.
- Fig. 2 Virtual displacement of a di-monopole by two different paths to the same final configuration.
- Fig. 3 Circuit of a monopole which is equivalent to the difference between the virtual displacements of Figs. 2a and 2b.

Fig. 1

Fig. 2

Fig. 3

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