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A MATHEMATICAL MODEL OF A UTILITY FIRM,

FINAL TECHNICAL REPORT.

on

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INDUSTRY FUNCTIONAL MODELING

PART III

Submitted to the
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by the

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MASTER

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4. DYNAMICS OF ECONOMIC SYSTEMS

4.0 OVERVIEW

This chapter of the report deals with what has been called "Phase II" of the project. The proposal for it was based on the conjecture that some of the models for the dynamical processes in an organization might be appropriate also for those that take place outside it, i.e., in the society at large. It was felt that, if true, an understanding of these processes would contribute to one of the environment in which a utility firm must do its planning, and it might help towards the conception of better plans.

The evidence developed during this study supports the conjecture. However, it would be going much too far to claim that it is conclusive. The nature of societal dynamical processes apparently is a problem that is largely unsolved and its solution would require an attack on an extremely broad front, by persons with expertise in many fields.

The attack contemplated under this project was a good deal narrower. It was based on the notion that many (though not all) of these processes were driven by micro-economic forces, i.e., by the strivings among the individual members of a society for improved economic conditions for themselves. This suggested drawing on a highly developed branch of mathematical economics, namely equilibrium theory. This theory had its beginnings two centuries ago. It

asserts, roughly speaking, that the strivings of the members of the society are reflected in the prices of the goods and services which those persons desire. The assertion more specifically is that prices of the various commodities will change until the demand for them levels off at a point of an equilibrium with their supply.

The main objective of mathematical economics was to show that a set of equilibrium prices actually exist. An existence proof, under conditions that are regarded as acceptably general, is of fairly recent vintage. More recent still are studies of related issues such as uniqueness and stability of the equilibrium, and of the price adjustment processes that lead to it.

Sect. 4.1 is a brief and rather superficial review of equilibrium theory and of the main underlying assumptions. It is a somewhat biased review. For one, it emphasizes those assumptions which we found inconsistent with our own work plan. For another, it stresses certain ideas and lines of reasoning which are not necessarily relied on by the majority of economists but which are compatible with the approach we had to take.

Our approach is developed in Sect. 4.2. The choice was governed by a fundamental mismatch between the techniques employed in organization theory and those in the main stream of equilibrium theory. The variables used in organization theory are basically discrete, namely the tasks which are carried out by the organization members. The basic

variables in equilibrium theory are the amounts in which the various commodities are traded or produced. Only a relatively small, and relatively recent, effort has been directed at economies with commodities that are discrete (i.e., that come only in indivisible units). After some exploratory study, we decided that we would be better to consider only economics in which all commodities consist of such units. The difficulties of developing an appropriate equilibrium theory seemed smaller in prospect than that of re-casting our work on organizations to conform with established practices in equilibrium theory.

Sect. 4.21 develops the basis of an equilibrium theory for "discrete exchange" economies. The term "discrete" is intended to indicate that all commodities are traded or produced in amounts that are integer multiples of certain units. This, we feel, reflects the situation in practice better than the traditional assumption. (Even liquids move through the economy in multiples of certain units.) An "exchange" (or "market") economy is one in which commodities change hands only by trading.

The more general case in which some commodities are manufactured from others is taken up in Sect. 4.22, under title of "production economies."

Sect. 4.22 finally discusses procedures which allow an equilibrium distribution of goods and services to be approached in an economy, at least in principle, if such a distribution exists in the first place. If it does not

exist, the procedures show that it does not. The main assumption that underlies these results is that the approach to an equilibrium is a competitive process in which the members of the economy engage, either individually or in coalitions with others. This assumption has been made also by economists, though apparently not by the majority. The competition is then modeled as a many-person game. It is similar to, but more general than, the games discussed in Sect. 3.2, and the appropriate equilibrium concept is then the so-called "core" of the game. The procedures described in Sect. 4.23 determine a solution in the core, or else show that the core is empty.

4.1 ECONOMIC EQUILIBRIUM THEORY--A REVIEW

Economic equilibrium theory deals with the question of whether or not a large collection of individuals, each with his own preferences and resources, can agree on a set of prices at which the acquisitions by each are balanced by the dispositions and production of all others. The possibility of such an equilibrium first occurred to Adam Smith [1] who advanced the notion that if the demand for any one commodity ever outran the supply of it, its price would surely rise to the point at which demand declined to equal the level of supply. This "invisible hand," in his terminology, prevented the conflicting desires of many from generating a situation of total chaos.

The questions of the existence, uniqueness, stability, computation, etc., of such an equilibrium have generated an enormous interest among mathematicians and mathematically inclined economists. However, only relatively recently have answers been obtained to many of these questions which are of sufficient rigor and generality to be considered satisfactory. K. J. Arrow was awarded the Nobel prize in 1978 for his contributions. A history of the work leading up to the solution to the equilibrium problem is given in [2] (Chap. 1).

The problem can perhaps be best explained by way of an example. Let the collection of individuals, the "economy" in the usual terminology, be made up of three members, Alfa, Bravo, and Charlies, who have three commodities (wine,

bread, and cheese) available for trade. This is one more member and one more commodity than that used for a similar example in [3] (Chap. 1). We follow that example further and assume that each member has an "initial endowment" of only one of these three commodities. Thus, Alfa starts out with three bottles of wine, Bravo with three loaves of bread and Charlie with three cakes of cheese.

In the usual treatment of the equilibrium problem, it is assumed that prices p_1, p_2, p_3 have somehow been established for the commodities. Alfa's initial endowment thus represents a monetary worth, $b^A = 3p_1$ which is called her "budget." It can be shown [2] (p. 20) that, for reasons which will not be important here, prices can be normalized so that their sum is equal to unity, and this is usually done in equilibrium studies regardless of the number of commodities that are being traded. Thus,

$$(4.1.1) \quad \sum p_i = 1, \quad p_i \geq 0,$$

where i ranges over all commodities, three in our example.

Given a set of prices, the members of an economy can now seek to trade some of their initial endowments in exchange for commodities held by others. If trading is all they do, as opposed to producing commodities, theirs is an "exchange economy." We start here by restricting our discussion to that case.

We suppose more specifically that the three members have done some trading and, as a result, Alfa has attained holdings in the amounts of x_1, x_2, x_3 in wine, bread, and

cheese, respectively. It is then assumed in equilibrium theory that the combined value of these holdings cannot exceed her original budget b^A :

$$(4.1.2) \quad p_1 x_1^A + p_2 x_2^A + p_3 x_3^A \leq b_1^A$$

It is further assumed that the new holdings are preferred by her over her original endowment, in the sense that her utility function $u^A(x_1, x_2, x_3)$ is greater when $x_i = x_i$ than when $x_1=3, x_2=x_3=0$. Finally, her holdings of any commodity cannot exceed the total original endowment, i.e.,

$$(4.1.3) \quad 0 \leq x_i^A \leq 3, \quad i = 1, 2, 3.$$

Analogous assumptions are made for the other two members of the economy. The generalization to economies with more than three members, or more than three commodities, or more varied endowments, is surely quite clear.

The equilibrium problem is then the following. Does there exist a set of prices, such as p_1, p_2, p_3 in our example, under which every member of an economy can maximize his utility by a suitable choice of his holdings, subject to a constraint (such as (4.1.2)) by his budget and to constraints (such as (4.1.3)) by the joint original endowments?

It is perhaps the main achievement of economic equilibrium theory to have proven the existence of such a set of prices, and to have done so not only for exchange economies of the kind discussed here so far but for economies that allow for the production of certain commodities from others rather than the mere trading of

them. This is a generalization to be discussed presently. The best known of several proofs which are now available was derived under certain assumptions, some of a technical and some of a conceptual nature [2]. One of the latter is important to our present discussion. It is that the holdings of a commodity by a member of an economy can be any real number in the range of possible holdings. In terms of our example, whatever trades Alfa has made, it need not have been in whole bottles of wine or whole loaves of bread or cakes of cheese. Wine could be traded in arbitrarily small drops, and bread and cheese in arbitrarily thin slices. This assumption can be defended at least up to a point but so can the opposite which allows trading only in certain minimal units. Both will be discussed in the next section.

An economic equilibrium is, in a way, the outcome of a competition and it is in fact often called "competitive" for that reason. The members of an economy in effect bid against each other for the goods they desire. One would accordingly think that any mathematical approach to the problem would rely on the theory that was specifically designed to deal with problems of equilibria in competitive situations, and that is game theory. Some economists have actually attacked the economic equilibrium problem via game theory [3]. Others have also found that computational ideas originally developed for game theoretical solutions carry over to equilibrium theory [4]. Although the majority of economists have used other techniques it will be important

for what follows to describe here briefly the minority approach.

The basis for this approach is the notion that the members of an economy are in effect engaged in a many-person game of roughly the kind introduced in Sect. 3.21 above. However, it is not a zero-sum game in general, nor is it symmetric. Moreover, the solution that is desired is not a Nash equilibrium which assumes that no cooperation develops among the players. In an economy, the members can typically guarantee higher utility levels for themselves if they enter into coalitions with others. The guarantee in this case means that they can be sure of these levels even if the non-members of the coalition join into another opposing coalition and play a two-person game against the first one.

The idea of a coalition which can guarantee each of its members a mean payoff that is better than any he could be sure of securing for himself (playing uncooperatively) has considerable intuitive appeal. It is on the other hand troublesome in a number of respects. One of the troubles is the question of how the individual payoffs ought to be determined. It is clear enough how to define the joint endowment of a coalition, namely as the sum of the endowments of all members. The coalition S made up of Alfa and Bravo in our example has the joint endowment of three bottles of wine and three loaves of bread. It is not clear, however, how the joint utility should be defined of this endowment or, for that matter, of any other holding $(x^A + x^B)$

for which their coalition can settle. However, if S actually does settle for this holding, as the result of their game against the "counter" coalition S' consisting only of Charlie, how will they divide their joint gain?

In game theory, and to a lesser extent in economics, the assumption is often made that individual utilities are "transferable." This means for most purposes that all holdings have a certain cash value and that a member of a coalition will be satisfied with receiving a "side payment," if necessary, in lieu of commodities. Assuming this kind of transferability makes the problem simpler, and we will make the assumption here that it is valid.

The arrangement resolves some but not all of the difficulty. Suppose that the coalition S has secured the mean gain $W(S)$ for itself and is now ready to divide it between Alfa and Bravo. How should this decision be made? In other words, how should $W(S)$ be split into proceeds z^A and z^B for Alfa and Bravo such that

$$z^A + z^B = W(S) \quad ?$$

Game theory is somewhat evasive in its answer to this question. It is, by and large, satisfied when it is able to say that such a split exists and that z^A and z^B are as good or better than the utility levels attainable by Alfa and Bravo if they contested Charlie uncooperatively.

In a game involving more than three players the comparison should be favorable not only against single players but against other coalitions. In principle, in

other words, the generic player Juliet of an N-person game would investigate the mean payoffs $W(S)$ of all coalitions S that she can possibly join. Suppose she found one, S^* say, and a division of utilities among its members

$$\sum_{J \in S^*} z^J = W(S^*) \quad (J \in S^*)$$

which had the property that, for any other S ,

$$(4.1.4) \quad \sum_{J \in S} z^J \geq W(S) \quad (J \in S) .$$

In such a case she could not hope to improve her share in a mean payoff by joining any coalition S other than S^* . This comparison would have to be with all other coalitions, including the one that involves only Juliet herself and the one involving all players. Juliet's search will thus cover all utility allocations which satisfy (4.1.4). They represent what might be called "equilibrium" allocations, in the sense that defections would be counterproductive from any coalition under which they are possible. The set of these coalitions is called the "core" of the game.

Unfortunately, there may be no such allocations. The core may be empty.

A main thrust in recent work on game theory has been at establishing conditions under which the core is non-empty and ways of determining it computationally [4], with and without the assumptions of utility transferability. Some of the impetus for this work has come from the suspicion that a connection might exist between the core of a game and the equilibrium of an economy as envisioned by Adam Smith. This suspicion was confirmed (chiefly, it seems, by the work of

Hildenbrand [3]) at least for exchange economies. It was shown that, under some convexity assumptions, the core of an exchange game is never empty and, as the number of members increase towards infinity, the allocations in the core approach the economic equilibria and vice versa.

This result has two curious features. For one, the economic approach to the equilibrium problem is squarely based on the concept of price for a commodity, while the game-theoretical approach introduces it as an optional afterthought. This feature has been used to argue that prices are not really fundamental to the equilibrium problem and that in fact, they do not make a great deal of sense except in very large economies. (In small economies they can be manipulated too easily by one member, hence are not an appropriate basis for an equilibrium characterization.) This has led to the conjecture that the core of a game might be a more fundamental concept than an economic equilibrium, and some of the recent work in the field has reflected this. The discussion in the next section will reflect it as well.

A second curious feature is the equivalence between the core of a game and the equilibria of an economy is the nature of the limiting process towards infinitely many members. It develops that that process must be executed in a rather special way which insures that the characteristics (i.e., the preferences) of every member recur, in the limit, infinitely often within the economy. It is possible that this feature is another prerequisite for the existence of

meaningful prices.

This discussion so far was limited to exchange economies. Production is unfortunately not a very natural activity in an economy such as the one in our example in which the commodities are wine, bread, and cheese. One must assume that it is somehow possible to manufacture, e.g., wine from bread and/or cheese. Such conversions could admittedly be rather miraculous but let it be assumed that these miracles actually do happen. In fact, in order to remain within the framework of equilibrium theory we assume that there exists a transformation function ϕ which defines the quantity x_1 of wine that can be produced according to

$$(4.1.5) \quad x_1 = \phi(x_2, x_3)$$

from the amounts x_2 and x_3 of bread and cheese. Under the usual assumptions of equilibrium theory x_1, x_2, x_3 are real numbers, subject only to constraints such as (4.1.2) and (4.1.3).

In equilibrium theory production is carried out by special members of an economy which are called "firms." The others, by contrast, are often called "households." Firms are different from households in the theory. For one, they can have no initial endowments. If Fox is the firm that manufactures wine according to the formula (4.1.5), it must buy its raw materials from the households of Bravo and Charlie. Moreover, a firm does not seek to maximize a utility of the same kind as a household. It seeks to maximize its "profit." In the case of Fox' operation, this

is the quantity

$$p_1x_1 - (p_2x_2 + p_3x_3)$$

which is the income from the sale of his output, minus the cost of his inputs.

This maximization is, of course, subject to the equation (4.1.5) which restricts the variables x_1, x_2, x_3 to lie within a certain set, the so-called "production feasibility set" for Fox. This set is assumed to be convex, in equilibrium theory. It is a rather curious assumption, in our opinion. There are very few functions ϕ , other than linear ones, that satisfy it. Its origin may, in fact, lie in a tradition in economics, perhaps initiated by Koopmans, according to which all production can be approximately represented by linear transformations. Some economists in fact, restrict themselves to linearity [4].

It may also be noted that the convexity assumption requires the retention of another, made above for exchange economies, namely the infinite divisibility of all commodities. Fox' wine manufacture can operate on arbitrarily small amounts of bread and cheese, and produce from them (not necessarily profitably) arbitrarily small amounts of wine.

In other words, the variables of the problem of an economic equilibrium, with or without production, are real numbers. The economies are, as we shall put it, "continuous." This is in contrast to the "discrete" economies to be discussed next in which the variables can

assume only integer values. In the language of economists, there are "large indivisibilities."

A final comment here concerns a second assumption, customarily made in work on equilibria in production economies, namely that, no matter how many commodities there are in an economy, the functions ϕ that govern production always are scalar. In the notation used earlier in this report, if there are m commodities in an economy, the functions ϕ are from R_+^m to R_+ . This assumption will also be found unenforceable in the sequel.

4.2 DISCRETE ECONOMIES

4.21 Exchange Economies

The main idea, as already mentioned, of Phase II of this project was to bring together concepts from economic equilibrium theory and mathematical organization theory, in the expectation that the combination would yield a reasonably persuasive model of some of the dynamics in economic processes. The first obstacle encountered in this undertaking was a fundamental mismatch between the two theories. One, namely equilibrium theory, is based on continuous variables, namely the commodities which undergo exchange and production. The other uses discrete inputs and outputs. To make them compatible, one of the two theories had to be put on the same footing as the other. After some thought and some mathematical experimentation, it was decided that the conceptual difficulties of such a change would be smaller if equilibrium theory were changed to discrete variables, rather than the other way round. It was discovered subsequently that similar modifications are being pursued by some mathematical economists [5], [6] though no doubt for different reasons, and, as far as we could determine, using different methods.

This section develops more particularly the main ideas for a discrete exchange economy. Production will be added in the next section. The development, for sake of simplicity, will be carried out here for the very simple exchange economy introduced in the preceding section. It

consists of three members (Alfa, Bravo, and Charlie) who have initial endowments of three bottles of wine, three loaves of bread, and three cakes of cheese, respectively. The exchanges that can take place among them can then be characterized by the same equations and inequalities as in the preceding section, i.e., by (4.1.1) through (4.1.3). The essential difference is that, in a discrete economy, the members can trade only in units, e.g., in bottles of wine, in loaves of bread, and cakes of cheese. It will develop, however, that this difference entails a second, perhaps more striking one. Prices of commodities do not seem to be as natural an adjunct of the equilibrium problem as in a continuous economy. The reason for this is that in a discrete exchange economy it seems more natural to define an equilibrium game-theoretically.

Such a definition presumes that of an "exchange" game. A discrete exchange economy is rather easily defined in this way. One must settle first what the (pure) strategies are that are available to its members. In our example, these can be characterized by triplets (y_1^J, y_2^J, y_3^J) , $J=A, B, C$, of positive integers. In (y_1^A, y_2^A, y_3^A) , the first integer y_1^A represents the number of bottles of wine which Alfa plans to keep, y_2^A , the number she offers to Bravo, and y_3^A the number she offers to Charlie. In (y_1^B, y_2^B, y_3^B) , y_2^B stands for the number of bread loaves which Bravo intends to retain, y_1^B and y_3^B for the numbers he offers to trade with Alfa and Charlie, respectively. Charlie's strategy options are denoted

analogously. Evidently, what is kept and what is offered by each member must add up to his initial endowment. Thus

$$y_1^J + y_2^J + y_3^J = 3, \quad (J=A,B,C).$$

An exchange, in our example of an economy, is the adoption by each of the three members of one among the pure strategies available to him. The offers made, in other words, would be accepted and the acceptances would, in turn, define the holdings with which the members would conclude the exchange. These holdings were denoted by triplets (x_1, x_2, x_3) in the preceding section. They are related to those introduced here by

$$(4.2.6) \quad \begin{aligned} x_1^A &= y_1^A, & x_2^A &= y_1^B, & x_3^A &= y_1^C, \\ x_1^B &= y_2^A, & x_2^B &= y_2^B, & x_3^B &= y_2^C, \\ x_1^C &= y_3^A, & x_2^C &= y_3^B, & x_3^C &= y_3^C. \end{aligned}$$

The holdings arrived at in this way determine the utilities or, in game-theoretical terminology, the "payoffs" which the members attach to them. They constitute the second portion of the specification of an exchange game (the first having been the pure strategies available to the players). These utilities (payoffs) can be arranged into arrays of the kind denoted with F,G,H in Sect. 3.2. They were four-dimensional there. However, since only three players are involved in the game under consideration, they are now three-dimensional and only three indices t, u, v (rather than the four in Sect. 3.2) are necessary to label the locations of the entries in F,G, and H. The index t would thus be used to label Alfa's strategies in F,G, and H.

However, as has just been said, it is convenient in the present instance to label her strategies by the triplets (y_1^A, y_2^A, y_3^A) , rather than the single index t , and that will be done in what follows. According to (4.2.6), the triplets (x_1^A, x_2^A, x_3^A) would be equally good labels.

The entries in the arrays are subject to no restriction, aside from having to be real numbers. In particular, they need not obey the zero-sum and symmetry relations (3.2.1) and (3.2.2). The solution concept introduced in Sect. 3.2, namely the Nash equilibrium, is also known to be ill-suited to the more general games considered here because the players can typically improve their mean payoffs by forming coalitions. One must accordingly take into account the possibility of such arrangements in seeking for the equilibria in a discrete economy. As explained in the preceding section, the concept of a "core" suggests itself as particularly appropriate.

Unfortunately, as will now be shown, the game representing a discrete economy may not have a core. To see this, suppose the utilities of players are linear in their holdings. This would mean that Alfa would assess the utility of a holding (x_1^A, x_2^A, x_3^A) as being

$$(4.2.7a) \quad F: \quad x_1^A f_1 + x_2^A f_2 + x_3^A f_3$$

in which f_1, f_2 , and f_3 are the utilities to her of having a bottle of wine, a loaf of bread, and a cake of cheese, respectively. She is indifferent to the holdings of the others. The sum in (4.2.7a) would thus be an element in her

utility array F. It would more specifically be the element in the location indexed by the triplets (y_1^A, y_2^A, y_3^A) , (y_1^B, y_2^B, y_3^B) , (y_1^C, y_2^C, y_3^C) that are compatible, via (4.2.7) with the holding (x_1^A, x_2^A, x_3^A) .

The elements of the utility arrays of Bravo and Charlie are analogously

$$G: x_1^B g_1 + x_2^B g_2 + x_3^B g_3 ,$$

(4.2.7b)

$$H: x_1^C h_1 + x_2^C h_2 + x_3^C h_3 .$$

The sums of the elements in the same locations of the three arrays are thus

$$x_1^A f_1 + x_1^B g_1 + x_1^C h_1 + x_2^A f_2 + \dots + x_3^C h_3 .$$

Suppose now that the preferences among the three commodities, hence the utilities per unit, are the same for all members, i.e.

$$f_i = g_i = h_i = u_i \quad (i=1,2,3) .$$

The sum is then

$$(4.2.8) \quad s = u_1 \sum_J x_1^J + u_2 \sum_J x_2^J + u_3 \sum_J x_3^J \\ = 3u_1 + 3u_2 + 3u_3 .$$

It is the same for all locations of the arrays F, G, H, or in other words, the game is of a kind called "constant-sum."

This is a slight generalization of the zero-sum games discussed in Sect. 3.2. It is known, however, that the core of a constant-sum game is either empty or else contains only the Nash equilibrium [7] (p. 124).

This result is not valid for our example only.

Proposition 4.2.1. If the members of a discrete economy

attach the same per-unit utility to every commodity, the game they play either has an empty core or else contains only one element, namely the non-cooperative Nash equilibrium.

Proof. The proof follows the line of reasoning we gave above for our example. The difference merely lies in the elements of the utility array for the generic economy member Juliet. They are of the form

$$\sum_i x_i^J u_i,$$

rather than (4.2.7), where i ranges over all commodities, and the sum s in (4.2.8) is

$$(4.2.9) \quad s = \sum_i u_i \sum_J x_i^J = \sum_i u_i e_i$$

where e_i is the combined endowment of all economy members in the i -th commodity. The proof is then completed by invoking the above-referenced result from game theory.

The proposition shows that, in discrete economies, the core of the game cannot be expected to be a satisfactory solution concept always and that in fact no equilibrium, in any reasonable sense of the term, may exist. Our counter-example is admittedly of a special sort, and it would be easy to rule it out by suitable assumptions regarding the members' utilities. It is unlikely, however, that this approach would be successful. Games other than the one developed here are known to have empty cores [3] (pp. 78-85).

The conclusion seems to be that it may not be a good idea to base a theory of the functioning of a discrete economy on the existence of an equilibrium.

4.22 Production Economies

In our discussion of Sect. 4.1 of continuous economies we criticized the inclusion of production in existing equilibrium theories on several counts. We consider several of the usual assumptions to be overly restrictive. Some of these assumptions introduce distinctions between firms and households. These strike us as artificial. They allow household to have endowments in all commodities while firms may have none. Households may have virtually arbitrary utilities while firms may operate under the profit motive only. Households are allowed to trade but not to manufacture; firms allowed to do only the opposite.

We also consider the current notion of production feasibility sets to be unnecessarily restrictive. These are the sets defined by the transformations

$$(4.2.10) \quad x' - \phi(x) = 0$$

which govern the manufacture of certain commodity bundles x' from others, viz. x . The feasibility set for (4.2.10) would consist of those combinations (x', x) which satisfy that equation. Equilibrium theory assumes these sets to be convex. It also assumes that the bundles x' consist of single commodities.

It will develop that in the discrete economies these assumptions are rather unnatural or even completely unenforceable. At least in some respects, therefore, the theory of such economies seems more realistic than that of their continuous counterparts. We will explain our case

based on the example of a three-member, three-commodity economy which we introduced in Sect. 4.1 (and we repeat here apologies for the lack of realism in our choice of raw materials and manufactures goods).

To start with, we assume production to be governed by (4.1.5). I.e., we assume that x_1^A bottles of wine can somehow be manufactured from x_2^A loaves of bread and x_3^A cakes of cheese according to the formula

$$(4.2.11a) \quad x_1^A = \phi(x_2^A, x_3^A)$$

in which ϕ is defined by the function table (table entries are x_1^A -values)

	x_2^A	x_3^A	1	2	3
	1		0	0	0
(4.2.11b)	2		1	1	1
	3		1	2	2

Production Table (i).

However, in order to hold the number of members of the economy down to three, we assume that Alfa does the manufacturing of wine (rather than the fourth member, Fox, we introduced in Sect. 4.1). The table says that Alfa's production process needs at least two loaves of bread in order to manufacture any wine at all, regardless of how many of the available cakes of cheese are used, but with three loaves and two or more cakes Alfa can produce two bottles. The table seems perfectly adequate at first glance as a listing of Alfa's production options. As will be seen presently, however, it is not.

An economy, according to the view advanced in the preceding section, is a game among its members. In order to define the game one must first specify the pure strategies that are available to each player and then the utilities to each of the outcome of every possible combination of strategies by the different players. In the present example, the specification of the pure strategies is simple for Bravo and Charlie. They have the same options in the present game as they did in the exchange economy, namely triplets (y_1^B, y_2^B, y_3^B) and (y_1^C, y_2^C, y_3^C) which represent their offers of exchange versus retention of their endowments.

Alfa now is a firm, however. Her strategies are different from what they were previously when she functioned as a household. She produces wine from bread and cheese. For her raw materials, she will presumably turn to Bravo and Charlie who offer her the amounts y_1^B and y_1^C , respectively. Suppose $y_1^B=3$, $y_1^C=2$. According to the function table, Alfa can accept these offers, i.e., set in (4.2.11)

$$x_2^A = 3, x_3^A = 2$$

and convert them into $x_1^A=2$ bottles of wine. Suppose, however, that $y_1^B=3$, $y_1^C=3$. In this case, presumably, Alfa can accept $x_2^A=y_1^B=3$ and $x_3^A=y_1^C=2$ as the inputs to her wine making process. One cake of cheese would be superfluous under this arrangement, to be dealt with in some way. The function table (4.2.11b) however, allows for no such way. The table must accordingly be expanded.

The expansion must actually be made in two directions of

which only one is pointed up by this example. The basic problem here is that raw materials may be offered to a firm in quantities which it cannot fully use. (They lie outside the "production feasibility set," in the terminology of Sect. 4.1.) The function table must specify the alternatives that are then open to the firm. Typical alternatives for Alfa's disposal of the extra cake of cheese are:

1. Put it in storage.
2. Remove it from the economy (i.e., destroy it or give it to charity).
3. Return it to Charlie.

In our example, she must select one of these options. In a more general she could choose two or three. She could, in other words, dispose of excess raw materials in all of these ways and would have the choice of how to apportion quantities to each.

For sake of brevity we will discuss here only the first option. (The others could be dealt with quite similarly.) The upshot of our discussion so far is that the table entries in (4.2.11b) are inadequate. They specify only the amounts of output produced from the inputs but not the disposal of excess amounts of the inputs. Thus, the entries should not be single numbers, as in (4.2.11b), but triplets as in the one shown below. In each, the first two numbers represent the amounts of bread and cheese that

	y_1^B	y_1^C			
		0	1	2	3
(4.2.12a)	0	(0,0;0)	(0,1;0)	(0,2;0)	(0,3;0)
	1	(1,0;0)	(1,1;0)	(1,2;0)	(1,3;0)
	2	(2,0;0)	(0,0;1)	(0,1;1)	(0,2;1)
	3	(3,0;0)	(1,0;1)	(0,0;2)	(0,1;2)

Production Table (ii)

Alfa must put in storage and the last the number of bottles of wine she produces. These last elements are the same as in (4.2.11b). Table (ii) specifies the options Alfa has in her handling of the inputs to the wine production. The function ϕ in 4.2.11a should thus be written

$$(4.2.12b) \quad (x_1^A, x_2^A, x_3^A) = \phi(y_1^B, y_1^C)$$

in which x_2^A and x_3^A are the excess quantities of raw materials stored by Alfa and x_1^A the amount she produces.

This is more general than the usual assumption for functions such as ψ , characterizing a production process.

The table also suggests that two other assumptions be abandoned which are customary in equilibrium theory. One is that firms be prohibited from having endowments in their case of raw materials. It is difficult to defend their being allowed to store such materials but being prohibited from having inventories of them to begin with. But if one allows a firm to have endowments, there seems to be no good reason for prohibiting it from trading in them. Thus, a second customary assumption seems unnatural in the present context, and perhaps not worth retaining. Firms, in other words, may have to be permitted the functions usually

reserved for households, and vice versa. Crosses between the two may be inevitable, and the distinction between them superfluous.

Table (ii), while more detailed than Table (i) in (4.2.11), is still inadequate in one respect. It does not specify what is to be done with the output of the production. For a complete definition of Alfa's pure strategies, one must specify also how much of the output is offered to Bravo and how much to Charlie. The entries in the table in other words cannot be merely triplets; they must be quadruplets. Moreover, these entries can no longer be indexed merely by y_1^B and y_1^C . For many combinations of values of these variables, Alfa has more than one way of allocating her production outputs. The table below illustrates this. It also shows that the function

y_1^B	y_1^C	0	1	2	3
0		(0,0;0,0)	(0,1;0,0)	(0,2;0,0)	(0,3;0,0)
1		(1,0;0,0)	(1,1;0,0)	(1,2;0,0)	(1,3;0,0)
2		(2,0;0,0)	(0,0;1,0)	(0,1;1,0)	(0,2;1,0)
			(0,0;0,1)	(0,1;0,1)	(0,2;0,1)
3		(3,0;0,0)	(1,0;1,0)	(0,0;2,0)	(0,1;2,0)
			(1,0;0,1)	(0,0;1,1)	(0,1;1,1)
				(0,0;0,2)	(0,1;0,2)

Production Table (iii)

Φ is not simple-valued, but set-valued. It relates the 16 strategies pairs that are available to Bravo and Charlie to the 24 that are available to Alfa, and it is of the form

$$(x_2^A, x_3^A; y_2^A, y_3^A) = \phi(y_1^B, y_1^C)$$

In this, y_1^B and y_1^C retain their meaning: they are the amounts of bread and cheese offered to Alfa. She puts x_2^A and x_3^A in storage and uses the rest, i.e., $(y_1^B - x_2^A)$ and $(y_1^C - x_3^A)$ towards her production of wine. She offers y_2^A bottles to Bravo and y_3^A to Charlie. (Both accept.)

The quadruplets $(x_2^A, x_3^A; y_2^A, y_3^A)$ specify the 24 pure strategies that are available to Alfa. Bravo and Charlie have the same 10 strategies as in the exchange economy discussed in the preceding section. The utility arrays F, G, and H, therefore, have $24 \times 10 \times 10$ entries each. Those in F, for instance, represent the utilities to Alfa of accepting raw materials in the amounts y_1^B, y_1^C , putting the quantities x_2^A, x_3^A into storage, and converting the rest into the manufactured product for distribution to Bravo and Charlie in the amounts of y_2^A and y_3^A . The entries in G and H are analogously defined.

It is no doubt clear that the construction of strategy lists developed here encounters no difficulties, other than those of scale, when extended to economies with more members, households as well as firms. It avoids the restrictions which were mentioned at the start of this section as being somewhat unattractive features of equilibrium theory when applied to discrete economies.

On the other hand, no examples were found during this study of discrete production economies with empty cores. The presence in these economies of firms deprives them of

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the kind of symmetries which led to the examples cited in the preceding chapter for exchange economies. It is suspected that situations that preclude the existence of equilibria (in the game-theoretical interpretation of the term) are somehow "rarer" in production, than in exchange, economies. However, no conditions are known to us which insure their existence.

4.23 Approach to an Equilibrium

Evidence was presented in the preceding two sections which indicates that, in a discrete economy, an equilibrium may not exist. The equilibrium concept in question is a game-theoretical one. The view is taken that the members of such an economy are competing with each other in an effort at maximizing their utilities. They do so regardless of whether they are "households" (which trade the goods they own) or "firms" (which manufacture certain goods from others) or crosses between the two. We mentioned that in such competitive situations it is often a good idea for members to form coalitions and to adopt joint strategies against all others. The mean payoff they jointly attain in this way, and distributed suitably among the coalition members, can make all of them better off than if they competed individually.

The problem, therefore, is to find a coalition, if it exists, and a suitable distribution of the payoff it can guarantee, which is more profitable to each of its members than any other possible coalition into which they could enter. If such an arrangement exists, it is in the nature of an equilibrium because no coalition member will have an incentive for seeking a change for the better.

The problem can be viewed as consisting of two parts. The first is a comparison of the joint mean payoffs that can be guaranteed by the various coalitions that can be formed among the members of an economy. The second is a comparison

of the distributions that can be made of that payoff among the coalition members.

Both problems take rather different forms depending on whether or not the utilities are "transferable" among the members, i.e., whether or not the joint gain can be freely distributed. In this project we have assumed transferability. It is the simpler and better manageable of the two assumptions.

With this in mind, we consider once more our example of the three-member, three-commodity economy. We treat first the exchange economy discussed in Sect. 4.21. There are seven coalitions possible in it, namely three consisting of one member each, to be denoted with A, B, C, and those made up of pairs, viz. AB, BC, and AC. Finally, there is the coalition ABC which comprises all three members.

A coalition can "guarantee" a certain outcome to its members if it assumes that all non-members join into a counter-coalition and it will have to play a two-person game against this combination of opponents. There are three such games possible in our example. One is A against BC, the second B against AC, and the last C against AB. The coalition ABC does not play a game. It is concerned only with the second problem mentioned above, namely the distribution of its combined holdings. This problem will be discussed presently.

In order to define the two-person games between coalitions, one must list the strategies that are open to

each player and the payoffs to each that result from the choice of one of these strategies by each. If the game is played over an exchange economy, the strategies available to, say, the coalition AB are, as in Sect. 4.21, the various ways in which the combined endowments can be divided between retention within the coalition on the one hand, and trade with C on the other. The strategies can thus be labeled by the pair of triplets (y_1^A, y_2^A, y_3^A) and (y_1^B, y_2^B, y_3^B) that were used in that section to index the strategies individually of Alfa and Bravo. The "strategy spaces" of AB, in other words, is the Cartesian product of those of Alfa's and Bravo's, and the number of strategies in it is the product, i.e., $10 \times 10 = 100$, of those in the factor spaces. The same observation applies if the game is over a production economy. If it is the one described in Sect. 4.22, the coalition AB can choose among $10 \times 24 = 240$ pure strategies which can be labeled by the pair $(x_2^A, x_3^A; y_2^A, y_3^A)$, (y_1^B, y_2^B, y_3^B) .

The coalition in any case acts like a single member that is allowed to trade as well as to manufacture. It is thus the kind of cross between a firm and a household that was mentioned in the preceding section. This is true even if the coalition is not allowed to trade away its inventory of raw materials, as was tacitly assumed here.

The utilities that a coalition attaches to the outcome of a play is the sum of the utilities of its players. That is, if AB "plays" the pair (y_1^A, y_2^A, y_3^A) , (y_1^B, y_2^B, y_3^B) in an exchange economy against C's (y_1^C, y_2^C, y_3^C) , then it earns the

sum of the payoffs which Alfa and Bravo would have earned by playing the same strategies individually. If these utilities are of the special form (4.2.7), the sums of the payoffs to AB and C can be the constants s in (4.2.8). The game between AB and C is then a constant-sum two-person game which is well-known to be equivalent to a zero-sum game [7] (p. 5). It can, therefore, be treated by the huge body of theory and computation that has been developed for such games.

The typical game between coalition and counter-coalition in an economy, however, will not be constant-sum but of a kind often called a "bi-matrix game." Solution algorithms exist for such games [8]. However, they are not well suited for our present purposes because they do not reflect the iterative nature of the process by which equilibria are likely to be approached in an economy. An algorithm that is of the desired nature was developed under this project. The underlying idea is the same as the one that led to the algorithm described in Sect. 3.22 which is due to Brown and von Neumann [9]. It is summarized in

Proposition 4.2.1. Let p, q be the probability vectors representing mixed strategies for the two players of a bi-matrix game with payoff matrices U and V . The transformations

$$\begin{aligned}
 p_i' &= (p_i + \rho \phi_i) / (1 + \rho \sum \phi_i) \\
 q_j' &= (q_j + \rho \psi_j) / (1 + \rho \sum \psi_j)
 \end{aligned}
 \tag{4.2.13a}$$

$0 < \rho < 1$

applied repeatedly converge on a pair p^*, q^* of equilibrium strategies for the players. In these equations, ϕ_i and ψ_j stand for

$$\phi_i = \sum_j u_{ij} q_j = \sum_j u_{ij} p_i q_j$$

(4.2.13b)

$$\psi_j = \sum_i v_{ij} p_i - \sum_i v_{ij} p_i q_j$$

when the right-hand sides are positive. Otherwise they vanish. The mean equilibrium payoffs for the two players are then $\sum_i u_{ij} p_i^* q_j^*$ and $\sum_i v_{ij} p_i^* q_j^*$.

Proof. The proof is in Appendix 4.2A.

This proposition suggests a principle by which a member of an economy, eg. Alfa, can calculate the worth to her of the various coalitions she can join. To do so, she could solve the two-person game for each such coalition by the repeated application of the transformation (4.2.13) and determine the mean payoff it can guarantee for itself. In the three-member example, she would have to solve three such games, namely those for the coalitions A, AB, and AC and determine the mean payoffs for each. In this way she would obtain three payoffs, $W^*(A)$, $W^*(AB)$, $W^*(AC)$, each calculated from the payoff matrix U appropriate to the game and calculated according to

$$W^* = \sum_i u_{ij} p_i^* q_j^*$$

from the equilibrium strategies p^*, q^* for that game.

This, however, would only be the first step in her decision of which coalition to join. She must also consider the question of whether she and the other coalition members

(if any) can divide their joint payoff in a way that is at least as profitable to each of them as they can arrange through any other coalition. In the terminology of Sect. 4.1, she must determine an allocation that is in the core of the game, and in fact, whether the game has a core in the first place. To do this, she must solve a linear programming problem.

This programming problem is implied by the line of reasoning developed in Sect. 4.1. Let z^A, z^B, z^C be a possible apportionment of the proceeds among Alfa, Bravo, and Charlie of the payoffs between a coalition and a counter coalition they have formed. According to that reasoning, the apportionment will be in the core if it satisfies (4.1.4) for all possible coalitions. In the present example these are

$$z^A \geq W^*(A), \quad z^B \geq W^*(B), \quad z^C \geq W^*(C)$$

(4.2.14a)

$$z^A + z^B \geq W^*(AB), \quad z^A \geq z^C \geq W^*(AC), \\ z^B \geq z^C \geq W^*(BC).$$

To these six inequalities, one must add an equation which says that if the three members form a single, unopposed coalition, their apportionments must add up to the combined utilities which we will denote $W^*(ABC)$. I.e.,

$$(4.2.14b) \quad z^A + z^B + z^C = W^*(ABC)$$

Thus, the core of the game is non-empty if there exists a triplet z^A, z^B, z^C which satisfies (4.1.14). Considering that three variables are required here to satisfy seven

relations, non-existence seems highly likely, offhand.

Suppose that such a triplet exists. It is then a feasible solution to the linear programming problem defined by (4.1.14), with an essentially arbitrary objective function. It is customary [10] to solve such problems by introducing slack variables into all inequality constraints, here equations (4.2.14a), and artificial variables into all constraints. The objective function is then the sum of the artificial variables, and it is minimized. If this minimum is zero, there exists a feasible solution. If not, the problem is infeasible.

Such a minimization can be carried out by the Simplex algorithm. However, it is well known that it can also be carried out by an iterative procedure which is a special case of the one described in Sect. 3.24. In this case actually somewhat more can be said [10] (Chap. 13) and [11].

Proposition 4.2.2. A linear programming problem in the variables $z_1, z_2, \dots, z_n$ with m inequality constraints is always convertible to a symmetric zero-sum two-person game, with a suitable pay-off matrix S , in which each player has $(n+m+1)$ pure strategies. The game can be so arranged that the solutions z_i^* of the programming problem are related to the components r_j^* of the mixed minimax strategy by

$$z_i^* = r_i^* / r_{n+m+1}^* ,$$

whenever $r_{n+m+1}^* \neq 0$. If $r_{n+m+1}^* = 0$, the feasible region is unbounded. The minimax solution r^* of the game can be determined iteratively by a repeated application of the

transformation ($0 < \rho' < 1$)

$$r_j' = (r_j + \rho' x_j) / (1 + \rho' \sum x_j).$$

(4.2.15)

$$x_j = \begin{cases} \sum s_{jk} v_k & \text{when positive,} \\ 0 & \text{otherwise} \end{cases}$$

Proof. See Appendix 4.2B.

The reason for introducing this iterative solution algorithm here is to indicate that a payoff allocation in the core of a multiperson game can be determined by the same general algorithmic procedure as the one used to determine the guaranteed gain of a coalition or, for that matter, for the optimizing control of the functioning of an organization. In the present case, one can view it in two ways. One is to assume that every member of an economy has carried out the determination of the gains attainable by all coalitions he can join or, in other words, that he has obtained the quantities denoted with $W^*(S)$ for the various coalition S in (4.2.14). He will perhaps have done so by means of the algorithm (4.2.13). With this behind him, he can determine core allocations using the algorithm (4.2.15) or else conclude from it that no such allocations exist.

The second point of view is that he carries out both determinations simultaneously. He can do so by using a slower time scale for the second algorithm than the first (by choosing $\rho' < \rho$). An argument similar to that discussed in Sect. 3.32 shows that the convergence properties of both can be preserved in this way.

4.24 Comments

The conclusion to which we were led in the preceding section do not add up to an equilibrium theory for discrete economies, at least not in the usual sense of the word. We did not give conditions under which an equilibrium exists and then prove its existence. Actually that had not been our aim in this project in the first place. We intended to develop a model for the dynamic processes within such an economy, on the assumption that these processes would carry the economy toward an equilibrium if one existed. These processes, we assumed, would presumably proceed regardless of whether or not it did. A model by which they might conceivably evolve was derived in the preceding section. It also bears out the conjecture mentioned in the Overview to this chapter, namely that this model would be quite similar to one for a process taking place within an organization. The resemblance is to the organizational control procedures developed in Sect. 3.32.

The approach we took to derive these results, however, eliminated a set of variables which dominated economic thinking about the equilibrium problem and which, in fact, gave rise to it to begin with. These are the prices of the commodities. Prices actually lose their importance whenever the equilibrium problem is formulated game-theoretically, as we mentioned already in Sect. 4.1. They can, however, be re-introduced in the end, as a re-interpretation of the probabilities that enter into the solution of the problem.

This re-interpretation is made possible by the normalization relation (4.1.1). In our solution, two kinds of prices appear. Those governing the trading outside a coalition, which can be identified with the components of the mixed strategy the coalition uses against the opposition. Here, the re-interpretation is immediate and plausible. The second kind of prices govern the distribution of the joint gain among the members of a coalition. Although mixed strategies enter into this distribution as well, by way of the game mentioned in Proposition 4.2.2, their re-interpretation as price systems is less clear. Eq.'s (4.2.13) and (4.2.15) in any case, characterize price adjustment processes.

We said in the Overview that our models are "conceivably" correct. We doubt that they actually are. The main reason for our doubt is the scale of the problem which these processes would in effect solve. Part of the problem, as was explained in the preceding section, would be the solution of a number of bi-matrix games. For an economy with n members, there would be $(2^n - 1)$ of these. The payoff matrices for each of the games would be huge. Even for the simple three-member production economy of our example, they would be 240×10 . For any realistic example they would be much larger. The linear programming problem would involve n variables and $2^n - 1$ constraints.

The astronomical scale of these problems prevents the dynamics of the processes of interest to be realized in

practice. A member of an economy could not possibly evaluate the performance of all possible coalitions he could join. In fact, most of those coalitions probably could not form to begin with because many of their potential members would not even know of their existence, let alone their performances. The execution of the evaluation algorithms (4.2.13) and (4.2.15) would require the members to be fully cognizant of the utility arrays of all members. In most cases in practice, people are not fully aware even of their own. The notion of a bounded rationality which overlay the work reported in Chap. 2 would have to be introduced in the problem of coalition formation as well.

There are, however, reasons other than the bounds on rationality that would inhibit such formations. In practice, some coalitions are prevented from forming by law, other are discouraged by taxation. The competitive situation that can develop in practice are thus probably rather few. On the other hand, they will rarely be in the nature of the simple games of our theory, namely those between coalitions and counter-coalitions.

Processes of a non-economic nature no doubt evolve at the same time as the economic ones. These, we feel, should be treated in a way similar to that we have given the socio-psychological characteristics of individuals in Chap. 2. Exploratory investigations were made of some of the issues mentioned here but none were carried far enough to deserve being reported. Their upshot was the remark made in

the Overview, namely that a much broader-based attack would be necessary on the problem that we were able to mount. Similar conclusions have been reached by others (see, e.g., [12], Chap. 10).

Appendix 4.2A A Solution Algorithm for Bi-Matrix Games

Brown and von Neumann [9] described a solution algorithm for two-person symmetric, zero-sum games which was generalized in Sect. 3.22 to many-person games that are also symmetric and zero-sum. The purpose of the discussion in this Appendix is to develop another generalization, namely to two-person games that need be neither symmetric nor zero-sum. Such games are generally called "bi-matrix" games (see, e.g., [7], p. 98).

Suppose that the two matrices of the game are A and B, and that x and y are the mixed strategy vectors used by the two players Alfa and Bravo. A and B are of the same dimension (mxn), x is an m-vector and y an n-vector.

A strategy pair x^*, y^* is a solution (equilibrium) for the game if

$$(A.1) \quad Ay^* \leq x^{*T}Ay^* , \quad x^{*T}B \leq x^{*T}By^*$$

where the superscript T denotes the transpose.

Lemma A1. If (x^*, y^*) is a solution for a bi-matrix game with matrices A, B, it is a solution also for the game with matrices $(A+aE)$, $(B+bE)$ where E is an (mxn)-matrix with 1's as elements, and a and b are arbitrary constants.

Proof. Using $(A+aE)$ on the left-hand side of (A.1) in place of A gives

$$\sum_j (a_{ij} + a) y_j^* = \sum_j a_{ij} y_j^* + a \sum_j y_j^* = \sum_j a_{ij} y_j^* + a$$

because $\sum_j y_j^* = 1$. On the right-hand side,

$$\sum_{ij} (a_{ij} + a) x_i^* y_j^* = \sum_{ij} a_{ij} x_i^* y_j^* + a$$

because also $\sum_i x_i^* = 1$. Thus if (A.1) is valid for A it is

valid also for $(A+aE)$. The analogous argument for B completes the proof.

Theorem A1. The transformation

$$(A.2a) \quad \begin{aligned} x' &= (x + \rho \xi) / (1 + \rho \phi), & \phi &= \sum \xi_i \\ (0 < \rho < 1) \end{aligned}$$

$$y' = (y + \rho \eta) / (1 + \rho \psi) \quad \psi = \sum \eta_i$$

with

$$(A.2b) \quad \xi = r(e(Ay - x^T Ay)) \quad , \quad \eta = r((x^T B - x^T By)e)$$

repeatedly applied converges on a solution. Here, e is a vector of suitable dimension and with 1's as elements, and $r(\cdot)$ is the ramp function

$$r(u) = \begin{cases} u & \text{for } u \geq 0, \\ 0 & \text{for } u < 0. \end{cases}$$

Proof. The proof is similar to that of Theorem 3.2.1.

One introduces the Lyapunov function

$$L(\xi, \eta) = \xi^T \xi + \eta^T \eta$$

and finds by an argument very much like the proof of that Theorem that

$$\begin{aligned} \xi'^T \xi' - \xi^T \xi &= 2\rho [\xi^T A \eta - \xi^T A y (\phi + \psi) \\ &\quad - x^T A \eta \phi + (x^T A y) (\phi + \psi) \phi] + o(\rho). \end{aligned}$$

The results for $\eta^T \eta$ is symmetric to this one, giving

$$\begin{aligned} L(\xi', \eta') - L(\xi, \eta) &= \\ &= 2\rho [\xi^T (A+B) \eta - (\xi^T A y + x^T B \eta) (\phi + \psi) - \\ &\quad - x^T A \eta \phi - \xi^T B y \xi + (\phi + \psi) (x^T A y \phi + x^T B y \psi)] + o(\rho). \end{aligned}$$

If A and B have only negative elements, the quantity in the brackets is negative, also. If on the other hand A or B

have positive elements, one can change them to $(A+aE)$ and $(B+bE)$, as in Lemma A.1, without affecting the solution. Suppose that had been done to start with and $a \leq 0$, $b \leq 0$ had been so chosen that $(A+aE)$, $(B+bE)$ have only negative elements. Thus, the expression in the bracket above can be arranged to be negative in any case.

From the fact that $L(\xi, \eta)$ is a Lyapunov function for small enough ρ , one now concludes as in Theorem 3.2.1 that the iterative application of (A.2) takes ξ and η asymptotically into null vectors. ¶

The results reported here, together with those in Sect. 3.22, strongly suggest that transformations of the general type of (A.1) could be used for the calculation of Nash equilibria in arbitrary many-person games. Unfortunately, an attempt during this project at proving this was unsuccessful.

Appendix 4.2B. Infeasibility of Linear Programs

The material in this appendix is for record only. It reports unpublished work by Dantzig [11] which may be useful towards the problem mentioned in Sect. 4.23, namely the existence of a core for a many-person game. Such an existence is assured if a set of linear inequalities, namely (4.21.4), has a solution. It is conjectured that it does if and only if one of the components in an associated game does not vanish. Dantzig's work supplies some support for the conjecture.

Somewhat more generally, suppose that the linear programming problem

$$Au \geq b, \quad \max c^T u$$

where A is an $(m \times n)$ -matrix, u and b are n -vectors, is to be solved. The associated game, as is well known [11], p. 290), is then a two-person, symmetric, zero-sum game with the matrix

$$\begin{bmatrix} & -A & b \\ A^T & & -C^T \\ -b^T & C^T & \end{bmatrix}.$$

Its minimax solution for the game is a probability vector p , $p^T = [x, y, z]^T$, which satisfies the inequalities

$$\begin{aligned} & -Ay + bz \leq 0, \\ (B.1) \quad & A^T x - Cz \leq 0, \\ & -b^T x + c^T y \leq 0. \end{aligned}$$

The quantity z is a scalar; the superscript T denotes a transpose.

Theorem B.1 Suppose that set $S = \{z \mid y \geq 0, Ay \geq b\}$ is non-empty. Then, if $z = 0$, z is unbounded.

Proof. Suppose $z = 0$. Since p is a probability vector, it follows that either $x > 0$ or $y > 0$ or both. Let $y > 0$. Since S is non-empty, there exists a y_0 such that $Ay_0 \geq b$, $y_0 \geq 0$. Hence for any scalar $\theta > 0$, $A(y_0 + \theta y) \geq b$ and $(y_0 + \theta y) \geq 0$. Since it is true for any θ , no matter how large it follows that S is unbounded.

Suppose next that $y = 0$ hence $x > 0$. Then $x^T b \geq 0$, and, from (B.1a) pre-multiplied by x^T

$$x^T A y_0 \geq x^T b,$$

hence $x^T A y_0 \geq 0$. On the other hand, by (B.1b), $x^T A \leq 0$ and consequently also $x^T A y_0 \leq 0$. This is a contradiction which implies $x = 0$. $\square$

4.3 REFERENCES

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5. PUBLICATIONS

Published

- P. Sen, Planning Under Uncertainty, Proc. IEEE Internat. Conf. on Cybernetics and Society, Seattle, WA, Oct. 82
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- P. Sen, Control and Planning with Set-Valued Uncertainty
- R. F. Drenick, Many-Person Games and Multilinear Programming
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