COMPARISON OF BOUNDARY ELEMENT
AND FINITE ELEMENT METHODS
IN TWO-DIMENSIONAL INELASTIC ANALYSIS

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ABSTRACT

The finite element method has been commonly used to solve boundary-value problems in inelastic deformation of metallic structures. Recently, Mukherjee and his coworkers applied the boundary element method to such problems. Planar time-dependent inelasticity problems were considered and a constitutive model with state variables was used to describe material behavior. In this paper, the accuracy and computational efficiency of these two methods are compared for certain selected planar problems. In order to make the comparison as meaningful as possible, 'in house' computer codes developed by the same group at Cornell, are used.

INTRODUCTION

Solutions of time-dependent inelastic boundary value problems for metallic structural components of arbitrary shape must generally be obtained by numerical methods. The finite element method (FEM) has been popular in recent years for the analysis of these nonlinear problems. Solutions to two and three dimensional problems have been obtained by the FEM and constitutive models for material behavior with separate descriptions for time-independent plasticity and time-dependent creep have been generally used (see reference¹ for a typical example).

The boundary element method (BEM - also called the boundary integral equation method) is an alternative general purpose method. While this method has been applied to many problems in elasticity (see, for example, references^{2,3}), applications to nonlinear inelastic problems have been relatively few. An interesting feature of this method is that the number of unknowns in the resultant algebraic system obtained by discretization is proportional to the number of boundary nodes rather than to the total number of nodes. Mukherjee and his coworkers have used the BEM to solve time-dependent inelastic deformation problems⁴⁻⁶. A combined creep-plasticity constitutive model with state variables, proposed by Hart⁷⁻⁸, was used in most of the numerical examples presented in references⁴⁻⁶. Hart's is one of a new generation of constitutive models using state variables that aim to represent inelastic deformation of metals and alloys more faithfully than is possible with the traditional approach.

A question of crucial practical importance is the evaluation of the accuracy and efficiency of the BEM in relation to the FEM for this class of problems. This paper presents a comparison of the two methods as applied

to several selected plane stress examples. The boundary element formulation uses a spatially linear description of the relevant variables over each boundary element and is capable of solving planar problems. The finite element formulation, on the other hand, uses linear strain triangles (LST) and is capable of solving planar and axisymmetric problems. Both these programs can use any of a number of newer constitutive models as long as it fits into the mathematical structure described in the next section. Hart's constitutive model is used in the numerical examples. Although a FEM program using constant strain triangles and Hart's model had been previously developed⁹, the present program used to generate the results in this paper is completely new. Thus, both the BEM and FEM programs discussed herein have been created by the same group in order to avoid programming bias and make the comparison as meaningful as possible.

A brief description of the structure of constitutive models with state variables is followed by an outline of the BEM and FEM formulations and solution strategy. Finally, numerical results using BEM, FEM and direct numerical integration (whenever possible) are compared for certain selected sample problems. The BEM and FEM programs, of course, are quite general and can solve problems for structural components of arbitrary shape subjected to general histories of loading.

MATHEMATICAL STRUCTURE OF STATE VARIABLE MODELS

The mathematical structure of many of the state variable models of inelastic deformation can be summarised by the following equations

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^n + \dot{\epsilon}_{ij}^T \quad (1)$$

$$\dot{\epsilon}_{ij}^n = h_{ij}(\sigma_{ij}, q_{ij}^{(k)}, T) \quad (2)$$

$$\dot{q}_{ij}^{(k)} = g_{ij}(\sigma_{ij}, q_{ij}^{(k)}, T) \quad (3)$$

$$\dot{\epsilon}_{kk}^n = 0 \quad (4)$$

Here $\dot{\epsilon}_{ij}^e$, $\dot{\epsilon}_{ij}^n$ and $\dot{\epsilon}_{ij}^T$ are the elastic, nonelastic and thermal strain rates respectively, σ_{ij} is the stress tensor, T the temperature and $q_{ij}^{(k)}$ are state variables. The number of state variables vary in the different models and they can be scalar or tensors. These state variables are assumed to completely characterize the present deformation state of the material and the history dependence of the rate of nonelastic strain upto the current time is assumed to be completely taken into account by their current values. It is important to note that the rates of the nonelastic strain and state variables at any time depend only on the current values of the stress, state variables and temperature. This fact plays a key role in the BEM and FEM formulations presented in the next two sections.

Hart's Constitutive Model

The specific equations for Hart's constitutive model are presented next. According to this model, the nonelastic strain is decomposed into two (time-dependent) components

$$\dot{\epsilon}_{ij}^n = \dot{\epsilon}_{ij}^a + \dot{\epsilon}_{ij}^p \quad (5)$$

where ϵ_{ij}^a is the anelastic strain, a stored strain which reflects the magnitude and direction of prior deformation history and ϵ_{ij}^p is the completely irrecoverable and path dependent permanent strain. The two state variables in the model are the anelastic strain and a scalar σ^* , called hardness, which is similar to an isotropic strain hardening parameter.

Let s_{ij} represent the deviatoric part of the stress tensor σ_{ij} . The deviatoric stress tensor is decomposed into two auxilliary tensors s_{ij}^a and s_{ij}^f

$$s_{ij} = s_{ij}^a + s_{ij}^f \quad (6)$$

The flow relations for the anelastic strain and the time rates of change of the permanent and nonelastic strains are

$$\dot{\epsilon}_{ij}^a = \frac{3}{2} \frac{\dot{\epsilon}^a}{\sigma^a} s_{ij}^a \quad \dot{\epsilon}_{ij}^p = \frac{3}{2} \frac{\dot{\epsilon}^p}{\sigma^a} s_{ij}^a \quad (7,8)$$

$$\dot{\epsilon}_{ij}^n = \frac{3}{2} \frac{\dot{\epsilon}^n}{\sigma^f} s_{ij}^f \quad (9)$$

where $\dot{\epsilon}^n$, $\dot{\epsilon}^p$, $\dot{\epsilon}^a$, σ , σ^a and σ^f are scalars defined as

$$\begin{aligned} \dot{\epsilon}^n &= \sqrt{\left(\frac{2}{3} \dot{\epsilon}_{ij}^n \dot{\epsilon}_{ij}^n\right)}, \quad \dot{\epsilon}^p = \sqrt{\left(\frac{2}{3} \dot{\epsilon}_{ij}^p \dot{\epsilon}_{ij}^p\right)}, \quad \dot{\epsilon}^a = \sqrt{\left(\frac{2}{3} \dot{\epsilon}_{ij}^a \dot{\epsilon}_{ij}^a\right)} \\ \sigma &= \sqrt{\left(\frac{3}{2} s_{ij} s_{ij}\right)}, \quad \sigma^a = \sqrt{\left(\frac{3}{2} s_{ij}^a s_{ij}^a\right)}, \quad \sigma^f = \sqrt{\left(\frac{3}{2} s_{ij}^f s_{ij}^f\right)}. \end{aligned} \quad (10)$$

The scalar invariants are related to each other through the uniaxial equations

$$\sigma^a = M \epsilon^a, \quad \dot{\epsilon}^n = \dot{a}^*(\sigma^f/M)^M \quad (11,12)$$

$$\dot{\epsilon}^p = \dot{\epsilon}^*(\ln(\sigma^*/\sigma^a))^{-1/\lambda} \quad (13)$$

$$\dot{\epsilon}^* = \epsilon_{ST}^*(\sigma^*/\sigma_s^*)^m \exp(Q/RT_B) \exp(-Q/RT) \quad (14)$$

$$\dot{\sigma}^* = \dot{\epsilon}^p \sigma^* \Gamma(\sigma^*, \sigma^a), \quad \Gamma(\sigma^*, \sigma^a) = (\beta/\sigma^*)^\delta (\sigma^a/\sigma^*)^{\beta/\sigma^*} \quad (15,16)$$

Equation (11) represents a linear anelastic element, (12) a nonlinear dash-pot, (13) and (14) a "plastic" element and, finally, (15) and (16) describe strain hardening. The flow parameters are M , M , m , λ , a^* and $\dot{\epsilon}_{ST}^*$ (at a reference hardness level σ_s^* and reference temperature T_B); β and δ are strain hardening parameters, R is the gas constant and Q the activation energy for atomic self diffusion. The material parameters, many of which are temperature dependent, are obtained from load relaxation and constant strain rate tension experiments. They are given in Appendix A of the EPRI report by Kumar and others⁹.

BOUNDARY ELEMENT FORMULATION

The Navier equations for the displacement rates for plane stress, in the presence of nonelastic and thermal strains, are^{5,6}

$$\begin{aligned} \dot{u}_{i,kk} + \frac{1+\nu}{1-\nu} \dot{u}_{k,ki} = & -\frac{\dot{F}_i}{G} + 2\dot{\epsilon}_{ij,j}^n + \frac{2\nu}{1-\nu} \dot{\epsilon}_{kk,i}^n \\ & + 2 \frac{(1+\nu)}{(1-\nu)} (\alpha \dot{T})_{,i} \quad i = 1,2 \end{aligned} \quad (17)$$

where F_i is the prescribed body force per unit volume, G and ν are the shear modulus and Poisson's ratio, α is the coefficient of linear thermal expansion and u_i is the displacement vector. Suitable traction and displacement boundary conditions must be prescribed. An integral representation of the solution for a point P on the boundary of the body is

$$\begin{aligned} (\delta_{ij} - c_{ij}) \dot{u}_j(P) = & \int_S [U_{ij}(P,Q) \dot{\tau}_j(Q) - T_{ij}(P,Q) \dot{u}_j(Q)] dS_Q \\ & + \int_V U_{ij}(P,q) \dot{F}_j(q) dV_q \\ & + \int_V \Sigma_{jki}(P,q) [\dot{\epsilon}_{jk}^n(q) + \delta_{jk} \alpha \dot{T}(q)] dV_q \end{aligned} \quad i,j,k = 1,2 \quad (18)$$

where P, Q are surface points, p and q are interior points, V is the volume and S the surface of the body. The kernels U_{ij} , T_{ij} and Σ_{jki} are known singular solutions due to a point load in an infinite elastic solid in plane stress. The traction is denoted by τ , δ_{ij} is the kronecker

delta, and the coefficients of c_{ij} are known functions of the included angle at the boundary corner at P , the angle between the bisector of the corner angle and the x_1 axis, and v . If the boundary at P is locally smooth, $c_{11} = c_{22} = 1/2$, $c_{12} = 0$. For a point p inside the body, the left hand side of equation (18) becomes $\dot{u}_i(p)$ and all P 's in the right hand side must be replaced by p 's.

Equation (18) is a system of integral equations for the unknown traction and displacement rates in terms of their prescribed values on the boundary, the prescribed body force and temperature rates, and the non-elastic strain rates. The unknown quantities only appear on the boundary of the body and the volume integrals are known at any time through the constitutive equations and prescribed histories of body force and temperature. A discretized version of equation (18) is obtained by dividing the boundary of the planar body S into straight boundary elements using N_b boundary nodes, and the interior into n_i polygonal elements. The components of traction, displacement and their rates are assumed to vary linearly over each boundary element with their values assigned at the nodes which lie on the intersection of the elements. The components of stress and state variables are assumed to be spatially uniform over an interior element. Discontinuities in traction across a node, if any, are taken care of by placing a 'zero length' element between two nodes and assigning different traction values at each of these nodes.

A discrete analog of equation (18) (with $\dot{F}_j = \dot{T} = 0$) has the form

$$\begin{aligned}
 (\delta_{ij} - c_{ij}) \dot{u}_j(P_M) &= \sum_{N_b} \int_{\Delta S_N} U_{ij}(P_M, Q) \dot{t}_j(Q) dS_Q \\
 &- \sum_{N_b} \int_{\Delta S_N} T_{ij}(P_M, Q) \dot{u}_j(Q) dS_Q \\
 &+ \sum_{n_1} \dot{\epsilon}_{jk}^n(q_n) \Delta \Sigma_{jki}(P_M, q_n) \\
 (i, j, k &= 1, 2, \quad M = 1, 2, \dots, N_b) \quad (19)
 \end{aligned}$$

where $\Delta \Sigma_{jki}(P_M, q_n) = \int_{\Delta V_n} \Sigma_{jki}(P_M, q) dV_q$ and $u_j(P_M)$ are the displacement components at the point P which coincides with node M . Substitution of the linear variation of $\dot{u}_i$ and $\dot{t}_i$ over each boundary element into equation (19) leads to a matrix equation of the type

$$[A]\{\dot{u}\} = [B]\{\dot{t}\} + \{\dot{b}\} \quad (20)$$

where the coefficients of matrices $[A]$ and $[B]$ contain integrals of the type

$$\int_{\Delta S_N} U_{ij}(P_M, Q) dS_Q, \quad \int_{\Delta S_N} U_{ij}(P_M, Q) s dS_Q$$

and the vector $\{\dot{b}\}$ involves $\Delta \Sigma_{jki}$ and the nonelastic strain rates. The vector $\{\dot{b}\}$ is known at any time through the constitutive equations and half of the total number of components of $\{\dot{u}\}$ and $\{\dot{t}\}$ are prescribed through the boundary conditions. The other half are the unknowns.

FINITE ELEMENT FORMULATION

The discretization process in this formulation is carried out by dividing the entire body into triangular finite elements. Linear strain triangles (LST) are used. Using a rate form of the principle of virtual work⁹⁻¹⁰, the equilibrium equation for an element, in the absence of prescribed body forces, can be written as

$$[k]\{\dot{\delta}\} + \{\dot{f}^n\} + \{\dot{f}^T\} + \{\dot{f}^B\} = 0 \quad (21)$$

where

$$[k] = \int_V [B]^T [D] [B] dV$$

$$\{\dot{f}^n\} = - \int_V [B]^T [D] \{\dot{\epsilon}^n\} dV$$

$$\{\dot{f}^T\} = - \int_V [B]^T [D] \{\dot{\epsilon}^T\} dV$$

and

$$\{\dot{f}^B\} = \int_S [N]^T \{\dot{i}\} dS$$

In the above equation, $[k]$ is the local stiffness matrix, $\{\dot{f}^n\}$ and $\{\dot{f}^T\}$ are the pseudonodal forces due to the nonelastic and thermal strain rates, and $\{\dot{f}^B\}$ is the equivalent nodal force due to the prescribed traction rate $\dot{i}$ on the boundary of the element. The volume integrals are evaluated over the volume of the element and the surface integral is evaluated over that part of the surface on which the traction is prescribed.

The vector $\{\dot{\epsilon}^n\}$ is interpolated linearly in each element for the evaluation of $\{\dot{f}^n\}$. The vector $\{\dot{\delta}\}$ denotes nodal displacement rates which are related to the displacement rate field through the shape function matrix $[N]$

$$\{\dot{u}\} = [N]\{\dot{\delta}\} \quad (22)$$

The local strain-displacement relation in terms of nodal displacement has the form

$$\{\dot{\epsilon}\} = [B]\{\dot{\delta}\} \quad (23)$$

and Hooke's law relating the elastic strain rates to the stress rates becomes

$$\{\dot{\sigma}\} = [D]\{\dot{\epsilon}^e\} = [D](\{\dot{\epsilon}\} - \{\dot{\epsilon}^n\} - \{\dot{\epsilon}^T\}) \quad (24)$$

Considering the entire body as an assembly of elements, the principle of virtual work and equation (21) leads to the following global equilibrium equation

$$[K]\{\dot{\delta}\} = -\{\dot{F}^n\} - \{\dot{F}^T\} - \{\dot{F}^B\} \quad (25)$$

The right hand side of equation (25) is known at any time through the constitutive equations, boundary conditions and the prescribed temperature history. The nodal displacement components are the unknowns.

SOLUTION STRATEGY-TIME INTEGRATION

Solution Strategy

The solution strategy is described in this section for the case of no prescribed body forces and a temperature distribution which is uniform in space and constant in time. However, prescribed histories of body forces and temperature can be easily included as discussed earlier in the paper.

The initial distributions of state variables (σ^* and ϵ_{ij}^a for Hart's model) are prescribed and the initial nonelastic strains are set to zero. The initial distribution of stresses and displacements is obtained from the solution of the corresponding elastic problem. This is obtained from the BEM or FEM equations as the case may be. The rates of nonelastic strain and the state variables at zero time are obtained from the constitutive equations (2-4). The solutions for the two schemes proceed as described in the next two subsections.

Boundary element method

The initial nonelastic strain rates and boundary conditions are used in equation (20), which is solved for the unspecified components of traction and displacement rates on the boundary of the body. The displacement and traction rates are thus determined at every boundary node at $t = 0$. These are now used in a discretized version of an equation analogous to (18) for an internal point p and the displacement rate field is obtained throughout the body. The total strain rates are obtained from $\dot{u}_i$ from the usual kinematic relations and then the stress rates throughout the body are obtained from the elastic strain rates from Hooke's law. The stress rates are obtained by analytical differentiation of the displacement rates in a manner discussed in reference⁵. Thus the initial rates of all the relevant variables

are now known and their values at a new time Δt are obtained by integrating forward in time. The rates are then found at time Δt and so on, and the time histories of all the variables are obtained. Since the state variable equations are generally stiff, choice of a suitable time-integration scheme is crucial. An Euler type strategy with automatic time-step control is used¹¹. A brief discussion of this method is presented later in this section.

Finite element method

The initial nonelastic strain rates and boundary conditions are used in equation (25), which is solved to obtain the initial rates of nodal displacements $\{\dot{\delta}(0)\}$. Next, the rates of displacement, total strain and stress are obtained at every node from equations (22-24). These rates are used to find the values of the variables at a new time Δt , and so on, and in this way the time histories of the relevant variables are obtained.

Time Integration

An Euler type strategy with automatic time step control, which is simple yet very efficient, is used to obtain the numerical results discussed later in the paper for both the BEM and FEM. A brief summary of the method in terms of a single differential equation

$$\frac{dy}{dt} = F(y,t) \quad (26)$$

is presented below. Further details and comparison with other numerical methods are available in reference¹¹.

The value of $y(t+\Delta t)$ in terms of $y(t)$ is

$$y(t+\Delta t) = y(t) + F(y,t)\Delta t \quad (27)$$

and the error at this step used for automatic time step control is defined as

$$e = \Delta t |\nabla F| / |y(t)| \quad (28)$$

where $\nabla F = F(y, t) - F(y, t - \Delta t_{\text{previous}})$, is the first backward difference of F . Two error parameters e_{max} and e_{min} are initially prescribed. The algorithm proceeds as follows:

$e_{\text{max}} < e$: replace Δt by $\Delta t/2$; recompute e .

$e \leq e_{\text{max}}$: accept Δt ; calculate $y(t + \Delta t)$

The time step for the next step, Δt_{next} , is decided according as

$e_{\text{min}} < e \leq e_{\text{max}}$: $\Delta t_{\text{next}} = \Delta t$

$e \leq e_{\text{min}}$: $\Delta t_{\text{next}} = 2\Delta t$

The problems at hand, of course, involve systems of such equations for the rates of variables at all the nodes. In this case the error e_i for the i_{th} variable is defined as an L^1 norm of the type

$$e_i = \frac{\Delta t \sum |\nabla F_k|}{\sum |y_k(t)|} \quad (29)$$

where the summation extends over the values of the i_{th} variable over all the nodes; and then

$$e = \max |e_i| \quad (30)$$

The algorithm then proceeds as described above.

NUMERICAL EXAMPLES - COMPARISON OF BEM, FEM AND DIRECT SOLUTIONS

Numerical results for several plane stress sample problems for 304 stainless steel plates at 200°C and 400°C, using Hart's constitutive model, are presented in this section. Comparisons of BEM, FEM and direct solutions are carried out.

Parameter Values

The values of the various parameters used in the numerical calculations, at 400°C, are given below. Those at 200°C are available in reference⁹.

$$\lambda = 0.15 \quad M = 7.8 \quad m = 5$$

$$M = 91.014 \text{ GPa} \quad (.132 \times 10^8 \text{ psi})$$

$$E = 168.24 \text{ GPa} \quad (.244 \times 10^8 \text{ psi})$$

$$\nu = 0.298 \quad \dot{a}^* = 0.69 \times 10^{25} \text{ sec}^{-1}$$

$$\dot{\epsilon}_{sT}^* = 1.269 \times 10^{-24} \text{ sec}^{-1} \quad \text{at} \quad \sigma_s^* = 68.95 \text{ MPa}$$

$$T_B = 673^\circ\text{K}$$

$$\beta = 1.234 \text{ GPa} \quad (.179 \times 10^6 \text{ psi}) \quad \delta = 1.33$$

The initial values of the state variables for an annealed sample are

$$\sigma^*(\underline{x}, 0) = 117.215 \text{ MPa} \quad (17 \text{ ksi})$$

$$\epsilon_{ij}^a(\underline{x}, 0) = 0$$

Numerical Results

The calculated uniaxial creep response of a plate subjected to a constant applied stress of 137.9 MPa (20 ksi) is shown in Figure 1. The direct solution is obtained by numerical integration of the one dimensional equations⁹. The results from the three methods coincide. Results for uniaxial constant strain rate extension at a strain rate of 0.02/min are shown in Figure 2. Again the results from the three methods very nearly coincide. Only a quarter of the plate is modelled here because of symmetry. Eight boundary nodes and one internal element is used in the BEM and two finite elements are used for the FEM calculations as shown.

Figures 3a and 3b show the BEM and FEM mesh, respectively, used to model a quarter of a circular plate with a concentric circular cutout under internal pressure increasing at a constant rate. The BEM calculation uses 30 boundary nodes and 20 internal elements while the FEM calculation has 48 elements and 117 nodes. Comparison of results is shown in Figure 3c. The BEM result is very close to the direct calculation while the FEM simulation has a maximum error of about 5% compared to the direct solution. For both the BEM and FEM calculations, the plane stress, as opposed to the axisymmetric option, has been used. The FEM mesh has fewer elements along the curved surface of the plate (compared to the BEM mesh) but has a linear (as opposed to uniform) description of strain on each internal element. The direct solution for this problem is obtained by the method discussed in reference¹².

The last problem considered is the inelastic response of a square plate with a circular cutout subjected to a constant remote uniaxial tensile stress of 68.95 MPa (10 ksi). Again, a quarter of the plate is modelled to take advantage of symmetry. The BEM mesh, shown in Figure 4a, has 37

boundary nodes and 20 internal elements while the FEM mesh, shown in Figure 4b, has 102 nodes and 41 elements.

The BEM and FEM results for displacement at the tip of the vertical diameter of the cutout and redistribution of σ_{22} along the line $x_2 = 0$ are shown in Figures 4c and 4d respectively. A direct solution cannot be obtained for this problem. The results obtained from the two methods are in fair agreement. The zero time elastic stress distributions in Figure 4d are within about 3% of the correct solution for the BEM and 6% for the FEM close to the cutout. The solutions are more accurate away from the cutout. The presence of stress concentration near the cutout causes substantial inelastic deformation in this region. This causes rapid unloading of stress close to the cutout while the stress distribution remains nearly unaltered beyond about twice the cutout radius. A vertical equilibrium check on the two long time solutions reveals that the FEM solution overpredicts the remote stress resultant by about 2% while the BEM solution underpredicts this quantity by about 2.8%. This is considered quite good considering that the BEM solution requires 367 and the FEM 420 time steps to reach the final simulated time. The exact long time stress distribution, of course, is not known.

Comparison of Computing Times

The CPU times on an IBM 370/168 for these problems are given in Table 1. The computing times are seen to be quite reasonable. It is seen that for the problems considered, the BEM program requires proportionately less time compared to FEM as the geometrical complexity of the problem increases. This trend was expected since the number of unknowns in the resultant algebraic system obtained by discretization is proportional to the number

of boundary nodes in BEM and to the total number of nodes in FEM. It is expected that the time advantage of the BEM relative to the FEM will become even greater for 3-D applications.

CONCLUSIONS

A BEM and a FEM formulation are presented in this paper for the solution of time-dependent inelastic boundary value problems using a constitutive model with state variables. The solution strategy, in either case, takes advantage of the determinate mathematical structure of the constitutive model. Only a single matrix inversion for the elastic solution is required (see equations (20) and (25)) followed by a march-forward time integration scheme. Both the BEM and FEM programs are quite general and can solve problems for structural components of arbitrary shape. Numerical results for several sample problems are obtained accurately and economically.

The BEM and FEM are both powerful general purpose methods for the solution of these complicated nonlinear problems of practical importance. The BEM, however, has several advantages over the FEM. One advantage, as mentioned earlier, is that the number of unknowns in the resultant algebraic system obtained by discretization is proportional to the number of boundary nodes in BEM as opposed to the total number of nodes in FEM. Another important advantage of the BEM is the ease of discretization and input data preparation. Even though discretization of the domain is necessary in the nonlinear problems considered in this paper, the topology of the internal elements is much simpler than for the FEM and the numbering of internal nodes or elements can be arbitrary without loss of computational efficiency. Finally, due to the nature of the BEM formulation, the stresses are continuous throughout the domain. They are generally more accurate than those obtained by the FEM and are easier to interpret.

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REFERENCES

1. Pan, Y.S. and Minami, H.M., "Inelastic analysis of nozzle-to-spherical shell attachment", Inelastic Behavior of Pressure Vessel and Piping Components, Chang, T.Y. and Krempl, E. ed., New York, 1978, American Society of Mechanical Engineers, 103-120.
2. Brebbia, C.A. (ed.), "Recent Advances in Boundary Element Methods", London, Plymouth, 1978, Pentech Press.
3. Cruse, T.A. and Rizzo, F.J. (eds.), "Boundary-Integral Equation Method: Computational Applications in Applied Mechanics", New York, 1975, American Society of Mechanical Engineers.
4. Kumar, V. and Mukherjee, S., "A boundary-integral equation formulation for time-dependent inelastic deformation in metals", Int. J. Mech. Sci., 19, (1977), 713-724.
5. Mukherjee, S. and Kumar, V., "Numerical analysis of time-dependent inelastic deformation in metallic media using the boundary-integral equation method", ASME Trans., J. Appl. Mech. 45, (1978), 785-790.
6. Morjaria, M. and Mukherjee, S., "Improved boundary-integral equation method for time-dependent inelastic deformation in metals", Int. J. Num. Meth. Engng. (In Press).
7. Hart, E.W., Li, C.Y. and Yamada, H., "Phenomenological theory: A guide to constitutive relations and fundamental deformation properties", Constitutive Equations in Plasticity, Argon, A.S. ed., Cambridge, Mass., 1976, M.I.T. Press, 149-197.
8. Hart, E.W., "Constitutive relations for the non-elastic deformation of metals", ASME Trans., J. Engng. Mat. Tech., 98, (1976), 193-202.
9. Kumar, V., Huang, F., Mukherjee, S. and Li, C.Y., "Theoretical and Experimental Analysis of Stresses in Reactor Components-Deformation in Type 304 Stainless Steel", EPRI final report for Contract No. RP 697-1, Department of Materials Science and Engineering, Cornell University, Ithaca, NY, 1979.
10. Zienkiewicz, O.C., "The Finite Element Method in Engineering Science". London, 1971, McGraw-Hill.
11. Kumar, V., Morjaria, M. and Mukherjee, S., "Numerical integration of some stiff constitutive models of inelastic deformation", ASME Trans. J. Engng. Mat. Tech. (In Press).
12. Mukherjee, S., "Thermoviscoplastic response of cylindrical structures using a state variable approach", Mechanical Behavior of Materials, Third International Conference on Mechanical Behavior of Materials, Cambridge, England, Smith, R. and Miller, K. ed., Oxford, 1979 (expected), Pergamon Press.

		Boundary Element Method			Finite Element Method		
	Problem	Boundary Nodes	Internal Elements	C.P.U. Secs	Nodes	Elements	C.P.U. Secs
1.	Uniaxial Creep	8	1	7.3	9	2	11.3
2.	Uniaxial extension	8	1	3.9	9	2	6.6
3.	Circular plate with circular cutout under increasing internal pressure	30	20	63	117	48	127
4.	Square plate with circular cutout under constant remote tension	37	20	91	102	41	205

Table 1. BEM and FEM program statistics.

Figure 1. Comparison of direct, BEM and FEM solutions for uniaxial creep at a constant stress of 20 ksi in an annealed 304 SS plate at 400°C. (1 ksi = 6.895 MPa).

Figure 2. Comparison of direct, BEM and FEM solutions for constant strain-rate extension in an annealed 304 SS plate at 400°C, $\dot{\epsilon} = .02/\text{min}$ (1. ksi = 6.895 MPa).

Figure 3a. BEM mesh for circular plate with concentric circular cutout under internal pressure. 30 boundary nodes and 20 internal elements.

Figure 3b. FEM mesh for circular plate with concentric circular cutout under internal pressure. 117 nodes, 48 elements.

Figure 3c. Comparison of direct, BEM and FEM solutions for circular plate with concentric circular cutout under internal pressure increasing at a constant rate. $\dot{p} = 10$ ksi/sec. Annealed 304 SS at 200°C. (1 ksi = 6.895 MPa).

Figure 4a. BEM mesh for square plate with circular cutout under uniaxial loading. 37 boundary nodes and 20 internal elements.

Figure 4b. FEM mesh for square plate with circular cutout under uniaxial loading. 102 nodes, 41 elements.

Figure 4c. Comparison of BEM and FEM solutions for displacement at A in an annealed 304 SS square plate with circular cutout under uniaxial tension at 400°C. $\sigma_{22}^{\infty} = 10$ ksi (1 ksi = 6.895 MPa, 1 in. = 2.54 cm).

Figure 4d. Comparison of BEM and FEM solutions for stress redistribution along BC in an annealed 304 SS square plate with circular cutout under uniaxial tension at 400°C. $\sigma_{22}^{\infty} = 10$ ksi (1 ksi = 6.895 MPa).