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TITLE: POINT VORTEX DYNAMICS: RECENT RESULTS AND OPEN PROBLEMS

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Point Vortex Dynamics: Recent Results and Open Problems

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Abstract

The concept of point vortex motion, a classical model in the theory of two-dimensional, incompressible fluid mechanics, was introduced by Helmholtz in 1858. Exploration of the solutions to these equations has made fitful progress since that time as the point vortex model has been brought to bear on various physical situations: atomic structure, large scale weather patterns, "vortex street" wakes, vortex lattices in superfluids and superconductors, etc. The point vortex equations also provide an interesting example of transition to chaotic behavior. We give a brief historical introduction to these topics and develop two of them in particular to the point of current understanding: (i) Steadily moving configurations of point vortices, and (ii) Collision dynamics of vortex pairs.

INTRODUCTION

In a seminal paper published in German in 1858, and in an English translation by P. G. Tait a few years later, Helmholtz set down the basic laws of *vortex dynamics*. One of these, the statement that vortex lines are material lines - or to quote Tait's translation "Each vortex-line remains continually composed of the same elements of fluid, and swims forward with them in the fluid" - in effect establishes vortices as the "particles" of fluid mechanics. This is nowhere more true than in two dimensions, where the vorticity of each fluid element is conserved in time, and where one can, thus, consider a model in which only a finite number of particles have any vorticity at all. Helmholtz, indeed, introduced just such a *point vortex model* in §5 of his paper. It provides a method by which the constraints of irrotational flow theory can be relaxed while not having to face up to the full complexity of viscous forces. As such it has enjoyed considerable development both purely theoretical - Batchelor (1967, Ch 7) uses the heading "Flow of effectively inviscid fluid with vorticity" for his discussion of this topic - and (later) as a basis for a variety of numerical procedures for flow simulation, now commonly referred to under the rubric *vortex methods*. For general reviews see, for example, Aref (1983, 1985) and Leonard (1980, 1985).

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The point vortex equations with which we are concerned here are most elegantly stated by considering the two-dimensional flow plane to be the complex z-plane and letting the vortices be represented by time dependent points $z_\alpha(t)$ in that plane; vortex $\alpha = 1, \ldots, N$ carries a constant circulation or strength Γ_α. The equations of motion are

$$\frac{dz_\alpha^*}{dt} = \frac{1}{2\pi i} \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta} \tag{1}$$

The prime on the summation sign denotes omission of the singular term $\beta=\alpha$; the asterisk denotes complex conjugation.

These equations continue to spawn intriguing new solutions, and thereby form a useful bridge between fluid mechanics and developments in other areas of physics and applied mathematics, notably the theory of dynamical systems. The exploration of such connections has often been stimulated by certain modelling situations. For example, before the advent of quantum mechanics W. Thomson, the later Lord Kelvin, advocated stationary vortex patterns as models of atoms and molecules. This led to detailed investigations by J. J. Thomson and others of the existence and stability of states of this type. Much later, laboratory observations of such vortex equilibria in superfluid ^4He (see Yarmchuk, Gordon and Packard 1979) led to further work along these lines (Campbell & Ziff 1978, 1979). We shall report on recent developments later in this paper. Similarly the observation of "vortex street" wakes stimulated research into the existence and stability of uniformly translating configurations of vortices starting with von Kármán's well known analysis in 1912 (see Lamb 1932, Ch 7) and continuing today.

Many current experimental results on vortex flows do not allow facile explanation in terms of point vortex dynamics since effects of three-dimensionality and/or viscosity are important. On the other hand, some of the results that we do have about the behavior of point vortices may provide inspiration for experimental studies. Certainly several of the most interesting are today without any clear laboratory realization. In this context the two-dimensional soap film experiments of Couder and collaborators (cf. Couder & Basdevant 1986) are worth mentioning.

This paper concentrates on two topics: First we discuss recent work on *steadily moving* (including completely stationary) *point vortex configurations*. We focus primarily on existence questions. Several open problems arise. Second, we report numerical results on the *collision of two vortex pairs*. We introduce the notion of *chaotic scattering*, or *chattering*, of one vortex pair off another, and discuss conjectures on this type of behavior. Again a number of unresolved questions arise.

We conclude by briefly mentioning two other areas where the concepts of

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dynamical systems theory may provide new approaches to point vortex motion and ultimately to more realistic vortex flow problems

VORTEX STATICS

The term is due to Lord Kelvin and embraces a wide variety of vortex motions in both two and three spatial dimensions where the configuration of vorticity is invariant in time and translates or rotates as a rigid body. Such configurations are of considerable interest since, on one hand, one can hope to determine them analytically, and, on the other, one can hope that being steady states they will be seen empirically. Stable configurations are particularly desirable, of course, but even unstable configurations are of interest. An unstable configuration may show up as a long-lived transient solution in a dynamically evolving system. The vortex sheet is a highly unstable configuration, yet it is difficult to think of one more frequently studied!

Even for point vortices the problem of determining all such patterns is far from simple and still largely open. Computer experiments have revealed many particular solutions, but there are significant gaps in our analytical understanding. Let us first recount some of the solutions that are available.

For identical vortices Kelvin, J.J.Thomson, Havelock and others studied simple, uniformly rotating configurations such as the open and face-centered vortex polygons. Extensive stability results are available for these states (Morikawa & Swenson 1971). More recently Campbell & Ziff (1978, 1979) have produced a catalog of the stable configurations for N identical vortices, where $1 \le N \le 30$. As N increases, the number of stable configurations at each N increases rapidly, but in the limit of an infinite vortex lattice only one stable configuration is known, namely the triangular lattice. Of considerable interest in condensed matter physics is

Open problem #1: Are there other stable lattices of point vortices than the triangular one? The long-range nature of the vortex-vortex interaction makes the question non-trivial.

It is not difficult to show directly from the point vortex equations (1) that for a configuration of n vortices of strength +1 and m vortices of strength -1 to form a completely stationary configuration, commonly *referred to as a vortex equilibrium*, n and m must be a successive *triangular numbers*, i.e. $n=j(j-1)/2$, $m=j(j+1)/2$ for some integer j.

Proof: Assume all left hand sides in (1) vanish. Multiply by $\Gamma_\alpha z_\alpha$ and sum on α. By expanding $\Gamma_\alpha z_\alpha$ as $\Gamma_\alpha (z_\alpha - z_\beta + z_\beta)$ and interchanging indices in the resulting sum we see that a stationary vortex configuration implies the constraint

$$\sum_{1 \le \alpha < \beta \le N} \Gamma_\alpha \Gamma_\beta = 0$$

or

$$\left(\sum_{\alpha=1}^{N} \Gamma_\alpha\right)^2 = \sum_{\alpha=1}^{N} \Gamma_\alpha^2$$

Now using that each Γ_α is either +1 or -1 we get $(m-n)^2 = m+n$, and setting $m-n=j$ gives the result stated ◊

The smallest such configuration (j=1) is a single point vortex. The next (j=2) is one vortex of one sign surrounded by an equilateral triangle of opposite signed vortices (Fig 1).

For larger j the determination and characterization of such states is not so simple. Tkachenko (1964) suggested introducing "generating functions" for individual species of point vortices. This is again done by considering the flow plane to be the complex z-plane, and defining polynomials P(z) and Q(z) of degree m and n, respectively, such that the roots of P (Q) define the positions of the vortices of strength -1 (+1). Tkachenko showed that P and Q then must satisfy the differential equation

$$PQ'' + QP'' = 2P'Q' \qquad (2)$$

where the primes indicate derivatives with respect to z. For a direct derivation of (1) from the point vortex equations see Aref (1986). Tkachenko solved (1) analytically for j=1,2,3,4. Beyond that direct analytical solution appears to become quite involved. Campbell & Kadtke (1987, see also Kadtke & Campbell 1987) solved (2) by computer algebra for larger values of j and also generalized the method to arbitrary species of singularities moving according to a velocity field that can be expressed as a polynomial in z.

Open problem #2: Is there a corresponding "generating function" formalism for the steadily rotating configurations of identical point vortices? (For collinear configurations the generating functions are Hermite polynomials.)

Remarkably a systematic analytical procedure exists for solving Eq (2) (Adler & Moser 1978, Bartman 1983): Consider the recursion-differential equations

$$P_{n-1}P'_{n+1} - P'_{n-1}P_{n+1} = P_n^2 \qquad (3)$$

(The original definition has a prefactor 2n+1 on the right hand side.) We may show that if (3) is satisfied for n=m and n=m+1 and

$$P''_n P_{n+1} + P_n P''_{n+1} - 2P'_n P'_{n+1} = 0 \qquad (4)$$

is satisfied for n=m, then (4) is also true for n=m+1. Thus, solutions of (2), $P=P_n$, $Q=P_{n+1}$

may be constructed inductively using (3)

Proof. The induction can be set up after noting the following identity

$$P_j P''_{j+1} P_{j+2} + P_{j+1} P''_{j+2} - 2P'_{j+1} P'_{j+2} =$$

$$P_{j+1} (P_j P''_{j+2} + P''_j P_{j+2} - 2P'_{j+1} P'_{j+1}) +$$

$$P_{j+2} (P''_j P_{j+1} + P_j P''_{j+1} - 2P'_j P'_{j+1})$$

$$z P_{j+1} (P_j P_{j+2} - P_j P_{j+2} - P^2_{j+1})$$

where the first term on the right hand side is

$$P_{j+1} (P_j P_{j+2} - P_j P_{j+2} - P^2_{j+1})$$

The induction is now clear □

Hence, if (3) is started off with the polynomials $P_0=1$ and $P_1=z$, the solutions can be found recursively by solving a *first order* ODE (i.e. Eq (3)) at each step. In this way solutions for large values of j can be generated. It may also be seen that there will be one arbitrary constant for each integration. Hence, for a given j the pair satisfying (2) will in general yield a *one-parameter family of solutions*. We are glossing over several nontrivial details here including the result that (3) can in fact be solved in terms of polynomials. Indeed, explicit formulae for the polynomials in terms of Wronskian determinants have been obtained (see Adler & Moser 1978). In Fig.1(b-d) we show examples of such stationary states for larger j. Note that although each vortex is completely stationary, *the configuration as a whole has a net circulation* so at large distances the flow field becomes that of a single vortex. Physically this is somewhat counterintuitive.

Open problem #3. Is there an analytic treatment possible of the *stability* of these equilibria? Numerical stability results are reported by Kadtke & Campbell 1987)

Open problem #4: The vortex circulation constraint governing these equilibria is the same as that necessary for *self similar collapse* and expansion motions (Aref 1979, Novikov & Sedov 1979, see also Aref 1983). Is there a generalization of the above to accomodate such time-dependent evolution?

A most interesting extension of Tkachenko's equation (2) noted by Bartman (1983) and Campbell & Kadtke (1987) is to the case of *uniformly translating* point vortex configurations. The case of a single pair, one positive and one negative vortex of the same absolute strength, was discussed already by Helmholtz (1858) as a simple two dimensional analog of the vortex ring. One can naturally inquire into generalizations of this situation. Are there similar states with two vortices of either sign? With three, etc.? It is clear on general grounds that there must be an equal number of +1 and -1 vortices for such a configuration to translate uniformly. It is easy to show directly from the point vortex equations (1) that for any configuration of point vortices to translate as a rigid body the total circulation must vanish.

Proof. Assume that all the left hand sides in (1) are equal to the same non zero complex number v^*. Multiply by Γ_α and sum on α. The double sum on the right hand side vanishes by antisymmetry in the indices α, β. We are left with the result that

$$v \sum_{\alpha=1}^{N} \Gamma_\alpha = 0$$

Thus, the sum of the vortex circulations must vanish for a rigidly translating configuration □

However, it is apparently difficult to extract the number of vortices of either sign, or, indeed, to see that there should be any restriction on this number.

For the case of translating vortices we can again introduce polynomial generating functions for the two species, P and Q, that are now of equal degree n. They satisfy a modified form of (2), viz

$$PQ'' + QP'' + 2\lambda(PQ' - QP') = 2P'Q' \qquad (5)$$

where λ is related to the velocity of translation of the configuration and the common magnitude of the circulation, Γ, of the vortices: $\lambda = 2\pi i v^*/\Gamma$. The polynomial solutions to this equation again involve the Adler-Moser polynomials (Bartman 1983).

A remarkable consequence of this observation is that the common degree n, i.e. the number of vortices of either species, must again be a triangular number, $n=j(j+1)/2$ for some j=1,2,3. Thus, try as we may, no uniformly translating states of a system of two +1 vortices and two -1 vortices can be found. (This particular result is, of course, not too difficult to verify directly.) However, for three vortices of either kind such states do exist. For four or five vortices of either species there are again no translating states. For six there are, etc. In fact, for each allowed number a one-parameter family has been found (see Fig 2). For large values of this parameter a configuration becomes a system of well separated pairs of approximately the same size and orientation. Nevertheless, the results just stated show that one cannot construct a uniformly translating state from an arbitrary number of such pairs, even if they are spaced very far apart. This remarkable *selection property* for point vortex states suggests that similar results must hold for the full two-dimensional Euler equations, and in this way points to an unexpected aspect of the dynamics of uniformly propagating vortex configurations.

Open problem #5 Is there an analytic treatment possible of the stability of these translating states? Numerical stability results are reported by Kadtke & Campbell 1987.

Open problem #6 Are there corresponding selection rules for periodic configurations of vortices in uniform translation? For example, are there vortex street patterns with a more complicated "basic cell" than the familiar von Karman street pattern?

VORTEX CHATTERING

The term *chattering* arises as a concatenation of chaotic and scattering. It refers to a notion that has arisen in several contexts. In the tracing of a ray of light upon successive reflections in mirrored surfaces, for example, one is really simulating the trajectory of an elastically deflected particle, a billiards problem with an open phase space "where rays come in from and can run off to infinity." The open phase space renders conventional concepts such as the Poincaré section less useful. "Almost all" rays will have only a finite number of intersection points with such a section. The ray tracing problem is a useful paradigm for the problem of chattering (Hénon 1988, Eckhardt 1987a).

Our topic in this section is how chaotic scattering applies to vortex interactions, in particular the collision of vortex pairs. Insight into this question may be obtained by considering Fig 3. We are looking at data from a series of scattering experiments. Two pairs consisting of vortices of strength ±1 and ±0.9 have been started on a collision course from a considerable separation. A parametrization of such initial states can be found such that only the relative angular momentum is changed in going from one to the next. The other integrals of the motion are unchanged. The relative angular momentum in turn is simply related to the offset of the direction of propagation of one pair relative to the other, i.e. this quantity is an *impact parameter* for the scattering process.

In Fig 3 this impact parameter is plotted along the abscissa. The ordinate describes a diagnostic quantity pertaining to the scattering event. Along it we measure the time from initiation of the motion until the separation of the two pairs again exceeds their initial separation. The absolute value of this quantity is clearly of limited interest, but the relative changes in it show up a profound structure. In Fig 3(a) we have a rather global view with the impact parameter ranging over a wide interval. The scattering time is a smooth function over most of this range but there is a spiky region. In Fig 3(b) we have expanded the impact parameter scale by a factor of 10 to capture this spiky structure. As we proceed from panel to panel in Fig 3 the abscissa is expanded by a factor of 10 zooming in on a small structure at the lower end of the spiky region. Note that the total impact parameter range in any given panel of Fig 3 is shown by a horizontal bar in the previous panel. We see that at each level of magnification the *scattering time* is flat over large intervals in scattering parameter. These flat portions or "plateaus" are interrupted by spikes, where the scattering time appears to change by some finite amount. The spiky structure of the scattering time graph apparently continues on ever finer scales of impact parameter. Note that peaks often appear taller at increased resolution. This is because the peak is in principle infinitely tall and improved resolution of the impact parameter allows us to approximate its exact location ever more closely.

Figure 4 shows a similar set of magnifications but at the upper edge of the spiky region. The scale along the abscissa in Fig 4(a) corresponds to that of Fig 3(c). The qualitative impression is the same. However, the peak scattering times that one is able to "hit" at a given resolution are substantially longer.

Figure 5 begins to explain in terms of real space trajectories of the vortices how the structure in Figs 3 and 4 appears. When two vortex pairs $\pm\Gamma$ and $\pm\Gamma'$ impinge on one another at close to head-on conditions they usually first suffer an *exchange scattering* in which non-neutral pairs $(+\Gamma,-\Gamma')$ and $(-\Gamma,+\Gamma')$ are formed. Since those pairs have a net circulation they propagate along (approximately) circular trajectories. In general, these circles intersect again, and at this point one of two things may happen: (E) the vortices can exchange back to their original partners, with which they then fly off to infinity, or (D) a *direct scattering* can take place in which the non-neutral pairs manage to negotiate one another but stay together. This argument can now be repeated until option (E) is realized (if ever). At each step in such a hierarchy there is a certain fraction of the direct scattering events that will lead to one more such event; the rest will be absolved by an exchange back to the original pairs. One would expect this fraction, f, to become independent of the level in the hierarchy so that the measure of incoming states, I_n at level n in this hierarchy that leads to further direct scattering satisfies

$$I_{n+1} = f I_n$$

This implies a structure of an n versus I graph of the type familiar in the Cantor set. The difference in scale along the ordinate of Figs 3 and 4, however, suggests that f is not independent of where we are along the impact parameter axis.

There are several additional details of this scenario, in large part envisioned already by Manakov & Shchur (1983), that we have explored. A comprehensive account appears in the recent paper by Eckhardt & Aref (1987); see also Eckhardt (1987b). Trajectory plots bearing out the analytical picture above are shown in these papers. The spiky nature of the graphs in Figs.3 and 4 is due to the availability for different values of the impact parameter of sequences ED^kE with different integers k, and the observation from the trajectories that each additional factor D adds essentially a fixed time increment to the scattering time. Thus, the plateaus in Figs 3 and 4 correlate with the exponent k in the sequence ED^kE.

In Fig 5(a-c) we show three sample trajectories with large values of k. Examples with k=1,2,3,4 may be found in Eckhardt & Aref (1987). The D^k portion of these trajectories is made up of sequences of three typical interaction patterns shown in Fig 5(d-f). The pattern in Fig.5(d) is particularly conspicuous. It occurs once in Fig.5(a), twice in Fig 5(b). We tend to think of the interaction in Fig.5(d) as being "stronger" than that in 5(e) which in turn is "stronger" than the one shown in 5(f).

One would like to rest assured that this complex behavior is reliable as far as numerical accuracy is concerned. In the simulations leading to the scattering data and trajectory plots shown here the four-vortex Hamiltonian is typically conserved to about one part in 10^5. In order to verify that this is sufficiently accurate we have systematically degraded the calculation. (We have also increased the accuracy of our time integration.) The objective is on one hand to have a calculation that is sufficiently accurate for the complex behavior seen to be trustworthy. On the other hand, too high accuracy makes the computations very costly. The results shown here are all from fully converged calculations in the sense that any increase of numerical accuracy does not change them. The main signature of a low accuracy calculation is that the plateaus in the scattering time plot are not flat. The peaks also shift somewhat but are retained even with relative accuracies in the Hamiltonian as low as 10%.

Open problem #7 Numerical experiments indicate that the scattering problem for two ±1 pairs is not chaotic (Eckhardt & Aref 1987). Elucidate.

Open problem #8 For the problem of two ±1 vortex pairs there are no steadily translating states. For the problem of *three* ±1 vortex pairs, however, such states do exist. Is this difference important in the context of vortex pair scattering? (Unstable steady states can lead to asymptotic "trapping" in a scattering process, i.e. play the role of a never-ending sequence EDDD ...)

CONCLUDING REMARKS: OTHER PROBLEMS

The two problem areas on which we have focussed above have little in common (Open problem #8 being a possible exception). Yet it is remarkable that in both of them a distinct and unexpected *discretized aspect* of the results emerges: In the problem of vortex statics we found unanticipated *selection rules* for configurations with triangular numbers. In the vortex scattering problem we discovered a remarkable *quantization* of levels in the scattering time. It is an intriguing question to what extent such results carry over to the dynamics of smooth vorticity distributions.

A number of problems of considerable practical significance seem to involve flows that are dominated by a small number of intense vortices. It is inevitable that in some of these cases effects discernible in the dynamics of a few point vortices will have close counterparts in a more realistic setting. Flow-structure interaction problems, in particular pressure fluctuations associated with impingement, come to mind. A related area in which chaotic motion would appear to have important consequences is in the radiation of sound from interacting vortices. The concept of *vortex sound* has a long and distinguished history (Powell, 1964). Regular vortex motion leads to vortex sound spectra consisting of discrete peaks. Chaotic vortex motion, by contrast, should lead to broad band spectra, that can appropriately be characterized as *vortex noise*.

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REFERENCES

Adler, M. & Moser, J 1978 On a class of polynomials connected with the Korteweg-de Vries equation. Commun. Math. Phys.61, 1-30.

Aref, H. 1979 Motion of three vortices. Phys. Fluids 22, 393-400.

Aref, H. 1983 Integrable, chaotic, and turbulent vortex motion in two-dimensional flows. Ann. Rev. Fluid Mech 15, 345-389.

Aref, H. 1985 Chaos in the dynamics of a few vortices - fundamentals and applications. In *Theoretical and Applied Mechancis,* F.I.Niordson and N.Olhoff eds., Elsevier, pp 43-68.

Aref, H. 1986 The numerical experiment in fluid mechanics. J. Fluid Mech.173, 15-41.

Bartman, A B. 1983 A new interpretation of the Adler-Moser KdV polynomials: Interaction of vortices. In (Proc. 2'nd Intern. Workshop on) *Nonlinear and Turbulent Processes in Physics,* R Z Sagdeev, ed. Harwood Academic Publ. pp. 1175-1181.

Batchelor, G K. 1967 *An Introduction to Fluid Dynamics.* Cambr. Univ. Press. 615pp.

Campbell, L J & Ziff, R M. 1978 A catalog of two-dimensional vortex patterns. Los Alamos Sci. Lab. Rep. No. LA-7384-MS. 40pp.

Campbell, L J & Ziff, R M. 1979 Vortex patterns and energies in a rotating superfluid. Phys. Rev. B 20, 1886-1902.

Campbell, L.J. & Kadtke, J.B. 1987 Stationary configurations of point vortices and other logarithmic objects in two dimensions. Phys. Rev. Lett 58, 670-673.

Couder, Y. & Basdevant, C. 1986 Experimental and numerical study of vortex couples in two-dimensional flows. J. Fluid Mech.173, 225-251.

Eckhardt, B. 1987a Fractal properties of scattering singularities. J. Phys. A (In Press).

Eckhardt, B. 1987b Irregular scattering of vortex pairs. Europhys. Lett. (In Press).

Eckhardt, B. & Aref, H. 1987 Integrable and chaotic motions of four vortices II: Collision dynamics of vortex pairs. Submitted to Proc. Roy. Soc. (London) A.

Helmholtz, H. von 1858 On integrals of the hydrodynamical equations which express vortex-motion. Transl. P.G.Tait, 1867, in Phil. Mag. (4) 33, 485-512.

Hénon, M. 1986 The inclined billiard. Observatoire de Nice preprint.

Kadtke, J. B. & Campbell, L. J. 1987 A method for finding stationary states of point vortices. Phys. Rev. A (in Press).

Lamb, H. 1932 *Hydrodynamics*. Republ. Dover Publ. 738pp. 6th ed.

Leonard, A. 1980 *Vortex methods for flow simulation*. J. Comput. Phys. 37, 289-335.

Leonard, A. 1985 Computing three-dimensional incompressible flows with vortex elements. Ann. Rev. Fluid Mech. 17, 523-559.

Manakov, S. & Shchur, L. N. 1983 Stochastic aspect of two-particle scattering. JETP Lett. 37, 54-57.

Morikawa, G. K. & Swenson, E. W. 1971 Interacting motion of rectilinear geostrophic vortices. Phys. Fluids 14, 1058-1073.

Novikov, E. A. & Sedov, Yu. B. 1979 Vortex collapse. Sov. Phys. JETP 50, 297-301.

Powell, A. 1964 Theory of vortex sound. J. Acoust. Soc. Amer. 36, 177-195.

Tkachenko, V. K. 1964 *Dissertation. Inst. Phys. Probl.*, Moscow, USSR.

Yarmchuk, E. J., Gordon, M. J. V. & Packard, R. E. Observation of stationary arrays in rotating superfluid Helium. Phys. Rev. Lett. 43, 214-217. Also Phys. Today 32, 21.

FIGURE CAPTIONS

Figure 1: Vortex equilibria, i.e. completely stationary configurations of point vortices. *(a)* A simple, well known example with one vortex of strength +1, three of strength -1. (c,d) Examples with 6 and 10 vortices of opposite sign. (b) A configuration with 10 and 15 vortices. For each pair of triangular numbers *there exists a one-parameter* family of equilibria. (After Campbell & Kadtke, 1987).

Figure 2: Steadily translating vortex configurations *made up of an equal number of* vortices of opposite strength. (a,b) Two *examples from a one-parameter family of solutions with three vortices of either sign.* (c,d) Examples with six vortices of either sign. The number of vortices of *either sign must be* triangular. (After Campbell & Kadtke 1987).

Figure 3: Scattering time versus impact parameter for the collision of a ± 1 *vortex pair* with a ± 0.9 pair. The scale along the abscissa is magnified by a factor of 10 from panel to panel. In panels (a)-(e) the entire impact parameter interval shown in the next panel is indicated by a bold horizontal bar.

Figure 4. Data similar to that in Fig.3 but for a region of the scattering range close to the "upper edge" of the chattering region shown in Figs.3(a,b). The magnification of the abscissa in panel (a) corresponds to Fig.3(c) and *increases by a factor* of 10 in the following panels.

Figure 5: (a-c) Two-pair collision sequences ED^kE with large k. The D^k portion of *these interaction patterns is made up of direct scattering processes such as* those shown in (d-f).

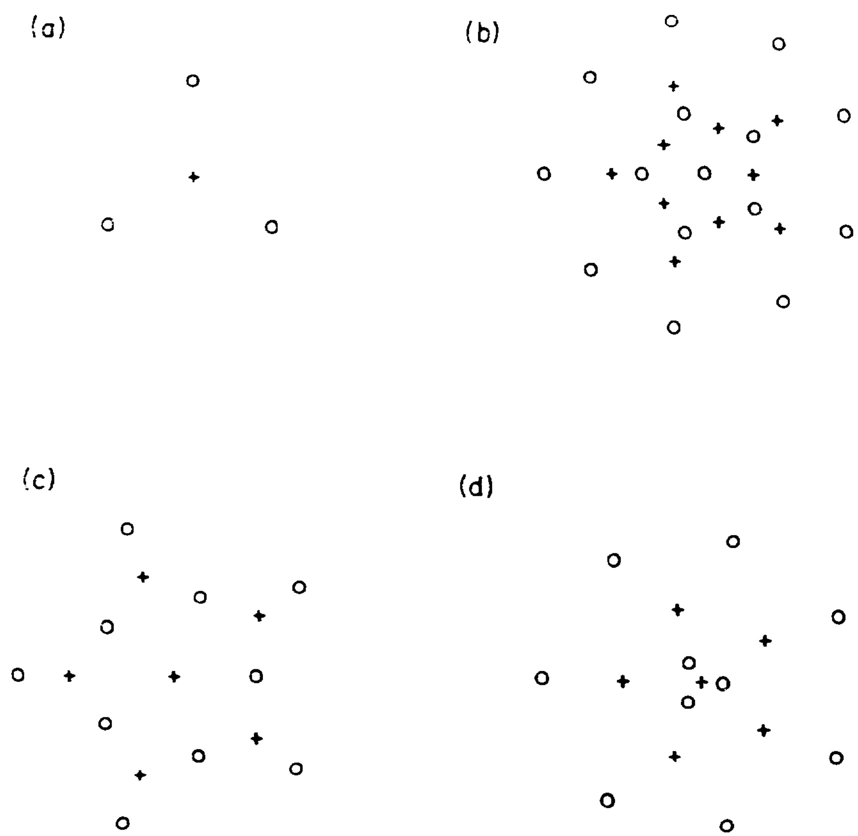


Fig. 1

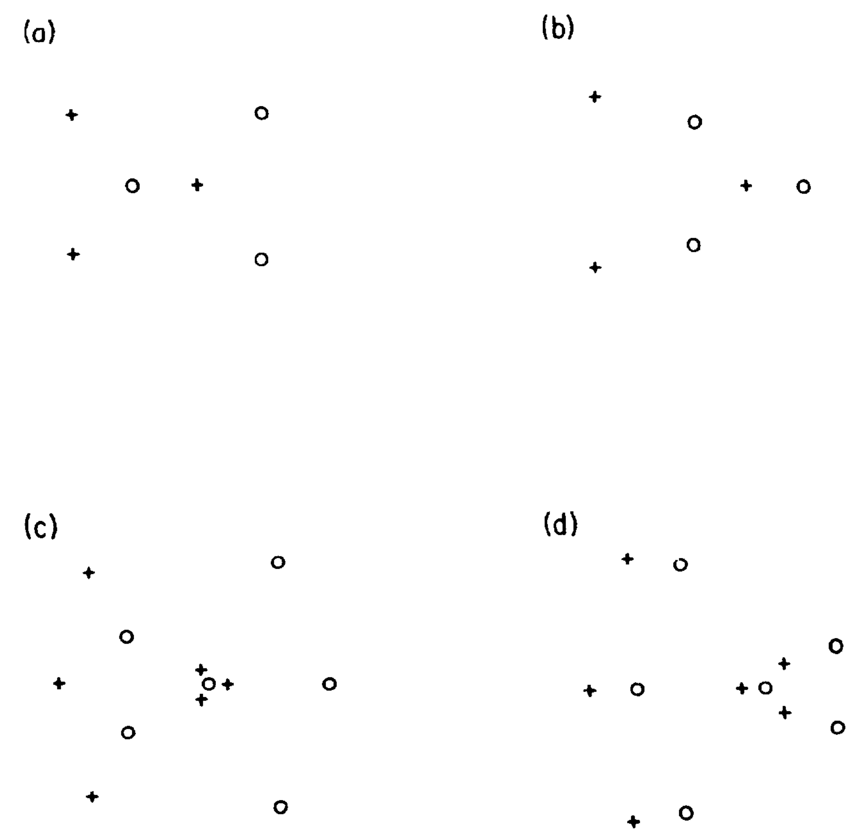


Fig. 2

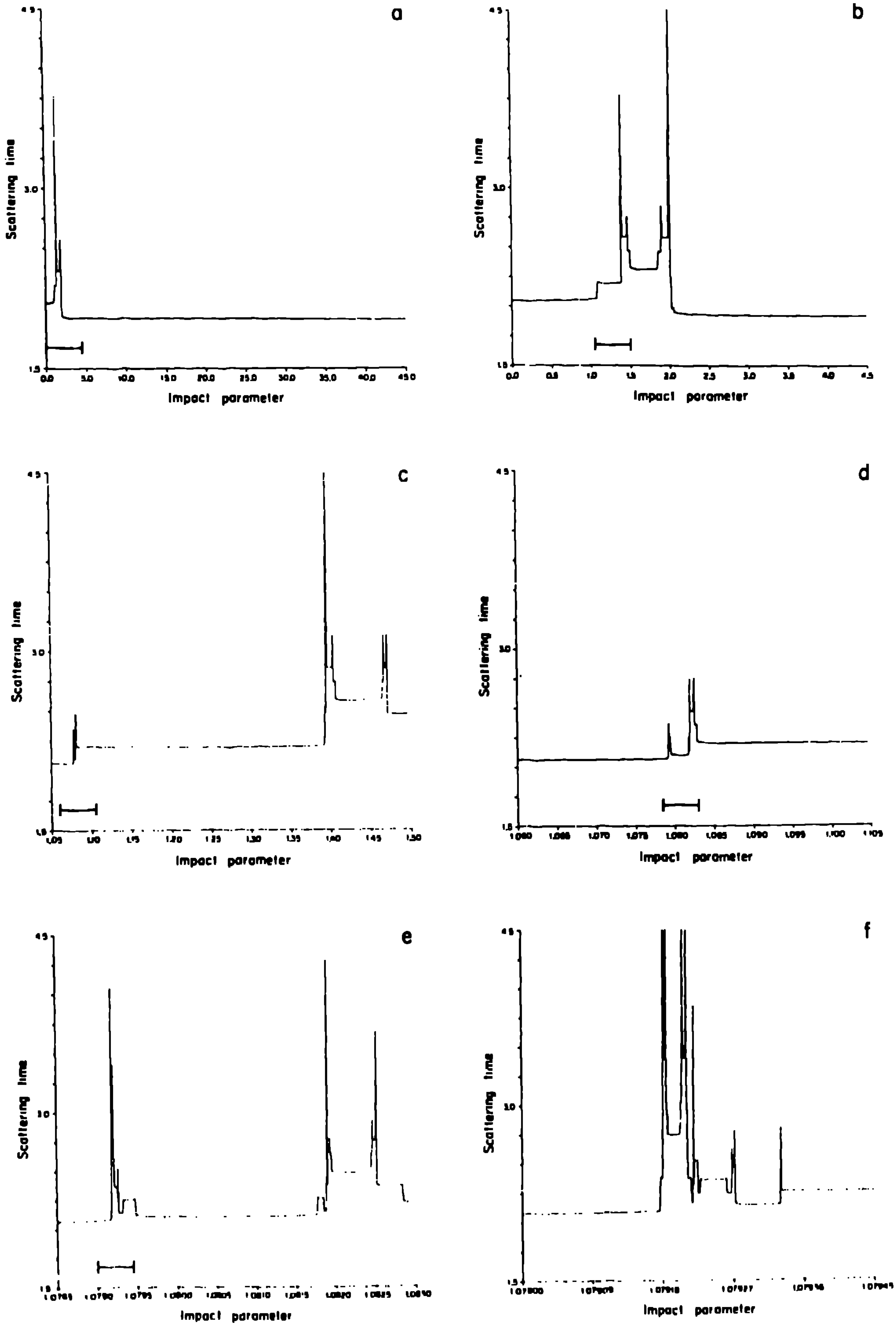
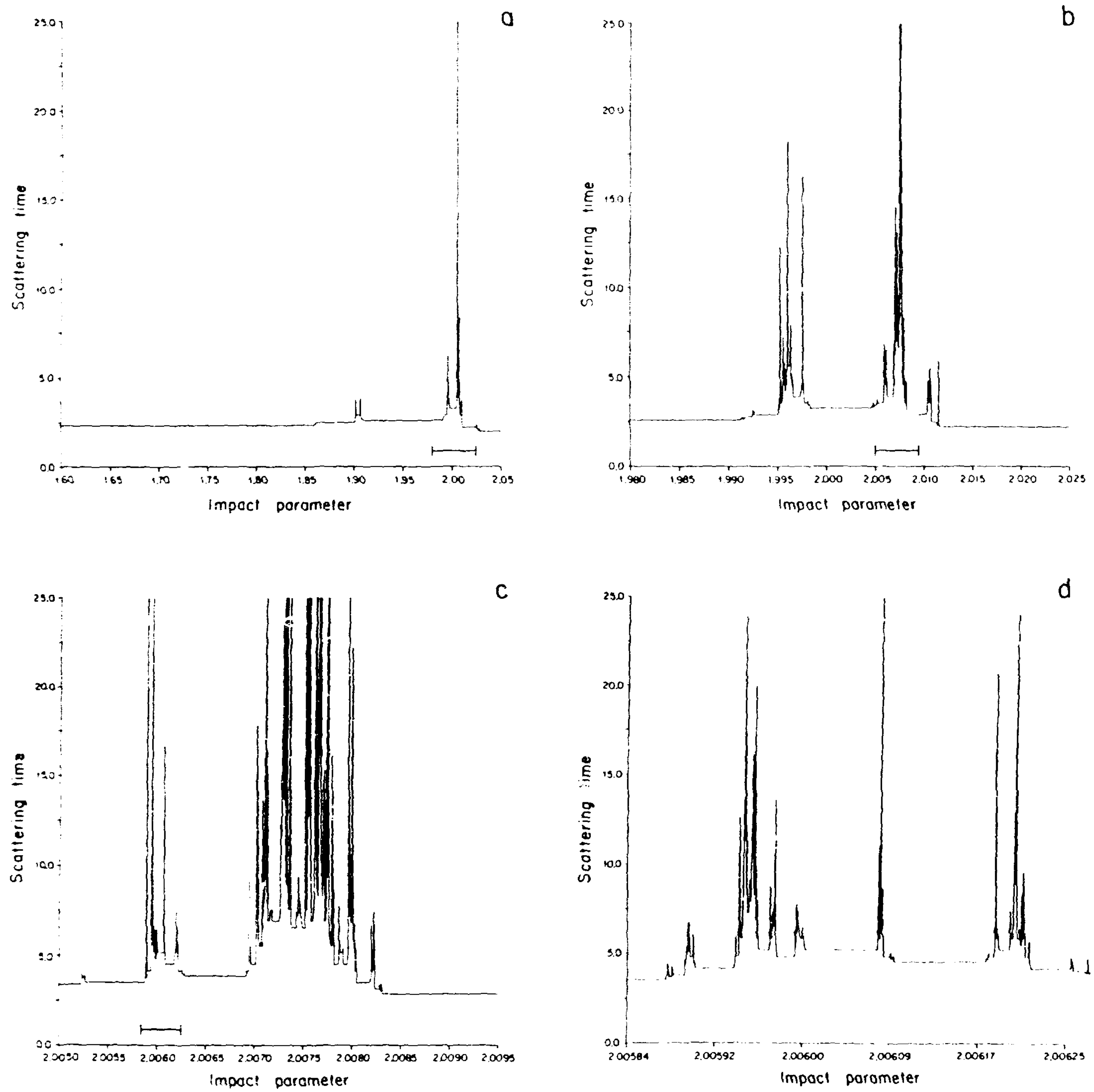

a
b
c
d
e
f
Scattering time
Impact parameter
4.5
3.0
1.5
0.0
5.0
10.0
15.0
20.0
25.0
30.0
35.0
40.0
45.0
0.5
1.0
2.0
2.5
3.5
4.0

Fig.3

Fig. 4



a
Scattering time
Impact parameter
b
Scattering time
Impact parameter
c
Scattering time
Impact parameter
d
Scattering time
Impact parameter

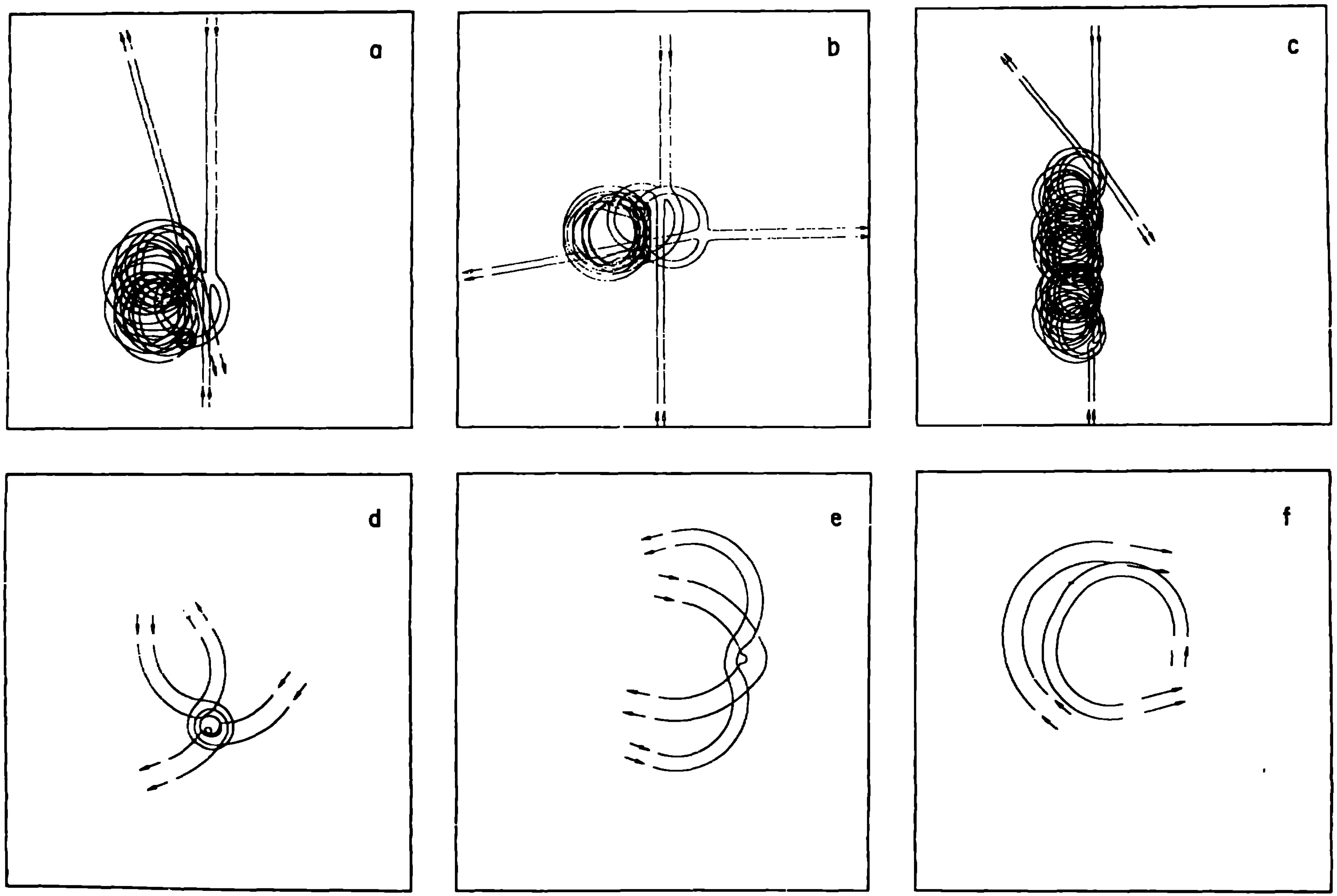


Fig 5