ASPECTS OF TOKAMAK TOROIDAL
MAGNET PROTECTION

by

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MASTER

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ABSTRACT

Simple but conservative geometric models are used to estimate the potential for damage to a Tokamak reactor inner wall and blanket due to a toroidal magnet field collapse. The only potential hazard found to exist is due to the MHD pressure rise in a lithium blanket. It was found that dividing the blanket into electrically insulated segments will reduce or eliminate this hazard because the radii and the conductivity of potential breakdown paths are reduced.

A survey is made of proposed protection methods for superconducting toroidal magnets. It is found that the two general classifications of protection methods are thermal and electrical. Computer programs were developed which allow the toroidal magnet set to be modeled as a set of circular filaments. A simple thermal model of the conductor was used which allows heat transfer to the magnet structure and which includes the effect of temperature dependent properties. To be effective in large magnets an electrical protection system should remove at least 50% of the stored energy in the protection circuit assuming that all of the superconductor in the circuit quenches when the circuit is activated. A protection system design procedure based on this criterion was developed.

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1. R. Sawdye, J. A. Sefcik and M. S. Kazimi, "Reliability Requirements for Admissible Radiological Hazards from Fusion Reactors," Trans. Am. Nucl. Soc., 27, 65-66, November 1977.
2. M. S. Kazimi et al., "Aspects of Environmental and Safety Analysis of Fusion Reactors," MITNE-212, Dept. Nucl. Eng., M.I.T., October 1977.
3. D. Dube, M. S. Kazimi and L. M. Lidsky, "Thermal Response of Fusion Reactor Containment to Lithium Fire," Paper presented at Third Top. Meeting on Fusion Reactor Technology, Santa Fe, N.M., May 1978.
4. R. W. Sawdye and M. S. Kazimi, "Application of Probabilistic Consequence Analysis to the Assessment of Potential Radiological Hazards of Fusion Reactors," MITNE-220, Dept. Nucl. Eng., M.I.T., July 1978.
5. D. A. Dube and M. S. Kazimi, "Analysis of Design Strategies for Mitigating the Consequences of Lithium Fire within Containment of Controlled Thermonuclear Reactors," MITNE-219, Dept. Nucl. Eng., M.I.T., July 1978.
6. R. W. Green and M. S. Kazimi, "Aspects of Tokamak Toroidal Magnet Protection," PFC/TR-79-6, Plasma Fusion Center, July 1979.

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NOMENCLATURE

A	vector potential (Chapter 2) cross-sectional area (1,3,4)	cm^2
B	magnetic flux density; magnetic field strength	Tesla
c	specific heat	$\text{J/g-}^\circ\text{K}$
C	rotational transform matrix	
D	electric displacement	coulomb/m^2
E	electric field intensity (2) energy (4)	volt/m Joule
E_D	total dissipated energy (4)	Joule
f	nondimensional structure fraction = A_s/A_c	
F	force	Newtons
g	element of the force matrix	N/amp^2
G	force matrix	
h	heat transmittance	$\text{W/cm}^2\text{-}^\circ\text{K}$
H	magnetic field intensity	amps/m
I	current	amps or amps/cm
ΔI	overcurrent	amps
I_{dn}	identity matrix	
J	current density (1,2,3) cost function (4)	amps/cm^2
J_c	critical current density	amps/cm^2
k	inductive coupling coefficient thermal conductivity	$\text{W/cm-}^\circ\text{K}$
l_p	length of perimeter between the conductor and the winding structure	cm
L	length of quenched conductor	cm
M	inductance	Henries or H/cm

M_{eq}	equivalent inductance based on the circuit's total dissipated energy	Henries
n	no. of toroidal coils	
N	no. of turns per coil	
N_l	no. of turns per unit length	
p	pressure	N/m^2 or psi
P	power (2,3,4) cooled perimeter per unit length (1)	Watts cm
Q	maximum allowable heat flux	
r	radius or radial direction	cm or m
R	resistance	ohm or ohm/cm
R_o	radius of breakdown path	m
t	time	sec.
t_d	discharge time	sec.
T	temperature	$^{\circ}K$
T_y	division point between low temperature and high temperature expressions for resistivity and specific heat	$^{\circ}K$
V	voltage	volts or volts/cm
W	energy (2,3)	Joules or Joules/cm
x, y, z	cartisian coordinates	
α	decay ratio = $\gamma \mu_o \sigma R_o^2$	
γ	decay constant	1/sec.
ϵ	electromotive force	volt
ξ	angle between two adjacent toroid coils	radians
θ	temperature difference (1) angle (2,3,4)	$^{\circ}K$ radians
λ	eigenvalue	
μ_o	magnetic permeability	$4\pi \times 10^{-7}$ H/m

ρ_m	mass density	g/cm^3
ρ_{charge}	charge density	coulombs/m^3
ρ_R	resistivity	
ρ_{R_1}	external dump resistivity = $R_1 A_c / L_c$	
τ	nondimensional time $t / \mu_0 \sigma R_0^2$	
σ	conductivity	mho/m
Φ	magnetic flux	Weber

Subscripts

l	faulted coil (4)
l,2	primary, secondary (1)
o,f	initial state, final state
c	conductor (4) coil (1)
d	discharge
D	dissipated
s	shell (3) structure (4)
m	magnet
i,j,k	summation indices

Superscripts

"	quantity measured on a per unit of conductor volume basis
i	force acting on the i^{th} coil (2)

Bar Notation

$\overline{x}$	Geometric Vector
$\hat{x}$	Geometric Unit Vector
$\underline{x}$	Matrix Vector
$\underline{\underline{x}}$	Matrix

Notes:

- 1) Mks units were used for induction calculations and cgs units were used for heat transfer and thermodynamic calculations.
- 2) A number of different coordinate systems were used. These and the dimensions that go with them are defined in figures where used.

I. INTRODUCTION

Thermonuclear fusion holds the promise of producing large amounts of useful energy from abundant sources: water and lithium. Fusion research for the past 25 years has concentrated on understanding the physics of fusion plasmas. Plasma containment times and temperatures have been improved to the point that workable fusion reactors might be designed by extrapolation of present experimental devices. One very successful device to date has been the Tokamak, a Russian invention, which contains the plasma in a toroidal shaped magnetic "bottle".

This bottle is formed by the superposition of three magnetic fields, toroidal, poloidal, and vertical. The toroidal field is formed by current carried by windings on a toroidal surface. The magnetic field in the toroid varies inversely with the distance from the toroid center and vanishes outside the toroid. In most devices the windings are collected into separate coils which are then placed so as to form a segmented toroid (see Figure 1.1). The poloidal field is produced by a current flowing through the plasma itself. This current is induced by transformer coils located in the center of the toroid. The net effect is a twisting of the toroidal field lines which stabilizes the plasma (see Figure 1.2). The vertical field is provided by coils above and below the toroidal field coils which further stabilize and position the plasma. The toroidal field is by far the largest of the three and the

Figure 1.1: Segmented Toroid for a Tokamak

Toroidal Field

Poloidal Field

Toroidal Field and
Poloidal Field Combined

Figure 1.2 Tokamak Magnetic Field Topology

toroidal field coils and their support are the largest structures in a Tokamak.

A number of design studies of possible Tokamak type fusion experimental power reactors (EPR) have been published recently by Princeton Plasma Physics Laboratory [1] (hereafter designated PPPL), the University of Wisconsin (UWMAK I and UWMAK III), Argonne National Laboratory [4] (ANL-EPR) and Oak Ridge National Laboratory [5] (ORNL-EPR). These studies are intended to identify and stimulate research into the engineering difficulties of designing fusion reactors. Among the fundamental requirements of many Tokamak EPRs are the large toroidal field coils operating at as large a field as possible and having a long duty cycle (on the order of minutes to a few hours). Water or liquid nitrogen cooled pulsed copper magnets such as those used in present experimental devices, are unworkable in an EPR sized device because of the enormous amount of power they would consume. The toroidal field coils must be constructed using superconducting conductors (either NbTi or Nb₃Sn). The poloidal and vertical field coils will also probably be built using superconductors.

In the remainder of this chapter, a survey of the necessity of stabilizing and protecting large superconducting magnets will be presented. A review of theory useful in the analysis of inductive circuits and linear magnetic devices is presented in Chapter 2. In Chapter 3 the results of exploring the possibility of interactions between the toroidal field and the reactor structure during a rapid toroidal field shutdown

are outlined. A detailed comparison of thermal and electrical protection methods is made in Chapter 4.

1.1 Superconductor Stabilization

The difficulty incurred when using superconductors is that they must be maintained at liquid helium temperatures (typically 4.2 °K). If for some reason the conductor warms to above its critical temperature, it spontaneously returns to the normal conducting state. A voltage develops across the normal section of the wire and energy is dissipated in the wire due to Joule heating. If no mechanism exists for rapid heat transfer away from the wire, then it will heat up rapidly and the normal region will tend to propagate along the wire. All of the energy stored in the magnetic field can be dissipated in the coils windings via this mechanism. This process is commonly referred to as a magnet "quench" and it illustrates the instability of superconducting wires near their critical temperatures. In small laboratory scale coils, quenches introduce inconvenience but not danger because the velocity of the normal front propagation is large compared to the size of the coil. This large propagation velocity is important because it causes the coil to warm up uniformly reducing thermal strains and resistive voltages across the coil. The heat input to initiate a quench may be a flux jump as magnetic flux moves into a conductor, a wire movement caused by the Lorentz force, or by a heat leak into the coil's cryostat.

The transition temperature is a function of conductor current density and the orientation and magnitude of the magnetic field that surrounds the conductor. This behavior is illustrated in Figure 1.3 [6]. Unfortunately, an increase in temperature, current, or magnetic field which does not exceed the constraining parameters may still cause a quench since for common superconducting materials the magnetic field diffusion rate is much larger than the thermal diffusion rate. In an ideal superconductor the magnetic field cannot enter the superconductor because surface currents are induced in the superconductor which create a field that exactly cancels the imposed field. In real superconductors, however, the limit on the current density (i.e., the critical current density J_c) limits the magnitude of the magnetic field which may be cancelled. For most superconductors this saturation field is less than 0.2 tesla. For imposed fields greater than the saturation field, the cancellation currents are forced to move into the conductor allowing the magnetic field lines to penetrate the superconductor. This flux motion is not a continuous one but takes place in small random increments. These increments are called flux jumps.

Because flux jumps release energy within the conductor, they form a mechanism for quenching even though the conductor remains below its critical current density. Assume a small flux perturbation releases energy in the conductor and causes a temperature rise ΔT_1 . This will cause a decrease in the

Figure 1.3 Temperature Dependence of Critical Current Density versus Applied Transverse Magnetic Field for
a) $\text{Nb}_{(x)}\text{Ti}$ b) Nb_3Sn

critical current,

$$\Delta J_c = \frac{dJ_c}{dT} \Delta T_1 \quad (1.1)$$

This in turn will allow further flux penetration releasing more energy and causing a further temperature rise ΔT_2 . If ΔT_2 is less than ΔT_1 , then the temperature perturbation will damp out and the superconducting state will continue as long as the final temperature reached is less than the transition temperature. This is called an incomplete fluxjump. If ΔT_2 exceeds ΔT_1 , or if the final temperature is greater than the transition temperature, then a complete or catastrophic fluxjump is said to occur. A conductor is said to be adiabatically stable if ΔT_2 is always less than ΔT_1 . This is also sometimes called intrinsic stabilization and it requires that the enthalpy of the superconductor be sufficient to absorb the fluxjump energy without quenching. As a general rule, superconductors should be made of filaments as small as possible in order to minimize the energy released in a fluxjump [6].

Whether a catastrophic fluxjump becomes a quench depends on the rate of heat transfer out of the conductor compared to the rate of energy dissipation in the conductor. The fluxjump becomes unstable and starts a quench if the energy production rate is greater than the heat transfer rate. Stabilization techniques are intended to allow fluxjumps

without quenching. For dynamic stabilization, the superconductor filament is covered with a small layer of normal metals. This added normal metal slows the rate of magnetic diffusion and increases the thermal diffusion rate, thus reducing the potential for catastrophic fluxjumps. Enthalpy stabilization is a more general case of intrinsic stabilization. In this case, materials with large specific heats such as lead or helium gas are placed in close thermal contact with the superconductor in order to absorb the fluxjump energy. None of the above methods is capable of preventing quenching due to wire movements, which release friction energy, or due to magnetic field or current pulses. As a result, coils designed using these stabilization techniques often exhibit a "training" phenomenon in which several quenches are required before stable operation at the design current is achieved. [6].

To prevent "training", large magnets are designed to be cryostable. This requires the superconductor to be placed in a large normal metal matrix. Ratios of superconductor mass to metal typically range from 1:10 to 1:100 and the normal metal is usually either copper or aluminum. Sufficient cooling is provided to prevent a temperature take-off even if the entire conductor current is carried in the normal metal. This requires that,

$$PQ \geq I^2 \rho_R / A \quad (1.2)$$

where P = cooled perimeter per unit length
 ρ_R = normal metal resistivity
 A = cross-sectional area of the conductor
 I = conductor current
 Q = maximum allowable heat flux

One of two types of cooling is used: Pool boiling of helium or forced flow of supercritical helium. For pool boiling, the maximum allowable heat flux is the critical heat flux at which the transition from nucleate boiling to film boiling occurs. For helium in contact with metal surfaces, this is usually taken to be 0.35 watt/cm^2 [7]. However, because the vapor produced is carried through the coolant channel, the critical heat flux is geometry dependent for cooling channels with large length to diameter ratios. For length to diameter ratios greater than 50, the critical heat flux approaches 0.15 watts/cm^2 [8,9]. Using forced flow supercritical helium eliminates the possibility of a sudden temperature jump because no peak heat flux limit exists. However, these conductors must be operated above the critical pressure (2.26 atm) and fluid pumping must be provided. An added design constraint is necessary. The coolant flow rate must be large enough so that the length of the normal region does not grow as it propagates downstream [10].

Several levels of stabilization are available to prevent quenching. Subdivision of the superconductor into small filaments minimizes the fluxjump energy release. Embedding the

filaments in a metal matrix damps out magnetic flux movements, provides good heat transfer away from the filament as well as a low resistance parallel current path. Cooling with a fluid of large heat capacity and good heat transfer properties prevents quench regions from propagating or persisting.

1.2 The Requirement of Active Protection

Designing a coil to be cryostable does not guarantee that a propagating quench cannot occur. A number of events can be postulated which constitute heat additions large enough to defeat cryostabilization. Among these are wire movements, conductor breaks, electrical insulation failure causing a short circuit, structural yielding, cryoinsulation failure, a dewar leak, loss of coolant, and internal coolant blockage due to a foreign object or a vapor block at a quenched region. The first six of these may cause a quenched region large enough and warm enough to drive the conductor beyond the cryostable region. Lack of coolant does not necessarily start a quench, but it allows quenched regions to propagate. Cryostable coils are stable under small disturbances but not necessarily stable under large disturbances. The toroidal field coils envisioned store energies on the order of tens to hundreds of gigajoules. This is more than enough energy to cause melting of portions of the coil or to accelerate coil fragments to very large velocities. It is therefore important that if a large disturbance occurs, the coil's energy be removed or dissipated safely.

A conservative requirement for such a system is that it be able to remove or dissipate a coil's energy in a time which is comparable to the time required for a quenched region to heat up to a damaging temperature. Cryostable coils are designed not to propagate quenches; so that when propagating quenches do occur, they propagate slowly. If the heat loss from the sides of the composite conductor are small, the propagation velocity is found to be [11]:

$$v = (I/Ac\rho_m)(\rho_R k/\theta)^{1/2} \quad (1.3)$$

where I = current (A)
 A = cross-sectional area (cm²)
 c = specific heat (J/g/°K)
 ρ_m = density (g/cm³)
 ρ_R = resistivity (Ω-cm)
 k = thermal conductivity (w/cm/°K)
 θ = the difference between the wire temperature
 and the critical temperature

For a copper stabilized conductor with a cross-sectional area of 2.5 cm² carrying a current of 10KA, the propagation velocity is approximately 10 m/s. Most reactors would require 12 to 24 coils each requiring perhaps 5 x 10⁴ meters of conductor. The time for a quench to propagate through an entire coil is then approximately 5000 sec (1.4 hr). While the quenched region propagates outward, the center of the quenched

region heats fairly rapidly. Studies performed at Brookhaven National Laboratory [12] using a three-dimensional heat transfer computer code shows that quenched regions will reach 300 °K within 40 to 50 secs after the quench begins in a UWMAK-I design coil. This is shown to cause damaging thermal strains within the coil. Only a tiny fraction of the coil's stored energy (less than 1%) is required to cause this temperature rise. It is obvious that quench propagation cannot be relied upon to safely dissipate such large coils' energy.

Safety systems may be divided into two types, active and passive. Passive systems are usually preferred because they have no moving parts and require no operator decision and are therefore less likely to fail when needed. Three passive methods have been suggested: a hard wired shunt resistor, inductive coupling to a secondary coil and quench propagation enhancement [13].

A shunt resistor, R_D , may be connected in parallel with the coil (see Figure 1.4). When the coil is superconducting, the resistor is shunted and it dissipates no energy. The only dissipation occurs in the internal resistance of the power supply, R_S . If a portion of the coil quenches, a resistance R_C appears within the coil. R_C is a function of the size and average temperature of the quenched region. The normal operating current is

$$I_O = \frac{V}{R_S} \quad (1.4a)$$

Obviously it is desirable to make R_s as small as possible. After R_c appears, the current in the circuit decays to

$$I_f = \frac{V}{\frac{R_c R_D}{R_c + R_D} + R_s} \quad (1.4b)$$

I_f can be minimized by letting R_D approach infinity, i.e., by removing R_D altogether. The minimum I_f is determined by the sum of R_c and R_s . For a small quench, R_c is small and R_s is designed to be small, therefore I decreases only slightly giving no protection.

If a very large quench R_c occurs, I_f is very small compared to I_0 , and the current decay can be modeled from I_0 to $I_f = 0$ with $V = 0$. In this case the energy dissipated in R_D is

$$E_D = \frac{R_s^2 R_D}{(R_D + R_s)(R_s R_D + R_c R_D + R_s R_c)} \frac{M I_0^2}{2} \quad (1.4c)$$

The energy dissipated in R_c is

$$E_c = \frac{R_c (R_s + R_D)}{(R_s R_D + R_c R_D + R_s R_c)} \frac{M I_0^2}{2} \quad (1.4d)$$

If E_D is minimized with respect to R_D , the optimal R_D is

$$R_D|_{\text{optimal}} = \frac{R_s}{\sqrt{\frac{R_s}{R_c}} + 1} \quad (1.4e)$$

For a large quench R_c is much greater than R_s and the optimal R_D is approximately equal to R_s . If $R_D = R_s$ is substituted into Equations (1.4c) and (1.4e), the ratio of E_D to

E_C is

$$E_D/E_C \approx 0.25 R_S/R_C \quad (1.4f)$$

This shows that R_S must be chosen to be large compared to a large quench R_C for a significant portion of the stored energy to be dissipated in R_D .

From the viewpoint of both the large and small quench, a large R_S is required to make this protection circuit effective. This large R_S can be obtained only by placing an active switch in series with R_S .

A secondary coil may be placed in close proximity to the coil to be protected (see Figure 1.5). For TF coil applications, the secondary coil would be made of normal metal and would have $M_1 = M_2$ and k , the inductive coupling coefficient approximately equal to 1. If $R_1 = R_2$, probably the best that may be hoped for, then energy extraction efficiency is 50%. This, given the enormous energy stored in the coil, is not good enough. Moreover, if R_1 is much less than R_2 , as seems likely in the early stages of a coil quench, the energy extraction efficiency is much lower.

Quench propagation enhancement is accomplished by placing materials with large thermal conductivities between windings. This causes the entire coil to be driven normal at a rate faster than the development of a hot spot. One possibility is the use of anodized aluminum disks as electrical insulation between coil winding pancakes. Good thermal contact is important because at

Figure 1.4 Passive Protection Using a Shunt Resistor

low temperatures, the increased thermal conductivity of the filler may be neutralized by contact resistance. Also, the very large size of toroidal field coil and the complete lack of thermal contact between individual coils may make this improvement insignificant [13].

Each of the passive protection methods described above appears to have limitations when applied to very large superconducting coils. However, each method can be improved by active intervention, i.e., the use of switches and heating wires, etc. They form the basis for the active methods described in Chapter 4.

1.3 An Overview of Proposed Tokamak Designs

Table 1.1 compares the important toroidal field magnet parameters for five reactor designs. UWMAK-I, UWMAK-III and PPPL represent designs of commercial size reactors producing roughly 2000 MWe. These are very large machines that represent the hoped-for result of present fusion research.

The ANL-EPR and ORNL-EPR designs represent demonstration reactors intended to prove the feasibility of fusion reactors. These reactors would produce roughly 200 MWe. General Electric [22], Westinghouse [23] and M.C.A. [24] have each designed a toroidal field magnet set of an EPR sized reactor using identical constraints. These designs point up the various alternatives available to designers.

Several general comments can be made. The commercial sized reactors enclose very large volumes ranging from 10 to

$$k = \frac{M_{12}}{M_{11} M_{22}}$$

$$\frac{E_{R_2}}{E_{\text{total}}} = \frac{k^2}{\left(1 + \frac{M_{11} R_1}{M_{22} R_2} \right)}$$

Figure 1.5 Passive Protection Using a Secondary Resistor

Table 1.1 Survey of Proposed Tokamak Fusion Reactor Toroidal Magnets.

	UWMAK I	UWMAK III	ANL EPR	ORNL EPR	PPPL
Thermal Power	4663 MW _t	4735 MW _t	638 MW _t	800 MW _t	5305 MW _t
No. of Coils	12	18	16	20	48
Horizontal Bore	14.8 m	12.75 m	7.78 m	7.4 m	12.06 m
Vertical Bore	21.5 m	24.08 m	12.6 m	9.7 m	19.0 m
Magnet Shape	Dee	Dee	Dee	Oval	Dee
Major Radius	13.15 m	10.13 m	6.62 m	6.75 m	9.97 m
Enclosed Volume	20400 m ³	14500 m ³	3380 m ³	3200 m ³	11900 m ³
Max. Field	8.66 T	8.75 T	10 T	11 T	16 T
Conductor Type	NbTi-Cu	NbTi-Al	NbTi-Cu	NbTi-Cu & Nb ₃ Sn-Cu	Nb ₃ Sn-Cu
Cond. X-sec. Area	5.98 cm ² to 1.76 cm ²	2.25 cm ²	32 cm ² to 49 cm ²	2.51 cm ² to 1.44 cm ²	3.60 cm ² to 1.44 cm ²
Operating Current	10212 A	9545 A	60 kA	19.5 kA	10 kA
Turns/coil	1440	960	140	104	657
Operating Temp.	4.2 °K	4.2 °K	3 °K	4 °K	4-8 °K
Cooling Type	pool boiling	pool boiling	pool boiling	forced flow He	not specified
Reinforcement Material	SS	2219 Al	SS	SS	SS
Winding Type	pancake disk	pancake disk	jelly roll	layered	pancake
Winding Width	1.91 m	.90 m	1.00 m	.62 m	.375 m
Thickness	.93 m	1.16 m	.73 m	.69 m	1.19 m
Conductor Length	1570 km	1120 km	80.6 km	624 km	1770 km
Stored Energy	158 GJ	108 GJ	30 GJ	29 GJ	250 GJ
E/V	7.76 J/cm ³	7.39 J/cm ³	8.88 J/cm ³	9.06 J/cm ³	21.1 J/cm ³
E/L	1008 J/cm	961 J/cm	3722 J/cm	465 J/cm	1412 J/cm
Adiabatic Ther- malization Temp.	180 °K	300 °K	100 °K	135 °K	250 °K

Note: SS = Stainless Steel

20 thousand cubic meters as measured within the centerlines of the TF magnet windings. EPR sized reactors are much smaller, enclosing 2 to 5 thousand cubic meters.

The maximum allowable field is determined by the type of superconductor used. Use of NbTi limits the maximum field to 8-10 tesla and the energy density to 6-8 J/cm³. Use of Nb₃Sn allows field strengths from 12-16 tesla and energy densities of 16-25 J/cm³. NbTi is presently available in several types of conductors and has been used successfully in many large magnets. Nb₃Sn is very brittle and very difficult to fabricate into commercially usable conductors. Researchers are presently looking for usable production methods.

The combination of differing volumes and differing energy densities cause the total stored energy in the toroidal magnetic field to range from 20 to 250 GJ. As a general rule, the protection problem becomes more difficult as the amount of energy stored increases. For comparison, the largest superconducting magnet built to date in the U.S. is the ANL 15 ft. bubble chamber magnet which stores 0.4 GJ [14].

The adiabatic thermalization temperature ($T_{\text{adiabatic}}$) is the temperature that the winding would reach if all of the field energy were dissipated uniformly in the conductor with no heat transfer to the coolant or structure. The low range of 100 - 300 °K suggests that the problem is not one of energy magnitude but one of localization. Concentration factors of 1/100 - 1/300 seem to be quite possible in unprotected toroidal field magnets, more than enough to cause severe local damage.

Many magnet designs call for use of graded conductors to minimize the amount of stabilizer used. The stabilizer cross section is varied to allow for the decrease in magnetoresistance from the inside of the coil to the outside. The differing cross sections will cause sizable differences in $T_{\text{adiabatic}}$ reached by different parts of the coil. This could lead to sizable thermal gradients and strains across the winding and again points up the problem of localization. The magnet protection systems in the above mentioned Tokamak designs are summarized in Table 1.2.

Table 1.2 Survey of Proposed Superconducting Toroidal Magnet Protection Methods

<u>Reactor Design</u>	<u>Protection Method</u>
UWMAK	The faulted coil is discharged across 10 kV source. The remaining coils are put in a persistent mode in a parallel circuit. $t_d \sim 30$ sec.
Argonne EPR	The entire magnet is discharged in series across a single resistor (.033 ohm, $V_{max} = 2$ kV). $t_d \sim 500$ sec.
Oak Ridge EPR	The magnet is divided into 4 parallel sub-circuits. Only the faulted coil is discharged across a resistor (.017 ohm, 1.3 kV _{max}). The remaining coils remain in the persistent mode in 4 circuits. $t_d \sim 10$ sec.
PPPL	Only the faulted coil is discharged across a resistor (1.2 ohm, 12 kV _{max}). The remaining coils are trimmed for continued reactor operation. $t_d \sim 80$ sec.
Magnetic Corp. of America	The entire magnet is discharged with a resistor between each coil (1.84 kV _{max}). $t_d \sim 120$ sec.
Westinghouse Electric Corp.	The entire magnet is discharged by connecting a resistor in parallel with each coil (905 V _{max}). $t_d \sim 300$ sec.
General Electric Co.	Thermal equilization is initiated using heating wires within the winding. The number of coils to be discharged is still open to question.

Note: t_d = discharge time

II. A REVIEW OF INDUCTIVE CIRCUIT ANALYSIS

The exact solution for time varying currents and magnetic and electric fields within any region requires the solution of Maxwell's equations. Even for simplified geometries this is a difficult task. Fortunately, in studying toroidal field interactions we may make two simplifying assumptions. First, for the phenomena of interest, the characteristic distances are short (on the order of meters) and the characteristic times are long (on the order of seconds). A characteristic velocity defined as the ratio of the two is much smaller than the speed of light. In this low frequency limit, fields and currents are said to be quasistatic, implying that local changes in the field or current act immediately throughout the region without any propagation delay. Second, for most of our work, we will assume that the dimensions of conductors perpendicular to current flow are small and well defined. This permits conductors to be treated as lumped elements allowing use of methods developed for analyzing electronic circuits. This assumption is quite good for dealing with interactions between toroidal field coils. It is not a very accurate assumption for dealing with the reactor structures. However, the results obtained using a simple lumped analysis in Chapter 3 will be shown to form an upper limit for energy deposition. These results will be augmented with analysis of a simple two-dimensional conductor to permit more accurate estimates of MHD pressure rises due to toroidal magnetic field collapse.

2.1 The Magnetic Field and Vector Potential

The general Maxwell equations are [15],

$$\bar{\nabla} \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} \quad (2.1)$$

$$\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad (2.2)$$

$$\bar{\nabla} \cdot \bar{D} = \rho_{\text{charge}} \quad (2.3)$$

$$\bar{\nabla} \cdot \bar{B} = 0 \quad (2.4)$$

In the absence of magnetic materials,

$$\bar{B} = \mu_0 \bar{H} \quad (2.5)$$

In the low frequency approximation

$$\frac{\partial \bar{D}}{\partial t} \ll \bar{J}$$

and equation 2.2 may be rewritten as

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} \quad (2.6)$$

These combined with Ohm's Law

$$\bar{J} = \sigma \bar{E} \quad (2.7)$$

and the Lorentz force law

$$\bar{\nabla} P = \bar{J} \times \bar{B} \quad (2.8)$$

form the basis for our analysis. It is convenient to use the zero divergence of the magnetic field to define a new function, the vector potential, $\bar{A}$, such that

$$\begin{aligned} \bar{\nabla} \cdot (\bar{\nabla} \times \bar{A}) &= 0 \\ \bar{B} &= \bar{\nabla} \times \bar{A} \end{aligned} \quad (2.9)$$

$\bar{A}$ may be any well defined vector function. Substitution of equation 2.9 into equation 2.6 and expanding yields

$$\begin{aligned} \bar{\nabla} \times (\bar{\nabla} \times \bar{A}) &= \mu_0 \bar{J} \\ \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \bar{\nabla} \cdot (\bar{\nabla} \bar{A}) &= \mu_0 \bar{J} \end{aligned}$$

As a simplification we may require that

$$\bar{\nabla} \cdot \bar{A} = 0$$

This choice of $\bar{\nabla} \cdot \bar{A}$ is commonly known as the Coulomb gauge (several other choices of gauge may also be used). The final result is:

$$\nabla^2 \bar{A} = \mu_0 \bar{J} \quad (2.10)$$

The particular solution of 2.10 may be shown to be [16]

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \int_{\text{Vol}} \frac{J(\bar{x}') d^3x'}{(\bar{x} - \bar{x}')} \quad (2.11)$$

for a known current distribution. For filamentary currents equation 2.11 becomes

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \oint_c \frac{Id\bar{\ell}(\bar{x}')}{(\bar{x} - \bar{x}')} \quad (2.12)$$

For simple geometries, Maxwell's equation (2.6) may be integrated directly to find $\bar{B}$. For most other problems it is easier to use equation 2.12 to calculate $\bar{A}$ and then use equation 2.9 to calculate $\bar{B}$.

One useful simple geometry is that of the continuously wound toroid illustrated in Figure 2.1. If equation 2.6 is integrated over any surface and use is made of Stoke's Theorem, Ampere's circuital law is obtained:

$$\oint_c \bar{B} \cdot d\bar{\ell} = \mu_0 \int_s \bar{J}_0 \cdot d\bar{S} \quad (2.13)$$

If the integration contour c is the path shown in Figure 2.1 and if it is noted that by symmetry, B is constant along c , then the integration yields

$$B = \mu_0 \frac{NI}{2\pi r} \quad (2.14)$$

Figure 2.1 A Continuously Wound Toroid

An extension of Equation 2.14 to an infinitely long solenoid having N_ℓ turns per unit length gives

$$B = \mu_0 N_\ell I \quad (2.15)$$

The magnetic field vanishes outside either the toroid or solenoids and Equation 2.14 or Equation 2.15 holds for any arbitrary cross-section. For the very long solenoid, the field in the winding with an exterior radius r_0 and carrying an average current density J , is

$$B = \mu_0 J(r_0 - r) \text{ (in the winding)} \quad (2.16)$$

The magnetic field of a circular filament may be calculated by using the vector potential and Figure 2.3. The circular filament of radius "a" lies in the xy plane. We wish to calculate the vector potential at any point P in the xz plane. Because the current I only has one component, in the ϕ direction, we deduce that A_r and A_z must be zero. A_ϕ may be calculated using equation 2.12 with the substitutions

$$\begin{aligned} d\vec{\ell} &= (-\sin \phi' \hat{x} + \cos \phi' \hat{y}) d\phi' \\ |x - x'| &= (a^2 + r^2 - 2ar \sin \phi \cos \phi')^{1/2} \end{aligned}$$

The integration needs to be performed only over the $\hat{y}$ component because the $\hat{x}$ component is symmetric and drops out.

Figure 2.2 Infinitely Long Solenoid

Figure 2.3 Coordinates for Calculating the Vector Potential and Magnetic Field for a Circular Filament

$$A_{\phi}(r, \theta) = \frac{I a \mu_0}{4\pi} \int_0^{2\pi} \frac{\cos \phi' d\phi'}{(a^2 + r^2 - 2ar \sin \theta \cos \phi')^{1/2}}$$

by letting

$$k^2 = \frac{4ar \sin \theta}{a^2 + r^2 + 2ar \sin \theta}$$

A_{ϕ} may be rewritten to be [16]

(2.17)

$$A_{\phi}(r, \theta) = \frac{I a \mu_0}{(a^2 + r^2 + 2ar \sin \theta)^{1/2}} \left[\left(\frac{2}{k^2} - 1 \right) K(k) - \frac{2}{k^2} E(k) \right]$$

where $E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi} d\phi$

and $K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$

are complete elliptical integrals which may be calculated from tables or by using series expansions. The magnetic field is found by taking the curl of $\bar{A}$ [17].

$$\bar{B}(r, \theta) = B_{\rho} \hat{\rho} + B_z \hat{z} \quad (2.18)$$

where $B_{\rho} = \frac{\mu_0 I}{4\pi} \cot \theta \left(\frac{k^2}{ar \sin \theta} \right)^{1/2} \left[\frac{(1 - \frac{1}{2}k^2)}{(1 - k^2)} E(k) - K(k) \right]$

and $B_z = \frac{\mu_0 I}{4\pi} \left(\frac{k^2}{ar \sin \theta} \right)^{1/2} \left[\frac{a^2 - r^2}{a^2 + r^2 - 2ar \sin \theta} - E(k) + K(k) \right]$

Note that B_ϕ is zero.

2.2 Electromagnetic Induction and Magnetic Field Energy

Eq. 2.1 states that time varying magnetic fields create "induced" electrical fields. The magnitude of these fields may be calculated by integrating Eq. 2.1 over any continuous surface s with an outer contour C :

$$\int_s \nabla \times \vec{E} \, ds = - \int_s \frac{\partial \vec{B}}{\partial t} \, ds$$

or

$$\int_C \vec{E} \cdot d\vec{l} = - \frac{\partial}{\partial t} \int_s \vec{B} \cdot d\vec{s}$$

For the varying field:

$$\epsilon = - \frac{\partial}{\partial t} \phi, \quad (2.19)$$

ϵ is the electromotive force around a contour completely enclosing a total magnetic flux ϕ . As a special case, consider a system of two current carrying loops. To find the electromotive force in loop 2 induced by varying the current in loop 1 we write

$$\epsilon_{21} = - \frac{\partial}{\partial t} \int_{S_2} \nabla \times \vec{A}_1 \cdot d\vec{S}_2$$

or
$$\epsilon_{21} = - \frac{\partial}{\partial t} \int_{C_2} \bar{A}_1 \cdot d\bar{\ell}_2$$

where ϵ_{21} is the voltage induced in loop 2 by changes of the current in loop 1. Multiplying the RHS through by I_1/I_1 yields

$$\epsilon_{21} = - \frac{1}{I_1} \int_{C_2} \bar{A}_1 \cdot d\bar{\ell}_2 \frac{dI_1}{dt}$$

or
$$\epsilon_{21} = -M_{21} \frac{dI_1}{dt} \quad (2.20)$$

where the quantity

$$M_{21} = \frac{1}{I_1} \int_{C_2} A_1 \cdot d\bar{\ell}_2 \quad (2.21)$$

is known as the mutual inductance and is a function of only geometric terms. Moreover, it can be shown that

$$M_{12} = M_{21} = \frac{\phi_{12}}{I_2} = \frac{\phi_{21}}{I_1} \quad (2.22)$$

The electromotive force induced in coils due to the changes in its own magnetic field is called self-inductance and it may be defined from

$$\epsilon_{11} = -M_{11} \frac{dI_1}{dt}$$

identically to Eq. 2.20.

It can be shown that the instantaneous power to maintain a magnetic field is

$$P_m = \int_{Vol} \bar{J} \cdot \frac{\partial \bar{A}}{\partial t} dv \quad (2.23)$$

The total energy stored in a magnetic field may be found by integrating Equation 2.23 over the time it takes to establish the present field. For linear, stationary media, i.e., for a conservative field, it can be shown that the total stored energy is

$$W_m = \frac{1}{2} \int_{Vol} \bar{J} \cdot \bar{A} dv \quad (2.24)$$

For a system containing N filamentary loops, the total stored energy becomes

$$W_m = \frac{1}{2} \sum_{K=1}^N \oint_{C_K} I_K \bar{A} \cdot d\bar{\ell}_K - \frac{1}{2} \sum_{K=1}^N \sum_{j=1}^N M_{Kj} I_K I_j \quad (2.25)$$

For a single loop Equation 2.25 becomes

$$W_m = \frac{1}{2} M_{11} I^2 \quad (2.26)$$

Equation 2.24 may be transformed into a more convenient form

$$W_m = \frac{1}{2\mu_0} \int_{Vol} B^2 dv \quad (2.27)$$

One such simple system consists of two infinitely long solenoids with the dimensions shown in Figure 2.4. All quantities are calculated on a per unit length basis. Equating Equations (2.25) and (2.27) gives

$$\begin{aligned} \frac{1}{2} M_{ii} (J_i t_i)^2 + M_{ij} J_i J_j t_i t_j + \frac{1}{2} M_{jj} (J_j t_j)^2 \\ = \frac{1}{2\mu_0} \int_0^{b_j} B(r)^2 (2\pi r dr) \end{aligned} \quad (2.28)$$

The magnetic field B can be calculated using Equations (2.15) and (2.16). Squaring B and performing the indicated integration and equating the coefficients gives

$$\begin{aligned} M_{ii} = \pi\mu_0 a_i^2 + \frac{2\pi\mu_0}{t_i^2} \left[\frac{b_i^2(b_i^2 - a_i^2)}{2} - \frac{b_i(b_i^3 - a_i^3)}{3} + \frac{(b_i^4 - a_i^4)}{4} \right] \\ M_{ij} = \pi\mu_0 \left[\left(1 - \frac{b_i}{t_i}\right) a_i^2 + \frac{(b_i^3 + 2a_i^3)}{2t_i} \right]; b_i < a_j \end{aligned} \quad (2.29)$$

A more difficult problem is finding the mutual inductance between two arbitrarily placed circular filaments. This may be accomplished by use of Equations (2.14) and (2.21) and the geometry shown in Figure 2.5. Loop 1 of radius a_1 lies in the xy plane centered at the origin 0. Loop 2 of radius a_2 lies in the x'y' plane which is translated and inclined with respect to the xyz axis. The position of loop 2 may be found by translating an axis set x'y'z' to the center of loop 2

Figure 2.4 The Magnetic Field Inside Two Infinitely Long Coaxial Solenoids

Figure 2.5 The Coordinate System and Rotational Transforms for Calculating the Mutual Inductance of Two Circular Filaments

b) Rotation Step 1

b) Rotation Step 2

Figure 2.5 (Con't.)

(x_1, y_1, z_1) . $x'y'z'$ is then rotated into $x''y''z''$. This rotation may be done in two steps. The first step rotates the axis set about the y' axis through an angle α so that x'' lies in the loop plane. The second step rotates the axis set about the new x'' axis through an angle γ so that the y'' axis also lies in the loop plane. The coordinate transformation may now be made using the rotational-transformation matrix C defined by [18]

$$\begin{bmatrix} \hat{x}'' \\ \hat{y}'' \\ \hat{z}'' \end{bmatrix} = \underline{\underline{C}} \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix}$$

For successive rotations the total rotational transformational matrix is the matrix product of the transformation matrix for each rotation. Using the transformation matrix given in Figure 2.5, the complete rotational transformation is

$$\begin{aligned} \hat{x}'' &= \cos \alpha \hat{x}' - \sin \alpha \hat{z}' \\ \hat{y}'' &= \sin \alpha \sin \gamma \hat{x}' + \cos \gamma \hat{y}' + \cos \alpha \sin \gamma \hat{z}' \\ \hat{z}'' &= \sin \alpha \cos \gamma \hat{x}' - \sin \gamma \hat{y}' + \cos \alpha \sin \gamma \hat{z}' \end{aligned} \quad (2.30)$$

We wish to perform the integration indicated in Equation 2.21 with an integration variable ϕ measured from x'' so that

$$\begin{aligned} d\bar{\ell}_2 &= a_2 d\phi [-\sin \phi \hat{x}'' + \cos \phi \hat{y}''] \\ &= a_2 d\phi [(-\sin \phi \cos \alpha + \cos \phi \sin \alpha \sin \gamma) \hat{x}' \\ &\quad + \cos \phi \cos \gamma \hat{y}' + (\sin \phi \sin \alpha + \cos \phi \cos \alpha \sin \gamma) \hat{z}'] \end{aligned}$$

To evaluate $\bar{A}_1$ we need to find r and $r \sin \theta$,

$$r = (x_2^2 + y_2^2 + z_2^2)^{1/2}$$

$$\text{where } x_2 = x_1 + a_2 (\cos \phi \cos \alpha + \sin \phi \sin \alpha \sin \gamma)$$

$$y_2 = y_1 + a_2 \sin \phi \cos \gamma$$

$$z_2 = z_1 + a_2 (\sin \phi \cos \alpha \sin \gamma - \cos \phi \sin \alpha)$$

$$\text{and } r \sin \theta = (x_2^2 + y_2^2)^{1/2}$$

The vector potential is

$$\bar{A}_1 = A_\phi(r, r \sin \theta) \sin \beta \hat{x} + A_\phi(r, r \sin \theta) \cos \beta \hat{y}$$

$$\text{where } \sin \beta = y_2 / r \sin \theta$$

$$\cos \beta = x_2 / r \sin \theta$$

The resulting integration is

$$M_{12} = \frac{a_2 N_1 N_2}{I_1} \int_0^{2\pi} [A_{\phi 1} \sin \beta (\sin \phi \cos \alpha - \cos \phi \sin \alpha \sin \gamma) + A_{\phi 1} \cos \beta \cos \phi \cos \gamma] d\phi \quad (2.31)$$

where N_1 and N_2 are the number of turns in coils 1 and 2 that the filamentary loops approximate. In general, this integral must be evaluated by numerical techniques. Appendix II documents a BASIC subroutine which evaluates the integral using Simpson's Rule. For a sector toroid where ξ is the angle between two coils and R_m is the major radius, the relationships

$$\begin{aligned} Z_1 &= R_m \sin \xi & \alpha &= \xi \\ X_1 &= R_m (1 - \cos \xi) & \gamma &= 0 \\ Y_1 &= 0 \end{aligned} \quad (2.32)$$

are useful.

The self inductance of coils that have a small winding cross-section may be calculated by approximating the coil with two filaments: one, at the cross-section center, or radius r and the other, at the inner edge of the cross-section, of radius $(r-a)$. The inductance for the two concentric circles is

$$M_{ii} = N^2 \mu_o (2r-a) \left[\left(1 - \frac{k^2}{2}\right) K(k) - E(k) \right] \quad (2.33a)$$

$$\text{where } k^2 = \frac{4r(r-a)}{(2r-a)^2}$$

and $E(k)$ and $K(k)$ are complete elliptical integrals.

If k is nearly unity, M_{ii} may be approximated by [16]

$$M_{ii} = N^2 r \mu_o \left[\ln \left(\frac{8r}{a} \right) - 2 \right] \quad (2.33b)$$

This is sometimes called the external inductance because it does not include the magnetic energy stored inside the conductors. However, for large N , the internal inductance is a very small fraction of the total inductance and it may be neglected.

2.3 Inductive Circuit Matrix Equation

Almost all of the building materials in presently proposed fusion reactor designs are non-magnetic. This allows us to use

a linear model to study the magnetic field interactions between toroid coils and between the coils and the reactor structure. Consider the schematic of a three coil toroid shown in Figure 2.6. Summing voltages around the first coil circuit gives

$$V_1 - R_1 I_1 - M_{11} \frac{dI_1}{dt} - M_{12} \frac{dI_2}{dt} - M_{13} \frac{dI_3}{dt} = 0 \quad (2.34)$$

Similar equations can be written for the other two coils. Using matrices, this equation can be expanded for the case of n inductive elements.

$$\underline{V} - \underline{\bar{R}} \underline{I} - \underline{\bar{M}} \frac{d}{dt} \underline{I} = 0 \quad (2.35)$$

where

$$\underline{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} \quad \underline{\bar{R}} = \begin{bmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_n \end{bmatrix}$$

$$\underline{I} = \begin{bmatrix} I_1 \\ \vdots \\ I_n \end{bmatrix} \quad \underline{\bar{M}} = \begin{bmatrix} M_{11} & & M_{1n} \\ & \ddots & \\ & & M_{nn} \end{bmatrix}$$

Note that $\underline{\bar{R}}$ is a diagonal matrix and that $\underline{\bar{M}}$ is a symmetric matrix. Equation 2.35 may be put into standard state equation form

$$\frac{d}{dt} \underline{I} = -\underline{\bar{M}}^{-1} \underline{\bar{R}} \underline{I} + \underline{\bar{M}}^{-1} \underline{V} \quad (2.36)$$

The inductance matrix $\underline{\bar{M}}$ may be used to simplify Equation 2.25 for the stored magnetic field energy

Figure 2.6 Three State Inductive Circuit

$$W_m = \frac{1}{2} \mathbf{I}^T \underline{\underline{M}} \mathbf{I} \quad (2.37)$$

A useful result may be demonstrated by rearranging the connections of the three coil circuit. The new circuit is shown in Figure 2.7. The magnetic energy of this circuit is

$$\begin{aligned} W_m &= \frac{1}{2} M_{11}^* I_1^2 + \frac{1}{2} M_{22}^* I_2^2 + M_{12}^* I_1 I_2 \\ &= \frac{1}{2} M_{11} I_1^2 + \frac{1}{2} (M_{22} + M_{33} + 2M_{23}) I_2^2 + (M_{12} + M_{13}) I_1 I_2 \end{aligned}$$

where (*) indicates the inductances for the new circuit.

Equating coefficients yields

$$\begin{aligned} M_{11}^* &= M_{11} & M_{22}^* &= M_{22} + M_{33} + 2M_{23} \\ M_{12}^* &= M_{12} + M_{13} \end{aligned}$$

In general, for n coils with $n-1$ coils wired in series

$$\begin{aligned} M_{11}^* &= M_{11} & M_{22}^* &= \sum_{i=2}^n \sum_{j=2}^n M_{ij} \\ M_{12}^* &= \sum_{j=2}^n M_{1j} \end{aligned} \quad (2.38)$$

If the $\underline{\underline{M}}$ and $\underline{\underline{R}}$ matrices are constant, an exact solution of equation 2.36 may be found. For a step change in $\underline{V}$ at time $t = 0$ with $\underline{I}(t=0) = \underline{I}_0$, the time response of the system is [20]

Figure 2.7 Two State Inductive Circuit

$$I(t) = \overline{\text{Exp}}(-\underline{\bar{M}} \underline{\bar{R}} t)^{-1} \underline{I}_0 + (-\underline{\bar{M}} \underline{\bar{R}})^{-1} [\text{Exp}(-\underline{\bar{M}} \underline{\bar{R}} t) - \underline{\bar{I}}_{dn}] \underline{\bar{M}}^{-1} \underline{V} \quad (2.39a)$$

where $\underline{\bar{I}}_{dn}$ = identity matrix

$\overline{\text{Exp}}(\underline{\bar{A}} t)$ = matrix exponential of $\underline{\bar{A}} t$

For calculation on a computer this equation is converted into the difference equation

$$I(n) = \overline{\text{Exp}}(-\underline{\bar{m}} \underline{\bar{R}} \Delta)^{-1} I(n-1) + (-\underline{\bar{m}} \underline{\bar{R}})^{-1} [\overline{\text{Exp}}(-\underline{\bar{m}} \underline{\bar{R}} \Delta) - \underline{\bar{I}}_{dn}] \underline{\bar{m}}^{-1} \underline{v}(n) \quad (2.39b)$$

where $I(0) = \underline{I}_0$

Δ = time step size

$t = n\Delta$

A BASIC subroutine for calculating $\overline{\text{Exp}}(-\underline{\bar{m}} \underline{\bar{R}} \Delta)^{-1}$ is documented in Appendix III. The system time response is obtained in a stepwise fashion. This gives an exact history of $\underline{I}$ as long as $\underline{v}$ changes only at the time steps. This solution is especially useful because any time step Δ may be used without creating numerical instabilities although care must be used when calculating $\overline{\text{Exp}}(-\underline{\bar{m}} \underline{\bar{R}} \Delta)$ for large values of Δ . Being able to use large time steps is important in studying systems which have very short time constants for states which cannot be neglected.

If $\underline{\bar{M}}$ or $\underline{\bar{R}}$ is not constant but time varying, then an exact solution of Equation (2.36) cannot usually be found. For this problem, any of the approximate numerical methods available may be used to solve Equation (2.36). A BASIC program for solving

Equation (2.36) using the fourth order Runge-Kutta approximation is documented in Appendix IV. The use of an approximate solution is particularly useful when material properties are temperature dependent which is common at cryogenic temperatures. The temperature dependence may be included by appending thermal model state equations to Equation (2.36).

If Eq. (2.36) is of second order with no applied voltage it is possible to obtain an analytical solution (it is algebraic complexity which make higher order solutions difficult to obtain). The state equation becomes

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \frac{1}{M_{11}M_{22} - M_{12}^2} \begin{bmatrix} -M_{22}R_1 & M_{12}R_2 \\ M_{12}R_1 & -M_{11}R_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \\ &= \begin{bmatrix} -a_{11} & a_{12} \\ a_{21} & -a_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \end{aligned} \quad (2.40)$$

The solution to Eq. (2.40) is

$$I_1 = \frac{I_o}{(\lambda_1 - \lambda_2)} \left[(\lambda_1 + a_{22} + a_{12})e^{\lambda_1 t} - (\lambda_2 + a_{22} + a_{12})e^{\lambda_2 t} \right] \quad (2.41)$$

$$I_2 = \frac{I_o}{(\lambda_1 - \lambda_2)} \left[(\lambda_1 + a_{11} + a_{21})e^{\lambda_1 t} - (\lambda_2 + a_{11} + a_{21})e^{\lambda_2 t} \right]$$

where

$$\lambda_{1,2} = \frac{1}{2(M_{11}M_{22} - M_{12}^2)} \left(M_{22}R_1 + M_{11}R_2 \pm \sqrt{(M_{11}R_1 - M_{22}R_2)^2 + 4M_{12}R_1R_2} \right)$$

By integration, the total energy dissipated in the resistors can be found to be

$$E_1 = \frac{R_1 I_0^2}{(\lambda_1 - \lambda_2)^2} \left[\frac{2(\lambda_1 + a_{22} + a_{12})(\lambda_2 + a_{22} + a_{12})}{\lambda_1 + \lambda_2} - \frac{(\lambda_1 + a_{22} + a_{12})^2}{2\lambda_1} - \frac{(\lambda_2 + a_{22} + a_{12})^2}{2\lambda_2} \right]$$

$$E_2 = \frac{R_1 I_0^2}{(\lambda_1 - \lambda_2)^2} \left[\frac{2(\lambda_1 + a_{11} + a_{21})(\lambda_2 + a_{11} + a_{21})}{\lambda_1 + \lambda_2} - \frac{(\lambda_1 + a_{11} + a_{21})^2}{2\lambda_1} - \frac{(\lambda_2 + a_{11} + a_{21})^2}{2\lambda_2} \right] \quad (2.42)$$

2.4 Forces Between Circular Coils

Consider a system of circular current carrying coils. We wish to find the force acting on any one coil due to the other coils. If the coil currents are maintained constant then it can be shown that

$$F_x = \frac{\partial W}{\partial x} \quad (2.43)$$

The energy W is the energy of the entire system. The differential change in position is applied only to the coil for which the force is desired and the force is in the same direction as the change in position.

As a numerical approximation, we may apply the central difference approximation to Eq. (2.43).

$$F_x^i = \frac{W^{i+1} - W^{i-1}}{2\Delta x} \quad (2.44)$$

where

$$W^{i+1} = W(x_i + \Delta x)$$

$$W^{i-1} = W(x_i - \Delta x)$$

Using matrix methods to describe the current system and substituting Eq. 3.37 into Eq. 2.44 it can be shown that

$$\underline{F}_x^i = \underline{I}^T \frac{(\underline{M}^{i+1} - \underline{M}^{i-1})}{4\Delta x} \underline{I} \quad (2.45)$$

If

$$\underline{G}_x^i = \frac{\underline{M}^{i+1} - \underline{M}^{i-1}}{4\Delta x} \quad (2.46)$$

then

$$\underline{F}_x^i = \underline{I}^T \underline{G}_x^i \underline{I} \quad (2.47)$$

This formulation may be used for any coordinate direction or for rotation about any coordinate axis. Note that $\underline{G}_x^i$ is a function of geometry only.

Because of the symmetry of the equilibrium positions of the toroidal field coils, $\underline{G}_x^i$ needs to be calculated only once for each displacement desired. A coordinate system is set centered on the equilibrium position of coil No. 1. The desired displacement is applied to that coil while the remaining (N-1) coils are maintained in their equilibrium positions. The only elements of the inductance matrix which change due to the displacement are the mutual inductances between coil No. 1 and the other coils. These are calculated

using Eq. 2.31. $\bar{G}_x^1$ may be written as

$$\bar{G}_x^1 = \frac{1}{4\Delta x} \begin{bmatrix} 0 & g_1 & g_2 & \dots & g_{n-1} \\ g_1 & 0 & 0 & \dots & 0 \\ g_2 & & 0 & & 0 \\ \vdots & & & & \vdots \\ g_{n-1} & 0 & 0 & \dots & 0 \end{bmatrix} \quad (2.48)$$

$$\text{where } g_j = M_{1(j+1)}^{i+1} - M_{1(j+1)}^{i+1} \quad (2.49)$$

Due to symmetry, if the force on the n^{th} toroidal coil is desired, Eq. 2.48 may be rearranged to be

$$\bar{G}_x^n = \frac{1}{4\Delta x} \begin{bmatrix} 0 & 0 & \dots & 0 & \overset{\text{n}^{\text{th}} \text{ column}}{g_{n-n+1}} & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots & & & \vdots & \\ 0 & 0 & & 0 & g_{n-1} & 0 & 0 & 0 & \\ g_{n-n+1} & g_{N-n-2} & \dots & g_{N-1} & 0 & g_1 & \dots & g_{N-n-1} & g_{N-n} \\ 0 & 0 & & 0 & g_1 & 0 & 0 & 0 & \\ \vdots & & & & \vdots & & & \vdots & \\ 0 & 0 & \dots & 0 & g_{N-n} & 0 & \dots & 0 & 0 \end{bmatrix} \quad \begin{matrix} \\ \\ \\ \text{n}^{\text{th}} \text{ row} \\ \\ \\ \end{matrix} \quad (2.50)$$

Since the $\bar{G}_x^n$ consists of mostly zeros, only the elements g_j need to be stored and a simple algorithm may be used to do the multiplication indicated in Eq. 2.47. The algorithm is

$$F_x^n = 2I_n \sum_{j=1}^{N-1} (g_j I_k) \quad (2.51)$$

Where $k = \begin{matrix} j+1 & \text{if } j+1 \leq N \\ j+i-N & \text{if } j+i > N \end{matrix}$

III. INTERACTIONS OF THE TOROIDAL FIELD AND REACTOR STRUCTURE

Immersed in the toroidal magnetic field is the reactor's inner structure: the first wall, blanket, and shielding. The structures are composed of conducting materials and may therefore interact inductively with changes in the toroidal field. The currents induced in the structure produce two effects: energy deposition in the conductor due to Joule heating and induced body forces or MHD pressure gradients due to the current-field cross product. In this chapter, an estimate of the magnitude of these effects for a complete toroidal field collapse will be made.

3.1 Structure Model

As a reference design of reactor structures, let us consider the UWMAK-III design [3]. A cross sectional view of UWMAK-III is shown in Figure 3.1. A material schematic is shown in Figure 3.2. To calculate the mutual inductance between individual toroidal field coils and structural components would be a very difficult problem. Instead, a simple but conservative model will be proposed.

The reactor is modeled as a circular toroid with an infinite major radius, i.e., as an infinitely long solenoid. The toroidal field coils become the outer windings of the solenoid and the reactor structure is modeled as a set of concentric cylindrical shells within the solenoid. The model

CROSS SECTION VIEW OF UWMAK III

Figure 3.1

A SCHEMATIC OF THE UWMAK III BLANKET AND SHIELD

ZONE COMPOSITION TABLE

I	97 % Li + 3 % Mo	VI	100 % S.S.
II	99 % C + 1 % Mo	VII	100 % C
III	100 % Mo	VIII	82 % Mo + 18 % $Al_2O_3 \cdot 2SiO_2$
IV	90 % Pb + 10 % S.S.	IX	72 % B_4C + 18 % $Al_2O_3 \cdot 2SiO_2$ + 10 % Mo
V	72 % B_4C + 10 % S.S. + 18 % $Al_2O_3 \cdot 2SiO_2$		

Figure 3.2

cross section, shown in Figure 3.3, is based on the outer blanket construction shown in Figure 3.2. The solenoid's radius is chosen to enclose the same cross sectional area as do the toroidal field magnets. The solenoid's initial current is chosen so that magnet energy stored per unit volume for the solenoid is the same as the average energy per unit volume for the toroid. Single turn inductances and resistances are calculated on a per unit axial length basis. The inductances are calculated using Equation (2.29). It should be noted that the mutual inductance between any two co-axial shells is only a function of the size of the inner cylinder. The mutual inductance between one shell and any outer shell is a constant. Therefore, only the self-inductance and a single mutual inductance needs to be calculated for each shell. These are given in the last two columns of Table 3.1 along with the other model parameters.

The model is conservative because the mutual inductances between the structure and the magnet are maximized. This maximizes the energy available for element heating. The induction of an element is proportional to the magnetic flux enclosed by the outer edge of the conductor. A more likely form of construction would be to separate the structural shells with strips of insulating material as shown in Figure 3.4. In this case, the enclosed flux is only that passing through the conductor itself. The mutual inductances are reduced and the effects reported below are largely mitigated.

1. Molybdenum
2. Lithium
3. Carbon
4. Lithium
- * 72% B_4C , 10% Stainless Steel, 18% $Al_2O_3 \cdot 2SiO_2$
5. Stainless Steel
- * 72% B_4C , 10% Stainless Steel, 18% $Al_2O_3 \cdot 2SiO_2$
6. Stainless Steel
- * 72% B_4C , 10% Stainless Steel, 18% $Al_2O_3 \cdot 2SiO_2$
7. Lead
8. Superconducting Magnet Coil

Figure 3.3 Eight Shell Reactor Blanket Model

Table 3.1

Energy Deposition Model Parameters

Region (i)	Material Type	Inner Radius (cm)	Outer Radius (cm)	Thickness (cm)	R (10^{-3} ohm - cm)	M_{ii} (10^{-3} H - cm)	M_{ij} (10^{-3} H - cm)
1.	Mo	529.7	530	.3	9.184	11.240	11.083
2.	Li	530	590	60	0.145	11.950	12.392
3.	C	590	620	30	6.052	14.204	14.453
4.	Li	620	624	4	2.416	15.241	15.274
5.	S.S.	644	664	20.	2.355	16.715	16.887
6.	S.S.	674	694	20.	2.465	18.292	18.472
7.	Pb	704	724	20.	0.821	19.939	20.127
8.	Magnet Coil	892	902.1	10.1	0	31.65	31.77

Material Properties Used:

Material Type	Resistivity (μ ohm cm)	Density (g/cm ³)	Specific Heat (J/g°K)
Stainless Steel (S.S.)	72.	8.02	2.523
Pb	23.	11.344	0.159
Li	15.54	0.436	3.56
Mo	5.2	10.2	0.251
Graphite	1375	2.25	0.712

B₄C and Al₂O₃SiO₃ were assumed to have infinite resistivities

Initial Magnet Current = 3.43 MA - turns/m

Total Stored Energy = 1.86 GJ/m

Ave. Magnetic Field = 4.31 tesla

= 7.45 J/cm³

a) largest possible
breakdown path

b) electrically insulated
segments

Figure 3.4: Blanket Segmentation

3.2 Energy Deposition in the Reactor Structure

To find the upper limit of energy deposition in the reactor structure, imagine that only one of the reactor structural shells (designated by the subscript s) is coupled inductively to the toroidal field coils (designated by the subscript m). This implies that no credit is given for energy deposition in the rest of the reactor structure, maximizing the energy deposited in the selected shell. Now suppose that the toroidal field is removed so rapidly that the magnetic field does not have time to diffuse out through the selected shell, i.e., that currents are induced in the shell which maintain the magnetic field inside the shell. This requires that the discharge time, t_d , be

$$t_d \ll \frac{M_{mm} M_{ss} - M_{ms}^2}{M_{mm} R_s} \quad (3.1)$$

For this case the maximum induced current in the shell is

$$I_s = B/\mu_o = \text{constant} \quad (3.2)$$

The total energy deposited in the shell then is

$$W_s = \frac{1}{2} M_{ss} \left(\frac{B}{\mu_o} \right)^2 \quad (3.3)$$

where both I_s and W_s are calculated per unit of axial length. These quantities are compiled for the first wall, lithium blanket and lead shielding in Table 3.2. If t_d

Table 3.2

Maximum Energy Deposition

Due to Magnetic Field Discharge in UWAK-III

(In this calculation it is assumed that only the affected shell is present in the magnetic field)

<u>Shell</u>	<u>Material Type</u>	<u>t_d (sec)</u>	<u>W_m (MJ/m)</u>	<u>ΔT (°K)</u>
First wall	Mo	2.26	661	2689
Blanket	Li	130	703	20.7
Lead Shield	Pb	13.8	1173	72.4

t_d = field diffusion time

W_m = total energy deposited in shell

ΔT = adiabatic temperature rise of shell

is greater than that prescribed by Equation 3.1 or if the other shells are also included, then the energy deposition will be reduced.

All of the shells may be included in the model by using Equation 2.35 to find 8 simultaneous ordinary differential equations. These can be solved exactly using the matrix exponential and the solution may be computed using the subroutine in Appendix III. The toroidal field is assumed to be removed by applying a discharge voltage across the magnet coil. The discharge voltage was varied to vary the discharge time. All results are given in terms of the discharge time. Current histories for the first wall, blanket, and lead shield are shown in Figures 3.5 - 3.7. Note that the lead shielding acts as a magnetic shield limiting the rate of magnetic field decay to the decay constant of the lead shielding. The energy deposition is summarized in Table 3.3 and Figure 3.8. Typical times for discharging the entire toroidal magnetic field are of the order of 300 to 1000 sec as will be shown in Chapter 4.

It can be seen that the energy deposition for the 8 shell model is less than the maximum achievable for all portions of the reactor and for all discharge times. In particular, the energy deposition in the first wall is sharply reduced. In practice, the first wall would be in good thermal and electrical contact with the lithium blanket so that most of the current would flow in the blanket and the worst case analysis cannot properly be applied to the first wall. This is important

Figure 3.5 Current Induced in the First Wall Due to a Toroidal Field Discharge

Figure 3.6 Current in the Lithium Blanket Due to a Toroidal Field Discharge

Figure 3.7 Current in the Lead Shielding Due to a Toroidal Field Discharge

Table 3.3
Energy Deposition in 8 Shell Model

Discharge Time (sec)	Total Energy Deposition (MJ/m)	Fraction of Total Energy (%)	ΔT (°K)
<u>First Wall</u>			
.81	6.13	.33	24.6
9.6	5.99	.32	23.5
32.2	5.58	.30	21.8
153	4.14	.22	16.2
1980	.69	.04	2.72
<u>Lithium Blanket</u>			
.81	478	25.7	14.1
9.6	467	25.1	13.8
32.2	439	23.6	13.0
153	327	17.6	9.64
1980	55	2.9	1.62
<u>Lead Shielding</u>			
.81	526	28.3	32.5
9.6	375	20.2	23.2
32.2	287	15.4	17.7
153	166	8.9	10.3
1980	25	1.4	1.6
<u>Total Structure</u>			
.81	1225	65.8	
9.6	1064	57.2	
32.2	919	49.4	
153	622	33.5	
1980	101	5.4

Figure 3.8 Energy Deposition in the Reactor Structure

because the first wall is highly radioactive and must not be damaged by magnetic field transients. These results would be further reduced if the toroidal field were only partially discharged, i.e., if only a single coil were discharged, or if the reactor structure were insulated to prevent breakdown across the current paths used for the model.

3.3 Induced Forces in the Reactor Structure

To find the upper limit on the induced magnetic force on a cylindrical shell, let us assume that the toroidal field is so rapidly discharged that the magnetic flux inside the shell does not have time to diffuse out through the shell. For this case, the magnetic field is

$$B = B_o \quad \text{inside shell}$$

$$B = B_o (r_o - r) \quad \text{in the shell material}$$

The current density in the shell is

$$J = \frac{B_o}{\mu_o (r_o - r_i)}$$

The field and current are shown in Figure 3.9. The cross product of current and field produces a stress gradient (pressure gradient if the conductor is fluid).

$$\bar{V}p = \bar{J} \times \bar{B} = \frac{B_o^2}{\mu_o} \frac{(r_o - r)}{(r_o - r_i)} \quad (3.4a)$$

Figure 3.9 Induced Magnetic Stress Gradient in a Cylindrical Shell

For fluids, the pressure at the outer edge of the shell may be obtained by integration of Equation 3.4a across the shell thickness. For a solid shell, Equation 3.4a should be included in the differential equation for stress in the shell which must then be solved. However, for thin shells, the integral of Equation 3.4a may be treated as a magnetic pressure and the usual stress equations for a thin walled pressure cylinder are then applied. The magnetic pressure is

$$p = \frac{1}{2} B_o^2 / \mu_o \quad (3.4b)$$

Note that the pressure is not a function of either the shell radius or the thickness. The requirement on the discharge time to reach this pressure is given by Equation 3.1. For $B_o = 4.31$ tesla, P is 7.4×10^6 N/m² (1072 psi).

For the case where the model is solved using n shells, the total applied magnetic stress to the i^{th} shell is

$$p = \frac{1}{2} \mu_o I_i^2 + \mu_o I_i \sum_{j=i+1}^n I_j \quad (3.5)$$

where I is in amp/m and the summation is taken over all shells exterior to the i^{th} shell. The results for the first wall and the lithium blanket are shown in Figures 3.10 and 3.11. The induced stress in the first wall is nearly eliminated by the shielding of the outer shells. The lithium blanket, however, still exhibits a large induced pressure rise. Therefore, it

Figure 3.10 Magnetic Pressure on the First Wall Due to a Toroidal Field Discharge

Figure 3.11 Magnetic Pressure on the Lithium Blanket Due to a Toroidal Field Discharge

is important to consider design choices which reduce this induced pressure.

One such design change would segment the blanket so that each segment is electrically insulated from the others. This requires changing our shell model from Figure 3.4a to Figure 3.4b. Let us consider one such segment and model it as a solid cylinder with its axis parallel to the applied field B_z . We wish to estimate the stress (pressure) induced when the applied field decays exponentially with a decay constant γ . The geometry and coordinate axis are shown in Figure 3.12.

It can be shown that Equations (2.1), (2.2) and (2.7) can be combined to derive the diffusion equation for magnetic field through a conducting medium,

$$\nabla^2 \bar{B} = \mu_0 \sigma \frac{\partial \bar{B}}{\partial t} \quad (3.6)$$

For our problem B_r and B_ϕ are zero and Equation (3.6) becomes

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial B_z}{\partial r} = \mu_0 \sigma \frac{\partial B_z}{\partial t}$$

This equation and the boundary conditions may be put in non-dimensional form so that the complete problem formulation may be written as

Figure 3.12 Infinitely Long Cylinder Coordinates

$$\frac{1}{r'} \frac{\partial}{\partial r'} r' \frac{\partial B}{\partial r'} = \frac{\partial B}{\partial \tau}$$

$$B(r, 0) = 1$$

$$B(0, \tau) = \text{finite}$$

$$B(1, \tau) = e^{-\alpha \tau}$$

(3.7)

where $B = B_z/B_0$ B_0 = initial applied field

$$\tau = t/\mu_0 \sigma R_0^2$$

$$r' = r/R_0$$

$$\alpha = \tau_{\text{diffusion}} / \tau_{\text{decay}} = \gamma \mu_0 \sigma R_0^2$$

The solution may be obtained using the method of variation of parameters. The solution is

$$B(r', \tau) = \sum_{n=1}^{\infty} \frac{2}{k_n J_1(k_n) (1 - k_n/\alpha)} \left(e^{-k_n^2 \tau} - \frac{k_n^2}{\alpha} e^{-\alpha \tau} \right) J_0(k_n r')$$

(3.8)

where J is Bessel's function of the first kind and k_n is the n^{th} root of $J_0(k_n) = 0$.

Formally the current density may be found from the curl of Equation 3.8. The magnetic pressure gradient is then the cross product of current and field which can be integrated across the cylinder radius to find the total pressure at the outside radius. The resulting series, however, converges very slowly and it is easier to formulate the spatial differential in Equation 3.7 using finite difference approximations. One can then derive a set of ordinary differential

equations in time which are solved using the Runge-Kutta program in Appendix IV. The curl and integration operations are done by central difference and Simpson's Rule equations respectively.

Typical field and current profiles are shown in Figures 3.13 and 3.14 for $\alpha=1000$, i.e., for a very rapid external field decay. Figure 3.15 shows that the maximum pressure decreases rapidly for α less than 2.4, the first root of $J_0(K) = 0$, and that potential damage from magnetic induced pressures can be eliminated by proper design.

3.4 Conclusion

It appears that energy deposition from toroidal field discharge should not cause any damage to the reactor structure. Field decay induced forces or pressures could reach damaging magnitudes if care is not taken to minimize the radius of possible breakdown paths. In particular, a design goal should be minimization of the quantity σR_0^2 , consistent with other design constraints where R_0 is the characteristic radius of the conductor perpendicular to the toroidal field.

Figure 3.13 Magnetic Field Decay in an Infinitely Long Cylinder.

Figure 3.14 Induced Current Density in an Infinitely Long Cylinder

Figure 3.15 Maximum MHD Pressure in an Infinitely Long Cylinder

IV. DESIGN OF MAGNET PROTECTION SYSTEMS

It was shown in Chapter 1 that passive protection systems alone are not sufficient to protect large superconducting toroidal field magnets from damage due to an unstable quench. These protection methods can be made much more effective if active switching is used. Sensitive instrumentation to detect quenches and a decision structure, either using computer logic alone or with plant operators in the loop, are required to activate the protection system. The possibility that the protection system will fail to function when needed through a failure of instrumentation, control, or actuators and the cost of such systems are the principal drawbacks of active protection. However, for very large magnet systems there appears to be no alternative.

Choice of a protection system should occur very early in the design process because this choice will affect the choice of internal structure, the material selection, and the external restraint required for design. Protection methods may be divided into two broad classes, electrical and thermal. This chapter will develop an evaluation criterion which will allow the designer, given a few basic system specifications, to choose between these two basic types of protection.

4.1 Thermalization

Thermalization protects a superconducting magnet from damage by causing the entire magnet to become normal conducting

and thereby distributes the magnet's stored energy uniformly throughout the entire magnet winding [12]. The magnet is protected because its large size and specific heat limit temperature differences within the winding to approximately 100 °K. Two methods are commonly proposed to cause the magnet to thermalize. In the first method, a resistive heating tape is interwoven with the conductor into the magnet winding. Driving current through the tape causes the conductor to quench, starting thermalization. Care must be taken in this design to prevent quenching due to induced currents during the energizing of the magnet. The second method requires dumping the liquid helium coolant out of the magnet cryostat and reinjecting warm helium gas to cause a uniform quench.

The main advantage of using thermalization is that the typical energy removal time is relatively short (on the order of 50-250 sec. for reactor sized toroidal field coils). As will be shown in Section 4.3 thermalization is the superior method for magnets having larger stored energy densities. Moreover, thermalization may be an effective means for dealing with accidental short circuits, and magnet power lead failures which are difficult to manage using electrical protection.

Thermalization, however, has several disadvantages. The thermalization process once started cannot be reversed but must be allowed to run its course before the magnet can be cooled down again and re-energized. This gives the operator very little flexibility, and makes false alarms very costly.

Finally, the thermalization process itself may damage the magnet due to temperature differences between the warm conductor and the cold support structure. It may also damage the magnet cryostat due to helium overpressures. Proper magnet design should be capable of minimizing this last problem.

Beyond designing the winding structure and cryostat to prevent damage during thermalization, the magnet designer has one major choice to make, that is the choice of discharging one or all of the coils of the toroidal field coil set. Discharging all of the coils in series has several advantages. Since the coils are discharged in series, no unbalanced forces will exist between coils during discharge. Also, as a result of using a series connection, only one set of power leads is needed through the cryostat which reduces the cryostat heat leakage and the possibility of power lead failure. Finally, the only switching needed is for starting the thermalization process and for isolating the power supply. The disadvantage of the series connection is that all of the magnet's stored energy is dissipated within the winding, maximizing the winding's final temperature, the temperature differences within the winding, and the subsequent cooldown time required before re-energizing the magnet. The cooldown time may not be a very important objection to the series connection for the first few experimental power reactors, because of the extensive investigations needed to find and correct the initiating cause of the shutdown.

Isolating and then thermalizing only the faulted coil greatly reduces the amount of energy dissipated within the coil because most (80%) of the coil's stored energy is transferred inductively to neighboring coils. However, a more complicated switch network and power leads for each coil are needed. The principal disadvantage, however, is that the unbalanced currents will cause large asymmetric forces which act on the coils next to the faulted coil. For example, for the UWMAK III design the force would be 684 MN acting in the toroidal direction on the two coils next to the faulted coil. (This is the worst case of the single coil discharge circuit described in the next section.) The external structure required to support this load is very large and it represents the principal trade-off that the designer must make in order to obtain lower internal winding temperatures during the thermalization process.

4.2 Electrical Protection

Electrical Protection requires switching external dump resistors into the magnet's electrical circuit. This causes the magnet's stored energy to be dissipated within the resistors or transferred inductively to neighboring coils. Dump resistors are usually specified rather than voltage sources because of the small value of the recovered stored energy. For very critical applications the voltage source might be used because it causes the current to ramp down at a constant rate whereas dump resistors produce exponential decay. Only

the use of dump resistors will be considered here.

The fraction of stored energy dissipated within the resistors depends on the exact protection circuit used. Four general types of circuits may be defined; the series, parallel, single coil discharge, and zero force circuits. These are shown schematically in Figure 4.1. In each case, the faulted coil is coil No. 1. These are not the only possible protection circuits but all circuits proposed to date are basically combinations of these circuits, usually designed to simplify the switching required.¹³ In this section, the trade-offs that the magnet designer can make in designing an electrical protection system will be discussed and analyzed.

The system response indices that a designer may consider in choosing a protection circuit are: 1) the total amount of the faulted coil's stored energy which must be dissipated, 2) the maximum size of the overcurrents induced in the neighboring coils, 3) the maximum forces induced due to unbalanced currents, 4) the switching complexity required, 5) the maximum voltage applied to the coil, and 6) the current decay time. Most of these indices are quantitative in nature and can be easily included in a cost or penalty function. Switch complexity is much more difficult to quantify, because it requires detailed information about a specific design. Thus, only a few general comments are made here about it. Switching complexity cannot be forgotten, however, and in particular designs it may be a dominant factor.

For a given coil set and protection circuit, the magnitudes of the response indices are functions of the resistances used

Figure 4.1 Electrical Protection Circuits

in the circuit. It can be shown that each of the first three response indices is independent of the absolute value of the circuit resistances and depends only on ratios of resistances. The system resistances therefore can be normalized with respect to the resistance connected to the faulted coil (R_1). The response indices are now functions of the normalized resistances and are independent of R_1 . The response indices may be used to define a cost function J .

$$J = a_1 F_{\max} + a_2 E_D + a_3 \Delta I_{\max} \quad (4.1)$$

where $F_{\max}$ = maximum unbalanced force induced during coil shutdown

E_D = total energy dissipated in R_1

$\Delta I_{\max} = I_{\max} - I_0$ = maximum overcurrent induced during coil shutdown

a_1, a_2 and a_3 = cost coefficients

The cost function J is independent of R_1 . The magnitude of R_1 determines the current decay time and the maximum voltages applied to the magnet coils. The shutdown time must be short enough and therefore R_1 large enough that most of the faulted coil's energy is removed before any damage occurs. Proper sizing of R_1 is the topic of Section 4.3. Making R_1 large implies large voltages to the magnet coil which might cause internal arcing. The maximum allowable voltage ($V_{\max}$) for a given winding structure type probably must be determined experimentally.

The above suggests a multi-step protection system design process. The designer first chooses a protection circuit and normalized resistances to minimize J or to meet design constraints. The methods to be developed in Section 4.3 are then used to estimate the necessary size of R_1 . Finally, R_1 is used to find the maximum voltages generated which can be checked against the maximum allowable voltage. If this voltage is exceeded, the designer has five choices. 1) repeat the above process, making alterations to J or relaxing design constraints. In essence, trading off E_D , $F_{\max}$ or $\Delta I_{\max}$ in order to improve $V_{\max}$; 2) redesign the winding structure and conductor insulation in order to improve the maximum allowable voltage; 3) raise the operating current by increasing the conductor size or by connecting several windings in parallel. This decreases the inductance of the coil and thereby improves the current decay time. The difficulty with this approach lies in designing reliable high current power leads; 4) Electrically subdivide a coil into two or more electric circuits. This increases the effective number of coils in the system and reduces the effective size of the faulted coil. The difficulties with this are that additional power leads are required and separate cryostats to provide physical isolation between subdivisions may be needed to handle quenches near the subdivision boundary. 5) Conclude the electrical protection will not work for the given coil set and a thermal protection system must be used.

To exemplify the trade-offs available in the first step of this design process, it is useful to consider a model of the UWMAK III toroidal field coil set. A single toroidal field coil is shown in Figure 4.2. To model these D-shaped coils as a set of circular coils it will be assumed that the major radius of the model is equal to the radius of the center of the midplane of the D-shaped coil. The minor diameter of the model is equal to the average of the horizontal and vertical bores. These two dimensions may be used with Eq. 2.31 to find the elements of the model inductance matrix. Comparison of the magnetic stored energy using this model with that calculated using Ref. 21 shows very good agreement. The magnet parameters and inductances are listed in Table 4.1.

The most general system equation is Eq. 2.35 with one current state for each coil and with $\underline{V} = \underline{0}$. In matrix form the state equations are

$$\underline{\bar{M}} \frac{d}{dt} \underline{I} = -\underline{\bar{R}} \underline{I} \quad (4.2)$$

which represents n state equations. When coils are interconnected, the number of state equations can be reduced by setting the interconnected coil currents equal. Repeated application of this procedure leads to the reduced state equations. If the resistances not in the faulted coil's circuit are relatively small, they may be neglected. This leads to a useful approximation for the faulted coil circuit state equation

UWMAK-III TOROIDAL FIELD MAGNET DESIGN

ALL DIMENSIONS IN METERS

Figure 4.2

Table 4.1

UWMAK III Toroidal Field Magnet Model Parameters

No. of Coils (n)	18
No. of Windings/Coil	960
Major Radius	10.1 m
Minor Radius	9.2 m
M_{11}	31.5 H
M_{12}	15.3
M_{13}	9.02
M_{14}	5.94
M_{15}	4.17
M_{16}	3.09
M_{17}	2.42
M_{18}	2.01
M_{19}	1.79
M_{110}	1.72

$$\frac{dI_1}{dt} = \frac{-M_{11}}{\text{DET } \underline{\bar{M}}} R_1 I_1 \quad (4.3)$$

$\underline{\bar{M}}$ in Eq. 4.3 is the reduced state equation induction matrix and R_1 is the sum of the resistances in the faulted coil circuit. This equation can be used to derive a useful first approximate for the total energy dissipated. The solution to Eq. 4.3 is

$$I_1 = I_0 e^{-\frac{M_{11} R_1}{\text{DET } \underline{\bar{M}}} t}$$

The power dissipated in R_1 is $R_1 I_1^2$ which when integrated gives the total energy dissipated, E_D :

$$E_D = \frac{1}{2} \frac{\text{DET } \underline{\bar{M}}}{M_{11}} I_0^2 \quad (4.4)$$

This may be used for sizing R_1 as described in Section 4.3.

For the series connection, Eq. 4.3 is the only state equation and it is therefore exact. Because the coils are connected in series, no unbalanced forces or induced over-currents occur during the shutdown cycle. This is the principal reason for using this circuit. Unfortunately, all of the magnet's stored energy (E_s joules per coil) is dissipated in the external resistors. For reactor sized magnets this means long shutdown times, on the order of 500 to 1000 seconds. If only a single external resistor is used this circuit requires a single set of power leads and only a few switches. However, this single resistor imposes a very large

voltage across the magnet leads which may cause arcing within the magnet winding. The voltage may be reduced by a factor of n by distributing resistors between the coils. However, the number of current leads and switches needed also increases by a factor of n thereby largely negating the advantageous switching simplicity of this circuit.

In the single coil discharge circuit, the faulted coil is connected to a dump resistor and is isolated from the other coils which remain connected in series. For this circuit, the two state equations are:

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = - \begin{bmatrix} R_1 I_1 \\ (n-1) R_2 I_2 \end{bmatrix} \quad (4.5)$$

The inductance elements may be calculated using Eq. 2.38. Their value for the UWMAK III model are given in Table 4.2. The factor $(n-1)$ assumes that the resistance in the second circuit is equally divided between coils.

The single coil discharge circuit allows most of the faulted coil's stored energy to be transferred inductively to neighboring coils. The minimum E_D ($R_2=0$) for this circuit may be on the order of 15 - 25% of the faulted coil's stored energy. For UWMAK III, the minimum E_D is 1.22 GJ or 21% of the faulted coil's stored energy. Because the remaining coils are in series the overcurrent in them is quite small. For $R_2=0$, $\Delta I_{\max}$ is 445 amperes for UWMAK III. R_2 may be chosen

to make the overcurrent zero by setting $dI_2/dt = 0$ at $t=0$. The condition for no overcurrent can be shown to be

$$\frac{R_2}{R_1} = \frac{M_{12}}{M_{11} \cdot (n-1)} \quad (4.6)$$

For UWMAK III, $R_2/R_1 = 0.19$ meets this condition.

The disadvantage of this circuit is that very large unbalanced forces act in the toroidal direction on the nearest neighbors to the faulted coil. For minimum E_D , the maximum force on the coils next to the faulted coils is 685 MN ($154 \cdot 10^6$ lb.f). It is possible to reduce this unbalanced force, at the cost of increasing E_D , by increasing R_2 . This trade off is shown graphically in Figure 4.3 for UWMAK III. It should be noted that implementing this trade off requires the ability to switch two different resistances into the circuit as needed.

It is possible to keep the unbalanced forces small by maintaining current symmetry around the toroid. The zero force circuit accomplishes this by connecting alternate coils in series. This assumes that there is an even number of coils in the toroid. The two state equations for this circuit are

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = - \begin{bmatrix} (n/2) R_1 I_1 \\ (n/2) R_2 I_2 \end{bmatrix} \quad (4.7)$$

The inductance matrix may be calculated by reduction of Eq. 4.2.

For UWMAK III, $M_{11} = M_{22} = 66.3H$ and $M_{12} = 54.4 H$.

Figure 4.3 Performance Variation of the Single Coil Discharge Circuit with the Normalized Second Coil Resistance

The minimum E_D for UWMAK III using the zero force circuit is 1.13 GJ or 18% of a single coil's stored energy. However, this induces a large overcurrent in the second circuit. For minimum E_D this overcurrent is 7600A or 80% over the operating current. It is possible for the designer to reduce this overcurrent at the cost of increasing E_D by increasing R_2/R_1 . This trade off is shown graphically in Figure 4.4 for UWMAK III. Again, it should be noted that implementing this trade off requires the ability to switch two different resistors into the circuit.

In each of the above three circuits we have the ability to trade off, or balance, at most only two out of the three elements of the cost function (Eq. 4.1). The parallel circuit is the most versatile of four circuits defined. It retains all n state equations of Eq. 4.2 and we may vary, at least theoretically, each of the n resistances independently to minimize the cost function or to meet design specifications and constraints. Indeed, we can choose the resistances so that the parallel circuit mimics the response of the three other circuits. The drawback to this flexibility is the complex switching required to implement it.

The absolute minimum E_D may be obtained using the parallel circuit with R_2 to $R_n = 0$. For UWMAK III $E_D = 1.01$ GJ, $F_{\max} = 743$ MN and $\Delta I_{\max} = 33\%$ which differ from the two coil discharge circuit by -17%, +9% and 560% respectively.

The performance of the parallel circuit may be improved by adjusting the normalized resistances R to minimize the cost

Figure 4.4 Performance Variation of the Zero Force Circuit with Normalized Second Coil Resistance

function. As an example of the results that can be obtained, it will be assumed that the designer wishes $\Delta I_{\max}$ to be constrained to less than 10%. To constrain $\Delta I_{\max}$, the last term in Eq. (4.1) is changed to

$$a_3 \cdot (\Delta I_{\max} - \Delta I_{\text{cst}}) \cdot \delta(\Delta I_{\max} - \Delta I_{\text{cst}})$$

where ΔI_{cst} = constrained value of $\Delta I_{\max}$

$$\begin{aligned} \delta(x) &= 0 \text{ if } x < 0 \\ &= 1 \text{ if } x > 0 \end{aligned}$$

and a_3 is made "large". For this problem, a_1 is set to 1.0 and then a_2 is varied in order to vary $F_{\max}$ and E_D . The results of these calculations are shown in Figure 4.5. The normalized resistances which achieve this minimization are represented by bar graphs in Figure 4.6. The sensitivity of the results with respect to the choice of a_2 should be noted for a_2 in the range 0.2 to 0.4. A better design and calculation procedure would probably be to constrain both $\Delta I_{\max}$ and $F_{\max}$ and then choose R to minimize E_D . While it may not be possible to implement the parallel discharge circuit it is still of interest because the response indices for this circuit, generated in the manner described above, represent the "best" that the designer can hope for using electrical protection. The relative performance of the four discharge circuits is summarized in Table 4.2.

4.3 Approximate Dump Resistor Sizing

The design criterion for a magnet protection system is that it must prevent magnet damage due to thermal strain caused

Figure 4.5 Optimized Performance of the Parallel Circuit

Figure 4.6 Optimized Parallel Circuit Resistances

Table 4.2

Comparison of Discharge Circuit Performance
for Minimum Discharged Energy for UWMAK III

Circuit Type	Minimum E_D (GJ/coil)	Overcurrent ΔI (Amps)	Maximum Intercoil Force (10^6 newtons)
Series	6.00	0	0
Single Coil Discharge	1.22	455	685
Zero Force	1.13	7600	0
Parallel			
(minimum E_D)	1.01	3150	743
(10% ΔI & minimum E_D)	1.47	950	558
(10% ΔI & minimum force)	3.14	950	113

by an unstable quench. A study by J. Powell, et al. [12] using a finite element heat transfer code shows that the greatest temperature difference and therefore the greatest thermal stress occurs between the conductor and the winding support structure.

One way to check that this design criterion is being met is to simulate the system by integrating the state equations of the system. Thermal states may be appended to Eq. 4.2 in order to find the temperature differences which will exist. As a simple approximation, two thermal nodes are defined: one in the conductor and the other in the winding structure. A single heat transmittance is defined between them. The conductor electric resistivity and specific heat and the structure specific heat are non-linear functions of temperature. They may be calculated using a BASIC subroutine described in Appendix I. The thermal state equations are:

For the conductor:

$$\frac{dT_c}{dt} = \frac{1}{\rho_{m_c} A_c C_c(T_c)} \left[\ell_p h(T_s - T_c) + \frac{\rho_R(T_c)}{A_c} I_1^2 \right] \quad (4.8)$$

For the structure:

$$\frac{dT_s}{dt} = \frac{1}{\rho_{m_s} A_s C_s(T_s)} \left[\ell_p h(T_c - T_s) \right] \quad (4.9)$$

where ρ_m = mass density
 ρ_R = resistivity
 T = temperature

A = cross-sectional area
 l_p = perimeter length
 c = specific heat
 h = heat transmittance
 L = length of quenched conductor
 subscript c = conductor node
 subscript s = structure node

The node placement for a UWMAK III type coil construction is shown in Figure 4.7. The voltage drop across the quenched region is included by changing the faulted coil's state equation (current state no. 1 of Eq. 4.2) to

$$\sum_{j=1}^n M_{1j} I_j = -[R_1 + L \rho_{R_c}(T_c)/A_c] I_1$$

The third major thermal component of the winding, the helium coolant, is not included in this simulation because only a small portion of the dissipated energy is transferred to the helium vapor due to its low heat capacity. As a check, it may be assumed that thermalization occurs and that all three components reach thermal equilibrium. Such a calculation, based on the UWMAK III winding, shows that less than 1% of the dissipated energy ends up in the helium vapor. This assumption is not appropriate if the designer is interested in finding the threshold between a stable and an unstable quench or if most of the coil's energy is dissipated in an external dump resistor. In the latter case the assumption of no helium is a conservative one and corresponds to the possibility of a loss of coolant initiating a quench. Finally, this assumption greatly simplifies the simulation and analysis.

Calculations using models of the UWMAK I and UWMAK III toroidal field magnets were performed. The model parameters

$$\text{assume } k_{\text{epoxy}} = 0.001 \text{ W/cm}^{\circ}\text{K}$$

$$h = k/d = 0.02 \text{ W/cm}^2\text{-}^{\circ}\text{K}$$

$$A_c = (1.5)(1.5) = 2.25 \text{ cm}^2$$

$$A_s = 2(.75)(2.5) + (1.5)(1.0) = 5.25 \text{ cm}^2$$

Figure 4.7 Winding Thermal Model for UWMAK III

Table 4.3

Base Case Parameters

	UWMAK I	UWMAK III
Total Stored Energy	158 GJ	108 GJ
Circular Model Major Radius	13.2 m	10.1 m
Minor Radius	9.25 m	9.2 m
No. of Coils	12	18
No. of Turns/Coil	2040	960
Conductor Length/Turn	64 m	65 m
Total Conductor Length	1567 km	1123 km
Total Inductance	2976 H	2371 H
2 Coil Discharge Inductances		
M_{11}	125.2 H	31.5 H
M_{12}	122.8 H	99.7 H
M_{22}	2605.0 H	2130.8 H
Operating Current	10212 A	9545 A
Conductor Material	NbTi-Cu	NbTi-Al
Structural Material	Stainless Steel	2219 Al alloy
Conductor X-sec. Area	3.35 cm ² (ave.)	2.25 cm ²
Structure X-sec. Area	4.92 cm ² (ave.)	5.25 cm ²
Conductor-Structure Perimeter	5.30 cm	4.50 cm
Heat Transmittance	0.01 W/cm ² -°K	0.02 W/cm ² -°K
Adiabatic Winding Thermalization Temperature	180 °K	300 °K

are given in Table 4.3. UWMAK I uses a copper stabilizer and a stainless steel winding structure. UWMAK III uses an aluminum stabilizer and an aluminum winding structure. The stored energy in UWMAK I is approximately 50% larger than that of UWMAK III. To investigate whether thermalization would provide magnet protection, the magnet coils were all assumed to be connected in series (which maximizes the energy dissipated within the winding) and the entire magnet was assumed to be driven normal. Figure 4.8 shows the temperature difference between the conductor and the winding structure as calculated using this model. The results suggest that the copper-stainless steel system is a better choice if this temperature difference is the only criterion because the effect of the larger heat capacity of copper outweighs the effect of aluminum's smaller resistivity. In both cases the maximum temperature difference is less than 100 °K which is probably acceptable.

When thermalization is used all of the coil array's stored energy, E_s , is dissipated internally within the winding. The purpose of an electrical protection system is to reduce the internal dissipation to an acceptable level. Energy dissipated in the external dump resistors is denoted, E_R . Energy dissipated within the coil winding is denoted, E_C . The sum of these two energies is the total energy dissipated, E_D . It is generally easiest to calculate these energies on a per coil basis. When this is done, then the difference between E_s and E_D is the energy transferred from the faulted coil to the neighboring coils.

In the most general case E_D depends on the size of the normal region as well as the circuit parameters. However, in two cases E_D can be calculated independent of the quenched region size. In the first case the quenched region resistance is assumed to be "small" relative to the external dump resistors. E_D here is equal to E_R and depends only on the circuit parameters. This E_D is calculated for various discharge circuits in Section 4.2. The assumption of a small quench is usually appropriate because most quenches are initiated locally and propagate slowly. This case is also satisfied when the protection system is sized to guarantee that E_R is large compared to E_C regardless of the quench size.

The second case occurs if all of the coils not connected to the faulted coil are placed in a persistent mode. In this case, E_D depends only on the inverse of the inductance matrix and is independent of the ratio of E_R to E_C . In the remainder of this section, E_D will be a protection circuit parameter calculated using one of the above two assumptions.

The single coil discharge circuit was chosen as an example to investigate the effectiveness of an electrical protection method for UWMAK I and UWMAK III. To minimize the amount of energy that must be dissipated the non-faulted coils were taken to remain in the persistent mode. The state equation for the faulted circuit is

$$\frac{d}{dt} I_1 = \frac{-M_{22}}{M_{11}M_{22} - M_{12}^2} (R_1 + L \frac{\rho_R(T_C)}{A_C}) I_1$$

The state equations for the temperatures of the conductor and winding structure are given by Equations (4.8) and (4.9). The inductances used are given in Table 4.3. Temperature dependent properties were calculated using the computer program given in Appendix 1. A number of calculations were performed using different sizes of the discharge resistor R_1 and the quenched conductor length L . The results are varied against the maximum discharge power ($P_D = R_1 I_O^2$) because the designer can vary this power by varying either the resistance or the operating current.

It is important to note that the quench size is an input parameter and that quench propagation and heat transfer along the conductor are not included in the model. Because of the very complex internal geometry of the magnet winding quench propagation is very difficult to model. The assumption of a non-propagating (i.e. adiabatic along the conductor length) quench provides a conservative estimate of conductor temperature because the energy available to raise the conductor's temperature is maximized. However, this model does not provide any information about internal coil voltages because L is an assumed value. Large voltages across the quenched conductor can cause internal arcing and serious magnet damage. Since quench propagation is not included, the resulting temperature histories are insensitive to L unless L is large enough to make the coil energy dissipation time scale of the same order of magnitude as the conductor temperature rise time scale.

It is interesting to compare two measures of protection system effectiveness based on two choices of quenched conductor length. One measure is the maximum temperature difference between the conductor and the winding structure during a small quench. This is denoted as the small quench ΔT . A small quench is approximated by assuming that a single coil turn quenches. A small quench maximizes ΔT because only an insignificant fraction of the total stored energy is used to produce a very large ΔT . The second measure is the fraction of energy dissipated in the coil to the total energy dissipated, E_c/E_D , when the entire coil quenches. The measure is denoted the coil dissipation fraction. Since E_D is a constant, this fraction is a measure of the electrical protection system's ability to remove E_D when it must compete for E_D against the largest possible quench. In either case the designer wishes to minimize the measure.

The effect of R_1 on the small quench ΔT and the coil dissipation fraction were calculated using the base case heat transmittance given in Table 4.3. The coil dissipation fraction was also calculated assuming that the heat transmittance was zero and infinity. The assumption of an adiabatic conductor is the most conservative assumption possible and it provides an upper limit on the discharge power required. The assumption of perfect thermal contact between conductor and structure is the most optimistic assumption possible and it provides a lower limit on the discharge power required. It is

interesting to note that both of these assumptions reduce separate thermal nodes to a single one. The results from these simulations are superimposed in Figures 4.9 and 4.10.

Special attention is now given to the behavior of the small quench ΔT . There appears to be a threshold value of discharge power above which no damage will occur and below which very serious damage will occur. Comparison of this threshold with the corresponding coil dissipation fraction (for the base case heat transmittance) suggests that the threshold coil dissipation fraction has a constant value of 50%. Repeated simulations using UWMAC I and UWMAC III and the series and parallel circuits with several heat transmittances shows that this is a general observation. Because this simulation ignores the effects of quench propagation along the length of the conductor, the initial temperature rise of the conductor is only a weak function of the assumed quench size. Therefore, a dump resistor sized to mitigate the large quench measure by removing at least 50% of E_D will also mitigate the small quench measure. Including the quench propagation would lower the small quench ΔT . Therefore, this simulation provides a conservative estimate of the required discharge power.

For designing a protection system, the designer would like to replace this computer simulation with a pencil and paper technique to estimate the required discharge power. Because material properties are very nonlinear at low temperatures there does not appear to be any simple technique for calculating the small quench ΔT . Fortunately because of its threshold

Figure 4.8. The Thermal Response of the Magnet Winding When All of the Coils Quench While Connected in Series

behavior the designer need only be able to assure that it will be "small". A design rule based on the coil dissipation fraction can be simply stated: The external discharge power must be large enough to remove at least one-half of the total dissipated energy, E_D , assuming the entire coil quenches at the beginning of the discharge cycle. An entire coil quench means that all of the windings in the circuit connected to the faulted coil quench. This design rule involves only integrated quantities and it is therefore much easier to calculate than one based on the small quench ΔT .

The total dissipated energy E_D is only a function of the discharge circuit as was noted earlier. E_D can be used to define a single equivalent inductance $M_{eq} = 2E_D/I_o^2$ which can be used to write a single electrical state equation. Using this equivalent inductance, the simulation has three state equations, one is electrical and the other two thermal. To be able to find an analytical solution to this system, the number of state equations must be reduced to two. This can be done by merging the two thermal nodes together. This new single node consists of the conductor and a portion of the winding structure which together have a single average temperature. For convenience, the cross-sectional area of the winding structure participating thermally can be defined as a nondimensional structure fraction, $f = A_s/A_c$. The structure fraction becomes a free variable replacing the heat transmittance. A useful correlation for estimating the structure fraction given the heat transmittance and the actual winding cross-section will be

developed later in this section.

Obtaining analytical results requires analytical expressions for the temperature dependent specific heats and resistivities. In order to obtain low order polynomial fits, two expressions for each property were derived. One expression is used below $T_g = 50$ to 70 °K where properties show much different behavior than they do above that temperature where a second expression is used. The expressions for specific heat and resistivity are of the general form

$$\begin{aligned} c &= \alpha_0 + \alpha_1 T + \alpha_2 T^2 + \alpha_3 T^3 \\ \rho_R &= \beta_0 + \beta_1 T + \beta_4 T^4 \end{aligned} \quad (4.10)$$

and are given in Table 4.4.

The system equations become

$$M_{eq} \frac{dI}{dt} = \frac{-L}{A_c} [\beta_0 + \beta_1 T + \beta_4 T^4] I - R_1 I \quad (4.11)$$

$$\frac{dT}{dt} = \frac{L/A_c}{\rho_{mC} LA_c} \frac{[\beta_0 + \beta_1 T + \beta_4 T^4] I^2}{[\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \alpha_3 T^3]} \quad (4.12)$$

where the specific heat coefficients become

$$\alpha_n = \alpha_n \Big|_c + f \frac{\rho_{mS}}{\rho_{mC}} \alpha_n \Big|_s \quad (4.13)$$

The system equations can now be manipulated to obtain an expression for the discharge power to satisfy the proposed rule of thumb. Dividing Eq. 4.11 by Eq. 4.12 yields

$$\frac{-M_{eq}}{A_c L} I \frac{dI}{dT} = \rho_{mC} \frac{(\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \alpha_3 T^3)}{(\beta_0 + \beta_1 T + \beta_4 T^4)} \quad (4.14)$$

Figure 4.9: Comparison of Protection System Effectiveness Measures for UWMAK I

Figure 4.10: Comparison of Protection System Effectiveness Measures for UWMak III

Table 4.4

Temperature Dependent Resistivities and Specific Heats

Aluminum $\rho_m = 2.71 \text{ g/cm}^3$

$$\rho_R = (10 + 7.94 \cdot 10^{-6} \cdot T^4) \cdot 10^{-9} \text{ ohm-cm} \quad 10 < T < 70$$

$$\rho_R = (-550 + 10.7 \cdot T) \cdot 10^{-9} \text{ ohm-cm} \quad 70 < T < 300$$

$$c = (15.4 - 2.53 \cdot T + .118 \cdot T^2 - 380 \cdot 10^{-6} \cdot T^3) \cdot 10^{-3} \text{ J/g-}^\circ\text{K} \quad 10 < T < 70$$

$$c = (-226 + 8.43 \cdot T - .0156 \cdot T^2) \cdot 10^{-3} \text{ J/g-}^\circ\text{K} \quad 70 < T < 300$$

Copper $\rho_m = 8.95 \text{ g/cm}^3$

$$\rho_R = (16 + 9.1 \cdot 10^{-6} \cdot T^4) \cdot 10^{-9} \text{ ohm-cm} \quad 10 < T < 50$$

$$\rho_R = (-258 + 6.62 \cdot T) \cdot 10^{-9} \text{ ohm-cm} \quad 50 < T < 300$$

$$c = (3.07 - .609 \cdot T + .0362 \cdot T^2 + 272 \cdot 10^{-6} \cdot T^3) \cdot 10^{-3} \text{ J/g-}^\circ\text{K} \quad 10 < T < 50$$

$$c = (-111.4 + 4.927 \cdot T - .01294 \cdot T^2) \cdot 10^{-3} \text{ J/g-}^\circ\text{K} \quad 30 < T < 200$$

Stainless Steel $\rho_m = 7.80 \text{ g/cm}^3$

$$c = (6.72 - .645 \cdot T + .047 \cdot T^2) \cdot 10^{-3} \text{ J/g-}^\circ\text{K} \quad 10 < T < 50$$

$$c = (-166.8 + 5.96 \cdot T - .0156 \cdot T^2) \cdot 10^{-3} \text{ J/g-}^\circ\text{K} \quad 50 < T < 200$$

Notes:

1) T is in $^\circ\text{K}$

2) Grunsison's Function, $G(T/\theta_R) = 48.94 (T/\theta_R)^3$, where $\theta_R/T > 6$ was used to derive the low temperature resistivities

Integrating Eq. 4.14 between I_0 and $I_f = 0$ and T_0 and T_f yields

$$\begin{aligned} \frac{1}{2} \frac{M_{eq}}{A_c L} I_0^2 = \rho_{mC} \left\{ \alpha_0 (T_f - T_0) + \frac{\alpha_1}{2} (T_f^2 - T_0^2) + \frac{\alpha_2}{3} (T_f^3 - T_0^3) + \frac{\alpha_3}{4} (T_f^4 - T_0^4) \right\} \\ + \frac{R_1 A_c}{L} \rho_{mC} \int_{T_0}^{T_f} \frac{(\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \alpha_3 T^3)}{(\beta_0 + \beta_1 T + \beta_4 T^4)} dT \end{aligned} \quad (4.15)$$

The LHS of Eq. 4.15 is the specific dissipated energy, E_D''' , based on the total conductor volume. The first term of the RHS of Eq. 4.15 is the total energy dissipated within the magnet winding. The second term is the total energy dissipated within the external resistor. For generality, $\rho_{R1} = R_1 A_c / L_c$ is defined to be the external specific resistivity. To satisfy the proposed rule of thumb for the required discharge power, it is required that one half of the dissipated energy be dissipated within the external resistance

$$1/2 E_D''' = \rho_{R1} \rho_{mC} \int_{T_0}^{T_f} \frac{(\alpha_0 + \alpha_1 T + \alpha_2 T^2 + \alpha_3 T^3)}{(\beta_0 + \beta_1 T + \beta_4 T^4)} dt \quad (4.16)$$

where

$$\begin{aligned} 1/2 E_D''' &= \rho_{mC} \left\{ \alpha_0 (T_f - T_0) + \frac{\alpha_1}{2} (T_f^2 - T_0^2) + \frac{\alpha_2}{3} (T_f^3 - T_0^3) + \frac{\alpha_3}{4} (T_f^4 - T_0^4) \right\} \\ &= \rho_{mC} G(T_f, T_0) \end{aligned} \quad (4.17)$$

For $T < T_Y$, $\beta_1 = 0$ and Eq. 4.16 becomes

$$\begin{aligned} \frac{1}{2} E_D''' = \rho_{R1} \rho_{mC} & \left\{ \frac{\alpha_1}{2\sqrt{\beta_O \beta_4}} \left(\tan^{-1} \frac{T_f^2}{\sqrt{\beta_O \beta_4}} - \tan^{-1} \frac{T_O^2}{\sqrt{\beta_O \beta_4}} \right) \right. \\ & + \frac{1}{4} \left(\frac{\alpha_O \xi}{\beta_O} - \frac{\alpha_2}{2\beta_4 \xi} \right) \left(\ln \frac{T_f^2 + 2\xi T_f + 2\xi^2}{T_f^2 - 2\xi T_f + 2\xi^2} - \ln \frac{T_O^2 + 2\xi T_O + 2\xi^2}{T_O^2 - 2\xi T_O + 2\xi^2} \right) \\ & + \frac{1}{2} \left(\frac{\alpha_O \xi}{\beta_O} + \frac{\alpha_2}{2\beta_4 \xi} \right) \left(\tan^{-1} \frac{2\xi T_f}{2\xi^2 - T_f^2} - \tan^{-1} \frac{2\xi T_O}{2\xi^2 - T_O^2} \right) \\ & \left. + \frac{\alpha_3}{4\beta_4} \ln \frac{\beta_O + \beta_4 T_f^4}{\beta_O + \beta_4 T_O^4} \right\} \end{aligned} \quad (4.18)$$

$$= \rho_{R1} \rho_{mC} H_1(T_f, T_O)$$

where $\xi = (\beta_O/4\beta_4)^{1/4}$. Care must be taken to use the branch of $\tan^{-1}$ from 0 to π . For $T > T_Y$, Eq. 4.17 becomes

$$\frac{1}{2} E_D''' = \rho_{mC} [G_1(T_Y, T_O) + G_2(T_Y, T_Y)] \quad (4.19)$$

where the subscripts 1 and 2 refer to using the proper coefficients for $T < T_Y$ and $T > T_Y$ respectively. Eq. 4.18 with $\beta_3 = 0$ and $\beta_4 = 0$ for $T > T_Y$ becomes

$$\frac{1}{2} E_D''' = \rho_{R1} \rho_{mC} [H_1(T_Y, T_O) + H_2(T_f, T_Y)] \quad (4.20)$$

where

Figure 4.11 The Required External Resistivity to Remove at Least 50% of the Dissipated Energy for Copper Stabilizer with a Stainless Steel Winding Structure

Figure 4.12 The Required External Resistivity to Remove at Least 50% of the Dissipated Energy for Aluminum Stabilizer with an Aluminum Winding Structure

$$H_2(T_f, T_Y) = \frac{\alpha_2}{2\beta_1} (T_f^2 - T_Y^2) + \frac{\alpha_1}{\beta_1} - \frac{\beta_o \alpha_2}{\beta_1^2} (T_f - T_1) \\ + \frac{\alpha_o}{\beta_1} - \frac{\beta_o \alpha_1}{\beta_1^2} + \frac{\beta_o^2 \alpha_2}{\beta_1^3} \ln \frac{\beta_o + \beta_1 T_f}{\beta_o + \beta_o T}$$

Given E_D''' and f we may solve Eq. 4.17 (4.19) for T_f and then use T_f in Eq. 4.18 (4.20) to solve for ρ_{R_1} . Figure 4.11 illustrates the results for copper conductor with a stainless steel winding structure. Figure 4.12 illustrates the results for aluminum conductor with an aluminum winding structure. For the UWMAK I example E_D''' is 142 J/cm^3 and for the UWMAK III example E_D''' is 87.1 J/cm^3 . For the adiabatic assumption ($f=0$) we find ρ_{R_1} to be $110 \times 10^{-9} \text{ ohm-cm}$ for UWMAK I and $120 \times 10^{-9} \text{ ohm-cm}$ for UWMAK III. The maximum discharge powers are 44.7 and 30.3 MW respectively. Comparison of these results with the adiabatic cases illustrated in Figures 4.9 and 4.10 show very good agreement.

To estimate the value of f which includes heat transfer between the conductor and the winding structure, a linear thermal state equation using average properties will be assumed. It will also be assumed that ρ_{R_1} has been chosen for the desired electrical protection system using the assumption of an adiabatic winding. The state equations for a thermal node in the conductor and in the winding structure become

$$\frac{dT_c}{dt} = \frac{P'''}{\rho_{m_c} \bar{c}_c} e^{\frac{-2P'''}{E_D'''} t} + \frac{h\ell p}{A_c \rho_{m_c} \bar{c}_c} (T_s - T_c) \quad (4.21)$$

$$\frac{dT_s}{dt} = \frac{h\ell_p}{A_s \rho_{m_s} \bar{c}_s} (T_c - T_s) \quad (4.22)$$

$P''' = \rho_{R_1} I_O^2 / A_C^2$ is the specific discharge power and because of the way ρ_{R_1} is calculated it is a good approximation for the average heat generation rate within the quenched conductor. $\bar{c}_c$ and $\bar{c}_s$ are the average specific heats for the conductor and the winding structure respectively. The factor 2 in the exponential of the first term of Eq. (4.21) reflects the imposed requirement that only 1/2 of the dissipated energy is dissipated in the external resistor.

Exact solutions for $T_c(t)$ and $T_s(t)$ can be found. Using the solution for $T_s(t)$ it may be shown that the rate of winding structure temperature rise is

$$\frac{dT_s}{dt} = \frac{P''' \left[\exp(-2P''' t/E_D''') - \exp\left(h\ell_p \left(\frac{1}{\rho_{m_c} \bar{c}_c} + \frac{1}{f\rho_{m_s} \bar{c}_s} \right) t/A_c \right) \right]}{\left[\rho_{m_c} \bar{c}_c + f_{\text{actual}} \rho_{m_s} \bar{c}_s \left(1 - \frac{2 \rho_{m_c} \bar{c}_c P'''}{E_D''' h\ell_p/A} \right) \right]} \quad (4.23)$$

For the assumption that the heat transmittance is infinite which is the assumption used to construct Figure 4.11 and 4.12, the rate of winding structure temperature rise is

$$\frac{dT_s}{dt} = \frac{P''' \exp(-2P''' t/E_D''')}{(\rho_{m_c} \bar{c}_c + f_{\text{effective}} \rho_{m_s} \bar{c}_s)} \quad (4.24)$$

If the second exponential in Eq. (4.23) is small, as is typically the case because the heat transfer time constant is much smaller than the electrical time constant for very large coils, then Eq. (4.24) will approximate the more accurate Eq. (4.23) if

$$f_{\text{effective}} = \left(1 - \frac{2 m_c \bar{c}_c P_{\text{adiabatic}}'''}{E_D''' h \ell_p / A_c} \right) f_{\text{actual}} \quad (4.25)$$

Eq. (4.25) provides the approximation for f to be used with Figures 2.11 and 4.12. If f_{eff} turns out to be less than zero then the winding should be assumed to be adiabatic. The average conductor specific heat can be calculated using Eq. (4.17) or (4.19) to calculate T_f . Both T_f and Eq. (4.10) can then be used to calculate the average specific heat. The results of such a calculation are illustrated for copper conductor with a stainless steel winding structure in Figure 4.13 and for an aluminum conductor with an aluminum winding structure in Figure 4.14. The value of f_{actual} should be used to calculate $\bar{c}_c$ in Eq. (4.25).

A straight forward method for sizing the required dump resistor can now be outlined. First the designer chooses the desired electrical protection circuit. Use is made of the common currents within the circuit to reduce the system equations as described in Section 4.2. The normalized resistances are used to calculate E_D''' . Second, the conductor is assumed to be adiabatic ($f=0$) and ρ_{R_1} is calculated. Third, the adiabatic value of ρ_{R_1} is used to estimate f using Eq. (4.25). The design value of ρ_{R_1} can now be found using the estimate of f . Only one

Figure 4.13 Estimation of the Copper Stabilizer Average Specific Heat

Figure 4.14 Estimation of the Aluminum Stabilizer Average Specific Heat

application of Eq. (4.25) should be made because repeated applications do not cause convergence to more accurate estimates of f .

To illustrate, the UWMAK I example used before will be duplicated for the base case. For this example $E_D''' = 142 \text{ J/cm}^3$, $h\ell_p/A_c = 0.159 \text{ w/cm}^3 \text{ }^\circ\text{Kg}$, $P''' = 1.04 \text{ w/cm}^3$, $\bar{c}_c = 0.94 \text{ J/g }^\circ\text{K}$, and $f_{\text{actual}} = 1.47$. Using these values, the new estimate for f from Eq. (4.25 is $f = .33$. For $f = .33$, the new ρ_{R1} is 70.8×10^{-9} ohm-cm. The corresponding discharge power is 29 MW. A comparison with Figure 4.9 shows very good agreement.

4.4 Summary

In this chapter a straight forward method for designing a magnet protection system has been outlined. This method may be used in the very early stages of the magnet design process and it allows the designer to explore the effect that magnet parameters such as conductor type, number of coils, winding structure and operating current have on the protection system design. From the examples used in this chapter, it can be seen that magnet protection can be made easier by limiting the individual coils to sizes as small as possible, by making the operating current as large as possible, and by making the heat transmittance between the conductor and winding structure as large as possible.

V. CONCLUSIONS AND RECOMMENDATIONS

The safety aspects of the toroidal magnet system of proposed fusion reactor designs is of concern because of the enormous stored energy in the magnet system and because these systems may be unstable when large disturbances occur.

Designing protection systems for these magnets is a difficult task because of the magnitude of energy stored and the rapidity with which it may be released. The largest superconducting magnets built to date are at the lower fringe of this protection problem.

Three types of problems due to a magnet failure may be postulated:

- 1) The toroidal field collapse may damage the reactor structure.
- 2) Internal heating of the magnet winding due to arcing or quenches may cause internal damage due to thermal strains.
- 3) Failure of the magnet structure may allow the magnet to burst or to move relative to the reactor as a unit.

In the case of bursting or gross movement, the stored energy is converted into mechanical work and kinetic energy. Protection against damage in the first two types of problems is the topic of this thesis.

An analysis has been developed which shows that even if very conservative assumptions are made, it is not possible to dissipate inductively enough of the toroidal field energy

within the reactor structure to cause damage to the reactor blanket or inner wall. It is possible that damaging stresses may be induced by the field collapse if care is not taken to keep the length of poloidal breakdown paths within the reactor small. To prevent this type of damage, the designer should minimize the quantity σR_o^2 , where σ is electrical conductivity and R_o is the radius of the possible breakdown path perpendicular to the toroidal field.

Providing protection against internal magnet damage due to an unstable quench necessitates careful magnet design. The design objective of such a system is that it should prevent damage due to thermal strains. Two types of protection systems have been proposed: thermal and electrical. If a thermal protection system is chosen, the magnet designer must assure that the conductor heat capacitance and the heat transmittance between the conductor and the structure are large enough to prevent thermal strains during the coil shutdown. If an electrical protection system is chosen, the dump resistor or voltage source must be large enough to dissipate at least half of the total energy released if the entire coil or coils protected quenched uniformly. An analysis for properly sizing a dump resistor to meet this criterion was outlined in Sec. 4.3. For the first experimental power reactor, thermal protection may be preferable since it is a more conservative choice which is simpler to implement and is capable of handling a greater variety of magnet accidents. Only after some operating

experience with these very large magnets has been gained, will the magnet designer be able to make a balanced assessment of the advantages gained by using electrical protection.

Three areas for future work in the area of magnet safety are recommended.

- 1) A thermal model of the internal magnet winding needs to be developed. Because of the intricacy of the internal structure, a major portion of the first effort in this direction should be to determine the minimum total size of the model which still accurately predicts experimental results rather than trying to expand the capabilities of existing thermal analysis codes. Two possible applications of the winding thermal model would be to explore the thermalization properties of different winding configurations (e.g. pancake, layer, solid disk, etc.) and to explore the transition point between stable and unstable quenches (e.g. Could increased coolant flow achieve recovery from an unstable quench?).
- 2) A method is needed for evaluating the cost of using complex switching schemes in order to improve the performance of electrical protection systems. The cost function should include the cost of installation and maintenance as well as the effect of the switching scheme and hardware on the system reliability.
- 3) A model needs to be developed which combines the structural and magnetic field dynamic interactions to

be applied to the magnet coils and their support structure. Such a model could be used to investigate the possibility of magneto-elastic buckling, to provide an estimate of the damage caused if a lateral support fails allowing gross coil movement and to provide an estimate of the size and kinetic energy of coil fragments caused by a coil burst. The model should include the effect of the mechanical work done in order to estimate the points at which fracture occurs.

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APPENDIX I

MAGNET TEMPERATURE DEPENDENT PROPERTIES

The specific heats and resistivities of materials used in superconducting magnet windings are temperature dependent at cryogenic temperatures. These dependences are illustrated graphically in Figures I.1 and I.2.

Although theoretic expressions are available which predict specific heats of metals at very low temperatures they are not particularly accurate over the whole range from cryogenic to room temperature. As a result the specific heats are calculated by second order interpolation between selected experiment measurements.⁴ The interpolation formula is

$$f(x_i+h) = f(x_p) + hf'(x_i) + \frac{1}{2}h^2 f''(x_i) \quad (I.1)$$

$$\text{where } f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}$$

$$f''(x_i) = \frac{2}{x_{i+1} - x_{i-1}} \left(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right)$$

i refers to the ith item in the look-up table. Table entries do not need to be equally spaced to use this formula.

The resistivity of metals may be calculated using Matheson's Rule which states that the resistivity is sum of terms, each of which is the function of one parameter. For example

$$\rho_R = \rho_{R0} (\text{impurity density}) + \rho_{R1}(T) + \rho_{R2}(B) + \text{etc.} \quad (\text{I.2})$$

since the magnet field seen by the conductor is very difficult to calculate and because ρ_{R2} is small compared to the other terms above about 30 °K, the last term in Eq. I.2 will be dropped. The first term ρ_{R0} , called the residual resistivity, is the dominant term at liquid helium temperatures and its value may vary over several orders of magnitude depending on the metal purity. The residual resistivity is often used as a measurement or specification of purity. The temperature dependent term may be calculated using Grunsision's Function which is tabulated in Ref. 3.

$$\rho_{R1}(T) = \rho_{R1} \text{TG} \left(\frac{\theta R}{T} \right) \quad (\text{I.3})$$

where $G \left(\frac{\theta R}{T} \right) = \text{Grunsison's Function}$

$\theta R = \text{scale temperature}$

$\rho_{R1} = \text{constant which fits experimental data}$

The following listing is a BASIC subroutine which calculates the specific heats of aluminum, copper and stainless steel and the resistivities of copper and aluminum. Note: the READ, DATA combination allow the initialization of the look-up table in much the same way as the DATA statement in FORTRAN.

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Figure I.1 Low Temperature Resistivities of Aluminum and Copper

Figure I.2 The Low Temperature Specific Heats of Aluminum, Copper, and Stainless Steel (SS)

```

100 REM      THIS SUBROUTINE PACKAGE CALCULATES THE TEMPERATURE DEPENDENT SPECIFIC
102 REM      HEATS FOR COPPER, ALUMINUM, AND STAINLESS STEEL AND RESISTIVITIES FOR
104 REM      COPPER AND ALUMINUM. THE PROPERTIES ARE CALCULATED USING SECOND ORDER
106 REM      INTERPOLATION FROM VALUES STORED IN THE ARRAY C. THE FORMAT OF EACH
108 REM      ROW OF C FOR THE FIRST SIX COLUMNS IS:
110 REM      T(AL), C(AL), T(CU), C(CU), T(SS), C(SS),
112 REM      1      2      3      4      5      6
114 REM
116 REM      THE ARRAY CONTAINS 30 ROWS OF DATA. THE RESISTIVITIES ARE CALCULATED
118 REM      USING MATHESON'S RULE AND GRUNSON'S FUNCTION. MATHESON'S RULE STATES
120 REM      THAT THE RESISTIVITY IS THE SUM OF A CONSTANT CALLED THE RESIDUAL
122 REM      RESISTIVITY WHICH IS A VERY STRONG FUNCTION OF MATERIAL IMPURITIES
124 REM      ALONE AND A TEMPERATURE DEPENDENT TERM WHICH GOES TO ZERO AS TEMPERATURE
126 REM      GOES TO ZERO. THE TEMPERATURE DEPENDENT TERM MAY BE CALCULATED USING
128 REM      GRUNSON'S FUNCTION.
130 REM
132 REM      
$$R = R_0 + R_1 T^* G(TG/T)$$

134 REM
136 REM      TG IS A SCALE TEMPERATURE, SIMILAR TO THE DEBYE TEMPERATURE, WHICH IS
138 REM      USED TO FIT GRUNSON'S FUNCTION TO EXPERIMENTAL DATA. THE LAST TWO
140 REM      COLUMNS OF C STORE GRUNSON'S FUNCTION AS A FUNCTION OF TG/T. THE FORM
142 REM      IS:
144 REM      TG/T, G(TG/T)
146 REM      7      8
148 REM
150 REM      THE CONSTANTS USED FOR THE RESISTIVITIES ARE:
152 REM
154 REM      CU      TG      K      R0      R1
156 REM      AL      320      K      .016      .006095
158 REM      AL      395      K      .010      .010000

```



```

168 REM
170 REM THE RESULTING VALUES FOR THE SPECIFIC HEATS ARE IN J/g-K AND
172 REM FOR RESISTIVITY IN MICRO-OHM-CM.
174 REM
176 REM THE SUBROUTINE PACKAGE CONTAINS FOUR SUBROUTINES:
178 REM
180 REM SUBROUTINE 1000 CALCULATES C AND R AND RETURNS TO THE
182 REM CALLING ROUTINE
184 REM
186 REM SUBROUTINE 2000 INITIALIZES THE ARRAY C
188 REM
190 REM SUBROUTINE 3000 FINDS THE PROPER TARGET PAIR IN THE ARRAY C
192 REM
194 REM SUBROUTINE 4000 DOES THE INTERPOLATION FROM THE ARRAY C
196 REM
198 REM VARIABLE LIST:
200 REM
202 REM FLAGS
204 REM F0=0 DO INITIALIZATION OF C
206 REM J1=1 ALUMINUM STABILIZER IS USED
208 REM =3 COPPER STABILIZER IS USED
210 REM J2=1 ALUMINUM STRUCTURE IS USED
212 REM =5 STAINLESS STEEL IS USED
214 REM
216 REM ARRAY POINTERS
218 REM I TARGET PAIR ROW POINTER
220 REM I1 SPECIFIC HEAT (STABILIZER) ROW POINTER
222 REM I2 SPECIFIC HEAT (STRUCTURE) ROW POINTER
224 REM I3 RESISTIVITY ROW POINTER
226 REM J COLUMN POINTER
228 REM
230 REM VARIABLES
232 REM C1 STABILIZER SPECIFIC HEAT
234 REM C2 STRUCTURE SPECIFIC HEAT
236 REM R RESISTIVITY

```

238	REM	T1	STABILIZER TEMPERATURE
240	REM	T2	STRUCTURE TEMPERATURE
242	REM	T	INTERPOLATION ARGUMENT
244	REM	C7	INTERPOLATION RESULT
246	REM	C3	R0, RESIDUAL RESISTIVITY
248	REM	C4	R1
250	REM	C5	TG
252	REM	C5, C8, C9	SCRATCH VARIABLES
254	REM	I5, I6, I7	
256	REM		

```

1000 REM
1001 REM      CALCULATE THE TEMPERATURE DEPENDENT PROPERTIES OF THE
1002 REM      WINDING MATERIALS
1003 REM
1004 IF F0<>0 THEN 1010
1005 GOSUB 2000
1006 REM
1007 REM      CALCULATE C1 FOR THE STABILIZER
1008 REM
1010 J=J1
1020 I=I1
1030 T=T1
1040 GOSUB 3000
1050 I1=I
1060 GOSUB 4000
1070 C1=C7
1075 REM
1080 REM      CALCULATE C2 FOR THE STRUCTURE
1085 REM
1090 J=J2
1100 I=I2
1110 T=T2
1120 GOSUB 3000
1130 I2=I
1140 GOSUB 4000
1150 C2=C7
1155 REM
1160 REM      CALCULATE THE STABILIZER RESISTIVITY R
1165 REM
1170 J=7
1180 I=I3
1190 T=C5/T1
1200 IF T=>60 THEN 1260
1210 GOSUB 3000

```

1220 I3=I
1230 GOSUB 4000
1240 R=C3+C4*T1*C7
1250 RETURN
1260 R=C3
1270 RETURN

```

2000 REM
2005 REM      INITIALIZE THE LOOKUP TABLE
2010 REM
2020 DELETE C
2030 DIM C(30,8)
2035 REM
2036 REM      INPUT SPECIFIC HEAT DATA
2037 REM
2040 FOR I=1 TO 30
2050 FOR J=1 TO 6
2060 READ C(I,J)
2070 NEXT J
2080 NEXT I
2090 DATA 3.07,4.463E-5,3,1.28E-5,3,3.35E-4
2100 DATA 3.786,5.997E-5,4,2.2E-5,4,4.49E-4
2110 DATA 4.4,7.987E-5,5,3.55E-5,6,6.86E-4
2120 DATA 5.382,1.072E-4,6,5.46E-5,8,9.34E-4
2130 DATA 6.003,1.245E-4,7,8.04E-5,10,0.00119
2140 DATA 8.426,2.567E-4,8,1.141E-4,12,0.00148
2150 DATA 10.479,4.133E-4,10,2.103E-4,14,0.00179
2160 DATA 13.726,8.095E-4,14,5.614E-4,18,0.00256
2170 DATA 15.017,0.0011,16,8.47E-4,22,0.00359
2180 DATA 18.095,0.001609,18,0.001232,26,0.00509
2190 DATA 20.003,0.002194,22,0.002384,30,0.0071
2200 DATA 27.01,0.0054,26,0.0041,35,0.0102
2210 DATA 31.61,0.009,31,0.00637,40,0.0138
2220 DATA 35.94,0.0133,35,0.01,50,0.022
2230 DATA 45.98,0.0273,40,0.0141,60,0.0311
2240 DATA 55.18,0.0429,45,0.01857,70,0.0404
2250 DATA 61.9,0.0546,55,0.02786,80,0.0497
2260 DATA 70.16,0.069,65,0.03676,90,0.0583
2270 DATA 79.35,0.0845,75,0.04484,123,0.081
2280 DATA 93.63,0.1069,85,0.05179,173,0.091
2290 DATA 111.17,0.1288,100,0.06018,223,0.099

```

```

2300 DATA 124.85,0.1434,120,0.06859,273,0.107
2310 DATA 143.55,0.1595,140,0.07466,373,0.118
2320 DATA 163.66,0.1736,160,0.07918,473,0.127
2330 DATA 183.79,0.1841,200,0.08513,673,0.136
2340 DATA 204.92,0.1932,250,0.08947,873,0.142
2350 DATA 278.57,0.2121,300,0.09227,1073,0.149
2360 DATA 473,0.235,477,0.096,0,0
2370 DATA 773,0.27,700,0.101,0,0
2380 DATA 923,0.305,1013,0.113,0,0
2381 REM
2382 REM      CONVERT SPECIFIC HEATS TO J/g-K
2383 REM
2390 FOR I=1 TO 30
2400 FOR J=2 TO 6 STEP 2
2410 C(I,J)=4.186*C(I,J)
2420 NEXT J
2430 NEXT I
2431 REM
2432 REM      INPUT GRUNSON'S FUNCTION
2433 REM
2440 FOR I=1 TO 30
2450 FOR J=7 TO 8
2460 READ C(I,J)
2470 NEXT J
2480 NEXT I
2490 DATA 0,1,0.5,0.9862,1,0.9465,2,0.8073,2.5,0.7205,3,0.6307
2500 DATA 3.5,0.5428,4,0.4608,4.5,0.3867,5,0.3217,5.5,0.2658
2510 DATA 6,0.2187,7,0.1471,8,0.09909,9,0.0674,10,0.04655
2520 DATA 11,0.03276,12,0.02353,14,0.01289,16,0.007584,18,0.00474
2530 DATA 20,0.003111
2540 DATA 26,0.001089,30,6.145E-4,34,3.724E-4,40,1.944E-4,48,9.375E-5
2550 DATA 52,6.806E-5,60,3.841E-5,64,0.02967,68,0.02328
2560 RESTORE
2570 I1=2
2571 REM

```

```
2572 REM      INITIALIZE ARRAY POINTERS AND FLAGS
2573 REM
2580 I2=2
2590 I3=2
2591 F0=1
2592 REM
2593 REM      INITIALIZE RESISTIVITY COEFFICIENTS
2594 REM
2600 IF J1=3 THEN 2650
2610 C3=0.01
2620 C4=0.01
2630 C5=395
2640 GO TO 2680
2650 C3=0.016
2660 C4=0.006095
2670 C5=320
2680 RETURN
```

```

3000 REM
3001 REM      FIND PLACE IN LOOKUP TABLE
3002 REM
3003 REM START SEARCH AT THE LAST TARGET PAIR FOUND.  THE ROW POINTER
3004 REM I POINTS TO THAT PAIR.
3005 REM
3010 I4=I+1
3020 IF C(I,J)<=T AND C(I4,J)>T THEN 3170
3030 IF C(I,J)>T THEN 3070
3040 I5=1
3050 I6=29
3060 GO TO 3090
3070 I5=-1
3080 I6=2
3090 I7=1
3100 FOR I=I7 TO I6 STEP I5
3110 I4=I+1
3120 IF C(I,J)<=T AND C(I4,J)>T THEN 3170
3130 NEXT I
3140 PRINT "ERROR IN TABLE READ ";I;J
3150 CLOSE
3160 END
3170 RETURN

```



```

4000 REM
4001 REM      INTERPOLATE
4002 REM
4003 REM DO SECOND ORDER INTERPOLATION ABOUT THE TARGET PAIR. THE
4004 REM LOOKUP TABLE MUST EXTEND AT LEAST ONE PAIR BEYOND THE
4005 REM TARGET PAIR IN ORDER TO ALLOW THIS INTERPOLATION.
4006 REM
4010 C8=T-C(I,J)
4020 I4=I+1
4030 I5=I-1
4040 I6=J+1
4050 C7=C(I4,I6)-C(I5,I6)
4060 C9=(C(I4,I6)-C(I,I6))/(C(I4,J)-C(I,J))
4070 C6=(C(I,I6)-C(I5,I6))/(C(I,J)-C(I5,J))
4080 C7=C(I,I6)+C8*(C7+C8*(C9-C6))/(C(I4,J)-C(I5,J))
4090 RETURN

```

APPENDIX II

A BASIC SUBROUTINE FOR CALCULATING THE MUTUAL INDUCTANCE BETWEEN CIRCULAR COILS

The inductance between two circular coils may be calculated using Eq. 2.31. The indicated integration can be done numerically using Simpton's Rule. Subroutine 1000 is a BASIC subroutine which does this integration. Subroutine 2000 calculates the integral argument at each integration step. The main program in this listing uses subroutine 1000 to calculate the elements of the complete inductance matrix. With a different main program subroutine 1000 may be used to calculate the elements g used to calculate the force between coils (Eq. 2.49). With the appropriate main program and a few minor changes, this subroutine could be used to calculate the mutual inductance between any two circular coils in any non-intersecting geometry. The elliptical integrals are calculated using series expressions given in Ref. 3. The series are

for $k < .8$

$$K = \frac{\pi}{2} (1+m) \left(1 + \frac{1^2}{2} m^2 + \frac{1^2 3^2}{2^2 4^2} m^4 + \frac{1^2 3^2 5^2}{2^2 4^2 6^2} m^6 + \dots \right)$$

for $k \geq .8$

$$K = \ln \frac{4}{k'} + \frac{1^2}{2^2} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} \right) k'^2 + \frac{1^2 3^2}{2^2 4^2} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} \right) k'^4 \\ + \frac{1^2 3^2 5^2}{2^2 4^2 6^2} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{2}{5 \cdot 6} \right) k'^6 + \dots$$

for $k < .8$

$$E = \frac{\pi}{2(1+m)} \left(1 + \frac{m^2}{2^2} + \frac{1}{2^2 4^2} m^4 + \frac{1^2 3^2}{2^2 4^2 6^2} m^6 + \dots \right)$$

for $k > .8$

$$E = 1 + \frac{1}{2} \left(\ln \frac{4}{k'} - \frac{1}{1 \cdot 2} \right) k'^2 + \frac{1^2 3}{2^2 4} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{1}{3 \cdot 4} \right) k'^4 \\ + \frac{1^2 3^2 \cdot 5}{2^2 4^2 \cdot 6} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{1}{5 \cdot 6} \right) k'^6 + \dots$$

where $k' = \sqrt{1-k^2}$ = complementary modulus

$$m = (1-k')/(1+k')$$

REFERENCES

- [1] J. D. Jackson, Classical Electrodynamics, 2nd edition, (New York: John Wiley & Sons, Inc., 1975) pp. 177-178.
- [2] Simon Ramo, John R. Whinnery, and Theodore Van Duzer, Fields and Waves in Communication Electronics, (New York: John Wiley & Sons, Inc., 1965) pp. 306-307.
- [3] Herbert Bristol Dwight, Tables of Integrals and Other Mathematical Data, 4th edition, (New York: The Macmillan Co., 1961).

```

100 REM
102 REM
104 REM
106 REM
108 REM
110 REM
112 REM
114 REM
116 REM
118 REM
120 REM
122 REM
124 REM
126 REM
128 REM
130 REM
132 REM
134 REM
200 PRINT "INPUT NO. OF COILS, MAJOR RADIUS (M), MINOR RADIUS (M),"
210 PRINT " AND THE NO. OF TURNS PER COIL"
220 INPUT N,R,A,N1
230 DELETE M,Y
240 DIM M(N,N),Z(3,3)
250 M=0
255 REM
260 REM
265 REM
270 SET DEGREES
280 PRINT "INPUT COIL SELF-INDUCTANCE ";
290 INPUT L
300 FOR I=1 TO N
310 M(I,I)=L
320 NEXT I
330 PRINT "M11 = ";L

```

```

335 REM
340 REM          CALCULATE THE MUTUAL INDUCTANCES
345 REM
350 P9=360/N
360 N9=INT(N/2)
370 FOR I=2 TO N9+1
380 F=(I-1)*P9
390 GOSUB 1100
400 PRINT "M1";I;" = ";L
410 FOR I1=1 TO N
420 J1=I1+I-1
430 IF J1<=N THEN 450
440 J1=J1-N
450 M(I1,I1)=L
460 M(I1,J1)=L
470 NEXT I1
480 NEXT I
490 END

```

```

1000 REM
1002 REM          CALCULATE THE MUTAL INDUCTANCE
1004 REM          USING THE VECTOR POTENTIAL
1006 REM
1008 REM          THIS SUBROUTINE CALCULATES THE MUTUAL INDUCTANCE BETWEEN
1010 REM ANY TWO CIRCULAR COILS WHICH CAN BE APPROXIMATED AS CIRCULAR
1012 REM FILAMENTS. THE NUMERICAL INTEGRATION IS DONE USING SIMPSON'S
1014 REM RULE. THE INTEGRAL ARGUMENT IS CALCULATED USING SUBROUTINE
1016 REM 2000. THE RESULTING INDUCTANCE IS IN HENRIES.
1018 REM
1020 REM VARIABLE LIST:
1022 REM R      TOROID MAJOR RADIUS (M)
1024 REM P      TOROID ANGLE (DEGREES)
1026 REM P1     INTEGRATION VARIABLE PHI
1028 REM P2     INTEGRATION STEP (MUST BE AN EVEN INTEGRAL SUB-
1030 REM          DIVISION OF 360 AND SMALLER THAN 180)
1032 REM P3     END OF INTEGRATION = 360
1034 REM A1     ALPHA
1036 REM B1     BETA
1038 REM A      RADIUS OF COIL 1 (M)
1040 REM B      RADIUS OF COIL 2 (M)
1042 REM X1     X1
1044 REM Y1     Y1
1046 REM Z1     Z1
1048 REM Q      INTEGRAL ARGUMENT RETURNED BY SUBROUTINE 2000
1050 REM L      MUTUAL INDUCTANCE BETWEEN COILS
1052 REM N1     NO. OF TURNS/COIL, ASSUMED EQUAL FOR BOTH COILS
1054 REM SCRATCH VARIABLES - C1-9, S1-4
1056 REM
1110 P2=10
1111 REM
1112 REM          CALCULATE POSITION OF COIL 2
1113 REM
1120 B1=0

```

```

1130 A1=P
1140 SET DEGREES
1150 Z1=R*SIN(P)
1160 X1=R*(1-COS(P))
1170 Y1=0
1171 REM
1172 REM      CALCULATE CONSTANTS IN INTEGRAL ARGUMENT
1173 REM
1180 S2=SIN(B1)
1190 S3=SIN(A1)
1200 C2=COS(B1)
1210 C3=COS(A1)
1220 B=A
1230 C4=B*S3*S2
1240 C5=B*C3
1250 C6=B*C2
1260 C7=B*S3
1270 C8=B*C3*S2
1280 P3=360
1281 REM
1282 REM      DO INTEGRATION USING SIMPSON'S RULE
1284 REM
1285 REM      CALCULATE ARGUMENT AT END POINTS
1286 REM
1290 P1=0
1300 GOSUB 2000
1310 L=0
1320 P1=P3
1330 GOSUB 2000
1340 L=L+0
1341 REM
1342 REM      CALCULATE ARGUMENT AT THE ODD POINTS
1343 REM
1350 FOR P1=P2 TO P3-P2 STEP 2*P2
1360 GOSUB 2000

```



```

1370 L=L+4*Q
1380 NEXT P1
1381 REM
1382 REM      CALCULATE ARGUMENT AT THE EVEN POINTS
1383 REM
1390 FOR P1=2*P2 TO P3-2*P2 STEP 2*P2
1400 GOSUB 2000
1410 L=L+2*Q
1420 NEXT P1
1421 REM
1422 REM      CALCULATE L AND RETURN
1423 REM
1430 L=L*PI*P2/180/3
1440 L=A*4.0E-7*N1+2*L
1450 PRINT "P = ";P;" L = ";L
1460 RETURN

```

```

2000 REM
2002 REM          CALCULATE INTEGRAL ARGUMENT
2004 REM
2006 REM          THIS SUBROUTINE CALCULATES THE INTEGRAL ARGUMENT.  THE
2008 REM          ELLIPTICAL INTEGRALS ARE CALCULATED USING SERIES SOLUTIONS.
2010 REM
2012 REM          VARIABLE LIST:
2014 REM          E2      SERIES SUMMATION TRUNCATION ERROR
2016 REM          X2      X2
2018 REM          Y2      Y2
2020 REM          Z2      Z2
2022 REM          R1      R
2024 REM          R2      R SIN(THETA)
2026 REM          G1      COS(BETA)
2028 REM          G2      SIN(BETA)
2030 REM          K1      COMPLEMENTARY ELLIPTICAL INTEGRAL MODULUS
2032 REM          K2      ELLIPTICAL INTEGRAL MODULUS
2034 REM          K        ELLIPTICAL INTEGRAL OF THE FIRST KIND
2036 REM          E        ELLIPTICAL INTEGRAL OF THE SECOND KIND
2038 REM          A2      PHI COMPONENT OF THE VECTOR POTENTIAL
2040 REM          M1      MODULUS FOR THE ELLIPTICAL INTEGRAL SERIES
2042 REM          SCRATCH VARIABLES T1,T2,N2,N3,C9,D7-9
2044 REM
2100 E2=1.0E-4
2101 REM
2102 REM          CALCULATE THE COEFFICIENTS NEEDED TO CALCULATE A2
2103 REM
2110 S1=SIN(P1)
2120 C1=COS(P1)
2130 X2=X1+C1*C5+S1*C4
2140 Y2=Y1+S1*C6
2150 Z2=Z1-C1*C7+S1*C8
2160 R1=(X22+Y22+Z22)10.5
2170 R2=(X22+Y22)10.5

```

```

2180 G1=X2/R2
2190 G2=Y2/R2
2200 C9=A↑2+R1↑2+R2*A↑2
2201 REM
2202 REM      CALCULATE THE ELLIPTICAL INTEGRAL MODULUSES
2203 REM
2210 K2=4*A*R2/C9
2220 K1=(1-K2)↑0.5
2221 REM
2222 REM      TEST FOR CHOICE OF SERIES
2223 REM
2230 IF K2>0.8 THEN 2540
2231 REM
2232 REM      CALCULATE K
2233 REM
2240 M1=(1-K1)/(1+K1)
2250 K=1
2260 T1=1
2270 T2=1
2280 N2=1
2290 N3=2
2300 T1=T1*N2↑2
2310 T2=T2*N3↑2
2320 K1=T1*M1↑N3/T2
2330 K=K+K1
2340 N2=N2+2
2350 N3=N3+2
2360 IF K1>E2*K THEN 2300
2370 K=PI*(1+M1)*K/2
2371 REM
2372 REM      CALCULATE E
2373 REM
2380 E=1+M1↑2/4
2390 T1=1
2400 T2=4

```

```

2410 N2=1
2420 N3=4
2430 T1=T1*N2↑2
2440 T2=T2*N3↑2
2450 E1=T1*M1↑N3/T2
2460 E=E+E1
2470 N2=N2+2
2480 N3=N3+2
2490 IF E1>E2*E THEN 2430
2500 E=PI*E/2/(1+M1)
2501 REM
2502 REM      CALCULATE THE PHI COMPONENT OF THE VECTOR POTENTIAL
2503 REM
2510 A2=((2/K2-1)*K-2*E/K2)/C9↑0.5
2511 REM
2512 REM      CALCULATE Q AND RETURN
2513 REM
2520 Q=A2*(G2*(S1*C5-C1*C4)+G1*C1*C6)
2530 RETURN
2531 REM
2532 REM      CALCULATE K AND E FOR K2>= 0.9
2533 REM
2540 D7=K1↑2
2550 D8=K1↑4
2560 D9=K1↑6
2570 K=(1+D7/4+9*D8/64+225*D9/2304)*LOG(4/K1)
2580 K=K-(D7/4+21*D8/128+195*D9/1536)
2590 E=1+(D7/2+3*D8/16+15*D9/128)*LOG(4/K1)
2600 E=E-(D7/4+15*D8/64+9*D9/64)
2610 GO TO 2510

```

APPENDIX III

A BASIC SUBROUTINE FOR CALCULATING THE MATRIX EXPONENTIAL

The linear system

$$\dot{\underline{x}} = \underline{\bar{A}} \underline{x} + \underline{\bar{B}} \underline{u} \quad (\text{III.1})$$

has the solution

$$\underline{x}(t) = \underline{\text{Exp}}(\underline{\bar{A}}t) \underline{x}(0) + \int_0^t \underline{\text{Exp}}(\underline{\bar{A}}(t-\tau)) \underline{\bar{B}} \underline{u}(\tau) d\tau \quad (\text{III.2})$$

$\underline{\text{Exp}}(\underline{\bar{A}}t)$ is the matrix exponential and for large systems it must be calculated numerically using the series

$$\underline{\text{Exp}}(\underline{\bar{A}}t) = \sum_{j=0}^{\infty} \frac{1}{j!} \underline{\bar{A}}^j t^j \quad (\text{III.3})$$

For large t , Eq. III.3 requires a large number of terms and it tends to become unstable numerically. A method derived using the Cayley-Hamilton theorem was proposed by E. J. Mastascasa¹ to reduce the number of terms required. This algorithm is described below and is the basis for the subroutines listed in this Appendix.

The series III.3 may be approximated by

$$\underline{\text{Exp}}(\underline{\bar{A}}t) \approx \sum_{j=0}^N \frac{1}{j!} \underline{\bar{A}}^j t^j = E_N(\underline{\bar{A}}) \quad (\text{III.4})$$

where A is $n \times n$ and $n < N$. Using the Cayley-Hamilton theorem it can be shown that

$$\underline{\bar{A}}^n = -p_1 \underline{\bar{A}}^{n-1} - p_2 \underline{\bar{A}}^{n-2} \dots - p_n \underline{\bar{I}} \quad (\text{III.5})$$

The $\underline{\bar{A}}^{n+q}$ term in Eq. (III.4) may be found by successive multiplication of Eq. (III.5) by $\underline{\bar{A}}$. Therefore, only the $(n-1)$ powers of $\underline{\bar{A}}$ need be computed. The characteristic equation of $\underline{\bar{A}}$ is

$$P(\lambda) = \lambda^n + p_1 \lambda^{n-1} \dots + p_{n-1} \lambda + p_n = 0 \quad (\text{III.6})$$

The p_n 's may be calculated using Boher's formula:

$$\begin{aligned} p_1 &= -T_1 \\ p_2 &= (p_1 T_1 + T_2) \\ p_3 &= -1/3 (p_2 T_1 + T_3) \\ p_n &= -\frac{1}{n} (p_{n-1} T_1 + p_{n-2} T_2 + \dots + p_1 T_{n-1} + T_n) \end{aligned} \quad (\text{III.7})$$

where $T_k = \text{trace } (\underline{\bar{A}}^k)$. The polynomial Eq. (III.4) may be expressed as

$$E_N(\underline{\bar{A}}) = Q(\underline{\bar{A}}) P(\underline{\bar{A}}) + R(\underline{\bar{A}}) = R(\underline{\bar{A}}) \quad (\text{III.8})$$

since $P(\underline{\bar{A}}) = 0$ by the Cayley-Hamilton theorem. $R(\underline{\bar{A}})$ is the remainder polynomial and is of degree $(n-1)$. The scalar version of Eq. (III.8) is

$$E_N(\lambda) = Q(\lambda) P(\lambda) + R(\lambda)$$

or

$$\frac{E_N(\lambda)}{P(\lambda)} = Q(\lambda) + \frac{R(\lambda)}{P(\lambda)} \quad (\text{III.9})$$

The polynomial $R(\lambda)$ is the remainder of the division of $E_n(\lambda)$ by $P(\lambda)$. The matrix exponential may now be expressed as

$$\overline{\exp}(\bar{A}t) = \sum_{j=0}^{n-1} r_j \bar{A}^j \quad (\text{III.10})$$

where r_j are coefficients of the reminder polynomial of the division

$$\frac{E_N(\lambda)}{P(\lambda)} = \frac{\sum_{j=0}^N \frac{1}{j!} t^j \lambda^j}{\sum_{j=0}^N p_{n-j} \lambda^j} \quad (\text{III.11})$$

If N is much greater than n Eq. III.10 affords a significant savings in the number of calculations required.

The algorithm may still go unstable numerically if t is large compared to the smallest eigenvalue of $\bar{A}$. If the smallest eigenvalue is much smaller than the other system eigenvalues, then the system size may usually be reduced and that eigenvalue eliminated improving the numerical stability. However, for the problem of interest in this thesis, the

modeling of the reactor structural shells, the smallest eigenvector is associated with the reactor inner wall. Since the reactor wall is very radioactive and of great interest for safety studies, it could not be ignored. The time step which may be used in Eq. 2.39 can be increased by making use of a scalar analog.

$$\exp(\text{ant}) = \prod_{n=1}^n \exp(\text{at}) \quad \text{III.12)}$$

Subroutine 2000 allows the user to make use of this algorithm for the matrix exponential.

Note: The statement

100 Z = X MPY Y

is the matrix multiplication of X and Y which is available on the Tektronic Microcomputer used for these calculations.

REFERENCES

- [1] Edward J. Mastascusa, "A Method for Calculating e^{AT} Based on the Cayley-Hamilton Theorem," Proceedings of the IEEE (Letters), July, 1978, pp. 1328-1329.

```

1000 REM      CALCULATE THE MATRIX EXPONENTIAL
1002 REM
1004 REM      THIS SUBROUTINE USES AN ALGORITHM WHICH IS DERIVED USING
1006 REM THE CAYLEY-HAMILTON THEOREM. THE ALOGORITHM WAS CHOSEN
1008 REM BECAUSE ONLY THE FIRST N-1 POWERS OF A NEED TO BE CALCULATED.
1010 REM THIS PROGRAM CALCULATES THESE TWICE. CONSIDERABLE COMPUTING
1012 REM TIME COULD BE SAVED IF THEY WERE SAVED (ON TAPE) FOR USE THE
1014 REM SECOND TIME THROUGH.
1016 REM
1018 REM VARIABLE LIST:
1020 REM
1022 REM      A      LINEAR SYSTEM MATRIX
1024 REM      E      MATRIX EXPONENTIAL
1026 REM      N      SYSTEM MATRIX SIZE
1028 REM      N9     NO. OF TERMS TO BE INCLUDED; MUST BE GREATER THAN N
1030 REM      P      CHARACTERISTIC EQUATION COEFFICIENT VECTOR
1032 REM      P5     COEFFICIENTS OF THE EXPONENTIAL SERIES AND THE
1034 REM              COEFFICIENTS OF THE REMAINDER SERIES VECTOR
1036 REM
1120 DELETE A1,A2,P,P5,E
1140 DIM A1(N,N),A2(N,N),P(N),P5(N+1),E(N,N)
1150 PRINT "INPUT TIME STEP AND NO. OF TERMS TO INCLUDE"
1160 INPUT T1,N9
1161 IF N<N9 THEN 1180
1162 N9=N+1
1163 REM
1170 REM CALCULATE CHARACTERISTIC EQUATION COEF. USING BOCHER FORMULA
1171 REM
1172 REM STORE THE IDENTITY MATRIX IN A1
1173 REM
1180 CALL "IDN",A1
1220 P=0
1221 REM
1222 REM APPLY THE BOCHER FORMULA

```

```

1223 REM
1230 FOR I1=1 TO N
1240 A2=A1 MPY A
1250 A1=A2
1260 P5(I1)=0
1261 REM
1262 REM COMPUTE THE MATRIX TRACE
1263 REM
1270 FOR I=1 TO N
1280 P5(I1)=P5(I1)+A1(I,I)
1290 NEXT I
1291 REM
1292 REM CALCULATE THE CHARACTERISTIC EQUATION COEFF.S
1293 REM
1300 P(I1)=P5(I1)
1310 IF I1=1 THEN 1350
1320 FOR I=1 TO I1-1
1330 P(I1)=P5(I)*P(I1-I)+P(I1)
1340 NEXT I
1350 P(I1)=-1/I1*P(I1)
1360 NEXT I1
1365 REM
1370 REM CALCULATE LAST N COEF. OF E(A)
1375 REM
1380 N7=1
1390 N8=0
1391 REM
1392 REM CALCULATE N9 FACTORIAL
1393 REM
1400 FOR I=1 TO N9
1410 N8=N8+1
1420 N7=N7*N8
1430 NEXT I
1431 REM
1432 REM COMPUTE THE LAST N+1 COEFF.S OF THE EXPONENTIAL SERIES;

```

```

1433 REM STORE THEM IN REVERSE ORDER IN P5
1434 REM
1440 FOR I=1 TO N+1
1450 P5(I)=T1↑(N9+1-I)/N7
1460 N7=N7/N8
1470 N8=N8-1
1480 NEXT I
1485 REM
1490 REM DO POLYNOMIAL DIVISION
1495 REM
1500 FOR I=1 TO N9-N+1
1510 X8=P5(I)
1511 REM
1512 REM DO ONE DIVISION, STORE THE REMAINDER COEFF.S IN P5
1513 REM
1520 FOR I1=1 TO N
1530 P5(I1)=P5(I1+1)-X8*P(I1)
1540 NEXT I1
1541 REM
1542 REM QUIT IF THE DIVISION IS COMPLETED
1543 REM
1550 IF N8<0.1 THEN 1600
1551 REM
1552 REM BRING DOWN TERM FROM EXPONENTIAL SERIES COEF.S TO COMPLETE
1553 REM THE REMAINDER
1554 REM
1560 P5(N+1)=T1↑(N9-N-I)/N7
1570 N7=N7/N8
1580 N8=N8-1
1581 REM
1582 REM REPEAT THE DIVISION PROCESS
1583 REM
1590 GO TO 1610
1591 REM
1592 REM END OF DIVISION PROCESS

```

```

1593 REM
1600 P5(N+1)=1
1610 NEXT I
1615 REM
1620 REM REMAINDER COEF. IN THE 1ST N PLACES IN P5
1625 REM
1630 REM CALCULATE THE MATRIX EXPONENTIAL
1635 REM
1640 CALL "IDN",A1
1650 E=P5(N)*A1
1660 FOR I=1 TO N-1
1670 A2=A1 MPY A
1680 A1=A2
1690 A2=P5(N-I)*A1
1700 E=E+A2
1710 NEXT I
1720 PAGE
1721 REM
1722 REM OUTPUT THE MATRIX EXPONENTIAL
1723 REM
1730 PRINT "EXP(A*DELTA T) = ",E
1740 PRINT "FOR DELTA T = ";T1
1750 PRINT "FOR NO. OF TERMS = ";N9
1760 DELETE A1,A2,P,P5
1770 RETURN

```

```

2000 REM
2010 REM          INCREASED TIME STEP OPTION
2015 REM
2020 PRINT "DO YOU WISH TO INCREASE THE TIME STEP?";
2030 INPUT W$
2040 IF W$="YES" THEN 2060
2050 RETURN
2055 REM
2060 PRINT "THE TIME STEP MAY BE INCREASED BY ANY MULTIPLE OF 10 BETWEEN"
2070 PRINT "10 AND 100. PLEASE INPUT THE DESIRED MULTIPLE. ";
2075 REM
2080 INPUT W1
2090 IF W1=100 THEN 2150
2100 FOR W2=10 TO 90 STEP 10
2110 IF W1=W2 THEN 2170
2120 NEXT W2
2130 PRINT "THE NUMBER YOU INPUT IS NOT A MULTIPLE OF 10 BETWEEN 10&100"
2140 GO TO 2020
2150 GOSUB 2360
2160 GO TO 2260
2170 GOSUB 2560
2180 IF W1=10 THEN 2260
2190 DIM E1(N,N),E2(N,N)
2200 E1=E
2210 E2=E
2220 FOR W2=20 TO W1 STEP 10
2230 E=E1*PY E2
2240 E2=E
2250 NEXT W2
2260 DELETE E1,E2
2270 T1=W1*T1
2280 PRINT "THE NEW TIME STEP IS = ";T1
2290 PRINT "DO YOU WISH TO SEE THE NEW MATRIX EXPONENTIAL? ";
2300 INPUT W$

```

```

2310 IF W#="YES" THEN 2330
2320 RETURN
2330 PAGE
2340 PRINT "EXP(A* ";T1;" ) =",E
2350 RETURN
2355 REM
2360 REM PRODUCES THE MATRIX EXPONENTIAL FOR T=100*T
2365 REM
2370 DELETE E1,E3,E4
2380 DIM E1(N,N),E3(N,N),E4(N,N)
2390 E1=E
2400 E3=E1 MPY E
2410 E=E3
2420 E4=E3 MPY E
2430 E3=E4
2440 E1=E4 MPY E3
2450 E=E1
2460 E3=E1 MPY E
2470 E=E3
2480 E1=E MPY E3
2490 E=E1
2500 E3=E MPY E1
2510 E=E3 MPY E1
2520 E1=E MPY E4
2530 E=E1 MPY E4
2540 DELETE E1,E3,E4
2550 RETURN
2555 REM
2560 REM PRODUCES THE MATRIX EXPONENTIAL FOR T=10*T
2565 REM
2570 DELETE E1,E2,E3
2580 DIM E1(N,N),E2(N,N),E3(N,N)
2590 E1=E
2600 E2=E1 MPY E
2610 E1=E2

```

2620 E3=E1 MPY E2
2630 E=E3
2640 E1=E MPY E3
2650 E=E1 MPY E2
2660 DELETE E1,E2,E3
2670 RETURN

APPENDIX IV

A BASIC PROGRAM FOR THE FOURTH ORDER RUNGE-KUTTA SOLUTION TO FIRST ORDER DIFFERENTIAL EQUATIONS

The initial value problem

$$\begin{aligned}\frac{d}{dx} \underline{y} &= \underline{f}(\underline{y}(x), \underline{u}(x), x) \\ y(0) &= \underline{y}_0\end{aligned}\tag{IV.1}$$

may be integrated approximately using the Runge-Kutta algorithm. For a constant integration step Δx the algorithm is

$$\underline{y}(x + \Delta x) = \underline{y}(x) + 1/6(\underline{k}_1 + 2\underline{k}_2 + \underline{k}_4)\tag{IV.2}$$

where $\underline{k}_1 = \underline{f}(x, \underline{y}(x), \underline{u}(x)) \Delta x$

$$\underline{k}_2 = \underline{f}\left(x + \frac{\Delta x}{2}, \underline{y}(x) + \frac{\underline{k}_1}{2}, \underline{u}\left(x + \frac{\Delta x}{2}\right)\right) \Delta x$$

$$\underline{k}_3 = \underline{f}\left(x + \frac{\Delta x}{2}, \underline{y}(x) + \frac{\underline{k}_2}{2}, \underline{u}\left(x + \frac{\Delta x}{2}\right)\right) \Delta x$$

$$\underline{k}_4 = \underline{f}(x + \Delta x, \underline{y}(x) + \underline{k}_3, \underline{u}(x + \Delta x)) \Delta x$$

This algorithm has the advantage of being self-starting and being relatively easy to program. Its principle disadvantage is that no attempt is made to optimize the integration step size or to check for numerical instability if the step size is too large. It may be used to simulate both linear and non-linear systems.

Notes: The WRITE @ 33: statements are used to output data to the cassette tape unit attached to the Tektronic 4051 Microcomputer used for running this program. The program listing contained in this appendix is set up to simulate the single coil discharge circuit for UWMAK I or UWMAK III. To run this model the temperature dependent properties program listed in Appendix I must be included starting at line 6000.

REFERENCES

- [1] M. L. James, G. M. Smith, J. C. Welford, Applied Numerical Methods for Digital Computation with FORTRAN, (Scranton, Pa.: International Textbook Co., 1967), p. 350.

```

100 REM      RUNGE-KUTTA SIMULATION MAIN PROGRAM
102 REM
104 REM      THIS PROGRAM USES THE FURTH ORDER RUNGE-KUTTA ALGORITHM TO
106 REM SIMULATE THE RESPONSE OF A SET OF FIRST ORDER ODE'S WITH INITIAL
108 REM CONDITIONS.  THE PROGRAM USES A FIXED TIME STEP BUT THE TIME
110 REM STEP CAN BE CHANGED AT THE END OF EACH OUTPUT PAGE.  OUTPUT
112 REM RESULTS CAN BE SAVED ON TAPE.  THE PROGRAM SUBROUTINES ARE:
114 REM
116 REM MAIN          100      MAIN PROGRAM
118 REM SUBROUTINE 1000      INITIALIZE THE MODEL PARAMETERS
120 REM SUBROUTINE 2000      CALCULATE THE STATE EQUATION DERIVATIVES
122 REM SUBROUTINE 3000      OUTPUT ROUTINE
124 REM SUBROUTINE 4000      FIXED STEP RUNGE-KUTTA INTEGRATION ROUTINE
126 REM SUBROUTINE 5000      OUTPUT PAGE HEADING
128 REM
130 REM SIMULATION PROGRAM VARIABLES:
132 REM
134 REM      PROGRAM FLAGS AND COUNTERS
136 REM      F<>0      STOP INTEGRATION FLAG
138 REM      F=0       CONTINUE INTEGRATION
140 REM      F0=0      DO DATA TABLE INIALIZATION
142 REM      P2<=1     PRINT OUTPUT LINE THIS STEP
144 REM      P2>1      DON'T PRONT OUTPUT LINE THIS STEP
146 REM      Z#="YES"  STORE OUTPUT ON TAPE
148 REM      Q6=0      DO END OF OUTPUT PAGE ROUTINE
150 REM
152 REM      VARIABLES
154 REM      N          NO. OF 1ST ORDER ODE'S
156 REM      H          INTEGRATION STEP SIZE
158 REM      X          INDEPENDENT VARIABLE
160 REM      Y          DEPENDENT VARIABLE VECTOR
162 REM      D          DERIVATIVE VECTOR
164 REM      E          NO. OF INTEGRATION STEPS PER OUTPUT LINE
166 REM      A          INITIAL VALUE OF X

```

168	REM	B	FINAL VALUE OF X
170	REM		
172	REM	MODEL VARIABLES:	
174	REM		
176	REM	Y(1)	FAULTED COIL CURRENT
178	REM	Y(6)	OTHER COILS IN SERIES CURRENT
180	REM	Y(2)	CONDUCTOR TEMPERATURE
182	REM	Y(3)	STRUCTURE TEMPERATURE
184	REM	Y(4)	TOTAL ENERGY DISSIPATED IN THE COIL WINDING
186	REM	Y(5)	TOTAL ENERGY DISSIPATED IN THE EXTERNAL RESISTANCE
188	REM	X	TIME
190	REM	U	FAULTED COIL'S DISCHARGE VOLTAGE
192	REM	R1	FAULTED COIL'S DISCHARGE RESISTANCE
194	REM	M1	M11
196	REM	M2	M12
198	REM	M3	M22
200	REM	M	M11*M22-M12*M12
202	REM	H1	HEAT TRANSMITTANCE
204	REM	R	CONDUCTOR RESISTIVITY
206	REM	C1	CONDUCTOR SPECIFIC HEAT
208	REM	C2	STRUCTURE SPECIFIC HEAT
210	REM	T1	CONDUCTOR TEMPERATURE
212	REM	T2	STRUCTURE TEMPERATURE
214	REM	H1	FRACTION OF COIL TURNS PARTICIPATING
216	REM	A1	FRACTION OF BASE CASE HEAT TRANSFER COEFFICIENT
218	REM	A2	FRACTION OF BASE CASE CONDUCTOR MASS/UNIT CONDUCTOR
220	REM		LENGTH
222	REM	A3	FRACTION OF BASE CASE STRUCTURE MASS/UNIT CONDUCTOR
224	REM		LENGTH
226	REM	R3	CONSTANT WHICH MULTIPLIES THE COIL RESISTIVITY TO
228	REM		GET THE TOTAL INTERNAL COIL RESISTANCE
230	REM	M4	TOTAL CONDUCTOR MASS QUENCHED
232	REM	M5	TOTAL STRUCTURE MASS PARTICIPATING
234	REM		
236	REM		

```

238 REM      DO PROGRAM INITIALIZATION
240 REM
250 PAGE
260 GOSUB 1000
270 PRINT "ENTER STOP TIME ";
280 INPUT B
290 PRINT "ENTER STEP SIZE AND NO. OF STEPS PER PRINT OUTPUT ";
300 INPUT H,E
310 P2=0
320 F0=0
330 F=0
335 REM
340 REM      SETUP TAPE FOR OUTPUT STORAGE
345 REM
350 PRINT "DO YOU WISH TO STORE OUTPUT ON TAPE? ";
360 INPUT Z$
370 IF Z$<>"YES" THEN 520
380 PRINT "INPUT DATA STORAGE FILE NO. ";
390 INPUT Q5
470 PRINT "INPUT FILE SIZE ";
480 INPUT Q6
490 FIND Q5
500 MARK 1.06
510 FIND Q5
515 REM
520 REM      SETUP DESIRED OUTPUT PAGE HEADING
525 REM
530 PRINT "DO YOU WISH TO ANOTATE THE OUTPUT HEADING? ";
540 INPUT Y$
550 IF Y$<>"YES" THEN 580
560 PRINT "INPUT ANOTATION. ";
570 INPUT X$
580 PAGE
581 REM
582 REM      WRITE PAGE HEADING

```

```
583 REM
590 GOSUB 5000
600 Q6=26
601 REM
602 REM      DO RUNGE-KUTTA SIMULATION
603 REM
610 GOSUB 4000
620 CLOSE
630 END
```

```

1000 REM
1001 REM      INITIALIZE THE MODEL PARAMETERS
1002 REM
1003 REM IF THE UNMAK III MODEL IS DESIRED INSERT THE LINE:
1004 REM      1005 GO TO 1500
1006 REM
1010 REM UNMAK I MODEL - TWO COIL DISCHARGE CIRCUIT
1011 REM
1020 A1=2
1030 A2=1
1040 A3=1
1050 R1=4.0E+7/10212/10212
1060 M1=125.2
1070 M2=122.8
1080 M3=2605
1090 M=M1*M3-M2+2
1100 U=0
1110 N1=1
1120 N=6
1130 A=0
1140 DELETE Y,D
1150 DIM Y(N),D(N)
1160 X=A
1170 Y(1)=10212
1180 Y(2)=10.1
1190 Y(3)=0
1200 Y(4)=0
1210 Y(5)=0
1220 Y(6)=10212
1230 H1=0.0531*64*100*2040*N1*A1*A2*10.5
1240 J1=3
1250 J2=5
1260 R3=64*100*2040*N1/3.348/A2*1.0E-6
1270 M4=64*100*2040*N1*29.96*A2

```



```

1280 M5=64*100*2040*N1*38.501*A3
1290 RETURN
1500 REM
1501 REM UMMAK III MODEL - TWO COIL DISCHARGE CIRCUIT
1502 REM
1510 A1=1
1520 A2=1
1530 A3=1
1540 R1=0
1550 M1=31.5
1560 M2=99.71
1570 M3=2130.8
1580 M=M1*M3-M2*2
1590 U=0
1600 N1=1
1610 N=6
1620 A=0
1630 DELETE Y,D
1640 DIM Y(N),D(N)
1650 X=A
1660 Y(1)=9545
1670 Y(2)=4.2
1680 Y(3)=4.2
1690 Y(4)=0
1700 Y(5)=0
1710 Y(6)=0
1720 H1=0.09*65*100*960*N1*A1*A2*0.5
1730 R3=65*100*960*N1*2.25/A2*1.0E-6
1740 M4=65*100*960*N1*6.231*A2
1750 M5=65*100*960*N1*14.538*A3
1760 J1=1
1770 J2=1
1780 RETURN

```

```

2000 REM
2001 REM      CALCULATE THE STATE EQUATION DERIVATIVES
2002 REM
2010 T1=Y(2)
2020 T2=Y(3)
2021 REM
2022 REM      CALCULATE THE MATERIAL PROPERTIES
2023 REM
2030 GOSUB 5070
2031 REM
2032 REM      CALCULATE THE DERIVATIVES
2033 REM
2040 R=R*R3
2050 U1=R*Y(1)
2060 X1=U1*Y(1)
2070 U5=R1*Y(1)+U
2090 D(1)=-M3*(U1+U5)/M
2090 C1=C1*M4
2100 D(2)=(X1+H1*(Y(3)-Y(2)))/C1
2110 C2=C2*M5
2120 D(3)=H1*(Y(2)-Y(3))/C2
2130 D(4)=X1
2140 D(5)=Y(1)*U5
2150 D(6)=M2*(U1+U5)/M
2160 RETURN

```

```

3000 REM
3001 REM      OUTPUT ROUTINE
3002 REM
3003 REM TEST FOR OUTPUT FLAG P2=1
3004 REM
3010 IF P2<=1 THEN 3050
3020 P2=P2-1
3030 RETURN
3031 REM
3032 REM WRITE OUTPUT LINE TO CRT SCREEN
3033 REM
3040 IMAGE40,30,5(1X,50.10),2(3E)
3050 PRINT USING 3040:X,Y(1),Y(6),Y(2),Y(3),Y(2)-Y(3),Y(4),Y(5)
3060 IF Z4<>"YES" THEN 3080
3061 REM
3062 REM WRITE OUTPUT LINE TO STORAGE TAPE
3063 REM
3070 WRITE 033:X,Y,Y(2)-Y(3)
3071 REM
3072 REM TEST FOR STOP CONDITIONS
3073 REM
3080 IF Y(3)>933 THEN 3110
3090 IF Y(1)<10 THEN 3110
3100 GO TO 3130
3110 F=10
3120 GO TO 3290
3121 REM
3122 REM TEST FOR END OF PAGE
3123 REM
3130 Q6=Q6-1
3140 IF Q6>0 THEN 3290
3141 REM
3142 REM DO END OF PAGE ROUTINE
3143 REM

```

```

3150 PRINT "JSTEP SIZE = ",H," SEC."
3160 PRINT "DO YOU WISH TO MAKE A COPY ? ";
3170 INPUT U$
3180 IF U$<>"YES" THEN 3200
3190 COPY
3200 PRINT "DO YOU WISH TO CHANGE STEP SIZE ? ";
3210 INPUT U$
3220 IF U$<>"YES" THEN 3250
3230 PRINT "ENTER NEW STEP SIZE AND NO. OF STEPS PER PRINT OUTPUT ";
3240 INPUT H,E
3241 REM
3242 REM OUTPUT NEW PAGE HEADING
3243 REM
3250 GOSUB 5000
3260 Q6=26
3261 REM
3262 REM OUTPUT FIRST OUTPUT LINE ON NEW PAGE
3263 REM
3270 GOSUB 3000
3280 REM
3281 REM RETURN TO THE CALLING ROUTINE
3282 REM
3290 P2=E
3300 RETURN

```

```

4000 REM
4001 REM      FIXED STEP RUNGE-KUTTA INTEGRATION ROUTINE
4002 REM
4010 DELETE P
4020 DIM P(5,N)
4030 X=A
4031 REM
4032 REM DO FIRST STEP
4033 REM
4040 GOSUB 2000
4041 REM
4042 REM OUTPUT DATA LINE
4043 REM
4050 GOSUB 3000
4060 REM
4061 REM TEST FOR INTEGRATION STOP
4062 REM
4070 IF F<>0 THEN 4320
4080 IF X=>B THEN 4320
4090 FOR P1=1 TO N
4100 P(1,P1)=H*D(P1)
4110 P(5,P1)=Y(P1)
4120 Y(P1)=P(5,P1)+P(1,P1)/2
4130 NEXT P1
4131 REM
4132 REM DO SECOND STEP
4133 REM
4140 X=X+H/2
4150 GOSUB 2000
4160 FOR P1=1 TO N
4170 P(2,P1)=H*D(P1)
4180 Y(P1)=P(5,P1)+P(2,P1)/2
4190 NEXT P1
4191 REM

```

```

4192 REM DO THIRD STEP
4193 REM
4200 GOSUB 2000
4210 FOR P1=1 TO N
4220 P(3,P1)=H*D(P1)
4230 Y(P1)=P(5,P1)+P(3,P1)
4240 NEXT P1
4241 REM
4242 REM DO FOURTH STEP
4243 REM
4250 X=X+H/2
4260 GOSUB 2000
4270 FOR P1=1 TO N
4280 P(4,P1)=H*D(P1)
4281 REM
4282 REM COMPUTE THE END OF TIME STEP VALUE OF Y
4283 REM
4290 Y(P1)=P(5,P1)+(P(1,P1)+2*(P(2,P1)+P(3,P1))+P(4,P1))/6
4300 NEXT P1
4301 REM
4302 REM REPEAT THE INTEGRATION CYCLE
4303 REM
4310 GO TO 4040
4320 RETURN

```

```
5000 REM
5001 REM      OUTPUT NEW PAGE HEADING
5002 REM
5010 PAGE
5020 IF Y$<>"YES" THEN 5040
5030 PRINT X$;"J"
5040 PRINT "      TIME      I1      I1      TEMP1";
5050 PRINT "      TEMP2      DELTA T      E COIL      E RES"
5060 RETURN
```