TITLE THE STATISTICAL PROPERTIES OF ELECTROMAGNETIC FIELDS
WITH APPLICATION TO RADIATION AND SCATTERING

AUTHOR(S) T. H. Lehman and E. K. Miller

SUBMITTED TO 1991 IEEE AP-S Symposium

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

By accepting this article for publication, the publisher recognizes that the U.S. Government retains a nonexclusive, royalty-free license to publish or reproduce the published form of this article, or to allow others to do so, for U.S. Government purposes.

The Los Alamos National Laboratory requests that the publisher identify this article as work performed under the auspices of the U.S. Department of Energy.

Los Alamos Los Alamos National Laboratory
Los Alamos, New Mexico 87545

THE STATISTICAL PROPERTIES OF ELECTROMAGNETIC FIELDS WITH APPLICATION TO RADIATION AND SCATTERING

T. H. Lehman, Consultant
3200 Carlisle NE
Albuquerque, NM, USA 87110

E. K. Miller
Los Alamos National Laboratory
Los Alamos, NM 87545

INTRODUCTION

As the size and complexity of electronic systems continue to increase, there is a growing need to characterize their electromagnetic behavior in statistical terms. For this to be useful, one should be able to establish the range of validity, and functional forms for distribution functions of the fields, using Maxwell's equations directly, as the number of samples required to do this is prohibitive from a measurement point of view. Concerning the statistical nature of electromagnetic fields, it is appropriate to quote from Beran and Parrent [1]: *"It should be made clear immediately that discussions of coherence or incoherence or partial coherence (statistics) need never enter a treatise on classical electromagnetic theory. In the classical electromagnetic theory governed by Maxwell's equations, one assumes that the electric field and magnetic field are measurable functions of position and time. In principle, one can, if one chooses, follow in detail the instantaneous variations of the electric or magnetic field, no matter how intricate the boundary conditions. In many cases, however, the procedure is extremely difficult and may, in fact, mask important physical effects . . . The situation is analogous to the one arising in classical statistical mechanics. Although in principle it is possible to follow in detail the dynamics of the system, it often proves much more useful to provide a statistical description."*

It is clear that statistical electromagnetics can be nothing more than a theory which describes the behavior of the average values of appropriate functions of the electromagnetic field. These functions must in some sense be measurable and, for example, could include all powers of the three components of both the electric and magnetic field; all powers of the electromagnetic field power density; or all the powers of the differences of the above functions when they are measured at different locations or different times (spatial and temporal correlation functions).

The explicit purpose for constructing these average values is to provide the capability for extracting useful and usable information from measured data for those cases where the intricate and complex nature of the measured fields masks this information. An additional consideration includes those cases where small variations in the system topology or small changes in the excitation produce large changes in the response at specific locations within the system, or when small changes in the measurement location produce similarly large changes in the response. From a purely deterministic point of view, these situations give rise to a problem set which is currently not resolvable.

If information is to be extracted from the average values of the appropriate functions of the electromagnetic fields, it is immediately obvious that the amount of such information will in some sense be a strong function of the number of the average values that are determined. This fact will certainly limit the utility of an average-value theory since there is in principle an infinite number of average values; for example, all powers of the x-component of the electric field. This limitation can be eliminated if it can be shown, *a priori* from the physics of the problem, that the average values of some of these functions are related to each other through well-defined relationships. This, of course, is found to be true in elementary statistical mechanics. We can imagine the state of statistical

mechanics if the velocity distribution of molecules in a box were to be a function of the shape of the box rather than being Maxwellian. Determining the relationships between different average values is an important step in any physics-based statistical theory. If these relationships exist and exhibit certain properties, then probability densities or distributions can be assigned to the field variables and the transition from an "average-value theory" to a "statistical theory" is complete.

For electromagnetics, no such average-value relationships have been demonstrated to exist. The derivation of these relationships and their range of validity are the subject of consideration here. For purposes of this discussion, we limit our attention to the fields inside a complex cavity, but with the ultimate goal of deriving the statistics of the far radiated and scattered fields of complex envelopes.

DESCRIPTION OF APPROACH

Consider a cavity with volume V which is source free and bounded by a perfectly conducting surface S of arbitrary shape. It is well known [Schiff (2)], that quantities such as the electric field $E(r,t)$ in the volume V , upon separating the space and time variations, can be expanded in a complete set of real vector functions $u_k(r)$ as

$$E(r,t) = \sum_{k=-\infty}^{+\infty} u_k(r) D P_k(t) \quad (1)$$

The cartesian components of the vector fields must satisfy both the second-order wave equation and the divergence equation, since there are no sources present, and $\nabla \times E(r,t)$ must vanish on the surface S , where $\hat{n}$ is an outward-pointing, unit normal. Using these properties and with the help of Green's theorem it can be shown that

$$\int_V u_k \cdot u_{k'} dv = \delta_{k',k} \quad (2)$$

where, following the usual convention, k is an eigenvalue and u_k is an eigenvector.

The quantities whose average values are of interest are functions of the electric field. Denoting spatial average values by $\langle \rangle$, then these functions are the three electric field components E_s , $s = x, y, z$ and all of their higher-order powers. These average values are defined by

$$\langle E_s^m \rangle = V^{-1} \int_V E_s^m(r) dv; \quad s = x, y, z \text{ and } m = 0, 1, 2, \dots \quad (3)$$

In order to evaluate the averages defined in Eq. (3), it is obvious from Eq. (1) that it will be necessary to evaluate the average values of all the powers $(u_{k,s})^m$ of the components of the eigenvector for all k and for all m . This is a monumental (essentially impossible) task, unless these integrals are straightforward and can be shown to be simply related for various values of m . In the next sections, we outline an approach that shows this is indeed the case for complex cavities and in fact, it is shown that the integrals are independent of k and that they are just the moments of a normal probability density.

A CONCEPTUAL APPROACH FOR FINDING THE EIGENVECTORS

It is well known that exact solutions of the vector Helmholtz equation are only possible for a very few canonically-shaped cavities. Thus, for cavities of arbitrary shape, no attempt is made to find exact solutions. Instead, we seek approximate expressions for the eigenvectors. These approximate expressions are developed by expanding the eigenvectors in a set of entire-domain basis functions (EDBF). Although the choice of the EDBF is somewhat arbitrary, we naturally choose basis functions that satisfy the requirements imposed on the fields as stated above.

The EDBF is developed by construction, using the following steps:

- 1) We specify a cartesian coordinate system for the space occupied by the

volume V .

- 2) We assume that the eigenvalue k of a specific eigenvector $u_k(r)$ is known.
- 3) We discretize the volume V into a large number N of cubes each having a volume $\Delta V = \Delta x \Delta y \Delta z = (\Delta x)^3$ such that $k\Delta x \ll 1$ and $V = N\Delta V$.
- 4) We project the six surface elements $\Delta S = (\Delta x)^2$ of each cube onto the walls of the cavity.
- 5) From the surface projections for each cube, we construct a rectangular cavity formed by the intersections of the six orthogonal planes containing each of the ΔS . The n 'th rectangular cavity constructed from the n 'th cube has dimensions X_n, Y_n, Z_n and the location of one of its corners with respect to the origin O of the coordinate system defined in step 1 is given by r_n . We refer to these rectangular cavities as "virtual" cavities.
- 6) For each virtual cavity, we choose the set of three integers (l, m, p) which yields the best approximation to the expression $(\omega_k/c)^2 = k^2 = \pi^2[(l/X_n)^2 + (m/Y_n)^2 + (p/Z_n)^2]$.
- 7) We write the eigenvector $u_k(r; n)$ for the n 'th virtual cavity assuming that each virtual cavity has perfectly conducting walls and that the components of the eigenvalues (i.e., the virtual cavity eigenvalue) are given by $k_x(n) = \pi l/X_n, k_y(n) = \pi m/Y_n, k_z(n) = \pi p/Z_n$.
- 8) We choose the resulting $u_k(r; n)$ to be the EDBF and write the solution for the eigenvector $u_k(r)$ as

$$u_k(r) = \sum_{n=1}^N u_k(r; n). \quad (4)$$

If the position vector r is considered to be random variable, then the expectation value of the amplitude of an eigenvector is identical to its average value or $E\{u_{k,s}\} = \langle u_{k,s}(r) \rangle$. Similar arguments hold for all the higher moments of $u_{k,s}(r)$, so that in general, after much mathematical manipulation [3], we find

$$E\{(u_{k,s})^n\} = \langle (u_{k,s})^n \rangle = \begin{cases} 1 \cdot 3 \cdot 5 \cdots (n-1) \sigma^n, & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}; s = x, y, z$$

But from Papoulis (4), this is the expression for the moments of a normal density. Therefore, the probability density function $f_u(u_{k,s})$ for the amplitude of any eigenvector $u_{k,s}(r)$ can be written as $f_{us}(u) = (3V/2\pi)^{1/2} \exp[-(3V/2)(u)^2]$. This is a one-parameter density function depending only on the cavity volume V .

From Eq. (1), the components of the electric field are seen to be given by expansions in terms of the eigenvectors, so that the three cartesian components of the electric field are also normal, independent, random variables with zero mean and variances $\sigma^2 = 1/(3V \sum [p_k(t)]^2)$, $k = -\infty$ to $+\infty$, from which it follows that

$$f_{E_s}(E_s) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[-\frac{E_s^2}{2\sigma^2} \right] \quad (5)$$

SOME REPRESENTATIVE RESULTS

Some results are presented in Fig. 1 of measurements made in a mode-stirred chamber (sometimes called a reverberation chamber) by Kaman Sciences [Smith (5)]. A total of 200 measurements were made at a frequency of 6 GHz of the power coupled to a shielded cable, each measurement corresponding to a new position of the paddle, or mode stirrer, separated by 1.8 degrees in rotation. The measured data (x) are compared with the predicted probability density (y). The latter is based on the theory presented above, recognizing the fact that since there are two independent polarizations for the electric field, the power then has a chi-squared distribution with two degrees of freedom. This is just an exponential distribution so that the probability density function for the

power, $f_p(p)$, is given by $f_p(p) = (1/M_p)\exp(-p/M_p)$, where M_p is the average power. By matching the average of the theoretical and measured density distributions, and using the transformation of random variables $y = 10\log(x)$ to employ a dB scale for the power, we obtain the results shown. Also shown in Fig. 1 are the corresponding cumulative distributions (with + denoting experimental and theoretical data), where even better agreement can be observed.

CONCLUDING OBSERVATIONS

The potential applications of statistical EM are varied and diverse. The most immediate utilization of the theory is to all aspects of reverberation-chamber testing, which could lead to better chamber and test design, improved data interpretation and storage, and more efficient characterization of the response of e equipment being tested. In addition, a rigorous theoretical description of a reverberation chamber's EM response would certainly give such measurements a firmer analytical basis.

Similar benefits should accrue in applications to radiation and scattered fields. For example, suppose the statistical distribution of the scattered fields from a large, complex target were to be known *a priori*. Then the number of measurements or computations needed to determine their expected value and variance, or to meet a stated confidence that the field did not exceed a certain level, could be substantially reduced, with a consequent decrease in measurement or computation cost.

Figure 1. A comparison of the probability densities and cumulative distributions for the power coupled to a shielded cable in a mode-stirred chamber measured by Kaman Sciences [5] compared with the theory presented here.

REFERENCES

1. Beran, M. J. and G. B. Parrent, 1964, Theory of Partial Coherence, Prentice-Hall, Englewood Cliffs, NJ.
2. Schiff, Leonard I. (1955), Quantum Mechanics, McGraw-Hill Book Company, Inc., New York.
3. Lehman and Miller (1990), "The Statistics of Electromagnetic Fields in Complex Environments," to be submitted for publication.
4. Papoulis, A. (1965), Probability, Random Variables, and Stochastic Processes, McGraw-Hill Book Company, New York.
5. Smith, Robert (1990), Private Communication, Kaman Sciences, Colorado Springs, CO.