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Semi-leptonic Decays on the Lattice : The Exclusive 0^- to 0^- Case

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ABSTRACT

We present our results for the meson form factors of several semi-leptonic decays. They are computed from the corresponding matrix elements evaluated on the lattice as ratios of Greens functions. The renormalization of the local operators is calculated nonperturbatively. The dependence of the form factors on the four momentum transfer q^2 is studied by injecting external three momenta to the initial and final state mesons. We study the pseudoscalar decays $K \rightarrow \pi l \nu$, $D \rightarrow K l \nu$, $D \rightarrow \pi l \nu$, $D_s \rightarrow \eta l \nu$ and $D_s \rightarrow K l \nu$ on different lattices. We also analyze scaling, finite size and SU(3) symmetry breaking effects. The uncertainties in some lattice parameters, e.g. a^{-1} , as a source of systematic errors in this calculation are discussed.

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1 Introduction and Motivation

In recent years a machinery has been developed to use the lattice technique for calculating weak matrix elements [1, 2, 3]. Once a reliable calculational method becomes available, we will be able to use a vast amount of experimental information to determine free parameters of the Standard Model and to test the limits of the theory in the search for new physics.

The common underlying idea of this type of calculation is well known. After the dynamics of the W boson is removed, we arrive at a low-energy effective theory formulated in terms of quark and lepton operators. The matrix elements of the quark operators are calculated using the lattice technique.

We shall concentrate here on the study of semi-leptonic decays, which are needed for the experimental determination of the elements of the Kobayashi-Maskawa mixing matrix.

The transition amplitude for the decay $B \rightarrow A l \nu$, which is controlled by the KM matrix element V_{ab} , is:

$$\mathcal{T} = \langle A l \nu | H_{eff} | B \rangle, \quad (1)$$

where B and A are arbitrary pseudoscalar mesons, l is a lepton and ν its anti-neutrino and $H_{eff} = \frac{G_F}{\sqrt{2}} J_{hadron}^\mu J_{lepton,\mu}^\dagger$ is the Hamiltonian for the effective theory.

After factoring the leptonic and hadronic currents we have:

$$\begin{aligned} \mathcal{T} &= \frac{G_F}{\sqrt{2}} V_{ab} H^\mu L_\mu^\dagger, \\ L_\mu &= \bar{v}(l^+) \gamma_\mu (1 - \gamma^5) u(\nu), \\ H^\mu &= \langle A | \bar{a} \gamma^\mu (1 - \gamma^5) b | B \rangle. \end{aligned} \quad (2)$$

We briefly review the phenomenology of semi-leptonic decays in the next section and then discuss how to calculate H^μ on the lattice in section 3. Section 4

discusses the renormalization of the vector current on the lattice. We then present our results for the decays $K \rightarrow \pi l \nu$, $D \rightarrow \pi l \nu$, $D \rightarrow K l \nu$, $D_s \rightarrow \eta l \nu$ and $D_s \rightarrow K l \nu$ in section 5. Conclusions and the outlook for the future are discussed in section 6.

2 Phenomenology

Since A and B are pseudoscalars H^μ is a four-vector and thus depends only on the two four momenta that characterize the states. The matrix element can then be parametrized in terms of form factors, *i.e.* functions of the scalar quantity q^2 which has the kinematically allowed range:

$$0 < q^2 \leq q_{\max}^2 = (m_B - m_A)^2 \quad , \quad q \equiv p_B - p_A \quad . \quad (3)$$

There are two conventional parametrizations:

•

$$H^\mu = f_+(q^2)(p_B + p_A)^\mu + f_-(q^2)(p_B - p_A)^\mu \quad (4)$$

•

$$\begin{aligned} H^\mu = & f_+(q^2) \left[(p_B + p_A)^\mu - \frac{m_B^2 - m_A^2}{q^2} (p_B - p_A)^\mu \right] \\ & + f_0(q^2) \frac{m_B^2 - m_A^2}{q^2} (p_B - p_A)^\mu \quad , \end{aligned} \quad (5)$$

$$(m_B^2 - m_A^2) f_0(q^2) = (m_B^2 - m_A^2) f_+(q^2) + q^2 f_-(q^2) \quad ,$$

$$f_0(0) = f_+(0) \quad .$$

Since $q = p_e + p_\nu$, the terms proportional to q^μ , *i.e.*, f_0 or f_- , contribute in proportion to the lepton masses and do not contribute significantly to the transition amplitude, except in the case of the heavy τ lepton.

The second parametrization uses the helicity basis, where f_+ is related to the exchange of a vector particle and f_0 to the exchange of a scalar particle. f_+ and f_0 are thus independent functions. Using this picture, if one assumes that the form factors are dominated by the poles of the corresponding resonances, the following functional behavior is expected in the limit of negligible widths of the resonances:

$$f_0(q^2) = \frac{f_0(0)}{1 - q^2/m_{0+}^2} \quad , \quad f_+(q^2) = \frac{f_+(0)}{1 - q^2/m_{1-}^2} \quad (6)$$

This simple picture certainly has limitations. For example in the case of a heavy meson decaying into a light one, *i.e.*, $B \rightarrow \pi e \nu$, the masses of the 0^- and 1^- resonances are not very different. The pole dominated form factor would vary very rapidly with q^2 near the endpoint. Another limitation is that for f_0 the 0^+ resonances are not known in most cases or only poorly established. On the lattice we can, in principle, determine form factors as a function of q^2 . Therefore, assumptions such as pole dominance are not needed. Indeed, lattice methods, again in principle, should provide a basis for testing such assumptions.

Many models [4] have been presented to calculate the form factors using some form of the quark potential model [5, 6, 7, 8, 9] or QCD sum rules [10, 11, 12] or the bag model [13]. In general the form factors are calculated at a fixed q^2 that is appropriate for the specific model and then extrapolated to the full range of q^2 by using nearest pole dominance. One model calculation [5] obtains an exponential form for the form factors, which follows from assuming a gaussian distribution for the spatial part of the meson wave functions. The predictions from different models do not differ much for most pseudoscalar to pseudoscalar transitions (see section 5) , except where the q^2 dependence of the form factors becomes important, for example in the cases of $D, B \rightarrow \pi$.

2.1 Experiments

In the following we will list some experiments that measured different semi-leptonic decays.

- Let us first describe an old experiment, which was reported by Donaldson *et al.* [14]. They measured the decay $K \rightarrow \pi \mu \nu$ and, using the Cabibbo angle, were able to deduce the form factors(s). More recently, chiral perturbation theory and $SU(3)$ symmetry have been used to calculate [15] the overall normalization (*i.e.* the value at $q^2 = 0$) of the form factors. Coupled with the experimental determination of the shape of the form factor, this procedure gives a precision calculation of the Cabibbo angle [16]. Note that when the final state lepton is a muon, both form factors of the semi-leptonic K decays can be determined from the experiment, since the muon mass is not negligible.

In section 5 we compare these experimental data with our lattice results.

- The decay $D \rightarrow K l \nu$ has been measured by three different experiments, Mark III [17], E 691 [18] and E 653 [19]. They find for the decay rate:

$$\Gamma(D^0 \rightarrow K^- l^+ \nu) = \begin{cases} (8.8 \pm 1.2 \pm 1.4) 10^{10} s^{-1} & \text{E 691} \\ (7.8 \pm 1.2 \pm 0.9) 10^{10} s^{-1} & \text{Mark III} \\ (5.1 \pm 1.5) 10^{10} s^{-1} & \text{E 653} \\ (7.0 \pm 0.9) 10^{10} s^{-1} & \text{average [20]} \end{cases} \quad (7)$$

We can express the decay rate in terms of the unknown parameters (assuming pole dominance):

$$\Gamma(D^0 \rightarrow K^- l^+ \nu) = |V_{cs}|^2 |f_+(0)|^2 1.53 10^{11} s^{-1} . \quad (8)$$

If we constrain V_{cs} by the Standard Model (assuming three generations of quarks and leptons and using unitarity of the KM mixing matrix) [16] to:

$$0.9733 \leq V_{cs} \leq 0.9754 \quad (9)$$

we can extract $f_+(0)$:

$$f_+(0) = \begin{cases} 0.75 \pm 0.05 & \text{E 691 \& Mark III} \\ 0.59 \pm 0.06 & \text{E 653} \\ 0.69 \pm 0.04 & \text{average} \end{cases} \quad (10)$$

Thus we see that an accurate calculation of $f_+(0)$ would give constraints on the Standard Model.

- Mark III [17] measured the Branching ratios:

$$B(D^0 \rightarrow K^- e^+ \nu_e) = (3.4 \pm 0.5 \pm 0.6) \% \quad (11)$$

$$B(D^0 \rightarrow \pi^- e^+ \nu_e) = (0.39^{+0.23}_{-0.11} \pm 0.04) \% .$$

As in eqn. (8), the branching ratios can be related to the KM matrix elements and the form factors at $q^2 = 0$. They find:

$$\left| \frac{V_{cd}}{V_{cs}} \right|^2 \left| \frac{f_+^\pi(0)}{f_+^K(0)} \right|^2 = (0.057^{+0.038}_{-0.015} \pm 0.005) . \quad (12)$$

Using the known values for V_{cs} and V_{cd} [16] we extract the following result on the ratio of form factors:

$$\frac{f_+^\pi(0)}{f_+^K(0)} = 1.06^{+0.35}_{-0.14} \pm 0.05 \quad (13)$$

The errors in this quantity are still very large, but with improving experimental measurements the deviation of this ratio from unity — a measure of SU(3) symmetry breaking — should eventually be measurable. Conversely, if the ratio of the form factors (13) could be calculated reliably, then together with precision experiments measuring the two modes it would allow one to determine $|V_{cd}/V_{cs}|$.

Only for the K decay is there experimental information available for detailed tests of the phenomenological models discussed above. However, this system is

very different from the heavy mesons. Since the models are often based on simplifications, it would be useful to have a method to distinguish among them. The lattice, at least in principle, provides such a method.

3 The Lattice

Following Wilson's formulation [21] for the lattice action $S = S_f + S_g(U)$ we write the gauge part as:

$$\begin{aligned} S_g(U) &= \beta \sum_{\square} \left(1 - \frac{1}{3} \text{ReTr } U_{\square}\right) , \\ U_{\square} &= U_{\mu}(x) U_{\nu}(x + \hat{\mu}) U_{\mu}^{-1}(x + \hat{\mu} + \hat{\nu}) U_{\nu}^{-1}(x + \hat{\nu}) , \end{aligned} \quad (14)$$

where $\beta = \frac{6}{g^2}$ is the gauge coupling. For the fermionic part $S_f = \bar{\psi} M \psi$ we take the inverse quark propagator M to be:

$$\begin{aligned} M &\equiv I - \kappa \epsilon \\ &= I - \kappa \left\{ \sum_{\mu} (1 - \gamma_{\mu}) U_{\mu}(x) \delta_{x+\hat{\mu}, x'} + \sum_{\mu} (1 + \gamma_{\mu}) U_{\mu}^{\dagger}(x') \delta_{x-\hat{\mu}, x'} \right\} , \\ &\text{with } \kappa = \frac{1}{2ma + 8r} . \end{aligned} \quad (15)$$

We omit, for notational simplicity, spin and color indices.

Note that the (lattice) fermion fields in this action have been scaled (compared to the continuum) by:

$$\psi^{\text{cont}} = \sqrt{\frac{2\kappa}{a^3}} \psi^{\text{lat}} . \quad (16)$$

The matrix element on the lattice is constructed in the usual way using a ratio of 2- and 3-point Green's functions. A pseudoscalar meson A with the quark flavors $\bar{a}q$ will be represented by the interpolating field with the corresponding quantum numbers $\chi_A(x) = \bar{a}(x) \gamma_5 q(x)$; its propagation is described by the 2-point function:

$$G_A(\vec{p}, t) = \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle 0 | \chi_A(\vec{x}, t) \chi_A^{\dagger}(0, 0) | 0 \rangle . \quad (17)$$

$$G_A(\vec{p}, t) \xrightarrow{t \text{ large}} \frac{C_A}{2E_A} e^{-E_A t} , \quad (18)$$

$$C_A = |\langle 0 | \chi_A(0) | A \rangle|^2 .$$

This notation [22] is a little different from what was previously [23, 24] used by our group. The old parameter Z_A is given by:

$$Z_A = \frac{C_A}{2E_A} . \quad (19)$$

Since C_A is a momentum independent constant, as follows from eqn. (18) using Lorentz invariance, the current notation is more convenient to use when we introduce momenta.

Defining the 3-point Green's function:

$$G_{AVB}^\mu(\vec{p}, t_A, t_B) = \sum_{\vec{x}_A, \vec{x}_B} \langle 0 | \chi_A(x_A) V^\mu(0) e^{i\vec{p} \cdot \vec{x}_B} \chi_B^\dagger(x_B) | 0 \rangle , \quad (20)$$

we can show that the desired matrix element can be expressed as the ratio:

$$|\langle A | V^\mu | B \rangle| = \left| \frac{G_{AVB}^\mu(\vec{p}, t_A, t_B)}{G_A(0, t_A) G_B(\vec{p}, t_B)} \right| (C_A C_B)^{1/2} \quad (21)$$

in the limit of large time separations $|t_A|, |t_B|, |t_A - t_B| \gg 0$. Because of translational invariance we can keep the current V^μ (the appropriate charge-changing vector current) at the origin. For practical reasons we always have one meson at rest. In this example A has no momentum; the discussion for the other case is analogous.

Table 1 lists all the lattices used for this analysis. The lattice spacing a sets the scale on the lattice and is a function of the gauge coupling β . The values for a^{-1} in table 1 are estimated from string tension measurements [25]. A $\sim 20\%$ uncertainty due to scaling violation is associated with a^{-1} and included in our estimate of systematic effects in section 5.1.

The momenta introduced on a space-time lattice are discretized:

$$\vec{p} = p_{\min}(m_x, m_y, m_z) \quad , \quad m_x, m_y, m_z \in \mathbb{Z}, \quad (22)$$

with $p_{\min} = 2\pi / N_s a$. N_s is the spatial length of the lattice. The minimal momenta that we can inject on our lattices are listed in table 1. For notational simplicity we write the momenta in units of $p_{\min}$, i.e., $(1, 0, 0)$. We will always take all possible permutations of $\vec{p}$ to average over.

The $16^3 \times 25$ and $16^3 \times 33$ lattices, both at $\beta = 5.7$, give us two *independent* sets of 16 configurations, i.e., a total of 32 configurations, to average over. They are analyzed together using only 25 timeslices.

3.1 Discrete Symmetries

This section closely follows the discussion in ref.[2] on how to use the discrete symmetries on the lattice for the Green's functions. It was shown there how the quark propagator in the background of a gluon configuration, $G(x, y, [A])$, is related to the propagator in the transformed background configurations $[A]^C$, $[A]^P$ and $[A]^T$ by the symmetries C , P and T .

Using these relations and assuming a symmetric background the application of C and P invariance to the 2- and 3-point functions as defined above yields:

$$G_A(\vec{p}, t) = G_A^*(\vec{p}, t) \quad , \quad (23)$$

$$G_{AVB}^4(\vec{p}, t_A, t_B) = G_{AVB}^{4*}(\vec{p}, t_A, t_B) \quad , \quad (24)$$

$$G_{AVB}^i(\vec{p}, t_A, t_B) = -G_{AVB}^{i*}(\vec{p}, t_A, t_B) \quad , \quad i = 1, 2, 3 \quad .$$

Our analysis of CP invariance tells us that the meson propagator and the 4-component of the 3-point function are real, and that the spatial part of the 3-point function is purely imaginary. Thus, as outlined in ref.[2], we will take only the real

or imaginary parts of the Green's functions into account, as if we were averaging over exactly CP symmetric background gluon configurations.

Similarly, if we apply T invariance to the Green's functions, we find:

$$G_A(\vec{p}, t) = G_A(\vec{p}, -t) \quad , \quad (25)$$

$$G_{AVB}^4(\vec{p}, t_A, t_B) = -G_{AVB}^4(\vec{p}, -t_A, -t_B) \quad , \quad (26)$$

$$G_{AVB}^i(\vec{p}, t_A, t_B) = G_{AVB}^i(\vec{p}, -t_A, -t_B) \quad , i = 1, 2, 3 \quad .$$

Thus, time reversal invariance tells us that the above Green's functions are (anti-) symmetric in time. Again, we shall use this result and (anti-) symmetrize the Green's functions accordingly.

4 Renormalization

In the preceding section we showed how to compute a matrix element of the vector current between two mesons, *i.e.*, $\langle A | V_\mu | B \rangle$. To be more precise, the vector current used in our calculation is the local current $V_\mu^{loc}(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$.

On the lattice, the (local) vector current $V_\mu^{loc} = \bar{\psi}\gamma_\mu\psi$ is not conserved with the lattice action as defined in eqn. (15), and we have to calculate the renormalization Z_V^{loc} . It was pointed out [26] that instead the current

$$\hat{V}_\mu = \frac{1}{2} [\bar{\psi}(x)(\gamma_\mu - r)U_\mu(x)\psi(x+\mu) + \bar{\psi}(x+\mu)U_\mu^\dagger(x)(\gamma_\mu + r)\psi(x)] \quad (27)$$

is conserved for degenerate bare masses and does not need renormalization. In the continuum limit the normal vector current is recovered and the on-shell matrix elements of the different vector currents are proportional:

$$\langle A | \hat{V}_\mu | B \rangle = Z_V^{loc} \langle A | V_\mu^{loc} | B \rangle \quad . \quad (28)$$

Unfortunately, there is a problem with the implementation of this non-local current into our calculation of the matrix element, as described in section 3. We take the current to be at the origin of the quark propagators, which makes it impractical to use an extended current like $\hat{V}_\mu$. Although it has the disadvantage that it needs renormalization, the local current is more economical in our case. It allows us to recycle old propagators that were used for other calculations and gives us more flexibility to study different reactions.

A perturbative calculation of Z_V^{loc} [27] yields at $\beta = 5.7$:

$$Z_V^{loc} = .80 \quad . \quad (29)$$

From direct computations of Z_V^{loc} [28, 29] it has been shown that the renormalization needs to be calculated nonperturbatively. This can be done using a Ward identity which relates the vector current to the corresponding scalar density [30]:

$$\langle A | \nabla^\mu \hat{V}_\mu | B \rangle = \frac{1}{2} \left(\frac{1}{\kappa_b} - \frac{1}{\kappa_a} \right) \langle A | \bar{a}b | B \rangle \quad . \quad (30)$$

Together with eqn. (28) we get:

$$Z_V^{loc} = \frac{\frac{1}{2} \left(\frac{1}{\kappa_b} - \frac{1}{\kappa_a} \right) \langle A | \bar{a}b | B \rangle}{i \sin aq^\mu \langle A | V_\mu^{loc} | B \rangle} \quad . \quad (31)$$

Note that in Euclidean space q^0 is imaginary. We see that the determination of Z_V^{loc} using the Ward identity involves the matrix element of the scalar density. This nonperturbative determination of the vector current renormalization is an alternative to a direct computation using the conserved current $\hat{V}_\mu$ and eqn. (28).

In table 2 we show our evaluation of Z_V^{loc} using the Ward identity. We have taken it into account in arriving at all our results in section 5.

If we compare in the table Z_V^{loc} from the $16^3 \times 25$ lattice at $\beta = 5.7$ to the $16^3 \times 40$ and the $24^3 \times 40$ lattice at $\beta = 6.0$, we see that for all processes the Z_V^{loc}

from the $\beta = 6.0$ lattices are closer to 1. If the contributions from $\mathcal{O}(a)$ effects are significant, then $Z_V^{l\phi c}$ should indeed be closer to unity at weaker coupling.

The masses used for the K decay are almost degenerate. We find our results for $Z_V^{l\phi c}$ to be compatible to the results obtained from a direct calculation of $Z_V^{l\phi c}$ using eqn. (28) [28, 29]. In the case of $K \rightarrow \pi$ we see a significant variation of $Z_V^{l\phi c}$ with momentum, but not for the D decays.

We have to conclude that our lattices at $\beta = 5.7 - 6.0$ give rise to significant $\mathcal{O}(a)$ effects that have to be corrected nonperturbatively.

5 Results

Table 3 lists all the lattice masses in units of a^{-1} (see table 1 for a conversion to GeV). Our results for the lattice form factors for the D decays are presented in tables 4,5 and 6. Table 7 lists the form factors for the decay $K \rightarrow \pi$ for different lattices. The extrapolated form factors for the various D decays considered here are found in tables 8,9 and 10. The final result for $f_+(0)$ together with a list of all systematic errors is presented in table 11 for all the D decays. Table 12 shows various model predictions in comparison with our lattice results for those decays.

The figures show the (dimensionless) form factors f_+ and f_0 or the ratio of form factors f_0/f_+ versus q^2/m_D^2 , so there are no large uncertainties coming from overall factors of a^{-1} .

The fits in the figures for the extrapolated form factors are done to the data from the $24^3 \times 40$ lattice at $\beta = 6.0$ only. The other lattices are used to estimate systematic errors due to finite size effects, scaling violation and lattice artifacts at finite momentum. We have two sets of lattices at two different couplings. At $\beta = 5.7$ we have $10^3 \times 20$ and $16^3 \times 25$ and at $\beta = 6.0$ we have $16^3 \times 40$ and $24^3 \times 40$.

We thus can estimate finite size effects at each coupling. The $24^3 \times 40$ and $16^3 \times 25$ lattices are similar in spatial volume so that by comparing dimensionless quantities from those two lattices we can estimate scaling violation effects.

As was stated earlier [22] we shall only use momenta $p/p_{\min} \leq \sqrt{2}$ in order to keep lattice artifacts due to finite momentum $pa \simeq 1$ under control.

It is necessary in our calculation to extrapolate our results to the physical cases, because the lightest quark propagators we can invert correspond to pion masses around $\sim 400 - 500$ MeV. We study the form factors as functions of the light hopping parameter. In order to control systematic errors due to extrapolation, we extract $f_+(0)$ and $f_0(0)$ in two different ways:

- The form factors for each momentum transfer are extrapolated to the appropriate hopping parameter, so we get the extrapolated form factors as functions of the momentum transfer q^2 . We then fit the form factors to pole dominance using the corresponding continuum resonance. For the scalar resonances there are no experimental data available, so we use the lattice value (see table 3) $m(0^+) = 2.8 \text{ GeV}$. (method I)
- The form factors for each light hopping parameter are fitted to pole dominance using the appropriate lattice resonances (see table 3). The results for $f_+(0)$ and $f_0(0)$ are then extrapolated in the light hopping parameter (method II). This is the method used in [31].

We also studied the ratio f_0/f_+ . We certainly lose information on the normalization of the form factors ($f_+(0) \equiv f_0(0)$), but we can test, for example, the idea of pole dominance which would imply:

$$\frac{f_0}{f_+}(q^2) = \frac{1 - q^2/m_{1-}^2}{1 - q^2/m_{0+}^2} . \quad (32)$$

Since the extrapolation to physical meson masses introduces an additional uncertainty, we also show the ratio for fixed κ 's, *i.e.* without extrapolation. This is compared to pole dominance using the masses of the 1^- and 0^+ resonances measured on the lattice and to a lattice version of the GISW model, where we used lattice values for the parameters in the model (the constituent quark masses and the wavefunctions at the origin). In the nonrelativistic quark model the pseudoscalar decay constant of a meson M is related to its wave function at the origin by:

$$\langle 0 | A_\mu | M \rangle = f_M p_\mu \quad , \quad f_M = \frac{2\sqrt{3}}{\sqrt{m_M}} |\psi_M(0)| \quad . \quad (33)$$

The constituent quark masses are taken from ref. [32] and the decay constants from ref. [33]. The lattice versions of pole dominance and of the GISW model are indistinguishable, which is an amusing coincidence due to the values of the lattice parameters.

Finally, we calculated the ratio $f_+^\pi(0)/f_+^K(0)$. This quantity is experimentally measured (see eqn. (13) in section 2.1). The deviation from unity of this ratio gives us a measure of SU(3) symmetry breaking effects. Table 13 lists our results for this ratio on the different lattices. One aspect of our calculation slightly complicates this matter. On the $24^3 \times 40$ lattice we calculated the form factors for $D \rightarrow K$ at $\vec{p}_D \neq 0$ only in the SU(3) limit (see table 6). At $\vec{p}_K \neq 0$ the form factors are calculated using non-degenerate hopping parameters. This was done because previous analysis for the decay $D \rightarrow K$ on other lattices (see tables 8,9) showed that at $\vec{p}_D \neq 0$ the difference between non-degenerate and degenerate κ values is numerically small, while the computational effort for a non-degenerate calculation at the above momenta would be considerable.

5.1 Systematic Errors

We use the decay $K \rightarrow \pi$ for an analysis of finite size effects, since it constitutes the lightest mesons on the lattice (see table 7).

Comparing the values of the form factors from the 16^3 lattices with $10^3 \times 20$, both at $\beta = 5.7$, we find that finite size effects do not seem to be important on the big lattice for meson masses down to $\sim 400 \text{ MeV}$. The comparison of the form factors from the two lattices at $\beta = 6.0$, i.e., $16^3 \times 40$ and $24^3 \times 40$ leads to a similar conclusion. Within the uncertainties involved our two largest lattices do not suffer significantly from finite size effects. As to scaling violation effects, we see a change in the values of the form factors going from $\beta = 5.7$ to $\beta = 6.0$ of about 30 %. The conclusion is that we have to take a systematic error due to scaling violation of 30 % into account for the form factors calculated from the lattices at $\beta = 5.7$. We shall take this as a rough estimate of the scaling violation effects from the $\beta = 6.0$ lattices compared to the continuum.

From tables 5 and 6 a comparison of the D system at zero momentum shows again that finite size effects are negligible in this calculation on the $24^3 \times 40$ lattice. However, if we inject momentum to the “pion” on the $16^3 \times 40$ lattice, the form factors show large fluctuations and the extrapolation to the chiral limit (see table 9) becomes unstable. We are injecting momentum (and hence averaging over smaller lattice volumes) with a particle that has already a noisy signal on the small volume of the $16^3 \times 40$ lattice. In the case of $D \rightarrow K$ the results from the $16^3 \times 40$ lattice are more stable and compare better to the $24^3 \times 40$ lattice. Here we only have one light hopping parameter to vary.

We obtain lattice meson masses for the D meson in the range $am \simeq 1.9 - 1.2$ as we vary the coupling between $\beta = 5.7 - 6.0$. The change in the values of

the form factors (see tables 4 and 6) however is roughly the same as in the case of K decays. That leads us to suspect that the dominant source of systematic error is scaling violation and not artifacts due to large values of am . Of course, the true size of these artifacts can only be reliably determined by repeating the calculation at weaker coupling. However, the present analysis, as well as earlier work on decay constants [33], may indicate that the artifacts due to $am \gtrsim \mathcal{O}(1)$ do not overwhelm the qualitative physical picture obtained at such masses.

In the following we list all the uncertainties that entered our analysis of systematic errors in table 11:

- We use two different methods to extrapolate to the physical cases (I and II). The comparison gives us an estimate of systematics due to extrapolation. We find this difference to be less than (or at most comparable to) the statistical errors.
- In order to account for the uncertainty in the scale, we vary a^{-1} at $\beta = 6.0$ over the range $a^{-1} = 1.7 - 2.0$ GeV. This affects the dimensionless form factors only indirectly, through the assignment of physical units to particle masses and q^2 values. We treat this error source as distinct from the “scaling violation” effects found by comparing $\beta = 6.0$ with $\beta = 5.7$.
- The hopping parameter for the charmed quark is varied over the range $\kappa_{ch} = .135 - .118$ (see table 3 for the corresponding meson masses) to account for the uncertainty in the scale.
- As described in the previous section, we calculate the renormalization of the local current nonperturbatively using the Ward identity. Into eqn. (31) enters the sine of the momentum transfer q^μ , rather than q^μ itself as in

the continuum. We take the variation of Z_V^{loc} due to this difference as an additional systematic error into account. For $a q \ll 1$ we are close to the continuum and this difference is negligible.

- As stated before we find scaling violations up to 30 % between $\beta = 5.7$ and $\beta = 6.0$. We include this error as an estimate of scaling violation effects (see table 11) for the $\beta = 6.0$ lattice. Clearly, a better estimate will come from future comparisons of $\beta = 6.0$ with still weaker couplings.

If we just consider each of the form factors separately, it is quite obvious from the figures and the above remarks that our statistical and systematic errors do not allow us, at this point, to seriously test the validity of pole dominance. Our attempts to extract a pole mass from the lattice data give us masses with huge uncertainties, so that we do not learn anything from them.

If we look at the ratio f_0/f_+ instead of the form factors directly we can improve on statistical and systematic errors. One expects the ratio f_0/f_+ to have smaller statistical errors because of the correlations between the configurations. We find that scaling violation errors now lie well within the statistical ones and are ≈ 10 % (see *e.g.* fig. 4). This is true also for the ratio $f_+^\pi(0)/f_+^K(0)$ (see table 13). An important result of this analysis is that the ratio f_0/f_+ serves better to test the q^2 dependence of the form factors because it is less sensitive to statistical and systematical uncertainties.

We find that the ratio f_0/f_+ does not agree well with the expectation from pole dominance. The major deviation appears to come from the q^2 dependence of f_0 . Indeed, one might anticipate large corrections to the pole dominance picture for the scalar intermediate state, which is a p-wave and therefore does not couple well to any localized current (the coupling vanishes in the non-relativistic limit).

For this reason, we do not consider the difference between $f_+(0)$ and $f_0(0)$ as a measure of a systematic error in our calculation of $f_+(0)$ – it is presumably due largely to the extrapolation of $f_0(q^2)$ to $q^2 = 0$ under the not-well-supported assumption of pole dominance. Indeed, without any pole dominance assumption there appears to be no inconsistency with the requirement $f_+(0) = f_0(0)$ (see *e.g.* figs. 4 & 7).

5.2 $K \rightarrow \pi l \nu$

Since there is considerable experimental data available for this decay, it provides us with a good test of the lattice technique.

Figure 1 shows the form factor f_0 for $K \rightarrow \pi$ in comparison with the experimental data from Donaldson et al. [14]. It is clear that the lattice results are reasonable, especially at $\beta = 6.0$. However, one would really like to study the difference of the form factors from unity (a measure of $SU(3)$ violation); the lattice is quite far from being able to provide that at present.

5.3 $D \rightarrow K l \nu$

Our results are shown in figures 2 and 3. We find for $f_+(0)$ (see table 11):

$$f_+(0) = 0.90 \pm 0.08 \pm 0.21 \quad , \quad (34)$$

where the first error is due to statistics and extrapolation and the second error is our estimate of systematics. This number should be compared to the corresponding experimental result given earlier in eqn. (10).

For the form factor f_0 (see figure 3) there are no experimental results available. The q^2 dependence of our Monte-Carlo data does not look like pole dominance,

although, because of the large error bars, they are still consistent with it. Experimentally the mass of the 0^+ resonance is not known. Our measurement of the scalar resonance on the lattice gives a value that is similar to estimates for this resonance from a potential model [7]. We used the lattice value (see table 3) for the 0^+ resonance and get:

$$f_0(0) = 0.70 \pm 0.08 \pm 0.24 \quad . \quad (35)$$

Table 12 shows a comparison of our results for the form factors with different model calculations. The q^2 dependence is not so significant in this decay, and all models give values for the form factors that are more or less consistent with the expected pole dominance shape within an uncertainty of $\sim 20\%$ and with our lattice results.

Results for the form factors of $D \rightarrow K$ (and $D \rightarrow \pi$) from a lattice calculation have also been presented [31, 34] using a method somewhat different from ours. They used a $20 \times 10^2 \times 40$ lattice at $\beta = 6.0$ and find $f_+^K(0) = 0.74 \pm 0.17$ for one set of configurations and $f_+^K(0) = 0.58 \pm 0.04$ for another. Assuming, as is suggested in ref. [34], that the statistical errors in the second set are considerably underestimated, the result from our group is consistent with theirs.

Figure 4 shows the extrapolated ratio f_0/f_+ together with the prediction from pole dominance and the GISW model. We find disagreement with both predictions. As mentioned above the violation appears to come mainly from $f_0(q^2)$.

5.4 $D \rightarrow \pi l \nu$

For the decay $D \rightarrow \pi$ little experimental information is available. A lattice calculation could ultimately serve here to distinguish between model calculations. Figures 5 and 6 show our lattice results for the form factors f_+ and f_0 . We find

for $f_+(0)$:

$$f_+(0) = 0.84 \pm 0.12 \pm 0.35 \quad , \quad (36)$$

and for $f_0(0)$:

$$f_0(0) = 0.62 \pm 0.06 \pm 0.34 \quad . \quad (37)$$

In table 12 we compare our results with different model calculations. We do not find good agreement with any of the listed results from the models, although our errors are too large at the moment for a meaningful comparison. Their predictions do not agree among themselves very well, because they used significantly different q^2 dependences. Here a reduction of the uncertainties in the lattice calculation would help to distinguish between them.

Figure 7 shows the ratio f_0/f_+ and the model predictions in comparison. The decay $D \rightarrow \pi$ is more sensitive to extrapolation than $D \rightarrow K$, leading to larger uncertainties. Still, away from the endpoint region our results do not agree very well with the model predictions. In figures 8,9 we show the ratio at fixed hopping parameters with lattice versions of the models. Because of the smaller errors here, the discrepancy between our lattice data and the predictions of the lattice version of the phenomenological models is somewhat more apparent.

A quantity for which there is experimental information [17] available is the ratio $f_+^\pi(0)/f_+^K(0)$. From table 13 we extract our final lattice result:

$$\frac{f_+^\pi(0)}{f_+^K(0)} = 0.93 \pm 0.10 \pm 0.13 \quad , \quad (38)$$

where a 10% uncertainty due to scaling violation is included in the systematic error (third number). This result agrees within errors with the experimental number for this quantity given earlier (see eqn. (13)).

5.5 $D_s \rightarrow \eta l \nu$

There is a contribution to this transition from disconnected graphs containing quark loops, which cannot be treated in a consistent fashion in the quenched approximation. In the flavor SU(3) symmetry limit these graphs cancel and only the connected piece contributes, which we can calculate using the same methods as for the other decays.

Figures 10 and 11 show our lattice results for the form factors f_+ and f_0 . There are no experimental data available for this decay. Our result for $f_+(0)$ is:

$$f_+(0) = 0.89 \pm 0.09 \pm 0.30 \quad . \quad (39)$$

For the scalar form factor we have:

$$f_0(0) = 0.70 \pm 0.08 \pm 0.25 \quad . \quad (40)$$

In Table 12 we compare our lattice results with model predictions. As in $D \rightarrow K$ the q^2 dependence is not very significant. Figure 12 shows the ratio of form factors in comparison with the pole dominance prediction. Again, this comparison shows more disagreement than the form factors alone, due presumably to $f_0(q^2)$.

5.6 $D_s \rightarrow K l \nu$

Like $D \rightarrow \pi$, this decay is controlled by V_{cd} . Since the final state meson is a kaon, it is like $D \rightarrow K$ not so sensitive to extrapolation and therefore better to study on the lattice than $D \rightarrow \pi$. Experimentally there is little known about this decay.

Figures 13 and 14 show the form factors, the ratio f_0/f_+ is graphed in figure 15. We find for $f_+(0)$:

$$f_+(0) = 0.84 \pm 0.08 \pm 0.18 \quad , \quad (41)$$

and for $f_0(0)$:

$$f_0(0) = 0.67 \pm 0.07 \pm 0.23 \quad . \quad (42)$$

6 Conclusions and Outlook

The analytical, phenomenological and lattice formulation have been presented for a study of several semi-leptonic decays. We show that an evaluation of the form factors of those decays is possible, although we have to deal with statistical and systematic errors from the numerical calculation.

The form factors are studied as functions of the momentum transfer. The lattice gives us here a severe limitation: the momenta are discretized and the continuum limit is possible only for small momenta, because finite momenta introduce additional lattice artifacts. We only use momenta that obey the continuum energy momentum relation. We always average over all possible permutations of the momenta, that helps in stabilizing the finite momentum data which otherwise show larger statistical fluctuations.

We find it useful to use the discrete symmetries for the calculation of the Green's functions to reduce statistical and systematical errors.

A study of the nonperturbative renormalization of the local vector current shows that $\mathcal{O}(a)$ effects are important on our lattices and would lead to significant errors, if not taken into account. As expected the renormalization is less important on a lattice at weaker coupling in agreement with previously found results [29].

The first case which we studied, the decay $K \rightarrow \pi$ is well known and therefore serves as a test of the lattice calculation. Finite size effects are most important for the light mesons. This case is also more sensitive to the extrapolation to the critical hopping parameter. We find at $\beta = 5.7$ a systematic error of $\sim 30\%$ due

to scaling violation compared to $\beta = 6.0$. All other systematic errors are similar to the statistical uncertainties. For the K decay we conclude that our best results of the form factors are consistent with the existing experimental data. However, one would require considerable improvement in statistics to be sensitive to deviations of the form factors from unity.

For the decay $D \rightarrow K$, we find $f_+^K(0) = 0.90 \pm 0.08 \pm 0.21$. This result is in fact closer to unity than to the experimental result (combined with a standard model analysis on the range of the KM mixing angle), but is consistent with experiment and also consistent with pole dominance. $f_0(q^2)$ does not fit well to pole dominance. If, however, we insist on fitting to pole dominance we find: $f_0^K(0) = 0.70 \pm 0.08 \pm 0.24$.

A comparison with existing model calculations shows agreement among them and with our results. This decay does not serve well to distinguish between different model calculations. We find that the ratio f_0/f_+ is, like many other lattice ratios (*e.g.* $f_+^\pi(0)/f_+^K(0)$), less sensitive to statistical and systematical uncertainties. In particular, the scaling violation for $\beta = 5.7 - 6.0$ is now only $\approx 10\%$. In comparison with model predictions we do find a discrepancy with the suggested q^2 dependences. This deviation from pole dominance can very likely be attributed to f_0 for which there is a small overlap of the 0^+ resonance with the local current.

The decay $D \rightarrow \pi$ shows more disagreement among the models. Our results are $f_+^\pi(0) = 0.84 \pm 0.12 \pm 0.35$ and $f_0^\pi(0) = 0.62 \pm 0.06 \pm 0.34$.

Our result for the ratio of form factors is $f_+^\pi(0)/f_+^K(0) = 0.93 \pm 0.10 \pm 0.13$, which is consistent within errors with the experimental number. Note that, because of the size of the errors, neither the theoretical nor the experimental result for this ratio gives clear evidence for SU(3) violations (deviations from unity). However, at this stage, the lattice calculation provides a weak one- σ bound on

SU(3) violations of $\leq 25\%$.

We conclude that overall the problems that we encounter with these calculations seem to be resolvable with time and computer power.

In an ongoing effort we are now analyzing lattices at $\beta = 6.3$ ($24^3 \times 55$) and $\beta = 6.4$ ($32^3 \times 65$). With these lattices we can test scaling violations for $\beta = 6.0$. Since their spatial volumes are smaller than on the other lattices, they are only suitable for heavy-light meson physics near the end point ($\vec{p} = 0$). At larger 3-momenta, we do expect appreciable finite size effects. Efforts will also be directed to explore [22] further semi-leptonic B decays, such as $B \rightarrow D, \pi$.

We estimate that, to reduce the discussed systematic errors in all these decays to the level of $\leq 10\%$, one needs a lattice that can accommodate the physical pion, *i.e.* $1/am_\pi \ll N$. Recent experience shows that a lattice volume of 16^3 is sufficient to accommodate a kaon at $\beta = 6.0$. At $a^{-1} \simeq 4 \text{ GeV}$ ($\beta \sim 6.4$), we would thus need a lattice of size $\sim 128^3 \times 256$ to treat a physical pion. This is of course not within reach of present-day computer resources. However, attempts [35] are currently underway to increase dramatically the computing power available for lattice calculations of such experimentally relevant quantities. Should those efforts bear fruit, they will make simulations of the desired quality feasible.

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lattice size		β (κ_{crit})	$a^{-1}(\text{GeV})$	spat. length (fm)	# of confs.	p_{min} (GeV)
gauge	fermion					
$16^3 \times 24$	$16^3 \times 25$	5.7 (.170)	~ 1	~ 3.2	16	.40
$16^3 \times 32$	$16^3 \times 33$	5.7 (.170)	~ 1	~ 3.2	16	.40
$10^3 \times 10$	$10^3 \times 20$	5.7 (.170)	~ 1	~ 2.0	18	.63
$16^3 \times 40$	$16^3 \times 39$	6.0 (.157)	~ 1.7	~ 1.9	19	.70
$24^3 \times 40$	$24^3 \times 39$	6.0 (.157)	~ 1.7	~ 2.8	8	.45

Table 1: List of lattices

process	lattice	β	κ 's	p / GeV	Z_V^{loc}
$K \rightarrow \pi$	$16^3 \times 25$	5.7	.155, .164	0.0	0.65 (05)
				0.39	0.38 (03)
				0.0	0.59 (07)
	$16^3 \times 40$	6.0	.160, .164	0.0	0.75 (03)
				0.67	0.13 (02)
	$24^3 \times 40$	6.0	.152, .154	0.0	0.79 (07)
$D \rightarrow K$	$16^3 \times 25$	5.7	.094, .160	0.0	1.17 (05)
				0.39	1.15 (04)
	$24^3 \times 40$	6.0	.118, .154	0.0	1.08 (07)
				0.67	1.11 (05)
$D \rightarrow \pi$	$16^3 \times 25$	5.7	.094, .164	0.0	1.06 (06)
				0.0	1.02 (08)
	$16^3 \times 40$	6.0	.118, .154	0.0	1.01 (05)
				0.0	0.92 (06)
	$24^3 \times 40$	6.0	.118, .155	0.0	1.08 (06)
				0.0	1.08 (07)
				0.0	0.95 (06)

Table 2: Z_V^{loc} computed with the Ward identity

<i>lattice</i>	κ_i	κ_f	$a m(0^-)$	$a m(1^-)$	$a m(0^+)$
$16^3 \times 25$	0.094	0.160	1.940 (09)	1.987 (13)	2.8 (07)
		0.164	1.896 (10)	1.948 (15)	2.8 (14)
		0.166	1.877 (11)		
extr.	0.094	0.170	1.850 (12)		
$16^3 \times 25$	0.155	0.160	0.799 (06)		
		0.164	0.724 (06)		
		0.166	0.688 (06)		
extr.	0.155	0.170	0.624 (07)		
$16^3 \times 25$	0.160	0.160	0.689 (06)		
		0.164	0.606 (06)		
		0.166	0.564 (06)		
extr.	0.160	0.170	0.480 (07)		
$16^3 \times 25$	0.164	0.164	0.513 (06)		
		0.166	0.410 (08)		
extr.	0.170	0.170	0.140 (10)		
$24^3 \times 40$	0.118	0.152	1.250 (08)	1.268 (10)	1.63 (27)
		0.154	1.213 (10)	1.231 (12)	1.63 (34)
		0.155	1.194 (12)	1.213 (14)	1.63 (40)
		0.156	1.174 (15)	1.193 (16)	1.60 (70)
extr.	0.118	0.157	1.155 (15)		
$24^3 \times 40$	0.135	0.152	0.896 (08)	0.930 (10)	
		0.154	0.855 (10)	0.900 (20)	
		0.155	0.834 (11)	0.870 (20)	
extr.	0.135	0.157	0.788 (12)		
$24^3 \times 40$	0.152	0.152	0.489 (08)		
	0.154	0.154	0.373 (07)		
	0.155	0.155	0.309 (06)		
	0.156	0.156	0.243 (04)		
extr.	0.157	0.157	0.080 (06)		
$24^3 \times 40$	0.154	0.155	0.342 (07)		
		0.156	0.311 (06)		
extr.	0.154	0.157	0.273 (07)		
$24^3 \times 40$	0.152	0.154	0.435 (08)		
		0.155	0.406 (08)		
extr.	0.152	0.157	0.339 (09)		

Table 3: The scalar, pseudoscalar and vector meson masses on various lattices

κ_i	κ_f	κ_{sp}	P_i	P_f	q^2 / m_π^2	$f_0(q^2)$	$f_+(q^2)$	$f_0/f_+(q^2)$	$f_-/f_+(q^2)$
.094	.160	.160	0.0	0.0	0.416	0.64 (08)			
			1.0	0.0	0.391	0.62 (08)	0.93 (16)	0.67 (06)	-0.75 (13)
			$\sqrt{2}$	0.0	0.365	0.61 (08)	0.91 (16)	0.68 (05)	-0.77 (12)
			0.0	1.0	0.316	0.63 (08)	0.90 (09)	0.70 (03)	-0.84 (09)
			0.0	$\sqrt{2}$	0.227	0.70 (09)	0.91 (10)	0.77 (04)	-0.90 (14)
.094	.164	.164	0.0	0.0	0.532	0.58 (09)			
			1.0	0.0	0.508	0.56 (09)	0.97 (29)	0.58 (10)	-0.77 (19)
			$\sqrt{2}$	0.0	0.485	0.54 (10)	0.93 (27)	0.59 (09)	-0.79 (18)
			0.0	1.0	0.401	0.58 (09)	1.01 (13)	0.58 (05)	-0.98 (11)
			0.0	$\sqrt{2}$	0.295	0.65 (13)	1.00 (15)	0.65 (06)	-1.11 (19)
.094	.166	.166	0.0	0.0	0.611	0.56 (10)			
			1.0	0.0	0.589	0.54 (11)	1.01 (45)	0.53 (17)	-0.75 (28)
			$\sqrt{2}$	0.0	0.566	0.52 (11)	0.97 (43)	0.53 (15)	-0.79 (26)
			0.0	1.0	0.456	0.57 (12)	1.15 (20)	0.50 (07)	-1.05 (14)
			0.0	$\sqrt{2}$	0.337	0.59 (20)	1.12 (25)	0.53 (10)	-1.33 (27)
.094	.160	.164	0.0	0.0	0.463	0.62 (09)			
			1.0	0.0	0.438	0.60 (10)	0.94 (27)	0.64 (10)	-0.74 (21)
			$\sqrt{2}$	0.0	0.413	0.59 (11)	0.91 (27)	0.65 (09)	-0.77 (20)
			0.0	1.0	0.348	0.62 (10)	0.96 (11)	0.65 (06)	-0.91 (14)
			0.0	$\sqrt{2}$	0.249	0.70 (12)	0.97 (13)	0.72 (06)	-1.01 (20)
.094	.160	.166	0.0	0.0	0.489	0.62 (11)			
			1.0	0.0	0.464	0.59 (12)	0.99 (40)	0.60 (14)	-0.78 (28)
			$\sqrt{2}$	0.0	0.440	0.58 (14)	0.97 (38)	0.60 (12)	-0.83 (25)
			0.0	1.0	0.365	0.63 (12)	1.03 (14)	0.61 (08)	-0.97 (20)
			0.0	$\sqrt{2}$	0.261	0.71 (17)	1.04 (18)	0.68 (08)	-1.13 (28)

Table 4: The form factors for D decays on the $16^3 \times 25$ lattice, $\beta = 5.7$

κ_i	κ_f	κ_{sp}	p_i	p_f	q^2/m_i^2	$f_0(q^2)$	$f_+(q^2)$	$f_0/f_+(q^2)$	$f_-/f_+(q^2)$
.118	.152	.152	0.0	0.0	0.362	0.91 (11)			
			1.0	0.0	0.314	0.78 (11)	1.00 (19)	0.77 (07)	-0.60 (17)
.118	.154	.154	0.0	0.0	0.462	0.89 (12)			
			1.0	0.0	0.419	0.77 (13)	1.09 (32)	0.70 (12)	-0.64 (26)
			0.0	1.0	0.202	0.63 (21)	0.78 (23)	0.81 (05)	-0.84 (21)
.118	.155	.155	0.0	0.0	0.529	0.90 (12)			
			1.0	0.0	0.490	0.77 (15)	1.18 (49)	0.65 (20)	-0.66 (38)
			0.0	1.0	0.223	0.43 (30)	0.54 (34)	0.80 (08)	-0.83 (31)
.118	.154	.155	0.0	0.0	0.490	0.92 (13)			
			1.0	0.0	0.449	0.78 (15)	1.11 (44)	0.71 (19)	-0.60 (39)
			0.0	1.0	0.210	0.56 (29)	0.68 (32)	0.83 (07)	-0.74 (27)

Table 5: The form factors for D decays on the $16^3 \times 40$ lattice, $\beta = 6.0$

κ_i	κ_f	κ_{sp}	p_i	p_f	q^2/m_π^2	$f_0(q^2)$	$f_+(q^2)$	$f_0/f_+(q^2)$	$f_-/f_+(q^2)$
.118	.152	.152	0.0	0.0	0.371	0.95 (14)			
			1.0	0.0	0.349	0.91 (13)	1.21 (23)	0.75 (07)	-0.60 (18)
			$\sqrt{2}$	0.0	0.327	0.89 (13)	1.14 (23)	0.78 (08)	-0.56 (20)
			0.0	1.0	0.261	0.89 (12)	1.17 (14)	0.76 (04)	-0.77 (14)
			0.0	$\sqrt{2}$	0.164	0.89 (12)	1.08 (15)	0.82 (02)	-0.92 (14)
.118	.154	.154	0.0	0.0	0.479	0.94 (13)			
			1.0	0.0	0.460	0.91 (12)	1.44 (30)	0.63 (09)	-0.72 (17)
			$\sqrt{2}$	0.0	0.440	0.90 (12)	1.34 (30)	0.67 (10)	-0.68 (21)
			0.0	1.0	0.339	0.86 (12)	1.28 (15)	0.67 (06)	-0.87 (15)
			0.0	$\sqrt{2}$	0.216	0.89 (14)	1.16 (19)	0.76 (04)	-0.99 (18)
.118	.155	.155	0.0	0.0	0.549	0.95 (11)			
			1.0	0.0	0.531	0.93 (11)	1.69 (41)	0.55 (11)	-0.79 (19)
			$\sqrt{2}$	0.0	0.514	0.92 (11)	1.52 (42)	0.60 (14)	-0.72 (24)
			0.0	1.0	0.390	0.85 (11)	1.39 (17)	0.61 (07)	-0.93 (14)
			0.0	$\sqrt{2}$	0.248	0.95 (19)	1.27 (24)	0.75 (06)	-0.95 (23)
.118	.156	.156	0.0	0.0	0.629	1.09 (09)			
			0.0	1.0	0.447	0.86 (13)	1.64 (32)	0.52 (06)	-1.02 (11)
			0.0	$\sqrt{2}$	0.295	1.16 (28)	1.51 (41)	0.77 (13)	-0.76 (44)
.118	.154	.155	0.0	0.0	0.509	0.99 (11)			
			0.0	1.0	0.359	0.89 (11)	1.34 (14)	0.66 (07)	-0.87 (16)
			0.0	$\sqrt{2}$	0.225	0.96 (18)	1.25 (22)	0.77 (06)	-0.93 (23)
.118	.154	.156	0.0	0.0	0.540	1.02 (08)			
			0.0	1.0	0.380	0.95 (12)	1.41 (21)	0.67 (08)	-0.80 (18)
			0.0	$\sqrt{2}$	0.233	1.14 (26)	1.43 (31)	0.79 (08)	-0.82 (31)
.118	.155	.154	0.0	0.0	0.515	0.93 (12)			
			0.0	1.0	0.368	0.80 (10)	1.27 (14)	0.64 (05)	-0.92 (13)
			0.0	$\sqrt{2}$	0.236	0.85 (14)	1.15 (19)	0.74 (04)	-1.03 (18)
.118	.156	.154	0.0	0.0	0.553	0.91 (11)			
			0.0	1.0	0.398	0.74 (09)	1.28 (15)	0.58 (06)	-0.99 (13)
			0.0	$\sqrt{2}$	0.255	0.86 (15)	1.19 (20)	0.72 (06)	-1.03 (23)
.135	.152	.152	0.0	0.0	0.206	1.09 (14)			
			1.0	0.0	0.160	0.94 (12)	1.06 (13)	0.89 (02)	-0.49 (10)
			$\sqrt{2}$	0.0	0.114	0.86 (11)	0.93 (12)	0.93 (02)	-0.44 (11)
.135	.154	.154	0.0	0.0	0.317	1.09 (13)			
			1.0	0.0	0.277	0.96 (11)	1.20 (14)	0.80 (04)	-0.59 (11)
			$\sqrt{2}$	0.0	0.238	0.89 (10)	1.05 (13)	0.84 (04)	-0.53 (13)
.135	.155	.155	0.0	0.0	0.396	1.11 (11)			
			1.0	0.0	0.361	0.98 (10)	1.34 (17)	0.73 (06)	-0.64 (13)
			$\sqrt{2}$	0.0	0.326	0.91 (09)	1.16 (15)	0.79 (06)	-0.56 (16)

Table 6: The form factors for D decays on the $24^3 \times 40$ lattice, $\beta = 6.0$

lattice	β	κ 's	m_K / GeV	m_π / GeV	$f_0(q_m^2)$	q_m^2 / m_K^2
$16^3 \times 25$	5.7	.155, .164	.724 (6)	.515 (7)	0.94 (10)	.083
$10^3 \times 20$	5.7	.155, .164	.76 (1)	.55 (2)	0.99 (12)	.074
$16^3 \times 40$	6.0	.152, .154	.756 (9)	.657 (9)	1.23 (14)	.017
$24^3 \times 40$	6.0	.152, .154	.739 (14)	.636 (13)	1.21 (13)	.020

Table 7: The form factors for $K \rightarrow \pi$ on different lattices

process	p_i	p_f	q^2 / m_π^2	$f_0(q^2)$	$f_+(q^2)$	$f_0/f_+(q^2)$	$f_-/f_+(q^2)$
$D \rightarrow \pi$	0.0	0.0	0.856	0.45 (16)			
	1.0	0.0	0.853	0.41 (17)	1.09 (80)	0.33 (30)	-0.80 (48)
	$\sqrt{2}$	0.0	0.849	0.37 (19)	1.04 (78)	0.32 (28)	-0.83 (47)
	0.0	1.0	0.559	0.52 (16)	1.27 (27)	0.35 (10)	-1.22 (18)
	0.0	$\sqrt{2}$	0.392	0.55 (23)	1.18 (29)	0.44 (11)	-1.49 (32)
	fit (I)		0.00	0.30 (13)	0.55 (13)		
$D \rightarrow K$	0.0	0.0	0.540	0.59 (13)			
	1.0	0.0	0.529	0.57 (16)	1.00 (57)	0.56 (21)	-0.78 (42)
	$\sqrt{2}$	0.0	0.517	0.54 (20)	0.99 (59)	0.54 (19)	-0.85 (41)
	0.0	1.0	0.393	0.62 (14)	1.08 (17)	0.56 (10)	-1.03 (25)
	0.0	$\sqrt{2}$	0.273	0.71 (18)	1.07 (20)	0.66 (09)	-1.18 (31)
	fit (I)		0.00	0.48 (12)	0.75 (12)		
$D \rightarrow K$ SU(3) limit	0.0	0.0	0.540	0.58 (09)			
	1.0	0.0	0.529	0.58 (09)	0.98 (33)	0.57 (12)	-0.76 (21)
	$\sqrt{2}$	0.0	0.517	0.54 (10)	0.95 (33)	0.56 (12)	-0.79 (21)
	0.0	1.0	0.393	0.59 (09)	1.02 (13)	0.59 (05)	-0.96 (11)
	0.0	$\sqrt{2}$	0.273	0.66 (12)	0.98 (14)	0.68 (05)	-1.06 (18)
	fit (I)		0.00	0.46 (07)	0.70 (09)		
$D_s \rightarrow \eta$	0.0	0.0	0.520	0.59 (08)			
	1.0	0.0	0.509	0.57 (09)	0.97 (30)	0.58 (11)	-0.76 (20)
	$\sqrt{2}$	0.0	0.498	0.55 (10)	0.94 (31)	0.58 (11)	-0.79 (20)
	0.0	1.0	0.392	0.59 (09)	1.02 (13)	0.59 (05)	-0.96 (11)
	0.0	$\sqrt{2}$	0.285	0.65 (13)	1.00 (15)	0.65 (06)	-1.10 (19)
	fit (I)		0.00	0.45 (07)	0.66 (09)		
$D_s \rightarrow \eta$	fit (II)		0.00	0.46 (07)	0.62 (08)		

Table 8: The extrapolated form factors from the $16^3 \times 25$ lattice at $\beta = 5.7$ for various processes

process	p_i	p_f	q^2 / m_π^2	$f_0(q^2)$	$f_+(q^2)$	$f_0/f_+(q^2)$	$f_-/f_+(q^2)$
$D \rightarrow \pi$	0.0	0.0	0.856	0.89 (22)			
	1.0	0.0	0.847	0.75 (28)	1.5 (10)	0.40 (44)	-0.77 (77)
	0.0	1.0	0.274	-0.06 (54)	-0.05 (65)	0.77 (18)	-0.81 (69)
	fit (II)		0.00	0.55 (12)	0.31 (38)		
$D \rightarrow K$	0.0	0.0	0.540	0.97 (15)			
	1.0	0.0	0.508	0.81 (23)	1.13 (70)	0.72 (34)	-0.50 (66)
	0.0	1.0	0.180	0.85 (22)	1.09 (25)	0.76 (04)	-1.12 (20)
	fit (I)		0.00	0.72 (11)	0.88 (18)		
	fit (II)		0.00	0.56 (14)	0.50 (38)		
$D \rightarrow K$ SU(3) limit	0.0	0.0	0.540	0.90 (12)			
	1.0	0.0	0.508	0.77 (15)	1.19 (48)	0.64 (20)	-0.67 (37)
	0.0	1.0	0.180	0.85 (17)	1.04 (17)	0.82 (04)	-0.84 (21)
	fit (II)		0.00	0.60 (10)	0.51 (25)		
$D_s \rightarrow \eta$	0.0	0.0	0.520	0.90 (12)			
	1.0	0.0	0.489	0.77 (15)	1.17 (45)	0.65 (18)	-0.66 (35)
	0.0	1.0	0.200	0.65 (21)	0.80 (22)	0.81 (05)	-0.84 (20)

Table 9: The extrapolated form factors from the $16^3 \times 40$ lattice at $\beta = 6.0$ for various processes

process	p_i	p_f	q^2 / m_π^2	$f_0(q^2)$	$f_+(q^2)$	$f_0/f_+(q^2)$	$f_-/f_+(q^2)$
$D \rightarrow \pi$ $\kappa_{ch} = .118$	0.0	0.0	0.856	0.95 (10)			
	1.0	0.0	0.852	0.97 (12)	2.44 (80)	0.20 (20)	-1.12 (31)
	$\sqrt{2}$	0.0	0.848	0.96 (14)	2.17 (84)	0.28 (26)	-1.03 (43)
	0.0	1.0	0.505	0.83 (14)	1.64 (30)	0.46 (07)	-1.09 (13)
	0.0	$\sqrt{2}$	0.314	1.07 (29)	1.47 (37)	0.69 (10)	-0.93 (39)
	fit (I)		0.00	0.61 (06)	0.81 (12)		
fit (II)			0.00	0.63 (06)	0.87 (10)		
$D \rightarrow \pi$ $\kappa_{ch} = .135$	0.0	0.0	0.856	1.14 (11)			
	1.0	0.0	0.850	1.09 (14)	2.00 (40)	0.36 (15)	-1.01 (30)
	$\sqrt{2}$	0.0	0.845	1.05 (13)	1.72 (36)	0.44 (17)	-0.88 (37)
	fit (I)		0.00	0.68 (07)	0.50 (10)		
	fit (II)		0.00	0.89 (08)	0.85 (13)		
$D \rightarrow K$	0.0	0.0	0.540	0.95 (11)			
	1.0	0.0	0.526	0.93 (11)	1.64 (38)	0.56 (11)	-0.79 (19)
	$\sqrt{2}$	0.0	0.511	0.91 (11)	1.50 (39)	0.60 (13)	-0.73 (24)
	0.0	1.0	0.357	0.89 (11)	1.33 (16)	0.67 (07)	-0.85 (16)
	0.0	$\sqrt{2}$	0.212	0.83 (14)	1.10 (19)	0.76 (03)	-1.02 (16)
	fit (I)		0.00	0.72 (08)	0.94 (08)		
fit (II)			0.00	0.68 (07)	0.86 (08)		
$D_s \rightarrow \eta$	0.0	0.0	0.520	0.95 (12)			
	1.0	0.0	0.506	0.93 (12)	1.59 (36)	0.58 (10)	-0.77 (18)
	$\sqrt{2}$	0.0	0.492	0.91 (11)	1.46 (37)	0.62 (12)	-0.71 (23)
	0.0	1.0	0.360	0.86 (11)	1.35 (16)	0.64 (05)	-0.90 (12)
	0.0	$\sqrt{2}$	0.229	0.95 (16)	1.24 (21)	0.76 (05)	-0.93 (21)
	fit (I)		0.00	0.71 (08)	0.92 (09)		
fit (II)			0.00	0.68 (07)	0.86 (08)		
$D_s \rightarrow K$	0.0	0.0	0.558	0.95 (11)			
	1.0	0.0	0.545	0.93 (11)	1.69 (40)	0.54 (11)	-0.81 (19)
	$\sqrt{2}$	0.0	0.533	0.92 (11)	1.54 (42)	0.58 (14)	-0.75 (25)
	0.0	1.0	0.386	0.77 (09)	1.27 (14)	0.60 (05)	-0.96 (13)
	0.0	$\sqrt{2}$	0.249	0.85 (14)	1.18 (19)	0.72 (05)	-1.03 (21)
	fit (I)		0.00	0.67 (07)	0.82 (07)		
fit (II)			0.00	0.67 (07)	0.86 (08)		

Table 10: The extrapolated form factors from the $24^3 \times 40$ lattice at $\beta = 6.0$ for various processes

	$D \rightarrow K$		$D \rightarrow \pi$		$D_s \rightarrow \eta$		$D_s \rightarrow K$	
	$f_+(0)$	$f_0(0)$	$f_+(0)$	$f_0(0)$	$f_+(0)$	$f_0(0)$	$f_+(0)$	$f_0(0)$
<i>result</i>	0.90	0.70	0.84	0.62	0.89	0.70	0.84	0.67
<i>stat. error</i>	0.08	0.08	0.12	0.06	0.09	0.08	0.08	0.07
<i>sys. errors:</i>								
extrapolation	0.08	0.04	0.06	0.02	0.06	0.03	0.04	0.00
var. of a^{-1}	0.04	0.03	0.03	0.02	0.04	0.03	0.04	0.03
var. of κ_{ch}	0.04	0.03	0.19	0.13	0.06	0.04	0.03	0.03
Z_V^{loc}	0.09	0.07	0.08	0.06	0.09	0.07	0.08	0.07
scaling violation	0.16	0.22	0.27	0.30	0.25	0.22	0.15	0.21
SU(3) limit	0.05	0.05			0.10	0.08	0.05	0.05
<i>total systematics</i>	0.21	0.24	0.35	0.34	0.30	0.25	0.18	0.23

Table 11: The results for the form factors at $q^2 = 0$ for various processes

<i>process</i>	<i>model</i>	$f_+(0)$	$f_+(q_m^2)$	$f_0(0)$	$f_0(q_m^2)$
$D \rightarrow K$	BSW, KS [7, 8]	0.76	1.32	0.76	1.05
	GISW, AW [5, 6]	0.77	1.15	0.77	0.80
	AEK, DP [10, 11]	0.6 – 0.75		0.6 – 0.75	
	Experiment	0.69 (04)			
	<i>this work</i> sys. error	0.90 (08) ± 0.21	1.64 (36)	0.70 (08) ± 0.24	0.95 (11)
$D \rightarrow \pi$	BSW, KS	0.69	2.62	0.69	1.35
	GISW, AW	0.51	1.28	0.51	0.53
	AEK, DP	0.6 – 0.75		0.6 – 0.75	
	<i>this work</i> sys. error	0.84 (12) ± 0.35	2.44 (80)	0.62 (06) ± 0.34	0.95 (10)
$D_s \rightarrow \eta$	BSW	0.72	1.32	0.72	1.03
	<i>this work</i> sys. error	0.89 (09) ± 0.30	1.59 (36)	0.70 (08) ± 0.25	0.95 (12)

Table 12: Various D decays in comparison

<i>lattice</i>	<i>method</i>	$f_+^\pi(0)/f_+^K(0)$
$24^3 \times 40$	I $\vec{p}_K \neq 0$ only	1.02 (12)
	II $\vec{p}_K \neq 0$ only	0.91 (08)
	I deg. for $\vec{p}_D \neq 0$ only	0.86 (11)
$16^3 \times 25$	I	0.73 (07)
	II	0.81 (08)
	I $\vec{p}_K \neq 0$ only	0.90 (07)
	II $\vec{p}_K \neq 0$ only	0.90 (08)
	I deg. for $\vec{p}_D \neq 0$ only	0.77 (06)
$16^3 \times 40$	II	0.61 (39)

Table 13: The ratio $f_+^\pi(0)/f_+^K(0)$ on various lattices

Figure 1: The form factor $f_0(q^2)$ vs. q^2/m_K^2 for $K \rightarrow \pi$

Figure 2: The form factor $f_+(q^2)$ vs. q^2/m_D^2 for $D \rightarrow K$

Figure 3: The form factor $f_0(q^2)$ vs. q^2/m_D^2 for $D \rightarrow K$

Figure 4: The ratio f_0/f_+ vs. q^2/m_D^2 for $D \rightarrow K$

Figure 6: The form factor f_0 vs. q^2/m_D^2 for $D \rightarrow \pi$

Figure 7: The ratio f_0/f_+ vs. q^2/m_D^2 for $D \rightarrow \pi$

Figure 8: The ratio f_0/f_+ vs. q^2/m_D^2 for $D \rightarrow \pi$ for fixed hopping parameters

Figure 9: The ratio f_0/f_+ vs. q^2/m_D^2 for $D \rightarrow \pi$ for fixed hopping parameters

Figure 10: The form factor f_+ vs. $q^2/m_{D_s}^2$ for $D_s \rightarrow \eta$

Figure 11: The form factor f_0 vs. $q^2/m_{D_s}^2$ for $D_s \rightarrow \eta$

Figure 12: The ratio f_0/f_+ vs. $q^2/m_{D_s}^2$ for $D_s \rightarrow \eta$

Figure 13: The form factor f_+ vs. $q^2/m_{D_s}^2$ for $D_s \rightarrow K$

Figure 14: The form factor f_0 vs. $q^2/m_{D_s}^2$ for $D_s \rightarrow K$

Figure 15: The ratio f_0/f_+ vs. $q^2/m_{D_s}^2$ for $D_s \rightarrow K$