Solar Oscillations, Gravitational Multipole Field of the Sun and the Solar Neutrino Paradox

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The visual solar oblateness work and the solar seismological work on the internal rotation of the Sun are reviewed and their implications concerning the static gravitational multipole moments of the Sun are discussed. The results of this work are quite deviant which is indicative of the complexity encountered and of the necessity for continued studies based on a diverse set of observing techniques. The evidence for phase-locked internal gravity modes of the Sun is reviewed and the implications for the solar neutrino paradox are discussed. The rather unique possibility for testing the relevance which the phase-locked gravity modes have to this paradox is also noted. The oscillating perturbations in the Sun's gravitational field produced by the classified internal gravity modes and the phase-locked modes are inferred from the observed temperature eigenfunctions. Strains of the order of $10^{-18}$ in gravitational radiation detectors based on free masses are inferred for frequencies near 100 μHz. The relevance of these findings is discussed in terms of a new technique for use in solar seismological studies and of producing background signals in studies of low-frequency gravitational radiation.
1. Introduction

The research on the gravitational multipole moments of the Sun has had a very interesting history in the 1960's, 1970's, and 1980's. The work in the 1960's started with the development of a unique instrument at Princeton University to measure the visual solar oblateness (Dicke 1972). Also in the 1960's at Wesleyan University and the University of Arizona, an astrometric instrument was designed and built in Arizona to track stars near the Sun and to make accurate measurements of the solar diameter. The program that is based on this second instrument is presently known by the acronym SCLERA. The design of the SCLERA instrument was strongly influenced by experiences gained on the Princeton system. In the 1960's and 1980's, visual solar oblateness observations were made with the Princeton instrument (Dicke and Goldenberg 1974; Dicke, Kuhn, and Libbrecht 1985, 1986) and in the 1970's and 1980's with the SCLERA telescope (Hill and Stebbins 1975a; Beardsley 1986).

Due to the effectiveness of a new observational technique which was introduced to reduce the deleterious effects of observing through the earth's atmosphere (Hill, Stebbins and Oleson 1975), the SCLERA instrument obtained such a good signal-to-noise ratio in solar diameter measurements that in 1974 the first evidence of global oscillations of the Sun was found (Hill and Stebbins 1975b). SCLERA has played a fundamental role in the development of the new field of solar seismology based on these oscillations.

In an unanticipated development, the SCLERA program returned to the field of experimental relativity with the realization that the internal

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1 SCLERA is an acronym for the Santa Catalina Laboratory for Experimental Relativity by Astrometry owned and operated by the University of Arizona.

2 The author was involved during 1963 and 1964 in the design and first phase of testing of the Princeton solar oblateness telescope (Dicke 1972).
rotation of the Sun could be inferred from properties of the normal modes of oscillation and that the primary source of a non-negligible gravitational quadrupole moment would likely be the internal rotation of the Sun. This development has led to a second independent way of inferring the gravitational quadrupole moment of the Sun and furnishes an opportunity for checking the results obtained using the more direct visual solar oblateness observations.

It now appears that the SCLERA program may be relevant to experimental relativity in yet another matter. The study of solar oscillations at SCLERA has led to the discovery of several different sets of phase-locked internal gravity modes (Hill 1986c). The phase-locked features indicate that the temperature eigenfunctions associated with these particular modes are relatively large in the Sun's core where the fusion of nuclear matter is accomplished. This not only has immediate implications for the solar neutrino paradox, but also for the potential detection of gravitational radiation. These phase-locked modes, in addition to the classified low-degree gravity modes, may produce oscillatory perturbations in the Sun's gravitational field sufficiently large to cause nontrivial background signals in the more sensitive gravitational radiation detectors which are being considered in the 1980's.

This is indeed an interesting history, starting with the work at Princeton University. We review in the following sections the work on the visual solar oblateness, the seismological work on the internal rotation of the Sun and the implications for the gravitational quadrupole moment of the Sun. In addition, the seismological work on the low-degree gravity modes and on the phase-locked gravity modes is discussed along with the predicted perturbations to the gravitational field and with the relevance of the phase-locked modes to the solar neutrino paradox. The equivalent strains produced by each of the two sources of gravitational field perturbations in gravitational radiation detectors are examined.
2. SCLERA Solar Seismological Program

The objective of a seismological program is the determination and subsequent inversion of the eigenfrequency spectrum of a given system to obtain information on the properties of that system. The determination of the eigenfrequency spectrum requires, in addition to the acquisition of the appropriate observations, the development of a mode classification program capable of classifying a large number of modes with widely different properties. This requirement indicates the need for the ability to classify both acoustic and internal gravity modes with low and intermediate order \( n \) and degree \( \ell \).

Another factor contributing to the success or failure of such a seismological project is the accuracy to which the eigenfrequencies are determined. In general, greater relative precision in eigenfrequency measurements is possible for those modes having longer coherence times. Using the coherence time of the 5-min oscillations as a reference, the acoustic modes and \( f \)-modes with periods longer than 5 min and gravity modes generally meet this condition.

The SCLERA observational and mode classification program has been developed to meet both of the above requirements: the low- and intermediate-order and degree acoustic, \( f \)- and gravity modes can each be detected and classified. This program is based primarily on the differential radius observations from SCLERA, supplemented with various types of Doppler shift and total irradiance observations from other observatories. The SCLERA diameter and differential radius observations yield information about the temperature eigenfunction of an oscillation at the extreme solar limb (Hill, Stebbins and Oleson 1975; Hill, Alexander and Caudell 1985). These observations provide the following categories of information for use in mode classification: (1) the multiplet fine structure of the eigenfrequency spectrum expected for a slowly rotating axisymmetric system, (2) the symmetry properties of the eigenfunction, (3) the parities of \( \ell \) and \( m \), (4) the magnitude of \( \ell \), (5) the \( \exp(i m \phi) \)-dependence of the eigenfunction, and (6) the \( \delta \)-dependence of the eigenfunction given by the spherical harmonic \( Y^m_{\ell} \).
It is apparent that this rather extensive list may permit not only the classification of modes but also the implementation of a number of tests of the accuracy of a given set of mode classifications. The availability of a set of independent tests has become a hallmark of the SCLERA mode classification program.

The results of the observational, analytical and theoretical programs appear in a series of works. These publications include a sizable amount of data, inferred information on the internal rotation of the Sun, evidence of deviations from asymptotic theory, apparent velocity hypothesis for detection of long-period gravity modes, evidence of mode coupling which may be relevant to the solar neutrino paradox, and the development of a new observational technique to detect solar oscillations with disk observations; the references to this work are Bos and Hill (1983); Hill (1984a,b, 1985a,b, 1986a,b,c); Hill, Alexander and Caudell (1985); Hill and Caudell (1985); Hill, Rabaey and Rosenwald (1986); Hill et al. (1986); Hill and Kroll (1986); Hill and Czarnowski (1986); Rabaey and Hill (1986); Oglesby (1986); Yi and Czarnowski (1986); Hill, Tash and Padin (1986).

It is quite common in the study of the solar 5 min oscillations to display the eigenfrequencies in a $k - \omega$ diagram. In practice, this corresponds to plotting the eigenfrequency as a function of the wavenumber $k$ which is $\omega \left[ l(l+1) \right]^{1/2}/R$ where $R$ is the solar radius. Such information, traditionally based on Doppler shift observations, is typically confined to frequencies between 2 mHz and 4 mHz.

The mode classification program at SCLERA has made it possible to extend the observed $k - \omega$ diagram from 2 mHz to 80 µHz for low- and intermediate-degree acoustic, $f$-, and gravity modes. The results of this mode classification program are summarized in Table 1, which shows the mode type, order, degree, and references where the analysis is presented. The domain of these results with respect to $n$ and $l$ is also indicated in Figure 1 in a $v_0, l - \lambda$ diagram analogous to the $k - \omega$ diagram discussed above. To date, 167 multiplets have been identified and $=1247$ resolved modes have been classified. In reference to the theoretical eigenfrequencies of the standard solar model of Saio (1982), the observed $m = 0$ eigenfrequencies are found to be $=10$ µHz above the theoretical values for the acoustic and $f$-
Table 1
Schedule of Classified Modes

<table>
<thead>
<tr>
<th>Mode Type</th>
<th>Order $n$</th>
<th>Degree $l$</th>
<th>Number of Multiplets</th>
<th>Number of Modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>acoustic</td>
<td>$12 &lt; n &lt; 27$</td>
<td>$0 &lt; l &lt; 6$</td>
<td>83</td>
<td>184</td>
</tr>
<tr>
<td>acoustic</td>
<td>1, 2, 3</td>
<td>$2 &lt; l &lt; 22$</td>
<td>30</td>
<td>293</td>
</tr>
<tr>
<td>gravity</td>
<td>0</td>
<td>$18 &lt; l &lt; 36$</td>
<td>19</td>
<td>378</td>
</tr>
<tr>
<td>gravity</td>
<td>$5 &lt; n &lt; 29$</td>
<td>$l = 15$</td>
<td>31</td>
<td>152</td>
</tr>
<tr>
<td></td>
<td>$n = 15$</td>
<td>$28 &lt; l &lt; 34$</td>
<td>4</td>
<td>240</td>
</tr>
</tbody>
</table>

$^a$Hill (1985a)
$^b$Hill (1984b)
$^c$Rabaey and Hill (1986)
$^d$Hill (1985b)
$^e$Hill (1986c)

modes and to be $\pm 1\ \mu$Hz above the theoretical eigenfrequencies for the low-degree gravity modes.

3. The Static Gravitational Multipole Moments of the Sun

Two independent techniques are currently being used to infer the static gravitational quadrupole and hexadecapole moments of the Sun with a potential accuracy that is relevant to a planetary system test of theories of gravitation (Will 1986). The first technique is based on a measurement of the visual shape of the Sun and the second technique relies on information obtained about the solar internal rotation derived in seismological studies of the Sun.

The determination of the multipole moments of the gravitational field from measurements of the Sun's shape relies on the validity of a relationship assumed between the visual shape and an associated gravitational equipotential surface. In the derivation of this relationship, it has traditionally been assumed that the solar surface rotates uniformly on "cylinders" so that the centrifugal force could be written as the gradient of a potential (cf. Dicke and Goldenberg 1974). It has been shown by Beardsley (1986) that such an assumption is more restrictive than required in relating the visual shape to an equipotential
surface. The derivation by Beardsley (1986) is based on the application of perturbation theory techniques developed in non-radial stellar pulsation theory and represents a significant refinement in placing the visual solar oblateness work on a less restrictive domain of validity.

There is an important distinction between the findings obtained from visual solar oblateness observations and those inferred from the internal rotation of the Sun. The information obtained from shape observations speaks to the properties of the gravitational potential directly, while the information obtained from the internal rotation addresses only the dynamical contributions of the internal rotation to perturbations of the gravitational potential. Thus, the availability of two types of findings offers the opportunity for an internal consistency check which may prove to be quite valuable.

Although the shape studies of the Sun do have the potential advantage of a more direct study of the Sun's gravitational potential, the interpretation of shape observations has proved to be nontrivial due to relative changes in the limb darkening function along two different diameters. On the other hand, the studies of the multiplet fine structure for information about the internal rotation have led to quite deviant results. The situation with regard to these two difficulties is reviewed in this section and it is clear that considerably more work is required before a resolution of the deviant findings may be found.

3.1 Visual Solar Oblateness and $P_4$ Shape Term Based on SCLERA 1983 Observations

The shape of the Sun has been under study for almost two and a half centuries, during which time the accuracy of the measurements performed has improved considerably. It is only within the last two decades, however, that these measurements have attained sufficient precision to have a potentially significant impact on solar physics and experimental relativity. The measurements of this class of precision are those based on the techniques of the Princeton 1966 solar oblateness work (Dicke and Goldenberg 1974; Dicke, Kuhn and Libbrecht 1985, 1986) and those based on the
techniques developed at SCLERA (Clayton 1973; Patz 1975; Hill and Stebbins 1975a; and Beardsley 1986).

One of the important lessons of the early 1970's was that extreme care must be exercised in relating observed visual solar oblateness to intrinsic solar oblateness. In particular, it was first noted by Durney and Roxburgh (1969) that a certain type of difference in polar and equatorial limb-darkening functions could pose a serious difficulty in determining the intrinsic oblateness from the Princeton type of solar oblateness observations. Considerations of the Durney-Roxburgh type of anomaly taught us in particular that detailed knowledge of such differences, if they exist, is required to relate the visual solar oblateness to the intrinsic oblateness obtained in the Princeton type of oblateness observations (Dicke 1973; Hill, Brown and Stebbins 1976). For the visual oblateness obtained using the techniques developed at SCLERA, however, sufficient information is usually available observationally to adequately treat the problems presented by a Durney-Roxburgh type of anomaly.

Visual oblateness observations were made in 1983 at SCLERA (Beardsley 1986) and extensive analysis of the SCLERA 1983 diameter observations has been made. Preliminary results are available from this analysis regarding the shape of the Sun. When the solar radius is expressed in terms of Legendre polynomials, the $P_2$ and $P_4$ coefficients in the solar shape can be evaluated using techniques developed at SCLERA. A preliminary value for the $P_4$ shape coefficient is $-1.8$ milliarcsec. Using this number and the distortion in the equipotential solar surface produced by differential rotation, an estimate for $J_4$ has been accomplished, where $J_4$ is defined by the equation

$$
\phi(r, \theta, \phi) = -\frac{GM_0}{r} \left[ 1 - \sum_{l=2}^{\infty} \left( \frac{R}{r} \right)^l J_l P_l(\cos \theta) \right], \ r > R (1)
$$

where $\phi$ is the gravitational potential, $G$ is the gravitational constant, $M_0$ is the mass of the Sun and $P_l$ is a Legendre polynomial.

The theoretical surface distortion was calculated by using the results of the theoretical analysis of Beardsley (1986) referred to in Section 3. Beardsley finds a value for $J_4$ of
\[ J_2 = -2.5 \times 10^{-6} \]  \hspace{1cm} (2)

A preliminary value of the difference in equatorial-polar differential radii \( \Delta r \) due to a \( P_2 \) shape term has similarly been found. The value is \( \Delta r = 33.6 \) arcsec. Combining this with the surface contribution of 8.3 milliarcsec due to rotation, the value

\[ J_2 = 5.2 \pm 1.7 \times 10^{-6} \]  \hspace{1cm} (3)

has been obtained and is included in the summary given in Table 2.

### 3.2 The 1983 and 1984 Princeton Type of Solar Oblateness Observations

Observations of the visual solar oblateness were obtained in 1983 and 1984 by Dicke, Kuhn, and Libbrecht (1985, 1986). These observations are of the Princeton type developed in the work of Dicke and Goldenberg (1974). They report two values for \( J_2 \) based on the 1983 observations: \((8.1 \pm 1.9) \times 10^{-6}\) and \((5.5 \pm 2.9) \times 10^{-6}\), depending on whether or not a certain type of systematic error is taken into account. For the 1984 observations, they report a value for \( J_2 \) of \((-1.3 \pm 0.9) \times 10^{-6}\). These findings are included in the summary given in Table 2.

The results of the analysis of these observations have led Dicke, Kuhn, and Libbrecht (1986) to suggest that the intrinsic visual oblateness and the gravitational quadrupole moment \( J_2 \) may be varying with the solar cycle.

Because of the important implications of such a suggestion and because of the vulnerability of the Princeton type of solar oblateness observations to the Durney-Roxburgh type of anomaly, these 1983 and 1984 observations have been reexamined (Hill and Beardsley 1986).

Quantitative work was incorporated in the analysis of the 1983 and 1984 Princeton type of solar oblateness observations by Dicke, Kuhn and Libbrecht (1985, 1986) for false oblateness signals due to a brightness signal. Using the findings of Dicke, Kuhn and Libbrecht (1985, 1986), it has been demonstrated observationally by Hill and Beardsley (1986) that the probability of limb-darkening differences not being present in the 1983 and
1984 Princeton type solar oblateness observations is $0.22$ and $= 2.4 \times 10^{-5}$, respectively. It was also demonstrated that the false oblateness signals due to brightness signals did not scale with other observables of the brightness signal. It is necessary for the scaling law to be valid for the brightness correction scheme used by Dicke, Kuhn and Libbrecht (1985, 1986) to work. The probabilities that the scaling law was valid in the 1983 and 1984 observations are $= 4.4 \times 10^{-2}$ and $= 2.6 \times 10^{-4}$, respectively. These findings along with the results of three other independent tests by Hill and Beardsley (1986) represent a statistically significant confirmation of the work of Hill and Stebbins (1975a,b) and Hill, Brown and Stebbins (1976) with regard to the existence of limb-darkening differences.

The demonstration of differences in limb-darkening functions and the invalidation of the scaling law leads to the conclusion that the associated systematic errors cannot be modeled without additional observational information. Based on the properties of this class of observations, it was concluded by Hill and Beardsley (1986) that systematic errors as large as 10 milliarcsec and possibly of either sign may be present in the final results of Dicke, Kuhn and Libbrecht (1985, 1986). Since there are no similar quantitative results available for brightness tests of the 1966 solar oblateness observations (Hill and Stebbins 1975b), this demonstration places even more severe limitations on the use of the 1966 Princeton visual solar oblateness as a measure of intrinsic solar oblateness.

These findings with respect to the existence of changes in limb-darkening functions and the potential size of resulting systematic errors considerably weaken the conclusions by Dicke, Kuhn and Libbrecht (1985, 1986) that they have obtained measures of the intrinsic solar oblateness and the gravitational quadrupole moment $J_2$ of the Sun at levels of accuracy relevant to a current planetary system test of theories of gravitation. The findings also cast doubt on their suggestion that the 1966, 1983, and 1984 Princeton type of solar oblateness observations indicate that the intrinsic oblateness and $J_2$ may be changing with the solar cycle.
3.3 Inferred Internal Rotation of the Sun

The identification of the multiplet line structure due to slow internal rotation of the Sun has presented many more problems than anticipated. In particular, there is widespread observational disagreement among the reported determinations of rotational splitting in which there are essentially three different sets of findings [see Hill, Bos and Goode (1982), Hill (1984b, 1985a,b) and Rabaey and Hill (1986) for one finding; Duvall and Harvey (1984), Brown (1985), and Libbrecht and Zirin (1986) for a second; and Duvall, Harvey, and Pomerantz (1986) for a third]. The first set of findings is based on the differential radius observations of typically longer period oscillations, the second set of findings is based on Doppler shift observations of the five min oscillations, and the third set of findings is based on intensity observations within an absorption line of the five min oscillations.

The essential features of the first set of findings imply an internal rotation $\Omega$ that is proportional to $1/r^2$ in the convection zone where $\alpha$ is close to the value of 2, $\Omega/2\pi = 3$ $\mu$Hz (synodic) in the radiative interior, and the $\theta$-dependence of $\Omega$ changes from that of the surface in the outer part of the convection zone. The second set of findings implies an internal rotation rate that is typically a little less than the surface rotation rate and that the latitudinal differential rotation is much less than at the surface for $0.3R < r < 0.7R$. The third set of findings also implies an internal rotation that is typically a little less than the surface rate but that the latitudinal differential rotation remains the same as that at the surface deep into the radiative interior. The values of $J_2$ inferred from these findings are included in the summary given in Table 2.

The dissonance of the information about rotational effects and the apparent grouping of the different findings according to observing technique demonstrates the merits of employing different observational techniques in testing the validity of a given interpretation of a set of observations. The need for continued diversity in the observational techniques of future studies is vital for the resolution of this difficulty and for the further development of solar seismology. It is in this spirit that a number of
different analyses have been undertaken at SCLERA with regard to the identification of the proper rotational splitting effects.

The analyses fall into several categories. They are: (1) the determination of the multiplet fine structure for low-degree five min modes, low-degree acoustic modes, intermediate-degree f-modes, and low- and intermediate-degree gravity modes using primarily the differential radius observations from SCLERA based on the SCLERA mode classification program; (2) the incorporation of observations obtained using different observing techniques into the SCLERA mode classification program; (3) the inferring of the implied internal rotation from these results and the determination of whether the observed multiplet fine structure is consistent with a single internal rotation curve for the Sun; (4) the comparison of the results obtained from one year's observations with those of another as a check on the mode classification program; and (5) the introduction of a new observational technique to enlarge the number of different types of observations used to study this problem.

The results from the works that fall into these five categories have not led to a resolution of the rotational splitting problem but have served to more clearly define the nature and extent of the difficulty.

The multiplet fine structure obtained at SCLERA for the low-order, low-degree acoustic modes, intermediate-degree f-modes, and intermediate-degree gravity modes obtained in the analysis of the 1979 differential radius observations is taken from Hill (1984b), Rabaey and Hill (1986), and Hill, Bos and Goode (1982), respectively. The multiplet fine structure obtained at SCLERA for the five min nodes has been obtained in an analysis (Hill 1985a) based on differential radius observations combined with a number of Doppler shift observations and total irradiance observations (Woodard and Hudson 1983a; Claverie et al. 1981; Grec, Fossat and Pomerantz 1983; Duvall and Harvey 1983; and Scherrer et al. 1983). The multiplet fine structure for the low-degree gravity modes obtained at SCLERA has been obtained in an analysis (Hill 1985b) based on differential radius observations combined with the differential velocity observations made at the Crimean Astrophysical Observatory (Kotov et al. 1983).
The degree of internal consistency of the multiplet fine structure obtained at SCLERA has been examined in a series of works (Hill et al. 1984; Hill, Rosenwald and Rabaey 1986; and Hill et al. 1986), in which each succeeding analysis included the input of a larger number of multiplets. In the most recent work (Hill et al. 1986), the multiplet fine structure for 30 low-order, low-degree acoustic modes, for 83 low-degree 5 min multiplets, and for two intermediate-degree gravity mode multiplets was used. These analyses indicate that the multiplet fine structure obtained at SCLERA for these multiplets is consistent with a single internal rotation curve for the Sun. They also indicate that the multiplet fine structure obtained for the intermediate-degree f-modes is consistent with this internal rotation curve already described in this section.

The efficiency of the SCLERA multiplet classification program that yielded the multiplet fine structure has also been tested by comparing the power spectra of observations obtained in different years and with different observing techniques. Three of these four tests furnish very critical tests of the multiplet classifications. This is because these tests are concerned with the low-order, low-degree acoustic modes which have a coherence time \(< 1\) y. Consequently, these tests are not seeking to determine if one power spectrum is random or not random relative to another power spectrum (which would not test the mode classifications), but are trying to ascertain if a power spectrum is random or not random relative to the frequency spectrum of the classified modes based on the 1979 observations. Positive results were obtained in each of the three tests, indicating a high level of proficiency for the mode classification program.

These series of analyses and tests at SCLERA have yielded an observationally based multiplet fine structure that is internally self-consistent. Further, the structure is observed to be stable from one year to another. There is one common feature of the observations used in the SCLERA analysis that may be relevant to defining the nature of the reported differences in the multiplet fine structure. That feature is that the differential radius observations, the differential velocity observations and the total irradiance observations are sensitive primarily to the temperature eigenfunctions of the oscillations.
3.4 Inferred Gravitational Quadrupole Moment of the Sun

The values for the gravitational quadrupole moment of the Sun $J_2$ inferred from visual oblateness observations and rotational splitting fine structure are summarized in Table 2. The entries in Table 2 are taken from Hill, Rabaey, and Rosenwald (1986) augmented with the latest results of Dicke, Kuhn, and Libbrecht (1985, 1986) and Heardsley (1986). A value of $J_2 = 3.4 \times 10^{-6}$ will introduce a Newtonian correction term to the observationally inferred general relativistic advance of Mercury's perihelion which is 1.0% of that predicted, for example, by Einstein's General Theory of Relativity. It is clear from the discussions in Sections 3.1, 3.2 and 3.3 and from the results in Table 2 that considerably more work is required before a resolution of the deviant findings may be found.

4. Detection and Classification of Low-Degree Internal Gravity Modes of the Sun

A proposal was made in 1978 (Douglass 1978) to use a satellite to measure any oscillating gravitational multipole moments of the Sun associated with the normal modes of oscillation that had been discovered only a few years earlier at SCLERA and at the Crimean Astrophysical Observatory. The importance of direct measurements of the associated gravitational moments was noted in offering another observational technique for use in seismological studies of the Sun. Estimates of the magnitude of these moments were subsequently made by Johnson et al. (1980) using both an approximate solar model and a Cowling polytropic model. They reported for both models that

$$\frac{(J_2')^2}{y_1(R)} \sim 10^{-3}$$ (4)

where $J_2'$ is the oscillatory amplitude of $J_2$ for a given mode of oscillation, $y_1 = \xi_r/r$ and $\xi_r Y_2^m$ is the radial component of the displacement eigenfunction.
Table 2
Gravitational Quadrupole Moment \( J_2 \)
of the Sun

<table>
<thead>
<tr>
<th>Rotational Splitting Fine Structure(^a)</th>
<th>( J_2 \times 10^6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duvall et al. (1984)</td>
<td>0.17 ± 0.04</td>
</tr>
<tr>
<td>Hill, Bos and Goode (1982)</td>
<td>5.5 ± 1.3</td>
</tr>
<tr>
<td>Hill et al. (1984)</td>
<td>4.5</td>
</tr>
<tr>
<td>Hill, Rabaey and Rosenwald (1986)(^c)</td>
<td>5.1 ± 1.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Visual Solar Oblateness</th>
<th>( T^d )</th>
<th>( J_2 \times 10^6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dicke and Goldenberg (1974)</td>
<td>1966</td>
<td>23.7 ± 2.3</td>
</tr>
<tr>
<td>Hill and Stebbins (1975a)</td>
<td>1973</td>
<td>1.0 ± 4.3</td>
</tr>
<tr>
<td>Dicke, Kuhn and Libbrecht (1985)(^ef)</td>
<td>1983</td>
<td>8.1 ± 1.9</td>
</tr>
<tr>
<td>Beardsley (1986)</td>
<td>1983</td>
<td>5.2 ± 1.7</td>
</tr>
<tr>
<td>Dicke, Kuhn and Libbrecht (1986)(^f)</td>
<td>1984</td>
<td>-1.3 ± 0.9</td>
</tr>
</tbody>
</table>

\(^a\) The value obtained by Gough (1982) is not included because it was based on a preliminary set of multiplet classifications which was in error (cf. Hill, 1984b).

\(^b\) Based on rotational curve of Hill et al. (1984).

\(^c\) The value of 7.7 ± 1.8 for \( J_2 \) reported by Hill, Rabaey and Rosenwald (1986) has been corrected for a factor of 2/3 omitted in their analysis.

\(^d\) Year of visual oblateness observations.

\(^e\) Two values are given based on whether or not a certain type of systematic error is taken into account.

\(^f\) See Section 3.2 for discussion of possible systematic errors introduced by observed limb-darkening function changes.

It is important to recognize that should it be possible to detect the normal modes of oscillation by measuring perturbations in the Sun's gravitational field, then these same perturbations may present a problem as a background signal in the more sensitive gravitational radiation detectors that are being considered. In this section we review the properties of the low-degree gravity modes that have been classified by Hill (1985b) and calculate in Section 6 the gravitational field perturbations associated with these modes. The relevance of these perturbations to the generation of background signals in gravitational radiation detectors is examined in Section 8.
The discovery that opened the way for classification of the low-degree gravity modes being considered here was made by Hill, Tash and Padin (1986). They hypothesized that the differential Doppler shift studies of the 160 min period oscillation made by the Crimean Astrophysical Observatory and the Stanford Observatory actually rely on apparent velocity signals -- that is, shifts which are due to the combined effect of the surface rotation of the Sun and the perturbations $\mathcal{J}_\lambda$ in the radiation intensity caused by global oscillations, rather than by physical displacement of the solar surface. This hypothesis has subsequently been confirmed in the work of Hill (1985b) where the differential velocity observations of Kotov et al. (1983) and the 1979 SCLERA differential radius observations (Bos and Hill 1983) were used in a combined analysis.

The result of this analysis was the classification of 31 gravity mode multiplets with $1 \leq \lambda \leq 5$ and $m = 0$ eigenfrequencies between 77.9 and 132.2 $\mu$Hz. Of a possible maximum number of 235 modes belonging to the 31 multiplets, 152 were classified. The observed eigenfrequencies for those classified modes with $\lambda = 2$ are listed in Table 3.

Also listed in Table 3 are the temperature amplitudes $(T'/T)Y^m_q(\pi/2,0)$ observed at the equatorial limb. These temperature amplitudes are derived from the observed differential radius amplitudes by using the spatial filter function $w(m)$ determined by Hill, Alexander and Caudell (1985). The function $w(m)$ gives the differential radius amplitude $\Delta r_1$ in terms of the perturbation $\mathcal{J}_\lambda$ at the equatorial limb and as a function of $m$.

It is important to note that two series of tests have been made of the Hill (1985b) mode classifications. The first series by Hill and Kroll (1986) is based on an analysis of the total irradiance power spectrum obtained by Woodard and Hudson (1983b) and the second series is based on an analysis of the power spectrum of the 1978 solar diameter observations made by Caudell et al. (1980). In the first series of tests, Hill and Kroll (1986) found that the probabilities for evidence of solar gravity modes not being present in the total irradiance spectrum or that the probabilities that the mode classifications of Hill (1985b) are wrong are 0.019 and 0.007, respectively. The two results are independent and based on the frequency properties of the spectra and on the spatial properties of the
Table 3

Observed Eigenfrequencies and Temperature Amplitudes\(^a\) for Classified \(l = 2\) Gravity Modes

<table>
<thead>
<tr>
<th>(n)</th>
<th>(m)</th>
<th>(v_{n,l,m}) (\mu)Hz</th>
<th>((T'/T)Y_{l}^{m}(\pi/2,0) \times 10^{5})</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>-2</td>
<td>137.43</td>
<td>3.7</td>
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</tr>
<tr>
<td></td>
<td>2</td>
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<tr>
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<td>114.63</td>
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<td>4.0</td>
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<tr>
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<td>95.24</td>
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</tr>
<tr>
<td></td>
<td>2</td>
<td>80.95</td>
<td>7.4</td>
</tr>
</tbody>
</table>

\(^a\)The \(T'/T\) are evaluated at an optical depth \(\tau_5 = 0.1\). The error associated with the temperature amplitudes is estimated to be approximately ± 35%. Entries are made only for those modes that have been identified in the observations.

\(^b\)The estimated error associated with the \(v_{n,l,m}\) is ± 0.03 \(\mu\)Hz.
eigenfunctions, respectively. In the second series of independent tests based on 1978 observations, it is found that the probabilities for evidence of gravity modes not being present are $2 \times 10^{-6}$, 0.004 and $1.9 \times 10^{-4}$ respectively. These three results are also independent and based on the frequency properties of the spectra, on the symmetry properties of the eigenfunctions and on the correlation between deviations of the 1978 measured peak frequencies from the frequencies of the classified modes and the ratio of amplitudes of the 1978 and 1979 signals.

It should also be noted that the data used in these two series of tests were taken in 1973 and 1980 while the data used in making the mode classifications were obtained in 1979. The positive correlation obtained in the tests discussed above demonstrates that the coherence time of the classified signals is $\geq 2$ yrs, a result consistent with the gravity mode classification.

5. Gravity Mode Coupling and the Solar Neutrino Paradox

The first evidence of the excitation of intermediate-degree gravity modes was obtained in the 1970's by Hill and Caudell (1979). They found evidence which indicated the excitation of a number of the gravity modes with frequencies near 250 $\mu$Hz and 370 $\mu$Hz and $20 < \ell < 40$. These results were obtained using the 1973 solar diameter observations. With the availability of the 1979 differential radius observations and their improved signal-to-noise ratio and the development of the SCLERA mode classification program, four gravity mode multiplets have been identified with $v_{n,\ell} = 350$ $\mu$Hz and with intermediate degree ($\ell = 30$). The preliminary results have been reported by Hill (1986c).

One of the more important effects of the presence of intermediate-degree gravity modes is to lower the effective central temperature of the Sun and thus to alter the production rates of solar neutrinos. This possibility was noted in the 1970's by Hill (cf. Barabanov et al. 1978; Thomsen 1978). The reduced core temperature would be, in part, a consequence of the additional energy transport mechanism furnished by the large-amplitude gravity modes.
Evidence of two features of mode coupling has been found in the investigation of four multiplets: the frequency patterns of the multiplet fine structure were found to consist of sections linear in m, and the phases of the oscillations were found to have a linear dependence on m. The phase versus m diagram results for the four multiplets represent highly statistically significant support for the phase-locking hypothesis, as well as the similar results for $v_{n,l,m}$ versus m diagrams for each of the four multiplets. It was also noted that the subsets of the members of a multiplet determined to be phase locked by examination of phase versus $m$ coincided with those found on examination of frequency $v_{n,l,m}$ versus m. This last result represents an important self-consistency test.

The evidence of phase locking of a single set of gravity modes leads to a modification of the neutrino production rates predicted by a standard solar model with no solar oscillations present. The evidence of two or more different sets of phase-locked gravity modes should lead to a low-frequency modulation of the neutrino production rates. The periods of modulation are determined by the sets of properties of the phase-locked gravity modes, while the amplitudes of modulation are determined by both the properties of the modes and the details of the neutrino production. Two facts -- that a modulation is a natural consequence of phase locking and that the periods can be derived from solar seismological studies -- place the mode-locking hypothesis in a unique position relative to the many other suggestions that have been offered for resolving the solar neutrino paradox: a periodic modulation of the neutrino production rates is not a natural consequence of most, if not all, of these other suggestions, and the relevance of the phase-locking hypothesis can be tested. Such a test can be made by looking for a correlation between the predicted periodic behavior of the neutrino production rates and their observed fluctuations.

The presence of the relatively large temperature eigenfunctions of phase-locked gravity modes for $0.05 \leq r/R \leq 0.45$ and the associated energy transport of the modes are both expected to alter the neutrino production rates. These two mechanisms lead to a modification of the effective temperature profile defined for a given process in the Sun. It is anticipated in particular that the former will be the more important process
for the modification of the production rate of the neutrinos from the decay of \(^8\)B.

The predicted periodicity of the enhanced neutrino production is obtained directly from the multiplet fine structure observed for the sets of phase-locked modes. The results of the analysis indicate a typical period for this periodicity of \(\approx 2\) y. Evidence of such a periodic behavior has been found, for example, by Sakurai (1984) and Haubold and Gerth (1985).

It should be noted that should a positive correlation be found between the properties of the solar oscillations and a periodicity in the neutrino production rate, it would have an impact on several areas of research. First, of course, such a result would confirm the presence of phase-locked gravity modes. Second, a positive result would provide the opportunity to infer in a quantitative manner the net contribution of the phase-locked modes to the solar neutrino paradox relative to other possible mechanisms such as neutrino oscillations. Third, and most important, the confirmation of the mode-locking hypothesis would demonstrate the value of solar seismology for future studies of solar neutrinos and the energy production processes in the Sun's core through the phase-sensitive detection of modulation in the neutrino production rates.

6. Periodic Variations of the Sun's Gravitational Field Due to Low-Degree Gravity Modes

The eigenfunctions of low-degree gravity modes have been calculated for radial orders \(1 \leq n \leq 12\). The theoretical eigenfrequencies for the \(l = 2, m = 0\) modes are listed in Table 4. These results are obtained assuming a linear adiabatic non-radial oscillation of the Sun. The variables used were the four dimensionless variables employed by Dziembowski (1971). With these variables, the gravitational field eigenfunction is obtained as a primary output. The theoretical results for \(T'/T\) and \(\phi'/\text{gr}\) are also given in Table 4. The outer boundary conditions used in these theoretical calculations correspond to an adiabatic oscillation in the atmosphere. The standard solar model of Saio (1982) was used for these calculations where \(X = 0.74, Y = 0.24\) and \(Z = 0.02\). This model is observed to fit quite well with the
Theoretical properties calculated using the standard solar model of Saio (1982).
Table 5

Predicted $J'_{n,2,m}$ and Strains $h_{n,2,m}$ for Classified l=2 Gravity Modes

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>$v'_{n,m}$ (MHz)</th>
<th>$J'_{n,2,m}$</th>
<th>$h_{n,2,m}$</th>
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<tbody>
<tr>
<td>7</td>
<td>-2</td>
<td>137.43</td>
<td>-5.7 x 10^{-12}</td>
<td>-1.05 x 10^{-18}</td>
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<td>1.63 x 10^{-18}</td>
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<td>132.15</td>
<td>4.9 x 10^{-10}</td>
<td>---</td>
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</tr>
<tr>
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<td>2</td>
<td>126.86</td>
<td>-5.9 x 10^{-12}</td>
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<td>2</td>
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<td>-5.7 x 10^{-12}</td>
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<td>97.76</td>
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<td>2</td>
<td>80.95</td>
<td>1.04 x 10^{-12}</td>
<td>-3.26 x 10^{-19}</td>
</tr>
</tbody>
</table>

a The accuracy of the $v'_{n,2,m}$ is estimated to be ± 0.03 MHz.

b The phase of $J'_{n,2,m}$ is defined in terms of $J'_{n,2,m,|m|}$ and relative to the temperature eigenfunction.

c The $h_{n,2,m}$ are evaluated at the maximum value of the declination $D_0$. For $|m| = 1$ and arbitrary value of $D$, the tabulated values for $h_{n,2,m}$ should be multiplied by $\sin2D/\sin2D_0$.  

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7. Low-Frequency Periodic Variations of The Sun's Gravitational Field due to Intermediate-Degree Gravity Modes

The observational evidence presented by Hill and Caudell (1979) and Hill (1986c) for the detection of intermediate-degree gravity mode multiplets with \( m = 0 \) eigenfrequencies approximately 100 \( \mu \)Hz below the maximum internal Brunt-Väisälä frequency of the Sun may, as noted in the Section 4, have interesting implications beyond relevance to the solar neutrino paradox. The lowest order nonlinear interactions between the modes of a multiplet will lead to one set of gravitational field perturbations with frequencies which are the sum of two eigenfrequencies and a second set which are harmonics of the rotational splitting of a multiplet. For the multiplets under consideration, the typical synodic value for this rotational splitting is 2.89 \( \mu \)Hz.

For the excitation of these modes and the effect which these modes have on the solar neutrino production rates, it is the nonlinear effects in the \( \varepsilon \)-mechanism that give rise to the most important contributions. In contrast, it is the nonlinear effects in the conservation of mass and conservation of momentum equations that are most important in perturbing the gravitational field. Thus, we evaluate the second order perturbations in terms of the first order perturbations in the conservation of mass and of momentum equations. In these expressions, the Lagrangian and Eulerian perturbations will be indicated by \( \delta \) and a prime, respectively, and the first and second order perturbations by subscripts \( 1 \) and \( 2 \), respectively.

From the continuity equation, we have

\[
\frac{\partial \rho_1^{\prime 2}}{\partial t} + \nabla \cdot (\rho_1^{\prime} \mathbf{v}_1^{\prime}) = -\nabla \cdot (\rho_1^{\prime} \mathbf{v}_1^{\prime}) \tag{5}
\]

where

\[
\mathbf{v}_i = \frac{\partial \xi_i}{\partial t}, \tag{6}
\]
the displacement eigenfunction is represented by $\xi_1$, and $\rho$ is the density. From the conservation of momentum equation, we have

$$\frac{\partial \vec{v}_2}{\partial t} + \rho \vec{v}_2 \cdot \vec{v} + \rho \phi_2 = -\rho_1 \frac{\partial \vec{v}_1}{\partial t} - \rho(\vec{v}_1 \cdot \vec{v})_1 - \rho_1 \phi_1'$$

(7)

where $\rho$ is the pressure. In Equations (5) and (6), $\rho_1'$, etc. actually represent sums over the interacting modes. At low frequencies, Equations (5) and (6) reduce to

$$\rho_2' + \nabla \cdot (\rho \vec{v}_2) = -\nabla \cdot (\rho_1' \vec{v}_1)$$

(8)

$$\nabla \rho_2' + \rho_2 \phi_2 + \rho \phi_2' = -\rho_1' \phi_1'$$

(9)

Assuming perturbations of the form

$$\rho_1'(t, r, \theta, \phi) = \rho_1(r) Y^m_l(\theta, \phi) e^{-i\sigma t},$$

(10)

the horizontal and radial components of Equation (9) reduce to

$$\rho_2' + \rho \phi_2' = -\rho_1' \phi_1' \sum_{m_1} Z(l, l, L; m_1, m_2, M)$$

(11)

$$\frac{d\phi_2'}{dr} + \rho \frac{d\phi_2'}{dr} + \rho \frac{d\phi_2'}{dr} = -n \rho_1' \frac{d\phi_1'}{dr} \sum_{m_1} Z(l, l, L; m_1, m_2, M),$$

(12)

where

$$n = \begin{cases} 2, & m_1 = m_2 \\ 1, & m_1 = m_2 \end{cases}$$

(13)

$$Z(l, l, L; m_1, m_2, M) = \int Y^m_l \ Y^m_l \ Y^M_L \ \sin \theta \ d\theta \ d\phi$$

(14)
L and M are the degree and angular order, respectively, of the second order perturbation and \( l, m_1 \), and \( l, m_2 \) are the respective degree and angular orders of the first order perturbations. The coefficient value of \( n = 2 \) in the right hand side of Equation (12) arises in the case of \( m_1 = m_2 \) because of the sum over interacting modes in Equation (9) for first order perturbations and because the radial dependence of \( \Phi'_1(r) \) is independent of \( m_1 \).

An equation in \( p'_2 \) and \( \Phi'_2 \) is obtained by using Equation (11) to obtain an expression for \( p'_2 \) in Equation (12). The resulting Equation is

\[
p'_2 \frac{d\Phi'_2}{dr} - \Phi'_2 \frac{dp'_2}{dr} = \left[ (1-n) p'_1 \frac{d\Phi'_1}{dr} + \Phi'_1 \frac{dp'_1}{dr} \right] \sum_{m_1} Z(l, l, L; m_1, m_2, M). \tag{15}
\]

Using the Poisson equation,

\[
\nabla^2 \Phi'_2 = 4\pi G p'_2, \tag{16}
\]

we obtain a second order nonhomogeneous differential equation for \( \Phi'_2 \):

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi'_2}{dr} \right) + \left[ \frac{L(L + 1)}{r^2} + 4\pi G \frac{dp'/dr}{d\Phi'/dr} \right] \Phi'_2 = \frac{4\pi G}{(d\Phi'/dr)} \left[ (1-n) p'_1 \frac{d\Phi'_1}{dr} + \Phi'_1 \frac{dp'_1}{dr} \right] \sum_{m_1} Z(l, l, L; m_1, m_2, M). \tag{17}
\]

The continuity equation was not required in the derivation of Equation (17) in the low-frequency approximation considered here. However, it has been included here primarily for applications where the frequency is not low. There is also a second reason motivated by the work of Dziembowski (1982), wherein he assumes that \( \xi'_2 = 0 \) in general. If this assumption were in fact valid, a much simpler expression for \( p'_2 \) could be obtained directly for Equation (8) with \( \xi'_2 = 0 \). However, the solution for the \( p'_2 \) does not agree with that obtained for Equations (15) and (16), demonstrating in general that \( \xi'_2 \neq 0 \), and must be retained in second order coupling calculations. The implications of the assumption that \( \xi'_2 = 0 \), such as made in the work of Dziembowski (1982), have not been considered here.
The perturbation $\Phi_2'$ external to the Sun can be written as

\[
\Phi_2' = \left( \frac{R}{r} \right)^{L+1} \int_0^R K_2 S \, dr
\]  

(18)

where

\[
S = \left( (1-n) \rho_1' + \Phi_1' \frac{d \rho_1'}{dr} \right) \sum_{m_1} Z(l, l, L; m_1, m_2, M),
\]

(19)

\[
K_2 = \frac{4 \pi G U_2 U_1 (R)}{(d \phi/dr)(dU_1/dr - U_1 dU_2/dr)}
\]

(20)

and $U_1$ and $U_2$ are two independent homogeneous solutions of Equation (17). The kernel $K_2$ is related to the kernel $K_{\Omega^2}$ used to calculate $J_2$ due to centrifugal forces produced by rotations. We have calculated $K_2$ using Equation (20) and the standard Solar model of Saio (1982); the results are shown in Figure 2. Also shown in Figure 2 for reference is $K_{\Omega^2}$ defined by the equation $J_2(d \phi/dr)_R = J_{\Omega^2} \Omega^2 dr/R$.

The primary contribution to $\int K_2 S \, dr$ arises from a part of the $\Phi_1' \frac{d \rho_1'}{dr}$ term in $S$ which can be written as

\[
\left( \frac{\Phi_1' \frac{d \rho_1'}{dr}}{\phi} \right)_\phi = y_1 y_3 \rho r \left[ g \frac{d}{d \ln r} \left( \frac{N^2}{g} \right) - N^2 \left[ 2 + \frac{N^2 r}{g} + \frac{rg}{c^2} (1 - \frac{g^2}{N^2}) \right] \right]
\]

(21)

where $y_1$ and $y_3$ are two of the variables used by Dziembowski (1971) defined as

\[
y_1 = \xi_r / r,
\]

(22)

\[
y_3 = \Phi'/gr,
\]

(23)

\[
g = d \phi/dr,
\]

(24)

\[
\frac{N^2}{g} = - \left( \frac{d \ln \rho}{dr} - \frac{1}{\Gamma_1} \frac{d \ln P_1}{dr} \right)
\]

(25)
The results for

\[ K'_2 = K_2 \rho \left\{ g \frac{d}{d \ln \rho} \left( \frac{N^2}{g} \right) - N^2 \left[ 2 + \frac{N^2 r}{g} + \frac{N^2 r}{c^2} \right] \left( 1 - \frac{a_2^2}{N^2} \right) \right\} \]  

are also included in Figure 2.

The eigenfunctions \( y_1 \) and \( y_3 \) have been calculated for \( \ell = 28 \) using the standard solar model of Saio (1982). The boundary conditions used are the same as the ones used in Section 6. Using those eigenfunctions, the kernel \( K_2 \) and Equation (21), the inferred values of \( J_{2,m} \) are

\[ J'_{2,m} = -4.4 \times 10^{-10} \sum_{m_1} Z(28, 28, 2; m_1, m_2, M). \]  

The normalization used in the \( y_1 \) and \( y_3 \) correspond to a maximum \( T'/T \) in the core of 0.01, a value not inconsistent with the evidence for phase-locked modes.

General expressions for \( Z(\ell, \ell, 2; m+M, m, M) \) can be obtained either from Messiah (1958) or Dziembowski (1982) and the results are

\[ Z(\ell, \ell, 2; m+M, m, M) = \begin{cases} (-1)^m \left( \frac{5}{4\pi} \right)^{1/2} \frac{[\ell(\ell+1) - 3m^2]}{(2\ell+3)(2\ell-1)}; M = 0 \\ (-1)^{m+1} \left( \frac{15}{8\pi} \right)^{1/2} \frac{(2m+1)[(\ell+m+1)(\ell-m)]^{1/2}}{(2\ell+3)(2\ell-1)}; M = 1 \\ (-1)^m \left( \frac{15}{8\pi} \right)^{1/2} \frac{[1(\ell+m+1)(\ell-m+1)(\ell-m)]^{1/2}}{(2\ell+3)(2\ell-1)}; M = 2. \end{cases} \]  

\[ \text{(30)} \]
It should be noted that a relative phase dependence on $m_1$ has been suppressed in equations containing $\sum_{m_1} Z$, which must be taken into account when the sum over $m_1$ is made, e.g., in Equation (29). A typical value of $Z$ for $\ell = 28$ is seen to be $-0.1$.

We thus conclude that the low-frequency perturbations due to the coupling of the intermediate-degree gravity modes are of the same order of magnitude as the perturbations ascribed to the low-degree gravity modes (cf. Table 5).


There are at the present time major ground-based projects in a number of countries to detect gravitational radiation. These laboratory efforts utilize basically two different approaches to look for strains caused by gravitational radiation. One approach employs massive resonant masses with mechanical Q's approaching $10^6$. Strain sensitivities ($h = \delta x/x$ where $x$ is a length) as low as $10^{-18}$ have been obtained at the typically kilohertz bar resonance frequencies. The other approach involves "free" masses whose separation is measured interferometrically. This approach permits a broadband response. To date, strain sensitivities of $10^{-18}/\text{Hz}^{1/2}$ have been achieved in the frequency range above 800 Hz utilizing 30 and 40 meter systems.

One of the objectives for future developments in that field is to lower the frequency range of low strain sensitivity. For example, this is seen in the study of Faller et al. (1985) where the properties of a large interferometer in space are examined. Their projected sensitivity is $10^{-19}/(\text{Hz})^{1/2}$ over a frequency range of 100 $\mu$Hz to 0.1 Hz.

The frequency range identified in the Faller et al. (1985) analysis includes the low-degree gravity modes that have been classified and discussed in Sections 4 and 6. We examine in this section the strains produced by these solar oscillations in interferometers. Also examined in this section are the very low-frequency (low-order harmonics of a frequency
at = 3 \mu K^{-1}

strains produced by the phase-locked modes discussed in Sections 5 and 7.

In this examination, the change in the longitudinal separation of two masses will be taken as a measure of a strain produced by perturbations in the near field of the Sun. The Sun's gravitational field is written as

$$\phi = -\frac{\text{GM}_0}{r} \left\{ 1 + \sum_{n=2}^{\infty} \left( \sum_{k=2}^{\infty} \left( \sum_{m=-k}^{k} \frac{J'_n,\ell,m}{r} \frac{p_{\ell}}{l} \cos[w_{n,\ell,m}t + m(\lambda - \lambda_{n,\ell,m})] \right) J_n,\ell,m \right) \right\}, \tag{31}$$

where $J'_n,\ell,m$ and $\lambda_{n,\ell,m}$ are constants, $\omega_{n,\ell,m}/2\pi$ is the eigenfrequency of the oscillation, and $\lambda$ is the heliocentric longitude. The rates of change of the oscillating elements for this $\phi$ are given by Lagrange's planetary equations (Smart 1953). Using these rates of change, the short period changes in longitude $\delta\lambda_{SP}$ are obtained and can be written as

$$\frac{\delta\lambda_{SP}}{\delta\lambda} = 2 \left( \frac{\text{GM}_0}{a} \right)^{1/2} \frac{1}{R} \sum_{n=2}^{\infty} \left( \sum_{\ell=2}^{\infty} \left( \sum_{m=-\ell}^{\ell} \frac{m(\ell + 1)}{\omega_{n,\ell,m} t + m(\lambda - \lambda_{n,\ell,m})} \frac{J'_n,\ell,m}{p_{\ell}} \right) \right) \frac{\delta\lambda_{SP}}{2} \left( \sin D \right) \cos \left[ \omega_{n,\ell,m} t + m(\lambda - \lambda_{n,\ell,m}) \right]. \tag{32}$$

where $\delta\lambda$ is the difference in longitude of the free masses not including the short period contributions, $a$ is the semi-major axis of the orbit of the two masses around the Sun, $\dot{\lambda} = d\lambda/dt$, and $D$ is the declination of the masses at time $t$. The contributions of $D$ have been neglected in this expression since the maximum value of $D$ is $7^\circ 15'$ for an interferometer in a near earth orbit. It has also been assumed that $\delta\lambda \ll 1$ and higher order terms in the eccentricity have been omitted.

The amplitude of the strains $\delta\lambda_{SP}/\delta\lambda$, defined as $h_{n,\ell,m}$, for a given perturbing field is therefore given by

$$h_{n,\ell,m} = 2 \left( \frac{\text{GM}_0}{a} \right)^{1/2} \frac{1}{R} \frac{m(\ell + 1)}{\omega_{n,\ell,m} t + m(\lambda - \lambda_{n,\ell,m})} \frac{J'_n,\ell,m}{p_{\ell}} \left( \sin D_0 \right) \right), \tag{33}$$

where $D_0$ is the maximum value of $D$. 

29
Using the inferred values for $J'_{n,l,m}$ obtained in Sections 6 and listed in Table 5 and the associated eigenfrequencies, the strain amplitudes $h_{n,l,m}$ produced by the low-degree gravity modes have been computed and are reported in Table 5. A typical value for $h_{n,l,m}$ is $10^{-18}$. The phases of the strains are relative to the phase of the temperature eigenfunction at the surface.

No entries for $h_{n,l,0}$ are given in Table 5 because the $m=0$ terms in Equation (32) are zero. However, the strains due to $m=0$ modes are not zero in general and the contributions of non-zero terms lead to strains estimated to be of the order of $10^{-19}$ for the classified $l = 2$ modes.

The strain amplitudes produced by the very low-frequency perturbations of the intermediate-degree gravity modes have also been evaluated for $L = 2$ and $m = 1$, and 2. Using the estimates obtained in Section 7 for $J'_{2,m}$, the results for $h_{2,m}$ are

$$h_{2,m} = \begin{cases} -2.5 \times 10^{-16} \sum_{m_1} Z(28,28,2; m_1,m_2,2) ; & m=2 \\ 6.2 \times 10^{-17} \sum_{m_1} Z(28,28,2; m_1,m_2,1) ; & m=1 \end{cases} \quad (34)$$

The frequencies for these two perturbations are 2.9 and 5.8 $\mu$Hz for $m = 1$ and 2, respectively.

The determination of whether or not there will be constructive interference or not in the sum $\sum_{m_1} Z$ will be made in a future detailed study of the properties of the mechanism which couples the phase-locked modes. Assuming for this discussion that there is not a strong contribution from constructive interference and taking as an estimate for $\sum_{m_1} Z$ the typical value for a single value of $Z$, we find that

$$h_{2,m} = \begin{cases} 3 \times 10^{-17} ; & m=2 \\ 10^{-17} ; & m=1 \end{cases} \quad (35)$$

Although the effective $J'_{2,m}$ for the harmonics produced by the phase-locked modes with frequencies = 700 $\mu$Hz were not evaluated here, a preliminary analysis suggests that the $h_{2,m}$ for these harmonics may be of
the order of $10^{-19}$ for $m = 2$ and 1 terms. These could also be one of the more important sources of perturbations to the gravitational potential with regard to the detection of low-frequency gravitational radiation.

In summary, the values obtained for the strain produced by classified modes are seen to be approximately a factor of $10^3$ larger than the strain sensitivity of $10^{-21}$ projected for the system studied by Faller et al. (1985). For the very low-frequency perturbations produced by the second order perturbations associated with the phase-locked gravity modes, the values estimated for the strains are found to be as large $10^4$ times the design sensitivity of $10^{-21}$ for periods of 2-4 days. In the 700 \mu Hz range, the contributions of the second order terms may be $10^2$ times greater than the design sensitivity of $10^{-21}$.

These findings indicate that there may be in the future interesting opportunities and applications for solar seismology with gravitational radiation detectors. The results of the calculations also indicate the potential value of the current solar seismology program in detecting low-frequency gravitational radiation.

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**Figure Captions**

Fig. 1. The theoretical $m = 0$ eigenfrequency spectrum based on the standard solar model of Saio (1982). The enclosed areas represent the domain with respect to $n$ and $\ell$ of the SCLERA mode classification program (see Table 1). The acoustic and gravity modes are indicated by positive and negative radial order numbers, respectively.

Fig. 2. The kernels $K_2$ and $K'_2$ used to calculate the perturbations in the gravitational potential and $K_2^2$ used to calculate $J_2$ defined as $J_2(\phi/dr)_R = \int K_2^2 \Omega^2 dr/R$. The units of the kernels are cm$^2$/sec$^2$. 
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Fig. 1.
Fig. 2.