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STOCHASTIC MOTION OF PARTICLES IN MIRROR MACHINES

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ABSTRACT

Several applications of stochasticity theory to particle motion in quiescent mirror machines are discussed. We briefly review the problem of magnetic moment jumps in a conventional mirror machine, and point out the role of stochasticity in extending the mirror loss cone. We consider magnetic moment jumps in a flat-bottomed magnetic well, such as the solenoid of a tandem mirror machine, and find that, for suitable choices of field parameters, the magnetic moment change per bounce passes through zero at certain values of energy or pitch angle. This system can be modelled by a simple, asymmetric, two-step map; numerical and analytic studies of the map indicate that phase space can contain stochastic layers, separated by nonstochastic layers that serve as barriers for diffusion. We study the motion of guiding centers in a nonaxisymmetric tandem mirror solenoid and show that ions in the Lawrence Livermore Laboratory Tandem Mirror Experiment (TMX) might be stochastic.

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STOCHASTIC MOTION OF PARTICLES IN MIRROR MACHINES

Mirror machines are conceptually based on the constancy of adiabatic invariants. In a conventional mirror machine, constancy of the magnetic moment causes particles with pitch angle greater than some minimum value (i.e., those outside the loss cone) to reflect as they move into a strong magnetic field region. Constancy of a longitudinal invariant confines particles radially. It is well known that these are not true invariants; as Chirikov has pointed out,¹ even in the absence of waves or collisions the existence of resonances between the gyro, bounce and drift motions of a particle preclude global calculation of a complete set of true invariants and can produce stochastic particle motion.

In this paper, we briefly review the role of stochastic motion in a conventional, quiescent (no waves), mirror machine and then concentrate on two examples of stochasticity that have arisen in connection with tandem mirror machines. The first, magnetic moment behavior in a flat-bottomed magnetic well, leads to consideration of a broader class of problems characterized by asymmetric, two-step mappings. The second, collisionless radial diffusion of guiding centers in a tandem mirror solenoid with nonaxisymmetric end plugs, is a perhaps mundane application of stochasticity theory but is, nevertheless, an important one for the tandem mirror program. The problem has been discussed before²; we present here a status report on the application to the Tandem Mirror Experiment (TMX) at Lawrence Livermore Laboratory (LLL).

Conventional Mirror Machines

We summarize here a few of the features of particle motion in conventional mirror machines. A much more extensive review is given in a recent paper by Chirikov.¹

A particle bouncing in a mirror field undergoes jumps in its magnetic moment μ . These may be regarded as the result of resonances of high harmonics of the bounce motion and the gyro motion.^{1,3} These jumps can

be quite accurately calculated from analytic theory, the net change per bounce, or in any other interval (s_1, s_2) along a field line, is

$$\frac{\Delta \mu}{\mu} \approx \left(\frac{v}{v_\perp} \right)_0 \operatorname{Re} [\exp(i\psi_0) \sum_j A_j \exp(i\Delta\psi_j)] \quad (1)$$

Here, ψ is the gyrophase and the subscript 0 denotes a value at a reference point (usually the minimum of B along a field line) designated as $s = 0$, where s measures arc length along a field line; the sum runs over contributing singularities of an integrand, most commonly zeroes of B in the complex s -plane; the coefficient A_j depends on the magnetic field configuration and the particular field line; and $\psi_0 + \Delta\psi_j$ is the (complex) gyrophase at the singularity,

$$\Delta\psi_j = \frac{q}{mc} \int_{s=0}^{s_j} ds \frac{B}{v_{\parallel}(\mu, W, s)} \quad (2)$$

where s_j is the location of the j^{th} zero of B and W is energy.

The number of jumps per bounce and where they occur are determined by the number and location of the zeroes of B . In particular, if we use the gyrophase ψ (rather than s) to label particle guiding center position along a bounce path, it is straightforward to show (and readily verified by numerical experiments) that the jumps occur at $\psi \approx \psi_0 + \operatorname{Re} \Delta\psi_j$, and that two neighboring jumps are resolvable if their separation in ψ is bigger than about 2π . The sum in Eq. (1) thus runs over zeroes that produce jumps in the interval (s_1, s_2) by this criterion.

These jumps can be stochastic. The most significant consequence is a finite extension of the velocity-space loss region beyond the usual mirror loss cone. This effect is due to the fact that $\exp(\operatorname{Im} \Delta\psi_k)$ is generally a strongly increasing function of pitch angle. We illustrate with an example, a particle near the axis of an axisymmetric machine with quadratic axial field variation $B(z, r=0) = B_0(1 + z^2/L^2)$. The location of the complex zeroes of B implies jumps at $z = 0$. The (area-preserving) mapping that takes μ from one bounce point to the next and the gyrophase from one midplane crossing to the next is given by

$$\bar{\mu} = \mu + \alpha \mu^{-1/2} \exp[-\kappa(\bar{\mu})/\epsilon] \cos \psi, \quad (3)$$

$$\bar{\psi} = \psi + \frac{1}{2\pi} \epsilon^{-1} \mu^{-1/2} (1 + \mu^{-1}) - \alpha \sin \psi \partial[\mu^{-1/2} \exp(-\kappa/\epsilon)] / \partial \mu, \quad (4)$$

where $\epsilon = v/\Omega_0 L$, Ω_0 is the gyrofrequency at $z = 0$, $\alpha = (3\pi/4) r_0/L$, μ is the normalized magnetic moment $(v_\perp^2/B)/(v^2/B_0)$, and

Fig. 1. Quadratic well mapping, $\epsilon^{-1} = 6$, $(3\pi/4)(r_0/L) = 0.4$. Stochastic region generated by 2000 iterates of a single particle. Nonstochastic plots each generated by 400 iterates of one particle.

$$\kappa(\mu) = \frac{1}{4\mu} \left[\left(\lambda + \frac{1}{\lambda} \right) \ln \frac{1 + \lambda}{1 - \lambda} - 2 \right], \quad (5)$$

with $\lambda \equiv \mu^{1/2} = (v_{\perp}/v)_0$. We can linearize this map in μ and rescale the μ variable to obtain a "standard map" of the type considered by Chirikov,⁵ $\bar{I} = I + K \cos \psi$, $\bar{\psi} = \psi + \bar{I}$, with mapping parameter $K = (\pi/4)\epsilon^{-1}(\alpha/\mu^2)(\mu+3) \times \exp(-\kappa/\epsilon)$. According to Chirikov,⁵ the mapping is predicted to be grossly stochastic or not according to whether $|K|$ is greater or less than 1. We note that, for our map, K decreases monotonically from ∞ to 0 as μ is increased (at constant energy) from 0 to 1. Hence there is a stochastic regime starting at $\mu = 0$ and extending to some value $\mu_c(r, v)$ at which $K = 1$. This is borne out by numerical experiment, as shown in Fig. 1. If we modify our model field to introduce a finite mirror ratio, by letting it be quadratic for $z < z_{\max}$ and flat for $z > z_{\max}$, then the classical loss cone is at $\mu_L = (1 + z_{\max}^2/L^2)$. When $\mu_c > \mu_L$, we have an extended loss region. The extent increases with energy.

Chirikov has pointed out¹ that the addition of quadrupole field components tends to increase the portion of phase space that is stochastic because the addition of a whole class of resonances involving harmonics of the drift frequency makes it easier for resonances to overlap; additionally, the possibility of Arnold diffusion is opened. However, even if the entire phase space were assumed stochastic, the effective size of the loss region would be limited because of the rapid decrease in the nonadiabatic diffusion coefficient $[\propto \exp(-2 \operatorname{Im} \Delta \psi_j)]$ with pitch-angle: at large enough $v_{\perp}/v$, stochastic diffusion would become smaller than

collisional processes. For parameters characteristic of LLL mirror experiments, the implied loss region is not much different in extent than that obtained from the axisymmetric stochasticity criterion.⁴

Magnetic Moment Jumps in a Tandem Mirror Solenoid

In a tandem mirror machine, each end of a solenoid is plugged by a magnetic mirror and an electrostatic potential is provided by an auxiliary mirror cell. Particles with energy below the confining potential energy are electrostatically confined; hence, the pitch-angle diffusion described in the last section is irrelevant. Machine parameters are chosen so that most particles are in this category, but designs are under consideration have an appreciable population of hot, hopefully magnetically confined, particles.

We model a tandem mirror solenoid field on axis by the expression

$$B(z) = B_0 R \left\{ \gamma \tanh \left[(z^2 - z_0^2)/\delta^2 \right] + 1 \right\}, \quad (6)$$

where $\gamma \equiv (1 - R^{-1})/\tanh(z_0^2/\delta^2)$, R is the mirror ratio at $z = z_0$, and the parameter δ determines the steepness of the magnetic field rise, as shown in Fig. 2. For small z_0/δ , there is, contributing to the change in magnetic moment, a single zero of B in the upper half complex z or ψ plane, giving rise to a single jump at $z = 0$. If z_0/δ is increased, the single zero splits into two at $\delta = \delta_c$, defined by $\gamma(\delta_c) = 1$. For $z_0/\delta > z_0/\delta_c$, two jumps per bounce occur; according to the criterion given in the preceding section, their separation increases with increasing z_0/δ . For large z_0/δ , the jumps occur near the ends of the solenoid. These predictions are born out by orbit calculations.⁶

Fig. 2. Field profiles for model given by Eq. (6) with $R = 8$.

Fig. 3. Comparison of analytic results (solid curve) of Eq. (7) for $\psi_0 = 0$ with numerical results (crosses) obtained by integrating particle orbits for several bounces and choosing the largest jump in μ , for $R = 8$, $z_0/\delta = 1.4$, $z_0 = 275$ cm, $B_0 = 1$ kG.

From Eq. (1) it follows that, in the regime $z_0/\delta > 1$, the magnetic moment change per bounce for a near-axis particle is

$$(\Delta\mu/\mu) \approx (3\pi\Omega_0 r_0/2v) \exp(-\text{Im}\Delta\psi) \cos(\text{Re}\Delta\psi) \cos \psi_0 \quad (7)$$

For values of energy or pitch angle such that $\text{Re}\Delta\psi = \text{odd integer} \times \pi/2$, $\Delta\mu$ is predicted to be zero independent of ψ . The approximations made in deriving Eq. (7) are best justified when $\epsilon \ll 1$ and $\text{Im} \Delta\psi \gg 1$, and when $\text{Re} \Delta\psi \approx 1$, which occur for δ slightly greater than δ_c (two almost merged jumps). As shown in Fig. 3, Eq. (7) appears to be in quite good agreement with orbit calculations.⁶ The small discrepancy in the location of the null appears to be due to the assumption, made in evaluating the integral that gives rise to Eq. (1), that the zeroes of B are sufficiently well separated to allow the contribution of each to be evaluated independently. Finite radius corrections⁷ to Eq. (7) are unimportant for the parameters of Fig. 3.

The null can be regarded as due to a cancellation of the two jumps when the gyrophase accumulated by a particle between the jump points is an odd multiple of π . Its existence suggests the possibility of a succession of barriers in phase space, regions of invariant K.A.M. curves, which particles cannot cross even though the region between barriers may be stochastic. Analysis of the problem indicates that such barriers are most likely to be seen when there is a high degree of asymmetry: a particle experiences two jumps in rapid succession (separated by a time of order of a gyroperiod), followed by a long time interval, followed by two jumps in rapid succession. Such a situation occurs, in the present problem, for $z_0/\delta \gg 1$, as well as for the case already considered; in a different context, it occurs in the problem of cyclotron wave resonances in a mirror

field, such as described by Smith et al.,⁸ when the wave frequency slightly exceeds the minimum gyrofrequency along a field line.

To study the problem in the more general context, we consider a very simple model system, an asymmetric two-step standard map defined by

$$\begin{aligned}\bar{I} &= I + k \sin \theta, & \bar{\theta} &= \theta + \lambda_1 \bar{I}, \\ \bar{\bar{I}} &= \bar{I} + k \sin \bar{\theta}, & \bar{\bar{\theta}} &= \bar{\theta} + \lambda_2 \bar{\bar{I}}.\end{aligned}\quad (8)$$

One can eliminate $\bar{\theta}$ and $\bar{I}$ from these equations to obtain a map relating double-bar quantities to unbarred quantities. If this new map is linearized in the action I , one can obtain a standard map (which now describes the net result of the two steps of the original map) provided that $k\lambda_1 \ll 1$. The stochasticity parameter K for the linearized map is $K = 2k(\lambda_1 + \lambda_2) \cos(\lambda_1 I/2)$. Hence, if $2k(\lambda_1 + \lambda_2) > 1$ and $k\lambda_1 \ll 1$, we expect to see stochastic regions around $\lambda_1 I = 2n\pi$, where n is any integer, and nonstochastic layers of width ΔI around $\lambda_1 I = (2n + 1)\pi$. Here $\Delta I \approx (2/\lambda_1) \left\{ \pi - 2 \cos^{-1} [2k(\lambda_1 + \lambda_2)]^{-1} \right\}$.

In Fig. 4, alternate steps of the two-step map Eq. (8) are plotted for several initial conditions. The nonstochastic bands are clearly visible. The criterion that the stochasticity parameter $K < 1$ is seen to underestimate the size of the nonstochastic layer. The stochasticity parameter is 4.2 in the middle of the stochastic regions.

Numerically, raising k , λ_1 , or λ_2 from the values of Fig. 4 eventually causes the nonstochastic layers to disappear. Analytically, we can estimate a criterion for disappearance from the condition that

Fig. 4. Alternate iterates of two-step map, Eq. (8), for $k = 0.1$, $\lambda_1 = 1.0$, $\lambda_2 = 20$. Each stochastic region is generated by 1500 two-step iterations of a single particle.

resonances on either side of the layer center overlap: we obtain the criterion $\pi k \lambda_1 \geq 1$. This agrees with numerical results as k is raised, but fails to predict the disappearance at high λ_2 (at $\lambda_2 \approx 72$ for $\lambda_1 = 1$, $k = 0.1$). An alternative criterion follows from the hypothesis that the layer vanishes when $\Delta I < k$; this criterion, $k^2 \lambda_1 (\lambda_1 + \lambda_2) > 1$, works well as any parameter is varied. The stochasticity parameter K in the middle of the stochastic region can become quite high (about 14 for $\lambda_2 = 70$, $\lambda_1 = 1$, $k = 0.1$) before the nonstochastic layers disappear.

We see that a system of this type can exceed a nominal (averaged) stochasticity criterion, be stochastic over an appreciable portion of the phase space, but still not have unbounded diffusion due to the existence of the barriers.

This analysis applies to systems with $1\frac{1}{2}$ or 2 degrees of freedom; in systems with more degrees of freedom, we speculate that the stochastic regions are connected in the full phase space, and that a particle can diffuse from one to the other much as particles move from the vicinity of the separatrix of one resonance to that of another in Chirikov's picture of Arnold diffusion.⁵

Radial Diffusion in Tandem Mirrors

Ultimately, we hope to have axisymmetric tandem mirror solenoids. However, present designs have quadrupole-stabilized mirror cells as plugs and, usually, another set of coils that gathers the fan shaped flux tubes emerging from the plug cells and recircularize them as they enter the solenoid. A result is that solenoid ions undergo a radial (cross-field) drift, proportional to $\cos 2\theta$, where θ is the azimuthal angle, at the ends of the solenoid. In most designs, the quadrupole fields are rotated by 90° at opposite ends of the solenoid, so that particles with negligible azimuthal drift experience near cancellation of radial displacements at opposite ends of the machine. However, as Ryutov and Stupakov² have pointed out, the displacements add in phase rather than cancel when the solenoid transit time is an odd multiple of a quarter azimuthal drift time. They show that the combination of radially extended resonant islands and velocity-space collisional transport can lead to significant radial diffusion, even if the islands are well separated; if the islands overlap, intrinsic stochasticity occurs and an even larger level of radial diffusion may result. In this section, we demonstrate that stochasticity may be of relevance for thermal particles in TMX.

The basic mapping describing the radial motion of guiding centers is

$$\begin{aligned} r_{n+1} &\approx r_n + (-)^n a(r_{n+1}, \epsilon, \mu) \cos 2\theta, \\ \theta_{n+1} &\approx \theta_n + b(r_{n+1}, \epsilon, \mu) - \frac{1}{2}(-)^n (\partial a / \partial r_{n+1}) \sin 2\theta, \end{aligned} \quad (9)$$

where ϵ, μ are energy and magnetic moment (taken here as constants). Linearizing in r and rescaling both variables, we can obtain a standard map with stochasticity parameter $K = 2a \partial b / \partial r$.

The step size per bounce a is approximately $a \approx (\rho r / L_{tr}) \kappa (v_\perp / v)$, where ρ is the gyroradius defined using the total velocity and midplane vacuum field, L_{tr} is the transition region length, and κ is obtained by fitting data from numerical integration of the drift equations. For TMX, we find $\kappa \approx 4.57 (v_\perp / v)^2 - 4.51 (v_\perp / v)^4$.

The azimuthal drift per bounce b is due to electric field ($\underline{E} \times \underline{B}$) and ∇B drifts. Ryutov and Stupakov¹⁰ estimated b based on an argument that the electrons in tandem mirror machines should be isothermal and have a radial (as well as axial) Maxwell-Boltzmann distribution. According to this argument, the presence of the quadrupole field and the asymmetry of the ambipolar potential in the plug mirror cells lead to highly noncircular drift surfaces and thus to rapid radial mixing of electrons. The radial electric field is then just $T_e \partial \ln n_e / \partial r$. Estimating the drift to be of order of the $\underline{E} \times \underline{B}$ drift, one then concludes that thermal ions in TMX are not stochastic. However, because the electron collision time in the TMX plugs is short compared to the azimuthal drift time, because most of the particles are confined to a region of radial and axial extent small compared to the vacuum field scale lengths, and because the quadrupole field is relatively weak in this region, we can make estimates indicating that radial mixing TMX of electrons should be small (radial confinement time $>$ axial confinement time). Preliminary results from the Livermore TMT transport code¹¹ indicate that, if electron radial transport is negligible and the ions are assumed nonstochastic, the ambipolar electric field is of order ϕ_e / r_p , where ϕ_e is the mid-solenoid ambipolar potential and r_p is the plasma radius. Estimating a gradient by $1/r_p$, we find that some TMX ions should be stochastic. As the issue is not completely settled, we estimate ion transport effects with both sets of electron assumptions.

We numerically iterate the mapping, Eq. (9), approximating b by the electric field contribution $(cL_s / Br) \partial \phi / \partial r$, where L_s is the solenoid length, and taking $\phi(r) = \Delta \phi \exp(-r^2 / L_r^2) + \text{const}$. We model the isothermal electron case by taking $\Delta \phi \approx T_e (1 - 1/e)^{-1} \approx 316$ eV for TMX, and the no-electron-transport case by taking $\Delta \phi \approx \phi_e \approx 1200$ eV for TMX.

Fig. 5. Iterates of the tandem mirror guiding center map, with $L_r = 30$ cm, $\Delta\phi = 1200$ eV. θ is plotted rotated by $\pi/2$ each iteration so that even and odd iterates can be plotted on the same figure.

We pick a particle with thermal energy and $v_1^2/v^2 = 0.5$ [near the peak value of $\kappa(v_1/v)$]. The results for the isothermal case show islands and nonisland orbits, but no stochasticity. The results for the no-electron-transport case are shown in Fig. 5. An appreciable portion of the plasma volume is stochastic. The $K = 1$ criterion is a good indicator of the stochasticity boundary. Less of the volume is stochastic at other pitch angles, where $\kappa(v_1/v)$ is smaller.

These radial diffusion results are obtained in the drift approximation in which the magnetic moment is constant and gyro motion is ignored. The resonance condition is $2\omega_\theta = (2m + 1)\omega_z$. Returning to the full picture of particle motion, we find a generalized resonance condition¹² of the form $2L\omega_\theta + M\omega_z + N\Omega = 0$ (with restrictions on allowed values of M). Here Ω is gyrofrequency. In a tandem mirror solenoid, $\omega_\theta \sim \omega_z$, and $\Omega \gg \omega_z$. However, the near-square well nature of the magnetic field profile causes copious generation of harmonics of ω_z . Thus, two quite different, but each important, classes of resonances exist: those with $N = 0$, $L \sim M \sim 1$, which are the ones considered in this section, and those with $N \neq 0$, $M \gg 1$, which are the ones responsible for magnetic moment jumps considered in the preceding section. In the presence of quadrupole field components, the latter can give rise to additional radial transport.

Summary

In a quiescent mirror machine, stochastic magnetic moment jumps extend the effective boundary of the velocity space loss cone. In a flat-bottomed magnetic well such as a tandem mirror solenoid, magnetic

moment jumps can be described by an asymmetric two-step map. Such maps can contain stochastic regions separated by nonstochastic barriers. Guiding-center ion motion in a tandem mirror solenoid with quadruple-stabilized end plugs may be stochastic in TMX. In all cases, the stochasticity arises from interaction of resonances of the gyro, bounce and drift motions.

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