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VARIATIONAL PRINCIPLES FOR THE RETURN CURRENT IN ENCEPHALOGRAPHY

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INTRODUCTION

The total electric current within the brain has been written as a sum of two terms, of distinctly different nature (Geselowitz, 1967; Barnard, 1967). The first, $\vec{J}^i$, is the 'primary' or 'impressed' intracellular current that flows within the neurons, and is the quantity of interest in neuroscience. This current leaks across the cell membrane, however, and since the cells are embedded in an electrically conducting medium the extracellular ('return') current flows through a large volume of this medium before returning to the cell. With the return current written as the product of the conductivity σ and the electric field intensity $\vec{E} = -\nabla V$, at low frequencies the electric potential V is determined by the continuity equation (Geselowitz, 1967).

$$\nabla \cdot (\vec{J}^i - \sigma \nabla V) = 0. \quad (1)$$

Although the three-dimensional partial differential equation (1) can be solved numerically for the potential V for any assumed functions $\vec{J}^i$ and σ , it is instructive to consider other mathematical formulations of the problem that are equivalent to Eq. (1). Some of these reformulations may provide some practical advantages, such as greater efficiency of the numerical solution. In addition, as will be shown below, some exact properties of the solution are exposed, which are not obvious by looking at Eq. (1).

One such reformulation of Eq. (1) that has been used recently in encephalography (Meijs, 1987; Härmäläinen, 1989) makes use of the assumption that the conductivity σ can be treated as a different constant in different regions, e.g., brain, skull, and scalp. With this assumption the three-dimensional partial differential equation (1) can be converted to a two-dimensional integral equation, where the integration runs over all the surfaces of discontinuity of σ (Geselowitz, 1967; Barnard, 1967).

We now consider some other reformulations of Eq. (1) based upon a variational principle (Heller, 1989).

VARIATIONAL PRINCIPLE

The variational principle states that the correct potential function $V(\vec{r})$ is the one that makes $P[V]$ a maximum, where

$$P[V] = - \int d^3r [2V \nabla \cdot \vec{J}^i + \sigma (\nabla V)^2] = 0. \quad (2)$$

A straightforward calculation of the first order variation δP produced by a change δV shows that if $\delta P = 0$ for arbitrary δV then (i) V satisfies Eq. (1) at every interior point, and (ii)

$$\vec{n} \cdot \sigma \nabla V = 0 \quad (3)$$

at every point on the surface. Equation (3) states that no current crosses the surface, and is the appropriate boundary condition provided the conductivity vanishes outside the surface.

It can also be shown that an extremum of P is actually a maximum; and furthermore, there is only one maximum, so the solution for V is unique (Heller, 1989). This is also important for numerical work since any change in V that increases P is a step in the right direction.

CONSERVATION OF ENERGY

One special variation of V about its correct value consists of multiplying the entire function by a constant. When this is done it leads to the result that at the correct solution $P[V]$ has the value

$$P_0 = - \int d^3 r V \nabla \cdot \vec{J}' - \int d^3 r \sigma (\nabla V)^2 \quad (4)$$

We shall now show that the first integral in Eq. (4) represents the rate at which the neurons do work in creating the current $\vec{J}'$, and the second integral in Eq. (4) represents the rate at which electrical energy is dissipated (ohmic heating) in the extracellular medium. Consequently, Eq. (4) represents the conservation of energy. To see this recall that for any current $\vec{J}$, $\int d^3 r \vec{J} \cdot \vec{E}$ represents the rate at which the electric field does work on the current. Since $\vec{E} = -\nabla V$, it takes only an integration by parts on the first integral in Eq. (4) to prove the two statements made above.

It is true, of course, that the conservation of energy, as embodied in Eq. (4), could have been written down *ab initio* without any reference to the variational principle, Eq. (2). But the variational principle leads naturally to this result, and that is one of the benefits of formulating the return current problem in this way. Eq. (4) can serve as a check on any numerical solution for V , no matter from what equation that solution was obtained.

POINT SOURCES

Equation (2) is not suitable for numerical work if the divergence of the primary current is concentrated at individual points, which is the case, for example, with a current dipole, because near such a point $\int (\nabla V)^2$ diverges. To overcome this problem write V as a sum of a known term V' and an unknown term ϕ , $V = V' + \phi$, where

$$V'(\vec{r}) = - \frac{1}{4\pi} \int d^3 r' \frac{\nabla' \cdot \vec{J}'(\vec{r}')}{\sigma(\vec{r}') |\vec{r} - \vec{r}'|} \quad (5)$$

and consequently $\nabla^2 V' = \nabla \cdot \vec{J}' / \sigma$.

With this decomposition, $P[V] = P' + p[\phi]$, where P' is known (in terms of $\vec{J}'$ and V') and $p[\phi]$ contains all the dependence on the unknown function ϕ .

$$p[\phi] = \int d^3 r [2\phi \nabla \sigma \cdot \nabla V' + \sigma (\nabla \phi)^2] \quad (6)$$

The return current problem has now been reduced to finding the function ϕ that makes $p[\phi]$ a maximum.

REGIONS OF CONSTANT CONDUCTIVITY

If it can be assumed that the conductivity is a different constant in different regions, e.g., brain, skull, and scalp, then the only places where $\nabla\sigma$ does not vanish are on the surfaces where σ jumps in value. Furthermore, $\nabla\sigma$ is normal to these surfaces, and Eq. (6) can be rewritten as

$$p[\phi] = - \sum_j \int dS \vec{n} \cdot [2(\sigma'_j - \sigma''_j) \nabla\phi + (\sigma'_j \nabla'\phi - \sigma''_j \nabla''\phi)] \phi - \sum_j \sigma_j \int d^3r \phi \nabla^2 \phi, \quad (7)$$

where we have followed the convention that the unit normal vector $\vec{n}$ points outward from the region designated with single primes, and into the region designated with double primes. The summation in Eq. (7) runs over all the surfaces of discontinuity of σ .

In trying to maximize $p[\phi]$ in either Eq. (6) or Eq. (7) there are no restrictions on the trial function ϕ . We know, however, from Eqs. (1) and (5) that the correct function ϕ satisfies $\nabla^2\phi = 0$ inside every region of constant σ , and consequently the sources of ϕ are all located on the surfaces separating regions having different values of σ . It makes sense, therefore, to parameterize the variational function ϕ in terms of a surface charge density τ ,

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_j \int dS' \frac{\tau(\vec{r}')}{|\vec{r} - \vec{r}'|}. \quad (8)$$

Since Eq. (8) satisfies $\nabla^2\phi = 0$ inside every region for any choice of τ , the final term in Eq. (7) can now be set equal to zero. The return current problem has now been reformulated into finding the surface charge density τ that makes $p[\phi]$ a maximum.

NUMERICAL METHODS

The original differential equation (1) for V can be solved directly, by a finite difference scheme, for example. If it is desired to model the primary current $\vec{J}^i$ with fine spatial resolution then this would require a very large number of grid points and lead, therefore, to a very large system of linear equations to be solved. The two-dimensional integral equation for the potential also leads to a system of linear equations (Meijs, 1987; Hämmäläinen, 1989), which would become large if more and more regions having different conductivity values are considered.

The variational principle can also be converted to a set of linear equations, by the method of finite elements, for example. Such a procedure has been used in electrocardiography (Yamashita, 1984; Pilkington, 1985), but which differs from Eqs. (2) or (6) in two important respects. In the ECG problem one tries to relate the potential on the surface of the heart to that measured on the body surface, and it is assumed that there are no current sources in the volume between these surfaces. The term linear in V in those equations is not present, therefore. A second difference between ECG and encephalography is that the boundary condition on the heart surface is not given by Eq. (3) since current can flow across that surface. Instead, the values of the potential there are taken as a constraint upon the allowed functions V .

There is another way to treat the variational principle, Eq. (7), that does not lead to a system of linear equations. It is based upon the integral equation for the surface charge density τ (Gelernter, 1964; Rush, 1966; Barnard, 1967), which can be written

$$\frac{1}{2\epsilon_0} \frac{\sigma' + \sigma''}{\sigma' - \sigma''} r(\vec{r}) = - \vec{n} \cdot \nabla V'(\vec{r}) + \frac{1}{4\pi\epsilon_0} \vec{n} \cdot \int dS' r(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}, \quad (9)$$

where $\vec{r}$ is a point on any of the surfaces of discontinuity of σ , and the integration runs over all the surfaces. One can, of course, try to solve Eq. (9) directly, in a manner analogous to solving the integral equation for V .

It is also possible to attempt a solution by iteration (Gelernter, 1964), but it is not clear whether this procedure will always converge. On the other hand, by combining Eqs. (7), (8), and (9), an iteration-variation procedure can be used, which systematically approaches the correct solution (Heller, 1989).

We intend to study the relative merits of a number of these techniques for finding the potential V that is produced by a variety of assumed primary currents J' and conductivity functions σ .

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