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GUIDING-CENTER HAMILTONIAN FOR LARGE
GYROEXCURSION PARTICLES IN
MIRROR CONFIGURATIONS

BY

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AN ANALYTICAL THEORY OF THE ADIABATIC INVARIANT OF THE HAMILTONIAN
IN A MAGNETIC FIELD

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An explicit expression is derived for the gyrocenter Hamiltonian H , valid for particles having gyroradius comparable to the system size, such as the high-energy ring particles in the Fermi acceleration at Jernell. The form of H is quite simple. It depends on the magnetic field strength B and its radial derivatives, evaluated at an average radius of the particle from the symmetry axis, and allows for a slow but otherwise arbitrary variation of B along the axis. The adiabatic invariant of the gyromotion is related not to the particle's full kinetic energy normal to the field, as is $(1/2)mv_{\perp}^2/B$ in small- ϵ theories, but rather to the small additional kinetic energy due to the radial wobble of the particle about a constant-radius orbit.

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2. From a valid guiding-center theory, we obtain a constant ϵ of the time scale in which gyromotion is averaged out, in which motion along B occurs. That is, one must have $\epsilon \ll 1$, where ϵ is the ratio of the gyroperiod τ_g to the characteristic time τ_B for variation of the magnetic geometry as a particle moves along B. (For smoothly varying B, τ_B may be taken to be the bounce time.) When $\epsilon \ll 1$, it is possible to find a good adiabatic invariant, the gyroaction J, regardless of the size of ϵ . Particles such as the ring particles in Saturn's ring, $\epsilon \approx 1$, and also $\epsilon \approx 1$.

In a previous paper¹ we have described a procedure for deriving K to any desired order in ϵ , for situations where the ordering $\epsilon \ll 1$ assumed in standard guiding-center theory is inapplicable. One concrete application of this procedure, the motion of ions in the Toramak sheath (for which $\epsilon \approx 1/3$), is given in Sec. I. Here another of the possible applications noted in Sec. I is carried out explicitly.

In Sec. II we describe in general terms the procedure to be used to find K . The details of this procedure are given in Secs. III through V, obtaining the expression (20), K , in Eq. (30) of Sec. V. These sections are largely mathematical in nature, with much of the interpretive discussion deferred until Sec. VI. That section covers several significant aspects of expression (20) for K are discussed.

One of these aspects concerns the physical interpretation of the gyromotion, the gyroaction $\mathcal{G}$. We find that $\mathcal{G}$ has quite different physical significance than $\mathcal{G}$ for the usual problem. Instead, $\mathcal{G}$ is a measure of a particle's kinetic energy about $\mathbf{B}_0$, where measures the small additional contribution to the kinetic energy due to the radial wobble in its orbit about an equilibrium, "guiding-center" radius, R_0 . Although the gyroaction is completely nonlocal in the sense that $\epsilon \ll 1$, in the sense that the particle makes only small excursions $\sim R$ from R_0 , the problem is local. This sort of locality is reflected in the expression obtained for K , which depends on the magnetic field and its derivatives in the vicinity of R_0 .

Finally, in Sec. VII, we briefly summarize the ground covered in the preceding sections, and discuss possible directions for extension of the work.

II. OVERVIEW OF THE PROCEDURE FOR OBTAINING K

The procedure for obtaining K from the original particle Hamiltonian H, via a sequence of canonical transformations, has been given in Ref. 1. For convenience, we now review this procedure, within the context of the more specialized problem considered here.

We work in the (right handed) cylindrical coordinate system (R, z, ϕ) , with R the distance from the symmetry axis (major radius), z the distance along that axis, and ϕ the toroidal angle (cf. Fig. 1). An axisymmetric magnetic geometry is assumed. Perturbations breaking this symmetry in ϕ could readily be incorporated into the formalism, however. This is further discussed in Sec. VII. The magnetic field is taken to be unsheared, having components only in the poloidal (R, z) plane. It is thus derivable from the single component $A_\phi = R^{-1} \psi$ of the vector potential A. The particle Hamiltonian H, the starting point of the transformation procedure to obtain K, is then given by ($m = c = 1$)

$$H(R, p_R; z, p_z; p_\phi) = \frac{1}{2} p_R^2 + U(R, z, p_z; p_\phi) \quad (1)$$

where

$$U = \frac{1}{2} p_z^2 + \frac{1}{2} R^{-2} \left[p_\phi - c A_\phi(R, z) \right]^2 + c' (R, z) \quad (2)$$

is the effective potential for $R(t)$. The magnetic field is then

$$\mathbf{B} = -R R^{-1} \mathbf{A}_z / \omega_c z - z R^{-1} \mathbf{A}_z / \omega_c R, \quad (3)$$

and the electric field is

$$\mathbf{E} = -\nabla \Phi(R, z) = -R^{-1} / \omega_c R - z^{-1} / \omega_c z. \quad (4)$$

The slow variation of the geometry parallel to $\mathbf{B}$ is manifested by multiplying z everywhere it occurs, while the dependence of Φ on R is arbitrary, corresponding to arbitrary λ .

Because of the axisymmetry, p_z is a constant, serving only as a parameter in the problem. In most of what follows, we shall accordingly suppress it. If the parallel variables $\mathbf{x}^0 = (z, p_z)$ were also constants of the motion, Φ would describe a one-dimensional oscillation in potential well $\Phi(R, \mathbf{x}^0)$. (The superscript "0" on $\mathbf{x}$ denotes that it is the "old," or "original" pair of parallel variables, as opposed to the superscripts "m" and "f" on $\mathbf{x}$, to be used later, which refer to "mixed" and "final" variables, respectively.) Because of the weak dependence of Φ on z , $\mathbf{x}^0$ is not constant, but has an $\mathcal{O}(\epsilon)$ jitter at the gyrofrequency Ω , in addition to a slower variation on the time scale of the bounce frequency. The nearness of the full $\epsilon \neq 0$ problem to the one-dimensional "frozen- λ " problem (which is exactly soluble) is useful as a guide on how to proceed, as is now described.

The transformation procedure consists of two steps. In the first step, a finite canonical transformation S of the mixed-variable type³ is made, to action-angle variables (λ^0, J^0) of gyration. S is chosen as the transformation which would solve the problem exactly if λ were frozen (i.e., if one had $\lambda = 0$). For $\lambda \neq 0$, the resultant Hamiltonian K^0 in terms of (λ^0, J^0) has an λ^0 dependence on J^0 , and so J^0 is not quite the desired adiabatic invariant J in terms of which K is expressed. However, J^0 lies close enough to J that K , J , and gyrophase λ can be obtained from J^0 , J^0 , and λ^0 by a sequence T of perturbative canonical transformations which remove the dependence of the Hamiltonian on gyrophase to successively higher orders in λ . Transformations T , comprising this second step of the transformation procedure, are readily induced using Lie methods.⁴

We now describe S more explicitly. From Eq. 1, the particle energy E is given by $E = H(R, p_R; \lambda^0)$. One solves this for p_R ,

$$p_R(R, E, \lambda) = \pm 2[E - U(R, \lambda)]^{1/2},$$

and defines the functional form of the action J^0 by $J^0(E, \lambda)$

$(2\pi)^{-1} \oint dR p_R(R, E, \lambda)$. One then inverts this, $J^0(E, \lambda) \rightarrow E(J^0, \lambda)$, and finally induces transformation S by the mixed-variable generating function $S(R, J^0; \lambda^m)$, given by

$$S(R, J^0; \lambda^m) = z p_z^0 + S_1, \quad (5)$$

$$S_1(R, J^0; \lambda^m) = \int^R dR' p_R(R', E(J^0, \lambda^m), \lambda^m). \quad (6)$$

Here $\lambda^m = (ez, p_z^0)$ is the mixed pair of parallel variables, with old coordinate z and new momentum p_z^0 , equal to p_z , but with part of the jitter due to $\epsilon \neq 0$ subtracted off. In the transformations T , one transforms from K^0, p_z^0 , and its conjugate coordinate z^0 to K, p_z , and its coordinate z . The final pair $\lambda^f = (z, p_z)$ of parallel variables is equal to λ^0 , minus the portion of λ^0 which jitters at the gyrofrequency.

It is $E(J^0; \lambda^0)$ which is the particle's true energy. The replacement $\lambda^0 \rightarrow \lambda^m$ has been made in constructing S because of the requirement that S be expressed in terms of mixed variables.

Following the usual prescription³

$$p = (\partial/\partial q) S(q, P), \quad Q = (\partial/\partial P) S(q, P), \quad (7)$$

one finds the transformation equations describing S . Of particular interest are

$$P_R = \partial S / \partial R = p_R(R, E(J^0, \lambda^m), \lambda^m) = [2(E(J^0, \lambda^m) - U(R, \lambda^m))]^{1/2}, \quad (8)$$

$$p_z = \partial S / \partial z = p_z^0 + \epsilon \partial S_1 / \partial ez \equiv p_z^0 + \epsilon^2 p_z(R, J^0, \lambda^m), \quad (9)$$

and

$$z = z^0 - \partial S_1 / \partial p_z^0 \equiv z^0 + \epsilon^2 z(R, J^0, \lambda^m). \quad (10)$$

Using Eq. (8) in (1), one obtains a non-canonical form H' for the Hamiltonian which we shall use in getting an explicit expression for K^0 (cf. Sec. IV.):

$$H(R, p_R; \lambda^0) = \frac{1}{2} p_R^2 + U(R, \lambda^0) = E(J^0, \lambda^m) - V(R, \lambda^m) + U(R, \lambda^0),$$

and thus

$$H(R, p_R; \lambda^0) = H'(R, J^0, \lambda^0, \lambda^m) = E(J^0, \lambda^m) + U(R, \lambda^0) - U(R, \lambda^m). \quad (11)$$

To obtain K^0 , one uses Eqs. (9) and (10) in H' to remove the variables λ^0 in favor of the new ones λ^f , getting in addition terms in λ^z, λ^p_z , which are higher order in λ . Since λ^z and λ^p_z are themselves functions of the mixed pair λ^m , this process must be applied iteratively, to obtain K^0 to as high order in λ as desired. In Sec. IV we carry out this procedure in detail. In Sec. V we then apply transformations T to obtain K from K^0 .

III. GENERATING FUNCTION S

In the discussion leading up to the definition, in Eqs. (5) and (6), of the generating function S , we saw that S was chosen to solve the related one-dimensional problem of a particle oscillating in a one-dimensional effective potential $U(R, \lambda)$, where the parallel variables λ are "frozen," i.e., taken as constant

parameters. (Note that this "freezing" is not an approximation of the time-dependence of the dynamical variables; it is simply a mathematical device to choose a useful transformation.) The formal procedure outlined in that discussion involves doing integrals (to obtain S_1) and functional inversions [to obtain $E(J^0)$ from $J^0(E)$] which may in practice be very difficult. In the case being studied here, the frozen- problem may be solved either by the generating-function approach, or more easily, using Lie methods. The latter approach yields $E(J^0)$ directly, without first having to compute $J^0(E)$, and enables the evaluation of S_1 by performing trivial integrals over gyrophase, rather than the more difficult integral over R in Eq. (6). In this section, we thus adopt the latter method in evaluating S , and the associated functions ψ_z , ϕ_z .

A. THE "FROZEN-" PROBLEM

We first need to solve the problem of particle motion in the one-dimensional potential well $U(R)$, where λ is constant. The motion is obtained as an expansion in the parameter λ , the ratio of the maximum excursion $\delta R_{\max}$ which the particle makes in R from the well bottom R_G , to the anharmonicity scale length L_1 of $U(R)$. $U(R)$ is expanded about $\delta R = R - R_G = 0$:

$$U(R|\lambda) = \sum_{m=0}^{\infty} U_m(R_G, \lambda) (\delta R)^m, \quad (12)$$

where by definition of R_G ,

$$\psi = U_1(R_G, \lambda) \exp(-R U(R)) \Big|_{R_G} \quad (13)$$

If only terms up to $U_2(\delta R)^2$ were kept in (1), the motion would be harmonic. Referring to this lowest-order problem as $\psi(\delta^2 \approx 1)$, we shall here solve for the motion up to $\psi(\delta^3)$, where δ may now be defined a bit more precisely as $(U_3/U_2)^{1/2} R_{\max}$. The procedure to higher order in δ is straightforward, using the Lie methods we shall introduce shortly. [In Ref. 2 the problem of ion motion in the Tormac sheath is treated in a manner completely analogous to the present exposition. The appropriate expansion parameter for the frozen- λ problem there is ϵ , and ψ is computed up to $\psi(\epsilon^3)$.]

Using the specific form (2) for $U(R)$, we evaluate the U_m 's we shall need. Setting $e\phi = 0$ for simplicity, one finds for U_0, U_1 :

$$U_0(R_G, \lambda) \equiv U(R, \lambda) \Big|_{R_G} = \frac{1}{2} p_z^2 + \frac{1}{2} R^2 \dot{\phi}^2 \Big|_{R_G}, \quad (14)$$

$$U_1(R_G, \lambda) = R \dot{\phi} (\partial/\partial R) (R \dot{\phi}) \Big|_{R_G} = R \dot{\phi} (-\dot{\phi} + \Omega_z) \Big|_{R_G} = 0 \quad (15)$$

where $\dot{\phi} \equiv R^{-2}(p_\phi - eA_\phi)$ follows from Hamilton's equations, and (restoring m, c) $\Omega_z \equiv eB_z(R)/mc$. The last equality in (15) follows from Eq. (13), which defines R_G . One sees from (15) that $R_G = R_G(\epsilon z, p_\phi)$.

There are two factors in Eq. (15), either of which may be set equal to zero to satisfy the final equality.⁵ Setting the

former of these to zero, $R \dot{\phi} = R^{-1}(p_{\phi} - eA_{\phi})|_{R_G} = 0$, corresponds to considering particles of small perpendicular energy $E_{\perp}$, and so small ω (including the so called "figure-8" particles), which we do not consider here. Setting the latter factor $(-\dot{\phi} + \omega_z)$ equal zero physically corresponds to particles with large enough $E_{\perp}$ that the magnetic force $R \dot{\phi} \omega_z$ at $R = R_G$, is just enough to cause the necessary radial acceleration $\dot{\phi}^2 R$, to keep the particle gyrating with gyroexcursion $R = R_G$. This is the case for the ring particles. Therefore,

$$(\dot{\phi} R (p_{\phi} - eA_{\phi}) / R) |_{R_G} = (-\dot{\phi} + \omega_z) |_{R_G} = 0. \quad (16)$$

Using Eq. (16), one evaluates U_2, U_3 , finding

$$U_2(R_G, \omega) = \frac{1}{2} (\dot{\phi} / \omega R)^2 U |_{R_G} = \frac{1}{2} \omega_z (\omega_z + R \dot{\phi} \omega_z / R) |_{R_G}, \quad (17)$$

and

$$U_3(R_G, \omega) = \frac{1}{6} (\dot{\phi} / \omega R)^3 U |_{R_G} = -\frac{1}{6} \omega_z (3R^{-1} \dot{\phi} \omega_z + \dot{\phi} \omega_z / \omega R - \dot{\phi}^2 \omega_z / \omega R^2) |_{R_G}. \quad (18)$$

Higher- m U_m are readily found, if needed. All are functions of the z component B_z of the magnetic field at $R = R_G$, and its radial derivatives.

To aid in visualizing the meaning of the mathematics, in

Fig. 2a we sketch a simple model for $p_z(p, z)$, for a fixed z , allowing for field reversal. We take

$$p_z = B_0(R/R_0 - 1), \quad (19)$$

where B_0, B_0 are parametrically dependent upon z . From Eq. (1), one has, in general,

$$A_z(R, z) = -\frac{1}{R} \int_0^R dR' R' B_z(R', z), \quad (20)$$

and so for this model,

$$A_z(R, z) = B_0 R^2 \left(\frac{1}{2} - R/3R_0 \right). \quad (21)$$

This is sketched in Fig. 2b. Figure 3a shows $p_z = eA_z(z)$ for A_z of the Eq. (21), and for p_z large enough that $p_z = eA_z$, and so $\dot{z}$, are positive for all R . This corresponds to the ring particles considered here. Finally, Fig. 3b is a sketch of the potential $U(R)$, for this illustrative model, and for the chosen value of p_ϕ . Rewriting condition (16) as

$$R_G^{-1} (p_z - eA_z) = -e\partial A_\phi / \partial R_G, \quad (22)$$

one sees from Fig. 3a that as p_ϕ is increased, the slope $-e\partial A_\phi / \partial R$ of $p_z - eA_z$ must increase, as must R_G , in order to maintain the equality in (22). That is, $\partial R_G(z, p_\phi) / \partial p_\phi > 0$.

We now return to arbitrary $B_z(z)$, and to the solution of the frozen- problem. We first make the transformation, canonical in the one degree of freedom phase space of the frozen- problem, from (P, p_θ) to action-angle variables (J, ψ) :

$$J = \frac{1}{2\pi} \oint (P, p_\theta) d\theta, \quad \psi = \frac{1}{2\pi} \oint (P, p_\theta) d\theta, \quad (23)$$

where the phase functions ψ and ψ_0 are defined as

$$\psi(\theta, z) = (2\pi B_0)^{1/2} \int_0^\theta \sin \left(\frac{R}{R_0} \right) d\theta, \quad (24)$$

$$\psi_0(\theta, z) = (2\pi B_0)^{1/2} \int_0^\theta \cos \left(\frac{R}{R_0} \right) d\theta. \quad (25)$$

Here $\frac{1}{R_0} = \frac{1}{2\pi B_0}$. The frequency ω_c is the gyrofrequency of the particle in (23). From Eq. (23), one sees that for a uniform $B_z = B_0$, $\frac{1}{R_0} = \frac{1}{2\pi B_0} = \frac{1}{2\pi B_0}$, i.e., a particle precesses exactly one revolution about $\theta = \theta_0$ each gyration. This is true since for a uniform B_z , the choice of origin $\theta = 0$ is arbitrary, so that a particle with orbit of gyroradius R_0 , but initial R coordinate $R_0 = R_1$ will simply gyrate at gyrofrequency ω_c in a circular orbit with center at $\theta = \theta_1$. The nonuniformity in B_z implies $B_0 \neq B_z$, and the corresponding gyroorbit will precess at the difference frequency.

Using Eqs. (23), (24) and (25) in (11), one finds the Hamiltonian $K^0(\psi, \psi_0)$ governing the frozen- problem:

$$K^0(\psi, j) = (U_0 + i_0 j) + \sum_{m=3}^{\infty} U_m r^m \sin^m \psi j^{m/2}. \quad (26)$$

The frozen- Hamiltonian K^0 has an unperturbed $[O(1)]$ part $K_0 = U_0 + i_0 j$ which is independent of gyrophase ψ , plus ψ -dependent terms $\sum_{m=3}^{\infty} U_m r^m \sin^m \psi j^{m/2}$, of order r^m . The motion may now be solved for perturbatively.

8. LIE METHODS

We employ Lie methods to carry out the perturbative solution. Here we will only sketch the basic mechanics of the technique. More detailed discussion and justification of Lie methods are given in Refs. 2, 4, and 6.

Given any phase function $G(\psi)$, the corresponding Lie operator $\tilde{G}$ is defined by

$$\tilde{G} = G, \quad (27)$$

where $\{ , \}$ denotes Poisson brackets, and ψ is the point in phase space. The utility of Lie methods rests on the fact that the operator

$$\exp \tilde{G} = 1 + \tilde{G} + \frac{1}{2} \tilde{G}^2 + \dots \quad (28)$$

induces a canonical transformation. Moreover, in contrast to the mixed variable prescription,³ the transformation equations from old to new variables, $\psi' = \exp(-\tilde{G})\psi$, or the inverse

transformations $U = \exp \tilde{G}$, are already in unmixed form. Perturbative canonical transformations have generators G which are small, and so the exponential operator in (28) may be approximated, to any desired order in the small parameter, by a truncated Taylor expansion.

For a Hamiltonian of the form of (26), i.e.,

$$h^0(j, \beta) = h_0(j) + \sum_{m=1}^{\infty} \epsilon^m h_m^0(j, \beta), \quad (29)$$

one wants to induce a sequence of transformations of type (28) to remove the β -dependence of the Hamiltonian to as high order in ϵ as desired. When all the β -dependence has been removed, the action variable j is just the invariant action one would obtain solving the problem by the alternative mixed-variable generating function method already described, and the Hamiltonian $h(j) = \sum_{m=0}^{\infty} \epsilon^m h_m(j)$ is just the effective particle energy $E(j)$ entering into the generating-function prescription. The inversion $j(E) \rightarrow E(j)$ needed there is thus completely circumvented in the Lie approach.

Here we shall apply this transformation procedure only to $h^0(j)$. The procedure to higher order is entirely analogous. In Sec. VII we shall use a slightly different variant of Lie methods to perform the second stage T of the transformation procedure, going to second order in ϵ .

We induce a transformation from old variables $x^0 = (x^0, j^0)$, to new variables $x^1 = (x^1, j^1)$, with the Lie generator $G_1(x)$:

$$x^1 = \exp(-\check{G}_1)x^0 = x^0 - \check{G}_1 x^0 + \frac{1}{2} \check{G}_1^2 x^0 - \dots \quad (30)$$

The new Hamiltonian k^1 is given by

$$\begin{aligned} k^1 &= \exp(-\check{G}_1)k^0 \\ &= k_0^0 + (k_1^0 + G_1 k_0^0) + \frac{1}{2}(k_2^0 + \check{G}_1 k_1^0 + \frac{1}{2}\check{G}_1^2 k_0^0) + O(\epsilon^2). \end{aligned} \quad (31)$$

Generator G_1 is determined by the requirements that the $O(\epsilon)$ part k_1^1 of k^1 be ϵ -independent, and that G_1 be single-valued in x . These conditions together require that k_1^1 , which, from Eq. (33), equals zero:

$$0 = \bar{k}_1^0 = k_1^1 = k_1^0 + \check{G}_1 k_0^0 = k_1^0 - \check{k}_0^0 G_1.$$

Solving for G_1 ,

$$G_1 = \check{k}_0^{-1} k_1^0 = - \left[\frac{dt}{0-\text{traj.}} \right] k_1^0 = - \check{k}_0^{-1} \left[\frac{d}{dt} \right] k_1^0. \quad (32)$$

Here we have used the fact that $\check{k}_0 = -\dot{x}, k_0$ is minus the total time derivative along the unperturbed trajectory. From Eq. (26), we read off k_1^0 :

$$k_1^0 = U_3 \bar{r}^3 \sin^3 \theta j^{3/2} = U_3 \bar{r} j^{3/2} (U_3/U_2) \sin^3 \theta. \quad (33)$$

Using this in (32), one finds

$$G_1 = \bar{r} j^{3/2} (U_3/U_2) [(3/4) \cos \theta - (1/12) \cos 3\theta]. \quad (34)$$

Since we are only keeping terms up to $O(\epsilon)$, k and its variables (θ, j) , are approximated by k^1 , and (θ^1, j^1) . Thus,

$$k = k^1 + O(\epsilon^2) = k^0 + O(\epsilon^2) = U_0 + G_0 j + O(\epsilon^2), \quad (35)$$

$$\begin{aligned} j &= j^1 + O(\epsilon^2) = (1 - \check{G}_1) j^0 + O(\epsilon^2) \\ &= j^0 - (\partial/\partial \theta^0) G_1(\theta^0, j^0) + O(\epsilon^2), \end{aligned} \quad (36)$$

and similarly

$$\theta = \theta^1 + O(\epsilon^2) = \theta^0 + (\partial/\partial j^0) G_1(\theta^0, j^0) + O(\epsilon^2). \quad (37)$$

These equations constitute the solution of the frozen- θ problem, to $O(\epsilon)$.

C. Calculation of S , δz , δp_z

As has been noted, the transformations T yield K as an expansion in $\hbar$. From Sec. III A, one sees that the frozen- $\hbar$ problem is solved as a further expansion in $\hbar$. Therefore K will emerge as a double expansion,

$$K = \sum_{\ell, m} \ell^m K_{\ell m}, \quad (38)$$

and similarly for K^0 . In this section we will calculate S_1 defined in Eq. (6) using the coordinate transformation (R, p_R) $\rightarrow$ (θ, j) obtained in the previous section:

$$S_1 = \int_{\theta}^R dR' p_R(R') = \int_{\theta}^R d\theta' (dR/d\theta') p_R(\theta', j^0 m). \quad (39)$$

From Eqs. (23) - (25), and (30), R and p_R are expressed as expansions in $\hbar$:

$$\begin{aligned} R = r(\theta^0, j^0) &= \exp(i\check{G}_1) r(\theta^1, j^1) \\ &= \exp(i\check{G}_1) r(\theta, j) + \text{higher order (h.o.)} \\ &= r(\theta, j) + i\check{G}_1 r(\theta, j) + \text{h.o.}, \end{aligned} \quad (40)$$

and similarly for p_R . Thus from (39) one sees that S_1 , as well as δz and δp_z defined in Eqs. (9) and (10), also emerge as

expansions in ϵ . We want to find K^0 , and so K , up to $O(\epsilon^2, \lambda)$. This will necessitate finding δz , δp_z only up to $O(\epsilon^0 = 1)$ (c.f. Sec. IV). Therefore we take the $O(\epsilon^0)$ part of expression (40) for δR and the corresponding term for p_R , and use (39) to evaluate S_1 to lowest order:

$$\begin{aligned} S_1(0, j|\lambda^m) &= \int_0^{\pi} d\alpha' [dr(\alpha', j|\lambda^m)/d\alpha'] p(\alpha', j|\lambda^m) \\ &= j(\pi + \frac{1}{2} \sin 2\pi). \end{aligned} \quad (41)$$

For this lowest-order evaluation of S_1 , corresponding to the harmonic approximation $(1/2)p_R^2 + U_2(\delta R)^2$ to the frozen- λ Hamiltonian, one can carry out the original integral over R in Eq. (6), of the form $\int dy (1 - y^2)^{1/2}$. One finds the same result as in (41), using Eq. (24) to transform from δR to R . For calculations of S_1 including the anharmonic corrections,² however the integral (6) rapidly becomes intractable, while the form (41) becomes no more difficult.

From Eq. (9), $\delta p_z \equiv \partial S_1 / \partial \epsilon z$, where R , j , and p_z are to be held constant. (We suppress the superscript "0" on p_z^0 for notational simplicity, whenever the distinction between the parallel variables $\lambda^f \equiv (\epsilon z, p_z)$ during the successive steps of transformations T is not crucial.) Therefore, in using $S_1(0, j|\lambda^m)$ from (41), care must be taken to keep the proper variables constant during differentiation. Doing this, one finds

$$\begin{aligned} (S_1^{(0)})_{R, j, p_z} = & -P_R(\theta, j, \lambda) [(R_G/\alpha z) p_z + (S_R/\alpha z)_{R, j, p_z}] \\ & + (S_1^{(0)}(z))_{R, j, p_z}. \end{aligned} \quad (42)$$

Of the three terms in (42), the second two are higher order in αz than the one in $(S_1^{(0)})_{R, j, p_z}$, and so are negligible, to the order we are working. Therefore

$$p_z (S_1^{(0)})_{R, j, p_z} = -P_R(\theta, j, \lambda^m) (R_G/\alpha z) + \text{h.o.} \quad (43)$$

The analogous expression, $\delta z = -\partial S_1 / \partial p_z = P_R(\theta, j, \lambda^m) (R_G/\alpha p_z)$, holds for δz . We saw following Eq. (15), however, that R_G is independent of p_z , and therefore

$$\delta z = 0. \quad (44)$$

This simplification occurs because $A_z = 0$. If the magnetic field were sheared, one could not set $A_z = 0$, and δz also would be nonzero.

IV. CALCULATION OF K^0

As described following Eq. (11), K^0 is obtained by using δz and δp_z in the mixed-variable form H' of the Hamiltonian, to eliminate the old variables in favor of the new. This iterative procedure is in principle executable to arbitrary order in both α and λ , yielding K^0 as a double expansion analogous to (38).

We carry out the expansion to $\mathcal{O}(c^2, \epsilon)$. The first iteration of the expansion of (11) is

$$\begin{aligned} H' &= E(\lambda^m) + U(R^m | \lambda^0) - U(R^m, \lambda^m) \\ &= E(\lambda^E) \\ &\quad + \epsilon P_z (\lambda / P_z) U(R^m, \lambda^m) \\ &\quad + (1/2) \epsilon^2 (P_z)^2 (\lambda / P_z)^2 U(R^m, \lambda^m) + \text{h.o.} \end{aligned} \quad (45)$$

We have suppressed some of the arguments here, for notational simplicity. We have also used $\delta z = 0$, so that no derivatives of E or U with respect to δz appear. The notation R^m denotes that the transformation S gives us R in terms of the mixed variables, $R = R^m = R_G(\lambda^m) + R(\cdot, j = \lambda^m)$. However, because $\delta z = 0$ to the order we are working, one has $\lambda^m = \lambda^f$ in (45), and no further iteration is needed. Finally, due to the simple dependence of U on P_z , Eq. (45) simplifies further:

$$\epsilon U / P_z = P_z, \quad \epsilon^2 U / P_z^2 = 1. \quad (46)$$

Using (46) in (45), and taking $\epsilon = k$ from (35), one reads off $K^0 = H'$ to $\mathcal{O}(c^2, \epsilon)$:

$$K^0 = K_{00} + \epsilon K_{01}^0 + \epsilon K_{10}^0 + \epsilon^2 K_{20}^0 + \text{h.o.}, \quad (47)$$

where

$$K_{00}(J^0, \dots) = K_0(J^0, \dots) = U_0 + \dots J^0, \quad K_{01}^0 = K_1 = 0,$$

$$K_{10}^0 = 2p_z p_z^0, \quad \text{and} \quad K_{20}^0 = \frac{1}{2} (2p_z)^2. \quad (48)$$

The superscripts "0" have been reinstated here to make clear that K^0 is to be evaluated at phase point $(\dots, J^0, \dots, p_z^0)$.

VI. CALCULATION OF K

In accordance with the program outlined in Sec. II, we now apply transformations T, eliminating the dependence of K^0 to $\mathcal{O}(\epsilon^2)$, and thereby obtaining K Eq. (40), the principal result of this paper.

To transform away the ϵ dependence, we use Lie methods, quite analogous to the manner in which the frozen- ϵ problem was solved. The Hamiltonian K^0 is of the form

$$K^0 = K_0(J, \dots, Z, p_z) + \sum_{\ell=1}^{\infty} \epsilon^{\ell} K_{\ell}^0(\dots, J, \dots, Z, p_z), \quad (49)$$

where $E(J, \dots) = K_0(J, \dots) = \sum_{m=0}^{\infty} \epsilon^m K_{0m}(J, \dots)$ and $K_{\ell}^0 = \sum_{m=0}^{\infty} \epsilon^m K_{\ell m}^0$. Expression (49) is analogous to expression (29) for k^0 . However, in contrast to (29), we note that the expansion parameter ϵ in (49) appears in two places, both multiplying the term K_{ℓ}^0 , analogous to (29), and also multiplying Z everywhere it occurs. Because of this additional ϵ dependence, expressions for terms in the Hamiltonian, of like order in ϵ , will be slightly different from those for the form (29).

To eliminate ϵ to $O(\epsilon^2)$, we transform K^0 twice, first acting on it with the operator $\exp \epsilon \check{W}_1$ to yield Hamiltonian K^1 , and then acting on K^1 with $\exp \epsilon^2 \check{W}_2$, to yield K^2 . Since we are only proceeding to $O(\epsilon^2)$, K^2 is an adequate approximation to K , i.e., $K = K^2 + O(\epsilon^3)$.

The generators W_1, W_2 of these transformations have the same functional dependences as the Hamiltonians they transform, $W_{1,2} = W_{1,2}(r, J, -Z, P_2)$. In contrast to the Frozen-1 problem, we are now transforming in the full two-degree of freedom phase space, and so the Lie operators have two parts. For operator $\check{W}_1$ for example, one has

$$\check{W}_1 = \check{W}_{10} + \epsilon \check{W}_{12}, \quad (50)$$

where $\check{W}_{10} = (W_1/\partial J) \partial/\partial J - (W_1/\partial J) \partial/\partial Z$, and $\check{W}_{12} = (W_1/\partial Z) \partial/\partial P_2 - (W_1/\partial P_2) \partial/\partial Z$ are both of order $\epsilon^0 = 1$.

Just as each K_ℓ is an expansion in ϵ , so is each W_ℓ . Because we are neglecting terms of order ϵ^2 and higher in K , it is sufficient to find just the $O(\epsilon^0)$ term in $W_{1,2}$.

Proceeding just as for k^1 in (31), one has

$$K^1 \equiv \exp(\epsilon \check{W}_1) K^0 = K_0 + \epsilon K_1^1 + \epsilon^2 K_2^1 + O(\epsilon^3), \quad (51)$$

where

$$K_1^1 \equiv K_1^0 + \check{W}_{10} K_0^0, \text{ and } K_2^1 \equiv K_2^0 + \check{W}_{10} K_1^0 + (1/2) \check{W}_{10}^2 K_0^0 + \check{W}_{12} K_0^0.$$

As for G_1 in Eq. (32), W_1 is chosen to be single-valued in θ , and to make K_1^1 θ -independent:

$$K_1^1 = \overline{K_1^0} = 0, \quad (52)$$

where the last equality follows from Eqs. (48), (43), and (25).

W_1 is therefore given by

$$\begin{aligned} W_1 &= \overline{K_0^{-1}} K_1^0 = - \frac{-1}{0} \frac{1}{J} \frac{1}{\rho} K_1^0 \\ &= - \frac{1}{0} P_z (R_G / 30Z) \bar{p} J^{1/2} \sin \theta = P_z (R_G / 30Z) \rho \left(\frac{1}{J} \right)^{1/2} \sin \theta. \end{aligned} \quad (53)$$

Equations (23) - (25) were used in obtaining the last form in (53).

Proceeding analogously, W_2 is determined by eliminating the θ -dependence from K_2^2 ,

$$K_2^2 = \overline{K_2^1} = \overline{K_2^0} + \frac{1}{2} \overline{W_1 K_1^0} + \overline{W_{12} K_0}, \quad (54)$$

where the relation $W_{10} K_0 = -K_1^0$, which follows from (52), has been used in simplifying the form for K_2^1 given after Eq. (51). Because K_0 is θ -independent, from (53) one sees that the last term in (54) equals zero. Using Eqs. (53), (48), (43), and (25), one readily evaluates the remaining two terms on the right-hand side of (54). Noting from (25) that $\bar{p}^2 = 2/J$, one finds

$$\overline{K_{20}^0} = \frac{1}{2} \overline{(P_z)^2} = \frac{1}{2} \frac{1}{0} J (R_G / 30Z)^2, \quad (55)$$

and

$$\frac{1}{2} K_1 = K_1^0 = -\frac{1}{2} P_Z^2 (R_G / \epsilon Z)^2. \quad (56)$$

Therefore, K_1^1 has the simple form

$$K_1^1 = \frac{1}{2} (J_0 J - P_Z^2) (R_G / \epsilon Z)^2. \quad (57)$$

Pulling the results of this section together, we write out the guiding - center Hamiltonian K , valid to $\mathcal{O}(\epsilon^2)$:

$$\begin{aligned} K &= K_{00} + \epsilon^2 K_2^2 + \text{h.o.} \\ &= (J_0 J + \frac{1}{2} P_Z^2 + \frac{1}{2} R_G^2 / Z^2) + \frac{1}{2} \epsilon^2 (J_0 J - P_Z^2) (R_G / \epsilon Z)^2. \end{aligned} \quad (58)$$

Equation (14) has been used here in writing out V_0 . In the following section, we provide some physical discussion of this remarkably simple result.

VI. DISCUSSION

The past few sections have been primarily mathematical in nature, aimed at obtaining K in Eq. (58). We now look at the various terms in K , and interpret their physical origin.

Contribution K_{00} is the $\mathcal{O}(\epsilon^0, \epsilon^0)$ part of K . However, because $F_{01} = K_{10} = 0$, K_{00} is equal to K to $\mathcal{O}(\epsilon, \epsilon)$. There are

three terms in this factor. To (11), (12), Hamilton's equations tell us that $\dot{z} = p_z/m$, so that the term $(1/2)mv_z^2$ is the gyro-averaged kinetic energy parallel to the symmetry axis, to this order.

In the standard, $v \rightarrow 0$ guiding-center Hamiltonian, the term $\frac{1}{2}mv_\perp^2$ in H_{GC} is equal to $\mu B = (1/2)mv_\perp^2$, i.e., μ is just one-half times the magnetic moment μ , and $\frac{1}{2}mv_\perp^2$ represents the perpendicular kinetic energy $E_\perp$ of the particle. This is not the case here. Instead, the bulk of $E_\perp$ is held in the term $(1/2)R_G^2 \frac{d^2\theta}{dt^2}$ arising from ψ_0 in (14). As one sees from the solution of the frozen- θ problem in Sec. III A, $\frac{1}{2}mv_\perp^2$ is only the small adiabatic contribution to $E_\perp$ coming from the radial "wobble" in a given ring particle's orbit. For a perfectly-centered circular orbit, $\dot{\theta}$ and $\ddot{\theta}$ are zero, though $E_\perp$ will still be large.

A. Adiabatic Invariant

The fact that the gyroinvariant J is related only to the radial wobble, and not to the full perpendicular energy, raises a further issue. One knows that in the small- v limit, the particle trajectories considered here will be governed by a small- v guiding center theory, in which $\mu = E_\perp/B_0$ is an adiabatic invariant. (Here B_0 is the magnetic field at the particle's guiding-center, which, for particles centered on the axis, is just B_z .) The question arises, then, if the corresponding term in $\mathcal{H}$ here, $\frac{1}{2}R_G^2 \frac{d^2\theta}{dt^2} = E_\perp/(eB_z/mc)$, is also an adiabatic invariant, in the limit $v \rightarrow 0$.

The answer to this question is "yes," we now show. We will show the invariance of α for a class of magnetic fields, (59), which contains the $\alpha = 0$ limit as a special case. We consider the set of vector potentials A_z which have the particular form

$$A_z(R, Z) = A_z(s) \Phi(R, Z), \quad (59)$$

where, in writing (59), these A_z keep the same shape, but expand or contract in Z according to whether the "stretching-factor" $s(Z)$ decreases or increases, respectively.

Writing condition (16) for P_z as $P_z = \partial A_z / \partial P_z$, $\partial / \partial P_z = \partial / \partial s$, $\partial / \partial Z = \partial s / \partial Z \partial / \partial s$, one sees that

$$(\partial / \partial Z) \alpha = -(\partial / \partial s) \alpha A_z(P_z, Z). \quad (60)$$

Therefore α is conserved if $A_z(P_z, Z) = 0$, i.e., if $P_z = R$, P_z varies with Z just so as to keep A_z constant. From Eq. (59) one sees that A_z is the magnetic flux contained inside a circle at a given P . Thus $A_z(R, Z)$ remains constant as Z varies if R varies to stay on the same field line. We conclude, then, that α is conserved if $R_G(Z, P_z)$ traces out a field line as Z is varied.

That this is the case for magnetic fields given by Eq. (59) can be seen from condition (22) for R_G . We follow a field line as $s(Z)$ changes from a value s_1 at Z_1 to s_2 at Z_2 .

Then $p_z = eA_z$ remains constant, while R changes by a factor s_1/s_2 . The left-hand side of (22) correspondingly changes by a factor s_1/s_2 . Since A_z has been compressed (or expanded) by a factor s_2/s_1 , the right-hand side also changes by a factor s_2/s_1 . Keeping R to stay on a field line therefore satisfies the condition for (22).

In the limit $\epsilon \rightarrow 0$, the magnetic field profile enclosed by a gyroradius is of the type (59), because $\epsilon \rightarrow 0$ implies that no variation of the profile takes place over a gyroradius. Only the strength of the field may vary as z does, and correspondingly the function $s(z)$ varies (as $B^{1/2}$).

B. Ponderomotive Term

We now discuss the contribution $-\frac{1}{2}K_z^2$ to E . This term has an origin quite analogous to the usual ponderomotive potential felt by a charged particle in a rapidly-oscillating electric field of amplitude E_0 , which has a slower spatial variation; both are effects at second order in the small parameter ϵ of the perturbation strength, arising from a nonzero time average on the rapid time scale of the problem.

If ϵ were equal zero, ψ^0 would be a constant, and the potential $U(R|\psi^0)$ in which the particle oscillates would have a time-independent form. For $\epsilon \neq 0$, however, the radial "wobble" in the well $U(R)$ is accompanied by a jitter δp_z in the parallel momentum p_z at the same frequency ω_0 as the wobble in the well. Therefore the shape itself has a time dependence. This changes

the frequency of oscillation in U_1 slightly:

$$\omega = \omega_0 (1 - \beta)(K - K_{00}) = (1/2) \omega_0 (K_0^2 - 2)^{-1/2}.$$

We have asserted that p_z given by Eq. (43), which contributes to K_0^2 , is just the oscillation due to $\beta \neq 0$ of p_z . In the derivation leading to Eq. (43), however, p_z was only defined as the variable difference $p_z - p_z^1$ coming from canonical transformation S . We now provide a physical derivation of the latter in p_z , and show that it is given by precisely expression (43), to $O(\beta)$.

We expand the function $\dot{p}_z(R, -z, p_z) = -\partial H/\partial z = R^{-2}[p_z - eA_z(R, -z)] e^{-A_z/z} = -p_R R^{-1}[p_z - eA_z(R, -z)]$ about $p_z = p_R - R_0 = 0$, to first order in βR . (Here $p_R = eB_R/mc = R^{-1} eA_z/z$.) Using (16), one finds

$$\dot{p}_z = \dot{p}_z(R_0 + \beta R) - \dot{p}_z(R_0) = -\beta R(\partial/\partial z) p_R(R)_{R_0} + O(\beta R)^2. \quad (61)$$

On the other hand, to lowest order in β , Hamilton's equation for p_R is

$$\dot{p}_R = -2U_2 \beta R = -\frac{1}{2} \omega_0^2 \beta R. \quad (62)$$

Comparing (61) and (62), and integrating once in time,

$$e\delta p_z = -p_R(e\partial/\partial z) R_0 \partial A_z/\partial R|_{R_0} \beta^{-2},$$

Using Eq. (17) for $U_2 = (1/2) \omega_0^2$,

$$E_z = -E_R(R_G) \cdot p_z^2 / R_G^2 / (z^2 + R_G^2 \cdot z^2 / R_G^2). \quad (63)$$

From condition (16), one can calculate $\cdot R_G / z = z$. The result

is

$$\cdot R_G / z = (R \cdot R_G^2 / R) / (z^2 + R \cdot z^2 / R) \cdot R_G. \quad (64)$$

Putting this into (42) and identifying p with p_R , one sees that (43) and (64) are the same expression. This was the result to be proven.

VII. CONCLUSION

We have derived the guiding-center Hamiltonian for particles having gyroexcursion comparable to the perpendicular magnetic scale length, valid to second order in the parameter ϵ of parallel inhomogeneity, and to first order in the parameter $\epsilon_\perp$, measuring the amount of radial wobble in the gyroorbit. The result [Eq. (53)] is quite simple, only slightly more complicated than the $\epsilon(\epsilon_\perp, \epsilon_\perp)$ expression. This simplicity is due in large part to the fact that the magnetic field is unsheared, so that one may take only one component A_ϕ of the vector potential to be nonzero.

Throughout this paper, axisymmetry of B was assumed, and accordingly $p_\phi = P_\phi$ was conserved. The same formalism could readily be adapted, however, to include the effects of ϕ -dependent perturbations, for example from small quadrupole fields, as studied by R. H. Cohen.⁷ Once the transformation S is defined, using $U(R)$ coming from only the axisymmetric fields, such

perturbations would simply appear as extra terms in the Hamiltonian K^0 . Given K^0 , in which the gyrophase is present, one could either proceed, as done here, to eliminate the θ -dependence, or else use this dependence to study resonance effects, as done in Ref. 7. In contrast to Ref. 7, where a quadratic z -dependence of B was assumed in order to be able to transform to action-angle variables of the bounce motion explicitly, the form for K derived here permits arbitrary dependence of B on z .

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FIGURE CAPTIONS

Fig. 1. Depiction of the geometry considered in the text, and the polar coordinates (R, z, ϕ) used to describe it.

Fig. 2a. Plot of the z-component $B_z(R)$ of the magnetic field, for the simple illustrative model of Eq. (19).

Fig. 2b. Sketch of the vector potential A_ϕ corresponding to Fig. 2a, and Eq. (21).

Fig. 3a. The function $p_\phi - eA_\phi(R)$ for the field of Fig. 1, and for p_ϕ large enough that $\dot{\phi} > 0$ for all R , corresponding to the ring particles.

Fig. 3b. Sketch of the effective potential $U(R)$, for the model of the preceding figures.

Fig. 1. 792372

Fig. 2. 792373

Fig. 3. 792371

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