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**NUMERICAL MODEL OF ELECTROMAGNETIC SCATTERING
OFF A SUBTERRANEAN 3-DIMENSIONAL DIELECTRIC**

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ABSTRACT

As part of the effort to develop On-Site Inspection (OSI) techniques for verification of compliance to a Comprehensive Test Ban Treaty (CTBT), a computer code was developed to predict the interaction of an electromagnetic (EM) wave with an underground cavity. Results from the code were used to evaluate the use of surface electromagnetic exploration techniques for detection of underground cavities or rubble-filled regions characteristic of underground nuclear explosions.

The interaction of EM radiation originating on the earth's surface with a subterranean cavity is calculated using an integral formulation of the problem. Given a primary source located on or above an interface separating a dielectric (air) from a dissipative half-space (earth), the primary and secondary (scattered from the cavity) fields are calculated. Specific attention is given to the vertical magnetic dipole source which is the characteristic field of a horizontal loop source, commonly used in exploration geophysics. Since experiments were to be carried out at NTS where the water table is very deep, the underground cavity is assumed to be empty, hence, is treated as an air-filled void. Computer models were calculated for ground conductivities .001 to .01 S/m which represent values found in a granitic rock to those in the alluvium encountered on the Flats. In this range of .001 to .01 S/m, the cavity is more easily detected at the higher ground conductivities of .01 S/m. This situation allows better contrast with the anomaly and is limited only by EM attenuation, which also increases with ground conductivity.

For OSI applications, this report evaluates the situation encountered at NTS; namely, that the water table is deep and the cavity-chimney structure is a resistive anomaly. In other regions of the world, it is expected that the

cavity would be water filled, hence, the cavity-chimney structure may be a conductive anomaly. This is a situation which should be studied in the future. However, it is the opinion of the authors that control source EM methods show promise as candidate OSI technologies.

This code also can be applied to applications other than the detection of voids. Any three-dimensional anomaly can be considered; the anomaly can be characterized by its electrical permittivity and conductivity which is included in the complex permittivity used in the code. A suggested application is for evaluating EM methods for use in site characterization.

1.0 INTRODUCTION

The U.S. and the U.S.S.R. have periodically conducted negotiations toward a Comprehensive Test Ban Treaty (CTBT) that would prohibit tests of nuclear weapons in all environments including underground. If a CTBT were to be achieved, it would probably include a provision for On Site Inspections (OSI's) to verify compliance. The OSI team would use a variety of technical methods to resolve the nature of an ambiguous event. With the growing nuclear proliferation problem, OSI technologies may also be useful in some cases in resolving questions about Nth country activities. The International Security Affairs Office of the DOE has supported a research effort in verification technologies in the LLNL Verification Program. Cavity detection by electromagnetic (EM) techniques is a candidate technology for OSI's. We have conducted a thorough theoretical study using computer calculation to evaluate the feasibility of detecting subterranean voids by EM induction methods.

Electromagnetic induction methods work on the general principle that dielectrics in the presence of a time-varying electromagnetic field will sustain electric polarization currents which in turn generate other electromagnetic fields. The mathematical formulation of the cavity problem (see Appendix C) includes a cavity current density related to the difference between the ground's complex electrical permittivity and that of the cavity's, which can be air-filled or water-filled. An observer will detect electromagnetic field variations which are due to both the source and the induced polarization currents in the buried cavity. Under certain circumstances, one can discriminate between the two field sources and, hence, detect the cavity.

One could speculate on a number of scenarios where one would use electromagnetic methods. Some of the variables would include ground conductivity, water table depth, and size and depth of the cavity. For this study, parameters were selected which included values characteristic of the Nevada Test Site (NTS). Since the water table is very deep at NTS, we expect the cavity to be more akin to an air-filled void than to a water-filled void. This cavity would act as a "resistive" rather than a "conductive" anomaly since the electrical conductivity of air is less than the conductivity of rock. Field experiments at NTS could provide verification of the technique and of the code.

2.0 METHOD

2.1 Background

In order to investigate the feasibility of detecting subterranean cavities using electromagnetic waves, we consider the half-space geometry in which the air and the earth are separated by a planar interface. A source, which may be electric or magnetic residing in the air on or above the interface, radiates a primary field. This primary field, which is unaltered by the presence of the buried inhomogeneity, is known from the results presented in Appendix B. This source produces a background field at a point of observation and an excitation field inside the cavity called the incident field. This subterranean incident field induces a polarization current inside the cavity which radiates a secondary or scattered electromagnetic field. Once this polarization current is known, the secondary field can be calculated from the equations in Appendix B. Along useful survey lines on the surface, the total field, the sum of the primary and secondary field, is measured with the view of detecting the presence of the cavity (assumed to be an air-filled void). In this theoretical investigation, we search for a source and receiver configuration which will allow the evaluation of some of the cavity parameters from an analysis of the secondary field as measured in conjunction with the primary field. This leads us to the consideration of the three region problems in the next section. The induced polarization current will be found using an integral equation approach.

2.2 Integral Equation

The geometry consists of the three regions indicated in Fig. 1, where the cavity has an arbitrary shape but uniform electrical parameters.

Figure 1. The 3-region problem assumes that magnetic permeability μ is constant.

The fields in each region are obtained by solving Maxwell's equations in each region subject to the requirement of the continuity of the tangential electromagnetic field across the interfaces separating regions a, b, and c. The starting point is Maxwell's equations in each region, which are written below using $e^{i\omega t}$ time dependence.

Region a

$$\nabla \times E_a = -i\omega\mu H_a \quad (1a)$$

$$\nabla \times H_a = i\omega\epsilon_a E_a \quad (1b)$$

$$\nabla \cdot (\epsilon_a E_a) = 0 \quad (1c)$$

$$\nabla \cdot (\mu H_a) = 0 \quad (1d)$$

Region b

$$\nabla \times E_b = -i\omega\mu H_b \quad (2a)$$

$$\nabla \times H_b = i\omega\epsilon_b E_b \quad (2b)$$

$$\nabla \cdot (\epsilon_b E_b) = 0 \quad (2c)$$

$$\nabla \cdot (\mu H_b) = 0 \quad (2d)$$

Region c

$$\vec{\nabla} \times \vec{E}_c = -i\omega\mu\vec{H}_c \quad (3a)$$

$$\vec{\nabla} \times \vec{H}_c = i\omega\epsilon_c\vec{E}_c \quad (3b)$$

$$\vec{\nabla} \cdot (\epsilon_c\vec{E}_c) = 0 \quad (3c)$$

$$\vec{\nabla} \cdot (\mu\vec{H}_c) = 0 \quad (3d)$$

Here:

$\vec{E}$ = electric field

$\vec{H}$ = magnetic field

ϵ = complex dielectric constant related to the permittivity ϵ^* and the conductivity σ according to

$$\epsilon = \epsilon^* - \frac{i\sigma}{\omega\epsilon^*} \quad (4)$$

μ = permeability of free space

The field quantities are total fields which include the primary field due to the prescribed source which excites the cavity. The source of this field is not included explicitly in this analysis. The goal is to obtain expressions for the total field inside the cavity and to relate these fields to the scattered fields which identify the presence and the nature of the cavity.

This three region problem can be reduced to a two region problem by modifying the cavity equations as described by Miller and Poggio [1]. By adding and subtracting $\epsilon_b E_c$ wherever the factor $\epsilon_c E_c$ appears in (3b) and (3c), one obtains in place of (3b) and (3c)

$$\vec{\nabla} \times \vec{H}_c = i\omega\epsilon_b \vec{E}_c + i\omega(\epsilon_c - \epsilon_b) \vec{E}_c$$

$$\vec{\nabla} \cdot (\epsilon_b \vec{E}_c + (\epsilon_c - \epsilon_b) \vec{E}_c) = 0$$

which allows the introduction of the current and charge (j_c , ρ_c) according to

$$j_c = i\omega(\epsilon_c - \epsilon_b) \vec{E}_c \quad (5a)$$

$$\rho_c = -\vec{\nabla} \cdot (\epsilon_c - \epsilon_b) \vec{E}_c \quad (5b)$$

Now the equations valid in region c may be written as

$$\vec{\nabla} \times \vec{E}_c = -i\omega\mu \vec{H}_c \quad (6a)$$

$$\vec{\nabla} \times \vec{H}_c = i\omega\epsilon_b \vec{E}_c + \vec{j}_c \quad (6b)$$

$$\vec{\nabla} \cdot (\epsilon_b \vec{E}_c) = \rho_c \quad (6c)$$

$$\vec{\nabla} \cdot (\mu \vec{H}_c) = 0 \quad (6d)$$

This set of equations describes the radiation by the current $\vec{J}_c$ embedded in a medium having the material parameters associated with region b. The equivalent set of Maxwell's equations obtained by combining those governing regions b and c is

$$\vec{\nabla} \times \vec{E}_a = -i\omega\mu\vec{H}_a \quad (7a)$$

$$\vec{\nabla} \times \vec{H}_a = i\omega\epsilon_a\vec{E}_a \quad (7b)$$

$$\vec{\nabla} \cdot (\epsilon_a\vec{E}_a) = 0 \quad (7c)$$

$$\vec{\nabla} \cdot (\mu\vec{H}_a) = 0 \quad (7d)$$

$$\vec{\nabla} \times \vec{E}_b = -i\omega\mu\vec{H}_b \quad (7e)$$

$$\vec{\nabla} \times \vec{H}_b = i\omega\epsilon_b\vec{E}_b + \vec{J}_c \quad (7f)$$

$$\vec{\nabla} \cdot (\epsilon_b\vec{E}_b) = \rho_c \quad (7g)$$

$$\vec{\nabla} \cdot (\mu\vec{H}_b) = 0 \quad (7h)$$

$$\vec{J}_c = i\omega(\epsilon_c - \epsilon_b)\vec{E}_b \quad (7i)$$

$$\rho_c = -\vec{\nabla} \cdot (\epsilon_c - \epsilon_b)\vec{E}_b \quad (7j)$$

$$\vec{\nabla} \cdot \vec{J}_c = -i\omega\rho_c \quad (7k)$$

The volume polarization current $\vec{J}_c$, which exists only inside the cavity region c , is a proper Maxwellian current on an equal mathematical footing with a conduction current (see Appendix C). For example, a discontinuity in $\vec{J}_c$ at a boundary gives rise to a surface charge obtained by carrying out the differentiation called for in (7k). Now we proceed by solving the two region problem containing a current in region c , which radiates without any further recognition of the presence of the cavity.

The scattering by a three-dimensional dielectric body has been solved in terms of an electric field volume integral equation by Livesay and Chan [2] in the absence of the air-earth interface and by Hohmann [3] in the presence of the interface. They formulate the integral equation in terms of a highly singular dyadic Green's function of the electric field type. The treatment of this singularity has been discussed by Van Bladel [4], Fikioris [5], Yaghjian [6], Lee, Boersma, Law and Deschamps [7], and others.

Here we formulate the solution in terms of the dyadic Green's function of the magnetic vector potential type. This approach is a generalization to three dimensions of Richmond's [8,9] whole-space two dimensional treatment of the scattering by a dielectric body. In addition, the treatment here will include the effects of the interface. The potential integral approach is numerically simpler because it focuses attention on well-behaved integrals involving the Green's function of the magnetic vector potential type. The differentiations called for in the field calculations are performed after the integration when numerically feasible in order to avoid raising the order of the singularity.

The solution to the problem of the radiation by an electric source in the presence of the half-space interface is given in Appendix A in terms of a potential integral solution. The fields are given by

$$\vec{H}^{SC} = \frac{1}{\mu} [\vec{\nabla} \times \vec{A}] \quad (8)$$

$$\vec{E}^{SC} = -\frac{i\omega}{k^2} [k^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})] \quad ; \quad k^2 = \omega^2 \mu \epsilon \quad (9)$$

where the vector potential of the magnetic type, $\vec{A}$, is written as

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \int_{\text{Cavity}} \vec{G}_e(\vec{r}, \vec{r}') \cdot \vec{J}_c(\vec{r}') dV' \quad (10)$$

in which the integration is carried out throughout the cavity volume. The explicit expressions for the dyadic Green's function $\vec{G}_e$ is found in Appendix A. The superscript appearing in $\vec{H}^{SC}$ and $\vec{E}^{SC}$ stands for scattered. It is a reminder that these are only the fields produced by polarization current $\vec{J}_c$. The total $\vec{E}$ is obtained by adding the incident field $\vec{E}^{inc}$ (the primary field) to the scattered one obtaining

$$\vec{E} = \vec{E}^{inc} + \vec{E}^{SC} \quad (11)$$

After inserting (10) and (9) into (11), one can write an explicit relation for $\vec{E}$ with help from (7i) as

$$\vec{E}(\vec{r}) = \vec{E}^{inc}(\vec{r}) - \frac{i\omega\mu}{4\pi k^2} (k^2 + \vec{\nabla} \cdot \vec{\nabla}) \int_{\text{Cavity}} \vec{G}_e(\vec{r}, \vec{r}') \cdot [i\omega(\epsilon_c - \epsilon_b)\vec{E}(\vec{r}')] dV' \quad (12)$$

This is an integral equation in the unknown total field $\vec{E}$ which has been formulated in terms of volume polarization currents. All that remains is

solving this equation for $\vec{E}$. This solution will be executed in the next section using the method of moments.

2.3 Moment-method Solution

The numerical problem will be solved using pulse function basis and point matching in the moment-method starting with the basic electric field volume integral equation

$$\vec{E}(\vec{r}) = \vec{E}^{inc}(\vec{r}) - \frac{i\omega\mu}{4\pi k^2} (k^2 + \vec{\nabla} \cdot \vec{\nabla}) \int_{\text{Cavity}} \vec{G}_e(\vec{r}, \vec{r}') \cdot \vec{J}_c(\vec{r}') dV' \quad (13)$$

where

$$\vec{J}_c = i\omega(\epsilon_c - \epsilon_b)\vec{E}$$

First divide the cavity into N cubic cells. Next, assume that the current is uniform within each cell by writing

$$\vec{J}(\vec{r}) = \sum_n \vec{J}(n) P(\vec{r}_n) \quad (14)$$

which is a three-dimensional pulse expansion. Here $\vec{J}(n)$ is the vector expansion for the current valid inside the n th cell. The spatial dependence for the current is contained in the pulse function defined by

$$P(\vec{r}_n) = P_x(\vec{r}_n) P_y(\vec{r}_n) P_z(\vec{r}_n) \quad (15)$$

where

$$P_j(\vec{r}_n) = \begin{cases} 1 & \text{for the } j\text{th cartesian coordinate residing in cell } n \\ 0 & \text{elsewhere} \end{cases} \quad (16)$$

Particular care must be exercised in treating the discontinuous nature of the current when applying the differentiation called for in the continuity equation which is

$$\vec{\nabla} \cdot \vec{J} = -i\omega\rho \quad (17)$$

In view of (14) and (17), it is clear each cell is modeled by a uniform current abruptly terminated at the faces. Consequently, the entire charge density resides at the surface as required in order to terminate the current density.

After inserting (14) into (13) one has

$$\vec{E}(\vec{r}) = \vec{E}^{inc}(\vec{r}) - \frac{i\omega\mu}{4\pi k^2} (k^2 + \vec{\nabla} \cdot \vec{\nabla}) \sum_n \left\{ \int_{\Delta v_n} \vec{G}_e(\vec{r}, \vec{r}'_n) P(n) dv' \right\} \cdot \vec{J}(n) \quad (18)$$

where each volume integral is performed over cell n . In order to produce a matrix equation, this equation is enforced at the center of each cell $\vec{r}_m$ (point matching), to obtain the matrix equation

$$\vec{E}(\vec{r}_m) = \vec{E}^{inc}(\vec{r}_m) - \frac{i\omega\mu}{4\pi k^2} (k^2 + \vec{\nabla} \cdot \vec{\nabla}) \sum_n \left\{ \int \vec{G}_e(\vec{r}_m, \vec{r}'_n) P(n) dv' \right\} \cdot \vec{J}(n) \quad (19)$$

valid for $m = 1$ to N . Hagmann [10] has considered the convergence of moment-method solution to the electric field volume equation using this method, collocation with pulse functions. This approach has been found to be satisfactory when the overall dimension of the scatterer is less than $100/k_0$ where k_0 is the free space propagation constant. In the application here, this criterion is easily met in all cases.

Since the source and observation points are both located inside the cavity, the dyadic Green's function $\hat{G}_e$ must be the one appropriate for the source below and the observer below the interface (SB08). An explicit equation for $\hat{G}_e$ is found in Appendix A as equation A.19, namely

$$\hat{G}_e = \begin{bmatrix} (G_{11} + G_{12} + U'_{11})\hat{x}\hat{x} & 0 & 0 \\ 0 & (G_{11} + G_{12} + U'_{11})\hat{y}\hat{y} & 0 \\ \frac{\partial W_{11}}{\partial x} \hat{x}\hat{x} & \frac{\partial W_{11}}{\partial y} \hat{z}\hat{y} & (G_{11} - G_{12} + k_2^2 V'_{11})\hat{z}\hat{z} \end{bmatrix} \quad (20)$$

where the Green's function, $\hat{G}_e$, and the Sommerfeld integrals U' , V' , and W are written out explicitly in Appendix A.

The moment-method solution to the scattering problem using pulse function basis and point matching is now available by combining (19) and (20) as a formal solution. The central problem in this method remains. It is the numerically efficient implementation of the integral over the Green's function and, more importantly, the differentiation of this integral as called for in (19). Appendix C is devoted entirely to the resolution of these questions.

3.0 NUMERICAL RESULTS

3.1 Terminology

The code calculates the electromagnetic fields which would exist at the surface of the earth in the presence of an EM source on the surface and a three-dimensional inhomogeneity (ex: the cavity) buried in the earth. These fields are referred to variously as the direct (primary) fields, the scattered (secondary) fields, and the total fields, which is the most relevant to experimental studies. In the absence of the scatterer, the fields are the direct (primary) fields. In the presence of the scatterer, the scattered (secondary) fields are added to the primary fields to produce the total fields. Clearly, the scattered (secondary) fields are the difference between the physically observed fields in the presence of and in the absence of the scattering void. The existence of the void is evident in the measured fields only when the scattered fields can be detected in the presence of the primary field. This occurs when the secondary field amplitude is comparable to that of the primary field or when the secondary field has a measurable component in a direction along which the primary field has a null.

Using the equations presented in the preceding section, the code can calculate the electric and magnetic fields for a vertical or horizontal, electric or magnetic point source. Specific antennas can be constructed from these point sources. This numerical effort concentrates on the vertical magnetic dipole source, specifically in the form of a horizontal loop, residing on the air-earth interface.

The geometry assumed is shown in Fig. 2, which is described in cylindrical coordinates with the origin centered on the source. The z coordinate, not shown, is vertical and pointing out of the plane of the

Figure 2. An aerial view of the geometry showing the vertical magnetic dipole source, the buried void, and the radial and circular survey lines used in the computation of the fields.

diagram. Angles, ϕ , are measured away from a line joining the source to the center of the scattering void. The radial coordinate, R , is measured from the source. Altogether, the physical system is symmetric under a reflection through the angle ϕ . In this coordinate system, the direct magnetic field from the source has a radial and a vertical component only. The associated electric field has only an azimuthal component.

All of the calculations presented here are for a 1000 Hz source, unless otherwise stated. All the individual components of the electric, $\vec{E}$, and the magnetic, $\vec{H}$, fields were computed by the code. In addition, specific ratios of field components were calculated. In particular, the ellipticity of the magnetic field has been plotted.

Using the notation

$$H_R = |H_R| e^{i\phi_R} \quad (21)$$

$$H_Z = |H_Z| e^{i\phi_Z} \quad (22)$$

the ellipticity, ϵ , as defined by Smith and Ward [11] (Fig. 3) is given by the relationship

$$\epsilon = \frac{|H_Z| e^{i(\phi_Z - \phi_R)} \cos \alpha - |H_R| \sin \alpha}{|H_Z| e^{i(\phi_Z - \phi_R)} \sin \alpha + |H_R| \cos \alpha} \quad (23)$$

In addition, an electric field ratio, which is closely related to the anomaly E field ratio reported by Cauterman, Nguyen, and Degauque [12], was found to be sensitive to the presence of the void. This anomaly ratio A_E is defined as

$$A_E = \frac{E_{\text{azm}}^{\text{secondary}}}{E_{\text{azm}}^{\text{total}}}$$

Figure 3. Ellipse parameters used in defining the $\vec{H}$ field ellipticity.

where $E_{\text{azm}}^{\text{secondary}}$ is the difference between the total azimuthal $\vec{E}$ field, $E_{\text{azm}}^{\text{total}}$, and the primary azimuthal $\vec{E}$ component.

Lastly, the ratio of the total radial magnetic field to the total vertical magnetic field, $H_{\text{rad}}/H_{\text{vert}}$, was plotted.

3.2 Conductivity

As noted in Fig. 2, the fields were calculated along radial survey paths, path A in the diagram, and along circular survey paths, path B in the diagram. These survey lines were chosen to take advantage of the symmetry of the physical configuration as described in cylindrical coordinates. Several plots have been included here as Fig. 4 through Fig. 6 to display the general nature of the computed fields. The details follow.

For a cubical scatterer with sides 60 meters long buried 20 meters below the surface, the field ratios are displayed in Figs. 4, 5, and 6. These plots correspond to a radial survey passing over the center of the cube which is located at a distance of 300 m from the source. Figure 4 is a plot of the ellipticity for a ground conductivity σ of .001 S/m and of .01 S/m in parts a) and b). Figure 4a shows a plot which differs little from one obtained using only primary field quantities. On the other hand, Fig. b) displays a noticeable variation indicating the presence of the scatterer in a host conductivity of .01 S/m. The ellipticity is particularly sensitive to the ground conductivity. The ratio of the radial magnetic field to the vertical magnetic field, $H_{\text{rad}}/H_{\text{vert}}$, shows a similar sensitivity. On the other hand, the anomaly $\vec{E}$ ratio, the ratio of the secondary to the total azimuthal $\vec{E}$ field, peaks just beyond the scatterer ($R = 300$) with little dependence of the host conductivity. A review of Figs. 4 through 6 shows that the scatterer is

easier to identify in the lossier earth over this range of conductivities, a result first noted by Hohmann [3]. Even though the attenuation of the fields increases with conductivity, the scattered fields are somewhat enhanced by the increase in conductivity because the magnitude of scattered fields is proportional to conductivity contrast between the void and the host medium.

3.3 Size and Shape

The size and shape of the void can be estimated to some degree from the circular survey indicated in the sketch below.

As indicated in the sketch, the survey path passes over the edges of the cubical and the spherical void at $\pm 8^\circ$ and $\pm 9^\circ$ respectively. In Figs. 7 through 10, the plots for the cubical and the spherical scatterer are labeled a and b respectively. The anomaly electric field ratio A_E is displayed in Fig. 7 for the case where the geometrical center of the void is located 50 m

(a) DISTANCE FROM THE SOURCE

(b) DISTANCE FROM THE SOURCE

Figure 4. Ellipticity plotted as a function of radial distance from the source for a survey path passing over the center of the void, 300 m from the source. (a) .001 S/m conductivity of the earth. (b) .01 S/m ground conductivity.

(a) DISTANCE FROM THE SOURCE

(b) DISTANCE FROM THE SOURCE

Figure 5. Ratio of the total radial to the total vertical magnetic field as a function of radial distance from the source (void is at $R = 300$) for a survey path passing over the center of the void for (a) .001 S/m conductivity of the earth and (b) .01 S/m conductivity of the earth.

(a) DISTANCE FROM THE SOURCE

(b) DISTANCE FROM THE SOURCE

Figure 6. The anomaly E field ratio A_E (Eq. 24) as a function of radial distance from the source for a survey line passing over the void. Two ground conductivities of (a) .001 S/m and (b) .01 S/m are considered.

below the surface, the top extending to 20 m below the surface. The plots show a ratio which is clearly measurable, since it indicates that the scattered fields are about 20% of the incident field. The nulls in the plots point at the extremities of the void. Thus the shape of the curves suggest the possibility of using this measurement to identify the shape of the scatterer. Figure 8 shows the azimuthal magnetic field. This is the scattered field which is the total field tangent to the survey path. The extent of the scatterer is evident in these plots but the difference between the cubical and the spherical scatterer plots lies primarily in the magnitudes. Figures 9 and 10 differs from Figs. 7 and 8 only in that the depth has been increased by 20 m. At this depth, it is no longer possible to differentiate between the cube and the sphere.

Altogether, Figs. 7 through 10 indicate that the extent and the shape of the void can be determined from either electric or magnetic field measurements. The measurements become more difficult to perform and to interpret with increasing depth. For voids whose size is of the order of $60 \times 60 \times 60$ m, the results are already questionable at a depth of 50 m.

3.4 Depth

Various field ratios are displayed in Figs. 11 through 16. They are ordered in a sequence with increasing depth to the center of the $60 \times 60 \times 60$ m cubical scatterer. The void is centered on a radial survey path from 200 to 400 m from the source. The void is "visible" at depths up to 60 meters in the plots of ellipticity, anomaly electric field ratio, and the normalized radial magnetic field. A cursory glance shows that none of these

(a) ANGLE FROM THE SOURCE

(b) ANGLE FROM THE SOURCE

Figure 7. The anomaly ratio A_E , (Eq. 24) of the electric field along a circular survey path passing over the center of a void ("zero angle") which is located 50 m below the surface. The earth parameters are $\epsilon = 10$, and $\sigma = .01$. (a) Cubical void. (b) Spherical void.

(a) ANGLE FROM THE SOURCE

(b) ANGLE FROM THE SOURCE

Figure 8. The total azimuthal magnetic field H_ϕ along a circular survey path passing over the center of a void which is located 50 m below the surface. The earth parameters are $\epsilon = 10$, and $\sigma = .01$. (a) Cubical void, 60 m x 60 m x 60 m. (b) Spherical void, radius ~ 37.2 m.

(a) ANGLE FROM THE SOURCE

(b) ANGLE FROM THE SOURCE

Figure 9. The anomaly ratio A_E of the electric field along a circular survey path passing over the center of a void which is located 70 m below the surface. The earth parameters are $\epsilon = 10$, and $\sigma = .01$. (a) Cubical void, 60 x 60 x 60 m. (b) Spherical void, radius ~ 37.2 m.

Figure 10. The total azimuthal magnetic field H_ϕ along a circular survey path passing over the center of a void which is located 70 m below the surface. The earth parameters are $\epsilon = 10$, $\sigma = .01$. (a) Cubical void, 60 x 60 x 60 m. (b) Spherical void, radius ~ 37.2 m.

(a) DISTANCE FROM THE SOURCE

(b) DISTANCE FROM THE SOURCE

Figure 11. Ellipticity for a radial survey path from a distance of 200 to 400 m from the source centered over the cubical scatterer. The earth parameters are $e = 10$, and $\sigma = .01$. (a) Depth to the center of the cube is 50 m. (b) Depth to the center of the cube is 60 m.

(a)

(b)

Figure 12. Ellipticity for a radial survey path from a distance of 200 to 400 m from the source centered over the cubical scatterer. The earth parameters are $\epsilon = 10$, and $\sigma = .01$. (a) Depth to the center of the cube is 70 m. (b) Depth to the center of the cube is 90 m.

Figure 13. Anomaly electric field ratio A_E (Eq. 24) for a radial survey path from a distance of 200 to 400 m from the source centered over the cubical scatterer. The earth parameters are $\epsilon = 10$, and $\sigma = .01$. (a) Depth to the center of the cube is 50 m. (b) Depth to the center of the cube is 60 m.

(a)

(b)

Figure 14. Anomaly electric field ratio A_E (Eq. 24) for a radial survey path from a distance of 200 to 400 m from the source centered over the cubical scatterer. The earth parameters are $\epsilon = 10$, and $\sigma = .01$. (a) Depth to the center of the cube is 70 m. (b) Depth to the center of the cube is 90 m.

(a) DISTANCE FROM THE SOURCE

(b) DISTANCE FROM THE SOURCE

Figure 15. Normalized radial magnetic field H_{rad}/H_{vert} for a radial survey path from a distance of 200 to 400 m from the source centered over the cubical scatterer. The earth parameters are $\epsilon = 10$, and $\sigma = .01$. (a) Depth to the center of the cube is 50 m. (b) Depth to the center of the cube is 60 m.

(a) DISTANCE FROM THE SOURCE

(b) DISTANCE FROM THE SOURCE

Figure 16. Normalized radial magnetic field $H_{\text{rad}}/H_{\text{vert}}$ for a radial survey path from a distance of 200 to 400 m from the source centered over the cubical scatterer. The earth parameters are $\epsilon = 10$, and $\sigma = .01$. (a) Depth to the center of the cube is 70 m. (b) Depth to the center of the cube is 90 m.

measurements serve as a useful diagnostic at the 90 m depth. For shallower depths, the void is clearly detectable using either electric or magnetic field measurements.

3.5 Survey Network

In the preceding sections, it has been shown that voids located at moderately large depths are detectable using a variety of electric and magnetic field measurements. In all these displays, the survey path, radial and circular, pass directly over the center of the void. However, in practice, this location is the unknown. As a consequence, this scheme is only useful if measurements made on survey paths near, but not directly over, the void can "point" at the anomaly. In this section, plots of field ratio will be displayed for paths passing directly over the void and for adjacent paths.

The radial paths corresponding to radial distances from 200 to 400 meters from the source are shown in the sketch below for the angles 0° , 3.75° , 7.5° , and 15° . Figures 17 and 18 display the plots of the ellipticity as a function of radial distance, while Figs. 19 and 20 display the anomaly E ratio along the same survey lines.

(a) DISTANCE FROM THE SOURCE

(b) DISTANCE FROM THE SOURCE

Figure 17. Ellipticity as a function of radial distance from the source with the buried void located at 300 m along the $\theta = 0$ line. The earth parameters are $\epsilon_r = 10$, and $\sigma = .01$ S/m. (a) $\theta = 0^\circ$ survey. (b) $\theta = 3.75^\circ$ survey.

Figure 18. Ellipticity as a function of radial distance from the source with the buried void located at 300 m along the $\theta = 0$ line. The earth parameters are $\epsilon_r = 10$, and $\sigma = .01$ S/m. (a) $\theta = 7.5^\circ$ survey. (b) $\theta = 15^\circ$ survey.

Figure 19. Anomaly electric field ratio shown as a function of the radial distance from the source with the buried void located at 300 m along the $\theta = 0$ line. The earth parameters are $\epsilon_r = 10$, and $\sigma = .01$ S/m. (a) $\theta = 0^\circ$ survey. (b) $\theta = 3.75^\circ$ survey.

Figure 20. Anomaly electric field ratio shown as a function of the radial distance from the source with the buried void located at 300 m along the $\theta = 0$ line. The earth parameters are $\epsilon_r = 10$, and $\sigma = .01$ S/m. (a) $\theta = 7.5^\circ$ survey. (b) $\theta = 15^\circ$ survey.

It is clear from these figures that a radial survey will detect the presence of the void if the survey line passes within 7.5° of the line which passes directly over the center of the void. This is equivalent to the statement that the survey line must pass over at least a corner of the cubical scatterer.

The circular paths from an azimuthal angle of -15° to $+15^\circ$, where the cavity is centered around 0° , are shown below for paths with radii of 200, 250, 300, 350 and 400 meters.

The normalized radial magnetic field and the anomaly electric field ratio are shown in Figs. 21 through 25. The radial magnetic field measurements are less

effective than the tangential electric field measurements. The anomaly electric ratio plots indicate that the void can be detected if the survey line passes within 50 m of the center of the scatterer.

The plots displayed in this section show that the void is detectable by either radial or circular surveys only if they pass near to the edges of the void. Consequently, for successful detection, the spacing between the lines of the survey must be comparable to the size of the scatterer buried at a moderate depth.

(a) ANGLE FROM THE SOURCE

(b) ANGLE FROM THE SOURCE

Figure 21. Field ratios along a circular survey line of radius 200 m. The void is centered at $\theta = 0^\circ$ and $R = 300$ m. (a) Normalized radial magnetic field H_{rad}/H_{vert} . (b) Anomaly electric field ratio A_E .

Figure 22. Field ratios along a circular survey line of radius 250 m. The void is centered at $\theta = 0^\circ$ and $R = 300$ m. (a) Normalized radial magnetic field. (b) Anomaly electric field ratio.

(a) ANGLE FROM THE SOURCE

(b) ANGLE FROM THE SOURCE

Figure 23. Field ratios along a circular survey line of radius 300 m. The void is centered at $\theta = 0^\circ$ and $R = 300$ m. (a) Normalized radial magnetic field. (b) Anomaly electric field ratio.

(a) ANGLE FROM THE SOURCE

(b) ANGLE FROM THE SOURCE

Figure 24. Field ratios along a circular survey line of radius 350 m. The void is centered at $\theta = 0^\circ$ and $R = 300$ m. (a) Normalized radial magnetic field. (b) Anomaly electric field ratio.

(a) ANGLE FROM THE SOURCE

(b) ANGLE FROM THE SOURCE

Figure 25. Field ratios along a circular survey line of radius 400 m. The void is centered at $\theta = 0^\circ$ and $R = 300$ m. (a) Normalized radial magnetic field. (b) Anomaly electric field ratio.

4.0 CONCLUSION

The interaction of electromagnetic (EM) radiation originating on the earth's surface with a subterranean cavity is calculated using an integral formulation of the problem and a moment-method solution. The numerical effort is concentrated on the vertical magnetic dipole source, which in exploration geophysics, takes on the form of a horizontal rectangular loop on the surface of the earth. In evaluating the behavior of ellipticity and magnetic field ratio $H_{\text{rad}}/H_{\text{vert}}$ in the conductivity range .001 to .01 S/m, it is found that the cavity is more detectable in the higher conductivity earth. Even though the electromagnetic wave is attenuated more with higher electrical conductivity, the earth shows greater contrast with the void if the earth has a higher conductivity. Detection sensitivity for both a radial survey scheme and a circular survey scheme are examined. Both magnetic receivers and electric receivers can detect the void if the survey passes over the void or at least pass near the edges of the void. However, the electric field measurements appear to be more sensitive than the magnetic field measurements; the electric receiver can be farther from the edge of the void than can be the magnetic receiver. Detection limitations are also a function of depth of burial in conjunction with size and conductivity contrast. Maximum detection limits for this study appear to be at 90 m depth for a 60 x 60 x 60 cube.

For application in OSI situations, computer models should be made for when the water table is shallow, namely, when the void is filled with water. This would most likely give a conductive anomaly. As for salt domes, the conductivity of salt is low if the salt is dry; it becomes very high if it gets wet. Many things could happen in a salt dome and would require a study by itself.

An interesting observation of how these predicted results compare with field experiments (see references 18, 19) is that the experimental results for a horizontal loop source show a stronger signature than predicted by the computer models. Perhaps the effects of the cavity extend to a greater distance than the actual physical dimensions of the cavity-chimney structure. In any case, these experimental results imply that the computer code predictions are conservative.

For other applications, this computer code offers a means of predicting the interaction of a 3-dimensional underground anomaly with EM waves for low frequencies (30 Hz to 10 kHz). The anomaly can cause a conductive or a resistive contrast with the ground. One obvious application is for evaluating detection methods of underground cavities for site characterization of a construction site. Another application includes evaluation of the effects of the air-earth interface for a subterranean antenna transmitting within the ground or to the surface. With further development, field estimates for low frequency radio transmissions near a land-sea boundary could be calculated.

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APPENDIX A

Half-space Green's Functions

Introduction

In all aspects of the scattering of electromagnetic waves by an obstacle embedded in a half-space, field computations are required which take into account the boundary conditions at the interface separating the upper and the lower half-spaces. The sources of these fields are electric or magnetic. The observation and source points may lie on the same or on opposite sides of the interface, above or below the interface. In all cases, the field computations proceed directly from Maxwell's equations. In their symmetrical time-harmonic form, they are

$$\vec{\nabla} \times \vec{E} = -\vec{J}_m + i\omega\mu\vec{H} \quad (A.1)$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_e - i\omega\epsilon\vec{E} \quad (A.2)$$

$$\vec{\nabla} \cdot \vec{D} = \rho_e \quad (A.3)$$

$$\vec{\nabla} \cdot \vec{B} = \rho_m \quad (A.4)$$

where the time dependence is $e^{i\omega t}$. $\vec{J}_m$ and $\vec{J}_e$ are magnetic and electric current sources respectively, connected to their associated charge densities ρ_m and ρ_e by equations of continuity. The magnetic source current is related to the magnetic polarization $\vec{M}$ by

$$\vec{J}_m = -i\omega\mu\vec{M} \quad (A.5)$$

The electric field $\vec{E}$ and the displacement field $\vec{D}$ are connected by

$$\vec{D} = \epsilon \vec{E} \quad (\text{A.6})$$

where the complex dielectric constant ϵ can be written explicitly in terms of the permittivity ϵ^* and the conductivity σ as

$$\epsilon = \epsilon^* \left(1 + \frac{i\sigma}{\omega \epsilon^*} \right)$$

The magnetic induction $\vec{B}$ is related to the magnetic field $\vec{H}$ by

$$\vec{B} = \mu \vec{H} \quad (\text{A.7})$$

where μ is the permeability of the medium.

In the calculations of interest here, the sources are either magnetic or electric depending upon the application. Consequently, the field computations will be divided into two sections in what follows. In both sections, the solution to the problem will be outlined in terms of dyadic Green's function of the vector potential type. The answers will be written in detail for further reference.

Green's Functions for Electric Sources

When only electric sources exist, Maxwell's equations reduce to

$$\vec{\nabla} \times \vec{E} = i\omega\mu\vec{H} \quad (\text{A.8})$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_e - i\omega\epsilon\vec{E} \quad (\text{A.9})$$

$$\vec{\nabla} \cdot \vec{D} = \rho_e \quad (\text{A.10})$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (\text{A.11})$$

In terms of the magnetic vector potential $\vec{A}$, the solution to these equations is given by

$$\vec{H} = \frac{1}{\mu} \vec{\nabla} \times \vec{A} \quad (\text{A.12})$$

$$\vec{E} = \frac{i\omega}{k^2} [k^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})] \quad (\text{A.13})$$

where $\vec{A}$ must satisfy the inhomogeneous wave equation

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}_e \quad (\text{A.14})$$

with $k^2 = \omega^2 \mu \epsilon$. In a homogeneous space, the solution to (A.14) is given by

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \int G_{\vec{r}}(\vec{r}, \vec{r}') \vec{J}_e(\vec{r}') dV'$$

where $G_{\vec{r}}$ is the free space Green's function which is

$$G_{\vec{r}}(\vec{r}, \vec{r}') = \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \quad (\text{A.15})$$

where $\vec{r}$ and $\vec{r}'$ locate the observation and source points, and the integration extends over the source region. In the presence of a half-space, $\vec{A}$ must be modified to account for the fact that $\vec{A}$ and $\vec{J}_e$ are no longer parallel in general. This requires the introduction of the dyadic Green's function $\vec{\vec{G}}_e$ to obtain the solution

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \int \vec{\vec{G}}_e(\vec{r}, \vec{r}') \cdot \vec{J}_e(\vec{r}') dV' \quad (\text{A.16})$$

which is written in cartesian components. By selecting x, y, and z directed point sources, the components of the vector $\vec{A}$ can be identified with those of

the dyad $\tilde{G}_e$. Consequently, in order to obtain an explicit equation for $\tilde{G}_e$, it is necessary to solve the classical boundary value problem consisting of the radiation by an x, y, or z directed electric dipole in the presence of a dissipative half-space. Equation (A.14) is the differential equation which provides the required general and particular solutions in each region. The appropriate boundary conditions on $\tilde{A}$ are found from (A.12) and (A.13) by requiring continuity of the $\tilde{E}$ and $\tilde{H}$ tangential components across the interface. All the details are given by Baños [13].

For the purpose of presenting the answer, the interface coincides with the x-y plane. Air (medium 2) and ground (medium 1) occupy the upper and the lower half-spaces respectively as indicated in the figure.

Figure A.1. Coordinate system showing the orientation of the half-space.

The solutions for $\tilde{G}_e$ differ according to the position of the source and observation points. The possibilities are:

1. Source above and observer above the interface (SAOA).
2. Source above and observer below the interface (SAOB).
3. Source below and observer below the interface (SBOB).
4. Source below and observer above the interface (SBOA).

For the SAOA case,

$$G_e = \begin{bmatrix} (G_{22} + G_{21}' + U_{22}')\hat{x}\hat{x} & 0 & 0 \\ 0 & (G_{22} + G_{21}' + U_{22}')\hat{y}\hat{y} & 0 \\ \frac{\partial W_{22}}{\partial x} \hat{z}\hat{x} & \frac{\partial W_{22}}{\partial y} \hat{z}\hat{y} & (G_{22} - G_{21}' + k_1^2 V_{22}')\hat{z}\hat{z} \end{bmatrix} \quad (A.17)$$

For the SAOB case,

$$G_e = \begin{bmatrix} (U_{21})\hat{x}\hat{x} & 0 & 0 \\ 0 & (U_{21})\hat{y}\hat{y} & 0 \\ \frac{\partial W_{21}}{\partial x} \hat{z}\hat{x} & \frac{\partial W_{21}}{\partial y} \hat{z}\hat{y} & (k_1^2 V_{21}')\hat{z}\hat{z} \end{bmatrix} \quad (A.18)$$

For the SB0B case,

$$G_e = \begin{bmatrix} (G_{11} + G_{12}' + U_{11}')\hat{x}\hat{x} & 0 & 0 \\ 0 & (G_{11} + G_{12}' + U_{11}')\hat{y}\hat{y} & 0 \\ \frac{\partial W_{11}}{\partial x} \hat{z}\hat{x} & \frac{\partial W_{11}}{\partial y} \hat{z}\hat{y} & (G_{11} - G_{12}' + k_2^2 V_{11}')\hat{z}\hat{z} \end{bmatrix} \quad (A.19)$$

For the SB0A case,

$$G_e = \begin{bmatrix} (U_{12})\hat{x}\hat{x} & 0 & 0 \\ 0 & (U_{12})\hat{y}\hat{y} & 0 \\ \frac{\partial W_{12}}{\partial x} \hat{z}\hat{x} & \frac{\partial W_{12}}{\partial y} \hat{z}\hat{y} & (k_2^2 V_{12}')\hat{z}\hat{z} \end{bmatrix} \quad (A.20)$$

Here G is a free space or image Green's function and U , V , U' , V' , and W are Sommerfeld integrals defined in terms of the indices i and j which specify the media according to Figure A.2 as:

$$G_{ii} = \frac{e^{ik_i R_i}}{R_i} \quad (A.21)$$

$$G_{ij} = \frac{k_i^2 - k_j^2}{k_i^2 + k_j^2} \frac{e^{ik_i R_j}}{R_j} \quad (A.22)$$

with $R_1 = [\rho^2 + (z-z')^2]^{1/2}$ and $R_2 = [\rho^2 + (z+z')^2]^{1/2}$

$$U(a,b,\rho) = \int_{-\infty}^{\infty} D_1 e^{-\gamma_1 a - \gamma_2 b} H_0^{(1)}(\lambda \rho) \lambda d\lambda \quad (A.23)$$

$$V(a,b,\rho) = \int_{-\infty}^{\infty} D_2 e^{-\gamma_1 a - \gamma_2 b} H_0^{(1)}(\lambda \rho) \lambda d\lambda \quad (A.24)$$

$$W(a,b,\rho) = \int_{-\infty}^{\infty} D_3 e^{-\gamma_1 a - \gamma_2 b} H_0^{(1)}(\lambda \rho) \lambda d\lambda \quad (A.25)$$

where

$$D_1 = \frac{1}{\gamma_1 + \gamma_2}$$

$$D_2 = \frac{1}{k_1^2 \gamma_2 + k_2^2 \gamma_1}$$

$$D_3 = \frac{\gamma_2 - \gamma_1}{k_1^2 \gamma_2 + k_2^2 \gamma_1}$$

and for the specific cases which are required in the complete development of the fields

$$U_{12} = U(-z', z, \rho)$$

$$U_{21} = U(-z, z', \rho)$$

$$V_{12} = V(-z', z, \rho)$$

$$V_{21} = V(-z, z', \rho)$$

$$W_{12} = W(-z', z, \rho)$$

$$W_{21} = W(-z, z', \rho)$$

$$W_{11} = W(-z-z', 0, \rho)$$

$$W_{22} = W(0, z + z', \rho)$$

For the primed integrals

$$U'(a, b, \rho) = \int_{-\infty}^{\infty} D_1^i e^{-\gamma_1^a - \gamma_2^b} H_0^{(1)}(\lambda \rho) \lambda d\lambda \quad (\text{A.26})$$

$$V'(a, b, \rho) = \int_{-\infty}^{\infty} D_2^i e^{-\gamma_1^a - \gamma_2^b} H_0^{(1)}(\lambda \rho) \lambda d\lambda \quad (\text{A.27})$$

where

$$D_1^i = \frac{1}{\gamma_1 + \gamma_2} - \frac{k_2^2}{\gamma_2(k_1^2 + k_2^2)}$$

$$D_2^i = \frac{1}{k_1^2 \gamma_2 + k_2^2 \gamma_1} - \frac{1}{\gamma_2(k_1^2 + k_2^2)}$$

Figure A.2 - Coordinate system showing the position of a source, its image, and the observation point.

In all the Sommerfeld integrals

$$\gamma_1 = [\lambda^2 - k_1^2]^{1/2}$$

$$\gamma_2 = [\lambda^2 - k_2^2]^{1/2}$$

while $H_0^{(1)}$ is the Hankel function.

The evaluation of the Sommerfeld integrals is the central problem in obtaining numerical results. These integrals are computed by numerical integration along contours in the complex λ plane as detailed by Lytle and Lager [14]. This integration scheme has been extracted from the Numerical Electromagnetics Code (NEC), a code which has been employed extensively in the analysis of the electromagnetic response of antennas and other metallic structures. It contains an accurate and efficient numerical approach to implement the rigorous Sommerfeld theory in which the contour integration is accomplished by adaptive interval-width Romberg integration as reported by Miller [15]. Details have been furnished by Burke, Miller, Brittingham, Lager, Lytle, and Okada [16].

The U , V , and W integrals are the fundamental Sommerfeld integrals given by Banos [13]. The U' and V' differ from those given in this reference according to

$$V_{ij} = V'_{ij} + \frac{2}{k_1^2 + k_2^2} G_{ij} \quad (\text{A.28})$$

$$U_{ij} = U'_{ij} + \frac{2k_1^2}{k_1^2 + k_2^2} G_{ij} \quad (\text{A.29})$$

where

$$G_{ij} = \frac{e^{ik_i R_j}}{R_j}$$

The modified Green's function, G_{ij}^i , are related to the usual "free space" image functions by

$$G_{ij}^i = \frac{k_i^2 - k_j^2}{k_i^2 + k_j^2} G_{ij} \quad (A.30)$$

These modifications are designed to improve the numerical integration properties of the V integrals by subtracting an analytically integrable singular function which is proportional to the "free space" image Green's function according to (A.28) and (A.29). The piece which is removed from the U and V integrals is grouped with the image Green's function to produce the modified version, G_{ij}^i , given by (A.30).

Green's Functions for Magnetic Sources

When only magnetic sources exist, Maxwell's equations reduce to

$$\nabla \times \mathbf{E} = -\mathbf{J}_m + i\omega\mu\mathbf{H} \quad (A.31)$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} \quad (A.32)$$

$$\nabla \cdot \mathbf{D} = 0 \quad (A.33)$$

$$\nabla \cdot \mathbf{B} = \rho_m \quad (A.34)$$

The solution to this set of equations can be written in terms of the vector potential of the electric type, $\mathbf{F}$, which allows computation of the fields according to

$$\mathbf{E} = -\frac{1}{\epsilon} \nabla \times \mathbf{F} \quad (\text{A.35})$$

$$\mathbf{H} = -\frac{i\omega}{k^2} [k^2 \mathbf{F} + \nabla(\nabla \cdot \mathbf{F})] \quad (\text{A.36})$$

where $\mathbf{F}$ is a solution to

$$\nabla^2 \mathbf{F} + k^2 \mathbf{F} = -\epsilon \mathbf{J}_m \quad (\text{A.37})$$

The solution is known to have the form

$$\mathbf{F}(\vec{r}) = -\frac{i\omega\mu\epsilon}{4\pi} \int \mathbf{G}_m(\vec{r}, \vec{r}') \cdot \mathbf{M}(\vec{r}') d\mathbf{v}' \quad (\text{A.38})$$

where use of equation (A.5) has been employed to relate the magnetic current $\mathbf{J}_m$ to the magnetic polarization $\mathbf{M}$. The solution for the magnetic Green's function follows from the procedure outlined for the electric Green's function $\mathbf{G}_e$. The result follows.

For the SAOA case,

$$\mathbf{G}_m = \begin{bmatrix} (G_{22} - G_{21}^1 + k_1^2 v_{22}) \hat{x}\hat{x} & 0 & 0 \\ 0 & (G_{22} - G_{21}^1 + k_1^2 v_{22}) \hat{y}\hat{y} & 0 \\ \frac{\partial W_{22}}{\partial x} \hat{z}\hat{x} & \frac{\partial W_{22}}{\partial y} \hat{z}\hat{y} & (G_{22} + G_{21}^1 + u_{22}) \hat{z}\hat{z} \end{bmatrix} \quad (\text{A.39})$$

For the SAOB case,

$$\mathbf{G}_m = \begin{bmatrix} (k_2^2 v_{21}) \hat{x}\hat{x} & 0 & 0 \\ 0 & (k_2^2 v_{21}) \hat{y}\hat{y} & 0 \\ \frac{\partial W_{21}}{\partial x} \hat{z}\hat{x} & \frac{\partial W_{21}}{\partial y} \hat{z}\hat{y} & (u_{21}) \hat{z}\hat{z} \end{bmatrix} \quad (\text{A.40})$$

For the SBOB case,

$$G_m = \begin{bmatrix} (G_{11} - G_{12} + k_2^2 V_{11}) \hat{\hat{x}}\hat{\hat{x}} & 0 & 0 \\ 0 & (G_{11} - G_{12} + k_2^2 V_{11}) \hat{\hat{y}}\hat{\hat{y}} & 0 \\ \frac{\partial W_{11}}{\partial x} \hat{\hat{z}}\hat{\hat{x}} & \frac{\partial W_{21}}{\partial y} \hat{\hat{z}}\hat{\hat{x}} & (G_{11} + G_{12} + U_{11}) \hat{\hat{z}}\hat{\hat{z}} \end{bmatrix} \quad (A.41)$$

For the SBOA case,

$$G_m = \begin{bmatrix} (k_1^2 V_{12}) \hat{\hat{x}}\hat{\hat{x}} & 0 & 0 \\ 0 & (k_1^2 V_{12}) \hat{\hat{y}}\hat{\hat{y}} & 0 \\ \frac{\partial W_{12}}{\partial x} \hat{\hat{z}}\hat{\hat{x}} & \frac{\partial W_{12}}{\partial y} \hat{\hat{z}}\hat{\hat{y}} & (U_{12}) \hat{\hat{z}}\hat{\hat{z}} \end{bmatrix} \quad (A.42)$$

where all the expressions have exactly the same meaning here as in the electric source case given earlier.

APPENDIX B

Half-space Fields

Introduction

In Appendix A, Green's functions were written for the vector potentials of the magnetic and the electric types. These Green's functions contain the solution to the half-space problem for point sources of the electric and the magnetic source types. This appendix contains a catalog of the field quantities which are derivable from these Green's functions.

Electromagnetic Fields From Electric Sources

For point electric sources, the fields are computed from

$$\vec{H} = \frac{1}{\mu} [\vec{\nabla} \times \vec{A}] \quad (B.1)$$

and

$$\vec{E} = \frac{i\omega}{k^2} [k^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})] \quad (B.2)$$

where

$$\vec{A} = \frac{\mu}{4\pi} \int \vec{G}_e \cdot \vec{J}_e dv \quad (B.3)$$

The Green's function $\vec{G}_e$ is known from equation (A.17) through (A.20) in Appendix A. In order to compute $\vec{E}$ and $\vec{H}$, it is convenient to introduce for the source $\vec{J}_e$ x, y, and z directed point sources. Then $\vec{A}$ can be computed according to (B.3). Since the Green's function is written in terms of cylindrical coordinates, the derivatives in equations (B.1) and (B.2) are most easily performed in cylindrical coordinates. The transformation to cartesian coordinates for the example of j-directed source is given by standard form:

$$V_{jx} = V_{j\rho} \cos\phi - V_{j\phi} \sin\phi$$

$$V_{jy} = V_{j\rho} \sin\phi + V_{j\phi} \cos\phi$$

$$V_{jz} = V_{jz}$$

where $\vec{V}$ stands for either $\vec{E}$ or $\vec{H}$. The field due to a j -directed source, j representing x , y , or z , is given by

$$\vec{V}_j = (V_{j\rho}, V_{j\phi}, V_{jz})$$

and

$$\vec{V}_j = (V_{jx}, V_{jy}, V_{jz})$$

in cylindrical and cartesian coordinates respectively.

In what follows, ρ and m are the electric and magnetic dipole strengths. The various Green's functions and Sommerfeld integrals are exactly those written in Appendix A. There are four cases:

1. Source above and observer above the interface (SAOA).
2. Source above and observer below the interface (SAOB).
3. Source below and observer below the interface (SBOB).
4. Source below and observer above the interface (SBOA).

Catalog of Fields From Electric Sources

$$E_{xx} = \frac{x^2}{\rho^2} E_{\rho}^H - \frac{y^2}{\rho^2} E_{\phi}^H$$

$$E_{xy} = \frac{xy}{\rho^2} (E_{\rho}^H + E_{\phi}^H)$$

$$E_{xz} = \frac{x}{\rho} E_z^H$$

$$E_{yx} = \frac{xy}{\rho^2} (E_{\rho}^H + E_{\phi}^H)$$

$$E_{yy} = \frac{y^2}{\rho^2} E_{\rho}^H - \frac{x^2}{\rho^2} E_{\phi}^H$$

$$E_{yz} = \frac{y}{\rho} E_z^H$$

$$E_{zx} = \frac{x}{\rho} E_{\rho}^V$$

$$E_{zy} = \frac{y}{\rho} E_{\rho}^V$$

$$E_{zz} = E_z^V$$

SAOA Case

$$E_{\rho}^H = \frac{i\omega\mu_1}{4\pi k_2^2} \left\{ \frac{\partial^2}{\partial \rho^2} [G_{22} + G_{21}^i + k_2^2 v_{22}^i] + k_2^2 [G_{22} + G_{21}^i + u_{22}^i] \right\}$$

$$E_{\phi}^H = - \frac{i\omega\mu_1}{4\pi k_2^2} \left\{ \frac{1}{r} \frac{\partial}{\partial \rho} [G_{22} + G_{21}^i + k_2^2 v_{22}^i] + k_2^2 [G_{22} + G_{21}^i + u_{22}^i] \right\}$$

$$E_z^H = \frac{i\omega\mu_1}{4\pi k_2^2} \left\{ \frac{\partial^2}{\partial z \partial \rho} [G_{22} + G_{21}^i - k_1^2 v_{22}^i] \right\}$$

$$E_{\rho}^V = \frac{i\omega\mu_1}{4\pi k_2^2} \left\{ \frac{\partial^2}{\partial r \partial z} [G_{22} - G_{21}^i + k_1^2 v_{22}^i] \right\}$$

$$E_z^V = \frac{i\omega\mu_1}{4\pi k_2^2} \left\{ \left(\frac{\partial^2}{\partial z^2} + k_2^2 \right) [G_{22} - G_{21}^i + k_1^2 v_{22}^i] \right\}$$

SAOB Case

$$\bar{E}_\rho^H = \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial^2 v_{21}}{\partial \rho^2} + u_{21} \right\}$$

$$\bar{E}_\phi^H = - \frac{i\omega\mu}{4\pi} \left\{ \frac{1}{\rho} \frac{\partial v_{21}}{\partial \rho} + u_{21} \right\}$$

$$\bar{E}_z^H = - \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial^2 v_{21}}{\partial z^2 \partial \rho} \right\}$$

$$\bar{E}_\rho^V = \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial^2 v_{21}}{\partial \rho \partial z} \right\}$$

$$\bar{E}_z^V = \frac{i\omega\mu}{4\pi} \left\{ \left(\frac{\partial^2}{\partial z^2} + k_1^2 \right) v_{21} \right\}$$

SBOB Case

$$\bar{E}_\rho^H = \frac{i\omega\mu}{4\pi k_1^2} \left\{ \frac{\partial^2}{\partial \rho^2} [G_{11} + G_{12} + k_1^2 v_{11}] + k_1^2 [G_{11} + G_{12} + u_{11}] \right\}$$

$$\bar{E}_\phi^H = - \frac{i\omega\mu}{4\pi k_1^2} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} [G_{11} + G_{12} + k_1^2 v_{11}] + k_1^2 [G_{11} + G_{12} + u_{11}] \right\}$$

$$\bar{E}_\phi^H = \frac{i\omega\mu}{4\pi k_1^2} \left\{ \frac{\partial^2}{\partial \rho \partial z} [G_{11} + G_{12} - k_2^2 v_{11}] \right\}$$

$$\bar{E}_\rho^V = \frac{i\omega\mu}{4\pi k_1^2} \left\{ \frac{\partial^2}{\partial \rho \partial z} [G_{11} - G_{12} + k_2^2 v_{11}] \right\}$$

$$\bar{E}_z^V = \frac{i\omega\mu}{4\pi k_1^2} \left\{ \left(\frac{\partial^2}{\partial z^2} - k_1^2 \right) [G_{11} - G_{12} + k_2^2 v_{11}] \right\}$$

SBOA Case

$$\bar{E}_\rho^H = \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial^2 v_{12}}{\partial \rho^2} + u_{12} \right\}$$

$$\bar{E}_\phi^H = - \frac{i\omega\mu}{4\pi} \left\{ \frac{1}{\rho} \frac{\partial v_{12}}{\partial \rho} + u_{12} \right\}$$

$$\bar{E}_z^H = - \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial^2 v_{12}}{\partial z^2} + \frac{\partial v_{12}}{\partial \rho} \right\}$$

$$\bar{E}_\rho^V = \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial^2 v_{12}}{\partial \rho \partial z} \right\}$$

$$\bar{E}_z^V = \frac{i\omega\mu}{4\pi} \left\{ \left(\frac{\partial^2}{\partial z^2} + k^2 \right) v_{12} \right\}$$

$$H_{xx} = \frac{xy}{\rho^2} (\bar{H}_\rho^H - \bar{H}_\phi^H)$$

$$H_{xy} = \frac{y^2}{\rho^2} \bar{H}_\rho^H + \frac{x^2}{\rho^2} \bar{H}_\phi^H$$

$$H_{xz} = \frac{y}{\rho} \bar{H}_z^H$$

$$H_{yx} = - \frac{x^2}{\rho^2} \bar{H}_\rho^H - \frac{y^2}{\rho^2} \bar{H}_\phi^H$$

$$H_{yy} = - \frac{xy}{\rho^2} (\bar{H}_\rho^H - \bar{H}_\phi^H)$$

$$H_{yz} = - \frac{x}{\rho} \bar{H}_z^H$$

$$H_{zx} = -\frac{y}{\rho} R_{\phi}^v$$

$$H_{zy} = \frac{x}{\rho} R_{\phi}^v$$

$$H_{zz} = 0$$

SAOA Case

$$R_{\rho}^H = \frac{p}{4\pi} \left\{ \frac{\partial}{\partial z} [G_{22} + G_{21}^1 + U_{22}^1] - \frac{1}{\rho} \frac{\partial W_{22}}{\partial \rho} \right\}$$

$$R_{\phi}^H = \frac{p}{4\pi} \left\{ \frac{\partial}{\partial z} [G_{22} + G_{21}^1 + U_{22}^1] - \frac{\partial^2 W_{22}}{\partial \rho^2} \right\}$$

$$R_z^H = -\frac{p}{4\pi} \left\{ \frac{\partial}{\partial \rho} [G_{22} + G_{21}^1 + U_{22}^1] \right\}$$

$$R_{\phi}^v = -\frac{p}{4\pi} \left\{ \frac{\partial}{\partial \rho} [G_{22} - G_{21}^1 + k_1^2 V_{22}^1] \right\}$$

SAOB Case

$$R_{\rho}^H = \frac{p}{4\pi} \left\{ \frac{\partial U_{21}}{\partial z} - \frac{1}{\rho} \frac{\partial W_{21}}{\partial \rho} \right\}$$

$$R_{\phi}^H = \frac{p}{4\pi} \left\{ \frac{\partial U_{21}}{\partial z} - \frac{\partial^2 W_{21}}{\partial \rho^2} \right\}$$

$$R_z^H = -\frac{p}{4\pi} \left\{ \frac{\partial U_{21}}{\partial \rho} \right\}$$

$$R_{\phi}^v = -\frac{pk_1^2}{4\pi} \left\{ \frac{\partial V_{21}}{\partial \rho} \right\}$$

SBOB Case

$$\bar{H}_\rho^H = \frac{p}{4\pi} \left\{ \frac{\partial}{\partial z} [G_{11} + G_{12} + U_{11}] - \frac{1}{\rho} \frac{\partial W_{11}}{\partial \rho} \right\}$$

$$\bar{H}_\phi^H = \frac{p}{4\pi} \left\{ \frac{\partial}{\partial z} [G_{11} + G_{12} + U_{11}] - \frac{\partial^2 W_{11}}{\partial \rho^2} \right\}$$

$$\bar{H}_z^H = -\frac{p}{4\pi} \left\{ \frac{\partial}{\partial \rho} [G_{11} + G_{12} + U_{11}] \right\}$$

$$\bar{H}_\phi^V = -\frac{p}{4\pi} \left\{ \frac{\partial}{\partial \rho} [G_{11} - G_{12} + k_2^2 V_{11}] \right\}$$

SBOA Case

$$\bar{H}_\rho^H = \frac{p}{4\pi} \left\{ \frac{\partial U_{12}}{\partial z} - \frac{1}{\rho} \frac{\partial W_{12}}{\partial \rho} \right\}$$

$$\bar{H}_\phi^H = \frac{p}{4\pi} \left\{ \frac{\partial U_{12}}{\partial z} - \frac{\partial^2 W_{12}}{\partial \rho^2} \right\}$$

$$\bar{H}_z^H = -\frac{p}{4\pi} \left\{ \frac{\partial U_{12}}{\partial \rho} \right\}$$

$$\bar{H}_\phi^V = -\frac{p}{4\pi} \left\{ \frac{\partial V_{12}}{\partial r} \right\}$$

Electromagnetic Fields From Magnetic Sources

For point magnetic sources, the fields are computed from

$$\bar{E} = \frac{1}{\epsilon} [\bar{\nabla} \times \bar{F}]$$

$$\bar{H} = -\frac{i\omega}{k^2} [k^2 \bar{F} + \bar{\nabla}(\bar{\nabla} \cdot \bar{F})]$$

where

$$\bar{F} = -\frac{i\omega\mu\epsilon}{4\pi} \int \bar{G}_m \cdot \bar{M} dv$$

The Green's function $\hat{G}_m$ is known from equations (A.39) through (A.42). The calculations are carried out in exactly the same manner as those for the electric source cases. The results are given in the notation used in the electric source case.

Catalog of Fields From Magnetic Sources

$$E_{xx} = \frac{xy}{\rho^2} (E_{\rho}^H - E_{\phi}^H)$$

$$E_{xy} = \frac{y^2}{\rho^2} E_{\rho}^H + \frac{x^2}{\rho^2} E_{\phi}^H$$

$$E_{xz} = \frac{y}{\rho} E_z^H$$

$$E_{yx} = -\frac{x^2}{\rho^2} E_{\rho}^H - \frac{y^2}{\rho^2} E_{\phi}^H$$

$$E_{yy} = -\frac{xy}{\rho^2} (E_{\rho}^H - E_{\phi}^H)$$

$$E_{yz} = -\frac{x}{\rho} E_z^H$$

$$E_{zx} = -\frac{y}{\rho} E_{\phi}^V$$

$$E_{zy} = \frac{x}{\rho} E_{\phi}^V$$

$$E_{zz} = 0$$

SAOA Case

$$E_{\rho}^H = \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial}{\partial z} [G_{22} - G_{21} + k_1^2 V_{22}'] - \frac{1}{\rho} \frac{\partial W_{22}}{\partial \rho} \right\}$$

$$E_{\phi}^H = \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial}{\partial z} [G_{22} - G_{21} + k_1^2 V_{22}'] - \frac{\partial^2 W_{22}}{\partial \rho^2} \right\}$$

$$E_z^H = - \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial}{\partial z} [G_{22} - G_{21} + k_1^2 V_{22}'] \right\}$$

$$E_{\phi}^V = - \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial}{\partial \rho} [G_{22} + G_{21} + U_{22}'] \right\}$$

SAOB Case

$$E_{\rho}^H = \frac{i\omega\mu}{4\pi} \left\{ k_2^2 \frac{\partial V_{21}}{\partial z} - \frac{1}{\rho} \frac{\partial W_{21}}{\partial \rho} \right\}$$

$$E_{\phi}^H = \frac{i\omega\mu}{4\pi} \left\{ k_2^2 \frac{\partial V_{21}}{\partial z} - \frac{\partial^2 W_{21}}{\partial \rho^2} \right\}$$

$$E_z^H = \frac{i\omega\mu}{4\pi} \left\{ k_2^2 \frac{\partial V_{21}}{\partial \rho} \right\}$$

$$E_{\phi}^V = - \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial U_{21}}{\partial \rho} \right\}$$

SBOB Case

$$E_{\rho}^H = \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial}{\partial z} [G_{11} - G_{12} + k_2^2 V_{11}'] - \frac{1}{\rho} \frac{\partial W_{11}}{\partial \rho} \right\}$$

$$E_{\phi}^H = \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial}{\partial z} [G_{11} - G_{12} + k_2^2 V_{11}'] - \frac{\partial^2 W_{11}}{\partial \rho^2} \right\}$$

$$E_z^H = - \frac{i\omega\mu}{4\pi} \left\{ \frac{\partial}{\partial \rho} [G_{11} - G_{12} + k_2^2 V_{11}'] \right\}$$

$$E_{\phi}^V = - \frac{i\omega\mu_1}{4\pi} \left\{ \frac{\partial}{\partial \rho} [G_{11} + G_{12}^i + U_{11}^i] \right\}$$

SBOA Case

$$E_{\rho}^H = \frac{i\omega\mu_1}{4\pi} \left\{ k_1^2 \frac{\partial V_{12}}{\partial z} - \frac{1}{\rho} \frac{\partial W_{12}}{\partial \rho} \right\}$$

$$E_{\phi}^H = \frac{i\omega\mu_1}{4\pi} \left\{ k_1^2 \frac{\partial V_{12}}{\partial z} - \frac{\partial^2 W_{12}}{\partial \rho^2} \right\}$$

$$E_z^H = - \frac{i\omega\mu_1}{4\pi} \left\{ k_1^2 \frac{\partial V_{12}}{\partial \rho} \right\}$$

$$E_{\phi}^V = - \frac{i\omega\mu_1}{4\pi} \left\{ \frac{\partial U_{12}}{\partial \rho} \right\}$$

$$H_{xx} = \frac{x^2}{\rho^2} H_{\rho}^H - \frac{y^2}{\rho^2} H_{\phi}^H$$

$$H_{xy} = \frac{xy}{\rho^2} (H_{\rho}^H + H_{\phi}^H)$$

$$H_{xz} = \frac{x}{\rho} H_z^H$$

$$H_{yx} = \frac{xy}{\rho^2} (H_{\rho}^H + H_{\phi}^H)$$

$$H_{yy} = \frac{y^2}{\rho^2} H_{\rho}^H - \frac{x^2}{\rho^2} H_{\phi}^H$$

$$H_{yz} = \frac{y}{\rho} H_z^H$$

$$H_{zx} = \frac{x}{\rho} H_z^V$$

$$H_{zy} = \frac{y}{\rho} H_z^V$$

$$H_{zz} = H_z^V$$

SAOA Case

$$\begin{aligned} \bar{H}_\rho^H &= \frac{m}{4\pi} \left\{ \frac{\partial^2}{\partial \rho^2} [G_{22} - G_{21}' + (k_1^2 + k_2^2) v_{22}' - u_{22}'] \right. \\ &\quad \left. + k_2^2 [G_{22} - G_{21}' + k_1^2 v_{22}'] \right\} \end{aligned}$$

$$\begin{aligned} \bar{H}_\phi^H &= -\frac{m}{4\pi} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} [G_{22} - G_{21}' + (k_1^2 + k_2^2) v_{22}' - u_{22}'] \right. \\ &\quad \left. + k_2^2 [G_{22} - G_{21}' + k_1^2 v_{22}'] \right\} \end{aligned}$$

$$\bar{H}_z^H = \frac{m}{4\pi} \left\{ \frac{\partial^2}{\partial r \partial z} [G_{22} - G_{21}' - u_{22}'] \right\}$$

$$\bar{H}_\rho^V = \frac{m}{4\pi} \left\{ \frac{\partial^2}{\partial \rho \partial z} [G_{22} + G_{21}' + u_{22}'] \right\}$$

$$\bar{H}_z^V = \frac{m}{4\pi} \left\{ \left(\frac{\partial^2}{\partial z^2} + k_2^2 \right) [G_{22} + G_{21}' + u_{22}'] \right\}$$

SAOB Case

$$\bar{H}_\rho^H = \frac{m}{4\pi} \left\{ \frac{\partial^2}{\partial \rho^2} [(k_1^2 + k_2^2) v_{21} - u_{21}] + k_1^2 k_2^2 v_{21} \right\}$$

$$\bar{H}_\phi^H = -\frac{m}{4\pi} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} [(k_1^2 + k_2^2) v_{21} - u_{21}] + k_1^2 k_2^2 v_{21} \right\}$$

$$\bar{H}_z^H = \frac{m}{4\pi} \left\{ \frac{\partial^2}{\partial \rho \partial z} [k_2^2 v_{21} - u_{21}] - k_1^2 \frac{\partial^2 v_{21}}{\partial \rho \partial z^2} \right\}$$

$$\bar{H}_\rho^V = \frac{m}{4\pi} \left\{ \frac{\partial^2 u_{21}}{\partial \rho \partial z} \right\}$$

$$\bar{H}_z^V = \frac{m}{4\pi} \left\{ \left(\frac{\partial^2}{\partial z^2} + k_1^2 \right) u_{21} \right\}$$

SBOB Case

$$\bar{H}_\rho^H = \frac{m}{4\pi} \left\{ \frac{\partial^2}{\partial \rho^2} [G_{11} - G_{12} + (k_1^2 + k_2^2) v_{11} - u_{11}] \right. \\ \left. + k_1^2 [G_{11} - G_{12} + k_2^2 v_{11}] \right\}$$

$$\bar{H}_\phi^H = -\frac{m}{4\pi} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} [G_{11} - G_{12} + (k_1^2 + k_2^2) v_{11} - u_{11}] \right. \\ \left. + k_1^2 [G_{11} - G_{12} + k_2^2 v_{11}] \right\}$$

$$\bar{H}_z^H = \frac{m}{4\pi} \left\{ \frac{\partial^2}{\partial \rho \partial z} [G_{11} - G_{12} - u_{11}] \right\}$$

$$\bar{H}_\rho^V = \frac{m}{4\pi} \left\{ \frac{\partial^2}{\partial \rho \partial z} [G_{11} + G_{12} + u_{11}] \right\}$$

$$\bar{H}_z^V = \frac{m}{4\pi} \left\{ \left(\frac{\partial^2}{\partial z^2} + k_1^2 \right) [G_{11} + G_{12} + u_{11}] \right\}$$

SBOA Case

$$\bar{H}_\rho^H = \frac{m}{4\pi} \left\{ \frac{\partial^2}{\partial \rho^2} [(k_1^2 + k_2^2) v_{12} - u_{12}] + k_2^2 k_1^2 v_{12} \right\}$$

$$\bar{H}_\phi^H = -\frac{m}{4\pi} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} [(k_1^2 + k_2^2) v_{12} - u_{12}] + k_2^2 k_1^2 v_{12} \right\}$$

$$\bar{H}_z^H = \frac{m}{4\pi} \left\{ \frac{\partial^2}{\partial \rho \partial z} [k_1^2 v_{12} - u_{12}] - k_2^2 \frac{\partial^2 v_{12}}{\partial \rho \partial z^2} \right\}$$

$$\bar{H}_\rho^V = \frac{m}{4\pi} \left\{ \frac{\partial^2 u_{12}}{\partial \rho \partial z} \right\}$$

$$\bar{H}_z^V = \frac{m}{4\pi} \left\{ \left(\frac{\partial^2}{\partial z^2} + k_2^2 \right) u_{12} \right\}$$

APPENDIX C

Numerical Implementation of the Moments - Solution

In considering the scattering from a three dimensional dielectric body of arbitrary shape in the presence of a half-space, an electric field volume integral equation has been derived in the text from a potential integral solution to Maxwell's equations. The equation is repeated here as

$$\mathbf{E}(\vec{r}) = \mathbf{E}^{inc}(\vec{r}) - \frac{i\omega\mu}{4\pi k_b^2} (k_b^2 + \vec{\nabla} \cdot \vec{\nabla}) \int_{\text{Cavity}} \mathbf{G}_e(\vec{r}, \vec{r}') \cdot \mathbf{J}_c(\vec{r}') d\mathbf{v}' \quad (C-1)$$

The polarization current is written in terms of the total electric field $\mathbf{E}$ inside the dielectric body as

$$\mathbf{J}_c(\vec{r}') = i\omega(\epsilon_c - \epsilon_b)\mathbf{E}(\vec{r}') \quad (C-2)$$

in which the subscripts c and b refer to the electrical parameters of the dielectric (cavity) and the host medium (below the surface) respectively. $\mathbf{E}^{inc}$ is a known incident field.

A moment-method expression for this volume integral equation, using pulse-function basis and point matching, follows immediately from (C-1) as

$$\mathbf{E}^{inc}(\vec{r}_m) = \mathbf{E}(\vec{r}_m) + \frac{i\omega\mu}{4\pi k_b^2} (k_b^2 + \vec{\nabla} \cdot \vec{\nabla}) \sum_{n=1}^N \left\{ \int_{\Delta v_n} \mathbf{G}_e(\vec{r}_m, \vec{r}_n') P(\vec{r}_n') d\mathbf{v}_n' \right\} \cdot \vec{J}(n) \quad (C-3)$$

where

$$\vec{J}(\vec{r}') = \sum_n \vec{J}(n) P(\vec{r}_n') \quad (C-4)$$

with $\vec{J}(n)$ the vector expansion coefficient valid inside cell n where $P(\vec{r}_n')$

has unity support. Here $\vec{r}_m$ is the center of cell m and $\vec{r}'_n$ is the position vector which covers cell n in the integration. A matrix equation follows by writing (C-3) for $m = 1$ to N, where N is the number of cells. In what follows, it is convenient to view (C-3) in component form

$$E_v^{inc}(\vec{r}_m) = E_v(\vec{r}_m) + \frac{j\omega\mu}{4\pi k_b^2} \sum_{n=1}^N \left\{ k_b^2 \delta_{v\alpha} + \frac{\partial^2}{\partial x_v \partial x_\alpha} \right\} \int_{\Delta v_n} G_{\alpha\beta}(\vec{r}_m, \vec{r}'_n) P(\vec{r}'_n) dv'_n \Bigg\} j_\beta(n) \quad (C-5)$$

Here the Greek indices α , β , and v assume the values x, y, and z. A repeated index implies a summation over x, y, and z.

This potential integral formulation automatically focuses attention on integral over the Green's function. There are two distinct volume integrals depending on whether or not a derivative is required on the result. The volume integral in the first term must be carried out in any treatment of this equation. However, the presence of the derivatives operating on the second term suggests the possibility of taking a derivative inside the integral with view to transferring some of the burden of the differentiability to the pulse expansion functions $P(\vec{r}'_n)$. Here the two approaches will be developed in detail. The evaluation of these two approaches is the key to the development of an efficient numerical implementation of (C-5).

Volume Integral Approach

If the differentiations called for in (C-5) are performed after the volume integrations over the Green's function, $P(\vec{r}'_n)$ can be set equal to unity at the outset. Then one has as the starting point

$$E_v^{inc}(\vec{r}_m) = E_v(\vec{r}_m) + \frac{j\omega\mu}{4\pi k_b^2} \sum_{n=1}^N \left\{ k_b^2 v_{v\beta}(m,n) + \frac{\partial^2}{\partial x_v \partial x_\alpha} v_{\alpha\beta}(m,n) \right\} j_\beta(n) \quad (C-6)$$

where it is only necessary to compute the single volume integral of the form

$$V_{\nu\beta}(m,n) = \int_{\Delta v_n} G_{\nu\beta}(\vec{r}_m, \vec{r}_n') dv_n' \quad (C-7)$$

with the integration extending over cell n . The numerical problem of computing this integral is considerably ameliorated by approximating the cubical cells by spheres having an equal volume following Richmond [9], Livesey and Chen [2], Hohmann [3], and Kim [17]. In addition, as a first approximation, the interface effects in the dyadic Green's function are neglected. This reduces the tensor volume integral to a diagonal one which requires evaluation of only the scalar term V given by

$$V(m,n) = \int_{\Delta v_n} G(\vec{r}_m, \vec{r}_n') dv_n' \quad (C-8)$$

where G is the usual free space Green's function of the form

$$G = \frac{e^{-ik_b |\vec{r}_m - \vec{r}_n'|}}{|\vec{r}_m - \vec{r}_n'|} \quad (C-9)$$

which is conveniently rewritten in terms of a spherical volume integral over a spherical Hankel function, $h_0^{(2)}$, as

$$V(m,n) = -ik_b \phi(\vec{r},n) \quad (C-10)$$

where

$$\phi(\vec{r},n) = \int_{\Delta v_n} h_0^{(2)}(k_b |\vec{r}_m - \vec{r}_n'|) r'^2 \sin\theta' d\theta' d\phi' dr' \quad (C-11)$$

In this notation (C-6) becomes for $m = 1$ to N

$$E_v(m) + \frac{\omega\mu}{4\pi k_b} \sum_{n=1}^N \left\{ k_b^2 \phi(m,n) + \frac{\partial^2}{\partial x_v \partial x_b} \phi(m,n) \right\} j_B(n) = E_v^{inc}(m) \quad (C-12)$$

where m stands for the physical point $\vec{r}_m$. As before

$$j_B(n) = i\omega(\epsilon_c - \epsilon_b)E_B(n) \quad (C-13)$$

The detailed evaluation of $\phi(mn)$ and its spatial derivatives is all that remains in the matrix fill. This procedure is the extension of Richmond's [9] two-dimensional treatment to three dimensions.

Computation of the Integral $\phi(m,n)$

The computation of $\phi(m,n)$ given by (C-11) as

$$\phi_m(r,n) = \int_{\Delta V_n} h_0^{(2)}(k_b |\vec{r}_m - \vec{r}'|) r'^2 \sin\theta' d\theta' d\phi' dr' \quad (C-14)$$

proceeds in a straight forward manner with help from the addition theorem for the spherical Hankel functions which is found in Harrington [20] in the form

$$\begin{aligned} \sum_{n=0}^N (2n+1) h_n^{(2)}(kr') j_n(kr) P_n(\cos\xi) & \quad r' > r \\ h_0^{(2)}(k|\vec{r}-\vec{r}'|) = & \\ \sum_{n=0}^N (2n+1) j_n(kr') h_n^{(2)}(kr) P_n(\cos\xi) & \quad r' < r \end{aligned} \quad (C-15)$$

with

$$P_n(\cos\xi) = \sum_{m=0}^N \epsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi - \phi')$$

The calculation differs according to whether the point of observation (the unprimed variable representing $\vec{r}_m$) is interior or exterior to the spherical volume of integration (the primed variable $\vec{r}'$ representing $\vec{r}'_n$).

For the interior field point case, the integration is carried out by dividing the region into two pieces which are separated by the radius of the point of observation $\vec{r}$ as indicated in the figure.

Figure C-1

In the notation

$$\phi(\vec{r}, n) = \phi_I(\vec{r}) + \phi_{II}(r)$$

After substituting (C-15) into (C-14), one obtains

$$\phi_{II}(r) = \sum_{n=0}^{\infty} \sum_{m=0}^n (2n+1) j_n(kr) \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) \iint_{r'=r}^{r'=a} r'^2 dr' \sin\theta' d\theta' d\phi' h_n(kr')$$

$$P_n^m(\cos\theta') \cos m(\phi - \phi')$$

The ϕ integration is carried out first using

$$\int_0^{2\pi} d\phi' \cos m(\phi - \phi') = 2\pi \delta_{m0}$$

then the summation over m in turn allows use of

$$\int_0^\pi \Theta P_n(\cos\theta) \sin\theta d\theta = 2\delta_{n0}$$

to obtain the last remaining integral which is

$$\phi_{II}(\vec{r}) = 4\pi j_0(kr) \int_r^a dr' r'^2 h_0^{(2)}(kr')$$

but it is known that

$$r^2 h_0 = \frac{1}{k} \frac{d}{dr} [r^2 h_1]$$

so immediately, with the subscript on the Hankel functions suppressed in the following,

$$\phi_{II}(\vec{r}) = \frac{4\pi}{k} j_0(kr) [a^2 h_1(ka) - r^2 h_1(kr)]$$

Using the other form of the addition theorem, the result for region I can be written by inspection

$$\phi_I(\vec{r}) = \frac{4\pi}{k} h_0(kr) [r^2 j_1(kr) - 0]$$

and the answer is

$$\phi(\vec{r}) = \frac{4\pi}{k} \left\{ j_0(kr) a^2 h_1(ka) - j_0(kr) r^2 h_1(kr) + h_0(kr) r^2 j_1(kr) \right\} \quad (C-16)$$

where $\vec{r}$ is any interior field point.

For the exterior field point case, the answer can be written immediately

as

$$\phi(\vec{r}) = \frac{4\pi}{k} h_0(kr) a^2 j_1(ka) \quad (C-17)$$

where $\vec{r}$ is any exterior field point.

For both cases, $\Phi(\vec{r})$ is only a function of r so the derivatives with respect to the cartesian coordinates can be written in the form

$$\frac{\partial^2 \Phi}{\partial x^2} = [\sin^2 \theta \cos^2 \phi] \frac{d^2 \Phi}{dr^2} + [\cos^2 \theta \cos^2 \phi + \sin^2 \theta] \frac{1}{r} \frac{d\Phi}{dr}$$

$$\frac{\partial^2 \Phi}{\partial y^2} = [\sin^2 \theta \sin^2 \phi] \frac{d^2 \Phi}{dr^2} + [\cos^2 \theta \sin^2 \phi + \sin^2 \theta] \frac{1}{r} \frac{d\Phi}{dr}$$

$$\frac{\partial^2 \Phi}{\partial z^2} = [\cos^2 \theta] \frac{d^2 \Phi}{dr^2} + [\sin^2 \theta] \frac{1}{r} \frac{d\Phi}{dr}$$

(C-18)

$$\frac{\partial^2 \Phi}{\partial x \partial y} = \sin^2 \theta \sin \phi \cos \phi \left[\frac{d^2 \Phi}{dr^2} - \frac{1}{r} \frac{d\Phi}{dr} \right]$$

$$\frac{\partial^2 \Phi}{\partial x \partial z} = \sin \theta \cos \theta \cos \phi \left[\frac{d^2 \Phi}{dr^2} - \frac{1}{r} \frac{d\Phi}{dr} \right]$$

$$\frac{\partial^2 \Phi}{\partial y \partial z} = \sin \theta \cos \theta \sin \phi \left[\frac{d^2 \Phi}{dr^2} - \frac{1}{r} \frac{d\Phi}{dr} \right]$$

In order to employ the relations given by (C-18), we need to write useful equations for the radial derivatives of Φ . When $m \neq n$, use of

$$\frac{1}{r} \frac{dh_0}{dr} = -k^2 \left[\frac{h_1}{kr} \right], \quad \frac{d^2 h_0}{dr^2} = k^2 \left[\frac{2h_1}{kr} - h_0 \right]$$

leads to the expressions

$$\frac{d^2 \Phi}{dr^2} = -\frac{4\pi(ka)^2}{k} j_1(ka) \left\{ h_0 - \frac{2h_1}{kr} \right\} = \left(\frac{4\pi a^2 j_1(ka)}{k} \right) \left\{ \frac{d^2 h_0}{dr^2} \right\}$$

(C-19)

$$\frac{1}{r} \frac{d\Phi}{dr} = -\frac{4\pi(ka)^2}{k} j_1(ka) \left\{ \frac{h_1}{kr} \right\} = \left(\frac{4\pi a^2 j_1(ka)}{k} \right) \left\{ \frac{1}{r} \frac{dh_0}{dr} \right\}$$

while we recall that

$$k^2 \phi = \frac{4\pi(ka)^2}{k} j_1(ka) \{h_0\} = k^2 \left(\frac{4\pi a^2 j_1(ka)}{k} \right) \{h_0(ka)\}$$

in terms of the spherical Bessel and Hankel function of order zero and one.

For the special case $m = n$, the $r \rightarrow 0$ limit can be carried out to find

$$\frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} = 0 \quad , \quad \alpha \neq \beta$$

and

$$\frac{d^2 \phi}{dr^2} = \frac{1}{r} \frac{d\phi}{dr}$$

so the remaining derivatives assume the simple form

$$\frac{\partial^2 \phi}{\partial x_\alpha \partial x_\alpha} = \frac{d^2 \phi}{dr^2}$$

Using equation (C-16) and the small argument expressions for the Bessel and Hankel functions, one finds that

$$\lim_{r \rightarrow 0} \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} = \delta_{\alpha\beta} \frac{4\pi}{k} \left\{ \frac{d^2 j_0}{dr^2} \right\} a^2 h_1(ka)$$

and as a final expression

$$\lim_{r \rightarrow 0} \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} = -\delta_{\alpha\beta} \frac{4\pi k}{3} a^2 h_1(ka) \quad (C-20)$$

while immediately

$$\lim_{r \rightarrow 0} \phi = \frac{4\pi}{k} \left[a^2 h_1(ka) - \frac{i}{k^2} \right]$$

Explicit Matrix Equation

With the information obtained in the preceding section, (C-12) and (C-13) can be combined with the results (C-17) through (C-20) to write the volume integral matrix equation in the explicit form

$$\sum_{n=1}^N \left\{ \left[\delta_{mn} + \delta(\phi_{xx}(m,n) + k_b^2 \phi(m,n)) \right] E_x(n) + \delta[\phi_{xy}(m,n)] E_y(n) + \delta[\phi_{xz}(m,n)] E_z(n) \right\} = E_x^{inc}(m)$$

$$\sum_{n=1}^N \left\{ \delta[\phi_{yx}(m,n)] E_x(n) + \left[\delta_{mn} + \delta(\phi_{yy}(m,n) + k_b^2 \phi(m,n)) \right] E_y(n) + \delta[\phi_{yz}(m,n)] E_z(n) \right\} = E_y^{inc}(m)$$

$$\sum_{n=1}^N \left\{ \delta[\phi_{zx}(m,n)] E_x(n) + \delta[\phi_{zy}(m,n)] E_y(n) + \left[\delta_{mn} + \delta(\phi_{zz}(m,n) + k_b^2 \phi(m,n)) \right] E_z(n) \right\} = E_z^{inc}(m)$$

for $m = 1$ to N . As notation

$$\delta = \frac{i(k_c^2 - k_b^2)}{4\pi k_b}$$

$$\phi_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

with x_j standing for x , y , or z .

We see that when the cubical volume integral which appears in the potential integral formulation is approximated by a sphere, the integrals can be carried out in closed form by elementary methods. In addition, the

derivatives of these integrals are available without any concern for the singular nature of the integrand. The entire matrix fill can be accomplished with a formula without resorting to any lengthy numerical computations. The volume integral itself is not sensitive to the difference between the cubical center and the spherical cell. However, the spatial derivatives of the integral are a function of the detailed cell shape for reasons which will be clarified in the next section. In view of this sensitivity to cell shape, an alternate approach will be developed in the next section.

Surface Integral Approach

As a starting point, we write (C-6) again

$$E_v^{inc}(\vec{r}_m) = E_v(\vec{r}_m) + \frac{j\omega\mu}{4\pi k_D^2} \sum_{n=1}^N \left\{ k_D^2 V_{v\beta}(\vec{r}_m, n) + \frac{\partial^2}{\partial x_v \partial x_\alpha} V_{\alpha\beta}(\vec{r}_m, n) \right\} j_\beta(n) \quad (C-22)$$

and concentrate on the term D involving the derivatives where with help from (C-5)

$$D = \frac{\partial^2}{\partial x_v \partial x_\alpha} \int_{\Delta v_n} G_{\alpha\beta}(\vec{r}_m, \vec{r}_n') P(\vec{r}_n') dv_n' \quad (C-23)$$

This expression can be rewritten by noticing that the derivative with respect to x_α can be taken inside the integral and changed to one with respect to x_α' using the symmetry of the Green's to obtain

$$D = - \frac{\partial}{\partial x_v} \int_{\Delta v_n} \frac{\partial}{\partial x_\alpha'} \left[G_{\alpha\beta}(\vec{r}_m, \vec{r}_n') \right] P(\vec{r}_n') dv_n'$$

Since no derivatives with respect to $P(\vec{r}'_n)$ are called for, it can be set to unity and Gauss' law, namely

$$\int_{\Delta v_n} \frac{\partial G_{\alpha\beta}}{\partial x'_\alpha} dv'_n = \int_{\Delta v_n} n_\alpha G_{\alpha\beta} dS'_n$$

employed to find

$$D = \frac{\partial}{\partial x_v} \left[\int_{S^-_\alpha} G_{\alpha\beta}(\vec{r}_m, \vec{r}'_n) dS'_n - \int_{S^+_\alpha} G_{\alpha\beta}(\vec{r}_m, \vec{r}'_n) dS'_n \right]$$

where the surfaces $S^\pm_\alpha$ lie in a direction whose normal is in the α^{th} direction. The - and + superscripts refer to the surfaces with the smallest and the largest values of x_α respectively. For this purpose, the cubical cells are aligned with their faces parallel to the cartesian coordinate system. As usual, the repeated Greek index α is summed over x, y, and z. As notation, let

$$\sum_\beta (m,n) = \int_{S^-_\alpha} G_{\alpha\beta}(\vec{r}_m, \vec{r}'_n) dS'_n - \int_{S^+_\alpha} G_{\alpha\beta}(\vec{r}_m, \vec{r}'_n) dS'_n$$

so that finally

$$D = \frac{\partial}{\partial x_v} \sum_\beta (m,n) \quad (C-24)$$

with the use of (C-13), (C-22) may be rewritten as

$$E_v^{\text{inc}}(\vec{r}_m) = E_v(\vec{r}_m) - \frac{(k_c^2 - k_b^2)}{4\pi k_b^2} \sum_{n=1}^N \left\{ k_b^2 v_{v\beta}(m,n) + \frac{\partial}{\partial x_v} \sum_\beta (m,n) \right\} E_\beta(n) \quad (C-25)$$

In accordance with the numerical approximation introduced in the volume integral approach, the effects of the interface are neglected. The surface integral term becomes

$$\sum_{\beta} (m,n) = \int_{S_{\beta}^{-}} G(\vec{r}_m, \vec{r}_n') dS_n' - \int_{S_{\beta}^{+}} G(\vec{r}_m, \vec{r}_n') dS_n'$$

and the volume integral reduces to

$$V(m,n) = \int_{\Delta V_n} G(\vec{r}_m, \vec{r}_n') dv'$$

and equation (C-25) can be rewritten in the explicit form

$$\begin{aligned} \sum_n^N \left\{ \left[\delta_{mn} + \alpha \left(\left(\frac{\partial}{\partial x^m} \sum_x (m,n) + k_b^2 V(m,n) \right) \right) \right] E_x(n) + \left[\alpha \frac{\partial}{\partial x^m} \sum_y (m,n) E_y(n) \right] \right. \\ \left. + \left[\alpha \frac{\partial}{\partial x^m} \sum_z (m,n) \right] E_z(n) \right\} = E_x^{inc}(m) \\ \sum_n^N \left\{ \left[\alpha \frac{\partial}{\partial y^m} \sum_x (m,n) \right] E_x(n) + \left[\delta_{mn} + \alpha \left(\left(\frac{\partial}{\partial y^m} \sum_y (m,n) + k_b^2 V(m,n) \right) \right) \right] E_y(n) \right. \\ \left. + \alpha \left[\frac{\partial}{\partial y^m} \sum_z (m,n) \right] E_z(n) \right\} = E_y^{inc}(m) \quad (C-26) \\ \sum_n^N \left\{ \left[\alpha \frac{\partial}{\partial z^m} \sum_x (m,n) E_x(n) \right] + \left[\alpha \frac{\partial}{\partial z^m} \sum_y (m,n) E_y(n) \right] \right. \\ \left. + \left[\delta_{mn} + \alpha \left(\left(\frac{\partial}{\partial z^m} \sum_z (m,n) + k_b^2 V(m,n) \right) \right) \right] E_z(n) \right\} = E_z^{inc}(m) \end{aligned}$$

where the remaining undefined symbol α is given by

$$\alpha = - \frac{(k_c^2 - k_b^2)}{4\pi k_b^2} \quad (C-27)$$

Conclusion

In this appendix, two forms of the moment-method matrix equation have been developed in a form suitable for numerical implementation. Both equations are founded on a pulse expansion of the polarization currents in the dielectric body and are known to converge to the correct physical answer in the limit of sufficiently small cell size as shown by Hagmann [10]. The two treatments differ in the treatment of the cell shape in the term which calls for spatial derivatives of the potential integral. The term involving the spatial derivatives is the scalar potential term. This is easily seen by taking a derivative inside the integral to operate on the current $P(\vec{r}'_n)$ and then introducing the continuity equation to rewrite the integral in terms of the surface charge density. Consequently, equations (C-21) and (C-26) will provide different answers when the termination of the uniform current by charges on a sphere looks different from those on a cube. A numerical comparison of the matrix elements in (C-21) and (C-26) shows that the surface integral form is required in considering the interaction between adjacent cells. The self-terms are equally well calculated by either procedure.

In the numerical implementation of the moments-method equation, (C-27) is used to calculate the nearest neighbor terms along the box diagonals of the matrix. Equation (C-21) was employed for all the other matrix elements because it is more efficient in the use of machine time. This rule was verified by checking the results using known test cases.

APPENDIX D

Units of Measure

The notation used in this report follows the conventions of SI (Système International d'Unités), which for our purposes, are the same as rationalized MKS units. The following quantities were mentioned in the text:

σ	Conductivity:	Siemens/meter = mho/meter
E	Electric Field Intensity:	volt/meter
J	Electric Current:	ampere/meter ²
B	Magnetic Induction:	tesla = 1 weber/meter ²
m	Magnetic Dipole Moment:	ampere-meter ²
ρ	Electric Dipole Moment:	ampere-meter