MODEL OF CURRENT ENHANCEMENT AT HIGH PRESSURE

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ABSTRACT

A model is proposed to account for the phenomenon of net current enhancement at high pressures recently observed on the Experimental Test Accelerator. The proposed mechanism involves energetic secondary electrons (delta rays) which are pushed forward by the self-magnetic field of the electron beam. For high current beams, the forward delta ray current can build up to a significant fraction of the beam current. Analytic calculations of the steady-state solution as well as the rate of buildup of the delta ray current are presented in this paper. In addition, numerical results from a nonlocal Boltzmann code, NUTS, are presented. The analytic and numerical results have many features which are in qualitative agreement with the experiments, but quantitative discrepancies still exist.

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I. Introduction

A new phenomenon has been discovered recently¹ by the ETA experimental team: as a high current electron beam propagates through a gaseous medium at high pressures, the net current peaks at values higher than the beam current. The following features characterize this phenomenon of current enhancement at high pressures:

(1) Current enhancement has been observed in several gases (nitrogen, air, neon). The net current is often enhanced by a significant fraction of the beam current, although the enhancement has never been observed to exceed a factor of two.

(2) Net current enhancement effects increase with increasing current.

(3) Net current enhancement effects increase with decreasing beam size.

(4) Associated with the net current enhancement, a broad radial profile of forward current was inferred from magnetic field measurements with B_{θ} probes.

(5) Net current enhancement is generally strongest at around 50 to 100 torr of N_2 .

Current multiplication at low pressures (< 1 torr) is a well-known phenomenon, which was studied in detail with the ASTRON accelerator.² There is little question that the mechanism for low pressure current multiplication is the two-stream instability.³ A characteristic signature of the two-stream instability is a high level of microwave activities. Recent microwave measurements⁴ have shown that in the high pressure regimes where current enhancement is observed, there is no significant microwave signal. Hence, two-stream instability is an unlikely candidate for the newly observed current enhancement phenomena.

In this paper, we propose a mechanism for net current enhancement at high pressures. The mechanism involves energetic secondaries (delta rays) which are pushed forward by the magnetic field of the beam. In this model, the most important delta rays are comparatively low in energy (tens to hundreds of keV, depending somewhat on the pressure). They are more abundant than the most energetic deltas, since the production cross section for deltas is inversely proportional to the square of the delta energy. The momentum transfer rates of these delta rays are low, and thus, they are able to respond to fields in a relatively frictionless manner.

These electrons are produced predominantly in the sideways direction (transverse to beam), constituting an outward radial current. The strong $\vec{j} \times \vec{B}$ force then converts the radial current to a forward axial current. In the strong magnetic field, the gyroradii of these electrons are relatively small. The pattern of current flow must therefore be determined by a self-consistent treatment of the entire ensemble as we have done in the model calculations presented in the following sections.

The results of this delta ray model are in qualitative agreement with the experimental observations. An alternative theory of delta rays could be proposed which does not rely on the magnetic field, but is simply based on the fact that the most energetic deltas have a forward momentum component at birth. Such a theory would not be able to account for the observed current and beam size dependence. Effects of the forward momentum component at birth are included in the present treatment.

The qualitative picture associated with our model is that the delta ray forward current is zero at beam head. As one moves into the beam body, the forward delta current increases. If the beam pulse is sufficiently long, the

forward delta ray current eventually reaches a steady state. (See Figure 1) The steady state occurs when the forward current generated by $\vec{J}_r \times \vec{B}_0$ is balanced by the finite (albeit small) frictional drag. In Sections II and III, we present a steady-state derivation of the delta ray current at saturation together with the asymptotic radial profile of the forward delta ray current distribution. Section II presents an exact Boltzmann solution assuming no energy loss. This solution provides an upper bound on the enhanced current. Section III gives an approximate solution for the more general case with finite energy loss. In Section IV, a linearly rising solution, which describes the early phase of the delta-ray current buildup, is derived. In Section V, we present a benchmark calculation from the code NUTS to compare with the analytic theories. The key results derived in the text are summarized in Section VI.

II. Steady-State Solution: No Energy Loss

In a previous report,⁵ we derived a Boltzmann theory for the beam-generated plasma channel. The theory treats the low-energy electrons and high-energy delta rays on the same footing. In our analytic treatment of delta rays, we will use as our starting point the Boltzmann equations derived in Ref. (5), but will make approximations appropriate to the high energy regime. In the two-term treatment of the Boltzmann equations, information about the delta rays is carried by three distribution functions F_0 , G_r , and G_z , which are functions of three independent variables: the radial coordinate r , the time in beam frame $x = ct - z$, and $y = \ln(\epsilon + \epsilon_s)$ where ϵ is the electron energy and ϵ_s is an externally specified scale parameter. F_0 is the density distribution in y space, and is related to the macroscopic plasma density by

$$n = \int dy F_0 \quad (1)$$

G_r and G_z carry information about the radial and axial flow of the secondary electrons, and are related to the currents by

$$J_r = ec \int dy G_r \quad (2)$$

$$J_z = ec \int dy G_z \quad (3)$$

To calculate the asymptotic delta ray current profile, the following approximations are made:

(1) Steady state

$$\frac{\partial F_0}{\partial x} = \frac{\partial G_z}{\partial x} = \frac{\partial G_r}{\partial x} = 0 \quad (4)$$

(2) No recombination effects (Recombination is a low energy phenomenon).

(3) Delta rays are generated by direct beam ionization only. Cascading effects as a source of delta rays are negligible.

(4) Negligible electric fields.

(5) Negligible plasma current from the slow electrons. Approximations (4) and (5) are reasonable in regimes well into the beam body, consistent with our steady-state assumption.

Under these approximations, the Boltzmann Equations (39) to (41) of Ref (5) reduce to a simple set of equations. The scalar equation is

$$\frac{1}{r} \frac{\partial}{\partial r} r G_r = \left(\frac{J_b}{ec} \right) n_g S^{dir} + \frac{\partial}{\partial y} \left(F_0 n_g S^{loss} \right) \quad (5)$$

where J_b is the beam current density, and n_g is the gas density. The two sets of atomic data input correspond respectively to electron production and energy loss processes with

$$S^{\text{dir}}(\epsilon) \equiv (\epsilon + \epsilon_s) \frac{d\sigma}{d\epsilon} , \quad (6)$$

where $\frac{d\sigma}{d\epsilon}$ is the differential ionization cross section by beam electron impact, and

$$S^{\text{loss}}(\epsilon) \equiv \frac{1}{(\epsilon + \epsilon_s)} \left(\frac{v}{c} \right) \sum_i \epsilon_i \sigma_i(\epsilon) . \quad (7)$$

In Eq. (7) the sum is performed over all inelastic processes with cross section σ_i and excitation energy ϵ_i . In the asymptotic state, the equilibrium in the density distribution of delta rays is maintained by a continuous source term from beam ionization balanced by two loss terms: an outflowing radial current in r space, and a downward flow in energy (or y) space.

The axial equation in the asymptotic limit becomes

$$\left(\frac{J_b}{ec} \right) n_g S_z^{\text{dir}} + \frac{eB_c}{mc^2} G_r = n_g S^m G_z . \quad (8)$$

The equilibrium in the axial component of momentum is reached by a balance of two source terms against the frictional drag. The first source term corresponds to the forward delta current at birth, with

$$S_z^{\text{dir}}(\epsilon) \equiv S^{\text{dir}}(\epsilon) \frac{v_{zb}}{c} \quad (9)$$

where v_{zb} is the forward component of the velocity for a secondary electron produced with energy ϵ . The second source term comes from a $J_r \times B_\theta$ drift. The drag term consists of an atomic data input

$$S^m(\epsilon) \equiv \frac{v}{c} \sigma_m(\epsilon) , \quad (10)$$

where $\sigma_m(\epsilon)$ is the momentum transfer cross section.

Finally, the equilibrium in the radial component of the momentum equation is described by

$$-\left(\frac{2\epsilon}{3mc^2}\right) \frac{\partial F_0}{\partial r} = \frac{eB_\theta}{mc^2} G_z + n_g S^m G_r, \quad (11)$$

where the outward pressure [LHS of Eq. (11)] is balanced by an inward magnetic force (from $\mathbf{J}_z \times \mathbf{B}_\theta$), and by frictional drag.

The four sets of atomic data input S^{dir} , S^{loss} , S_z^{dir} , and S^m for air are displayed in Figure 2. For convenience of analysis, we have also made power-law fits to the high energy end of the data. These are given by

$$S^{\text{dir}}(\epsilon) = \frac{3.26 \times 10^{-18}}{\epsilon}, \quad (12)$$

$$S_z^{\text{dir}}(\epsilon) = 6.3 \times 10^{-24}, \quad (13)$$

$$S^m(\epsilon) = \frac{3.5 \times 10^{-14}}{\epsilon^{3/2}}, \quad (14)$$

$$S^{\text{loss}}(\epsilon) = \frac{7 \times 10^{-15}}{\epsilon^{3/2}}, \quad (15)$$

where all the S functions have the dimensions of cm^2 , and ϵ is the energy in eV.

We first present the exact solution to Eqs. (5) to (11) for the forward delta ray current in the limit of $S^{\text{loss}} \rightarrow 0$. An approximate correction for finite S^{loss} will be calculated in Section III. The solution for the forward delta ray current derived in this section represents a rigorous upper bound to Eqs. (5) to (11).

In the limit of $S^{\text{loss}} \rightarrow 0$, Eq. (5) can be easily integrated to obtain

$$G_r = \frac{I_b(r)}{2\pi r e c} n_g S^{\text{dir}}, \quad (16)$$

where the enclosed current is defined as

$$I_b(r) \equiv 2\pi \int_0^r dr' r' J_b(r') . \quad (17)$$

Equation (8) gives the axial current distribution function, G_z , in terms of G_r . Combining it with Eq. (16), we obtain

$$G_z = \left(\frac{eB_\theta}{mc^2} \right) \left(\frac{S^{dir}}{S^m} \right) \frac{I_b(r)}{2\pi r ec} + \left(\frac{J_b}{ec} \right) \left(\frac{S_z^{dir}}{S^m} \right) . \quad (18)$$

The forward delta ray current is obtained by integrating G_z over y ;

$$J_\delta = ec \int_{y_c}^{y_{max}} G_z dy , \quad (19)$$

where the lower limit y_c represents the low energy cutoff for delta rays.

The equation for the asymptotic delta ray forward current is then given by

$$J_\delta = \left(\frac{eB_\theta}{mc^2} \right) \frac{I_b(r)}{2\pi r} D_0 + J_b D_1 , \quad (20)$$

where the two dimensionless constants are

$$D_0 = \int_{y_c}^{y_{max}} dy \left(\frac{S^{dir}}{S^m} \right) , \quad (21)$$

$$D_1 = \int_{y_c}^{y_{max}} dy \left(\frac{S_z^{dir}}{S^m} \right) . \quad (22)$$

In Fig. (3a) and (3b), we plot D_0 and D_1 as a function of y_c , with y_{max} fixed at $y_{max} = 12.9$. y_c is determined to be the value above which

S_{loss} is negligible. Within the context of the present section, we have no rigorous procedure for calculating y_c . Our best estimate of the dimensionless constants are $D_0 = .08$ and $D_1 = .019$, based on a value of $y_c \sim 10$.

In Eq. (20), the magnetic field B_θ arises from the beam current as well as from the delta ray current. The low-energy plasma current is assumed to be negligible. Then, we have

$$\frac{eB_\theta}{mc^2} = \frac{2}{I_A r} (I_\delta(r) + I_b(r)) , \quad (23)$$

where, for the purpose of this paper, we define I_A as

$$I_A = \frac{mc^3}{e} = 17 \text{ kA} \quad (\text{with no factor of } B_Y) \quad (24)$$

and the delta current enclosed is defined similarly as

$$I_\delta(r) = 2\pi \int_0^r dr' r' J_\delta(r') . \quad (25)$$

Equation (20) can now be viewed as a condition which determines the delta current and magnetic field profile self-consistently. It is readily solved by rewriting the equation as a first order differential equation in $I_\delta(r)$;

$$\frac{d}{dr} I_\delta(r) = \frac{2D_0}{I_A} \left[\frac{I_b^2(r)}{r} + \frac{I_b(r) I_\delta(r)}{r} \right] + 2\pi r D_1 J_b(r) . \quad (26)$$

The solution to Eq. (26) for the Bennett case is derived in Appendix 1. In terms of the scaled radial coordinate,

$$u \equiv \frac{r^2}{a^2} , \quad (27)$$

where a is the Bennett radial scale length, the delta current profile is given by

$$\frac{I_o(u)}{I_{b0}} = \left[\frac{(1+u)^{\kappa}}{(1+\kappa)} - 1 + \frac{\kappa}{(1+\kappa)(1+u)} \right] + \frac{D_1}{(1-\kappa)} \left[(1+u)^{\kappa} - \frac{1}{(1+u)} \right] \quad (28)$$

In Eq. (28) I_{b0} is the total beam current, and the dimensionless parameter, κ , is given by

$$\kappa \equiv D_0 \frac{I_{b0}}{I_A} \quad (29)$$

In Figure 4, we plot the net current profile as a function of r/a for several values of κ . We note several properties of the solution:

(1) For parameters consistent with the ETA experiments, the forward delta current is a significant fraction of the beam current.

(2) The delta current fraction is a monotonically increasing function of κ , and therefore of the beam current I_{b0} .

(3) The delta current is an increasing function of the radial coordinate, implying a broad distribution of forward current.

(4) For a given wall size r_w , the smaller the beam size a , the larger the scaled radial variable,

$$u_w = \frac{r_w^2}{a^2} \quad (30)$$

and hence the larger the delta current fraction measured at the wall.

The term proportional to D_1 represents the contribution to the current from the forward component of the deltas at birth. In the special limit of $\kappa = 0$ (no magnetic field effects), the solution reduces to

$$\frac{I_{\delta}(u)}{I_{b0}} = D_1 \frac{u}{1+u} = D_1 \frac{I_b(u)}{I_{b0}} \quad (31)$$

The delta current is then simply a small fraction of the beam current. The dependence of the forward current on beam current, beam size, and the broad radial profile are all lost. In the general case of a finite κ , the D_1 term represents a small contribution to the forward current.

An approximate solution that is useful for predicting the net current enhancement measured at the wall can be obtained by considering the limits

$$(1) \quad u_{\omega} \gg 1$$

$$(2) \quad \kappa \ln u_{\omega} \ll 1$$

The asymptotic delta ray current fraction is then given by

$$I_{\delta 0} \equiv I_{\delta}(r_{\omega}) = D_0 \frac{I_{b0}^2}{I_A} \left[\ln \left(\frac{r_{\omega}^2}{a^2} \right) - 1 \right] + D_1 I_{b0} \quad (32)$$

One may question if the broad radial profile of delta currents is a feature peculiar to the Bennett, which has a characteristic long tail. In Appendix 1, we obtain the solution for a truncated Bennett. Table 1 gives some illustrative numerical examples to indicate the dependence of the delta ray current on the beam profile. The results of Appendix 1 show that: (1) The broad radial profile of delta currents is a general feature independent of details of beam current profile. (2) For a fixed total current, maximal current enhancement occurs when the beam is "tightly packed" (small cutoff-radius).

The calculations up to this point have been performed under the assumption that there is no low energy plasma current. Effects of non-vanishing plasma current can be estimated by replacing Eq. (23) with the more general expression

$$\frac{eB_\theta}{mc^2} = \frac{2 I_n(r)}{I_A r} \quad (23')$$

An approximate solution is readily obtained if one assumes that the net current has the same radial profile as the beam profile. It is given by

$$I_{\theta 0} = D_0 \frac{I_{n0} I_{b0}}{I_A} \left[\ln \left(\frac{r_w^2}{a^2} \right) - 1 \right] + D_1 I_{b0} \quad (32')$$

where I_{n0} is the net current measured at the wall.

In addition to the forward delta ray current, it is also of interest to ask what the delta ray density is. Since we already have explicit forms for the current distribution functions G_r and G_z [Eqs. (16) and (18)], it is straightforward to integrate the radial momentum equation [Eq. (11)] to obtain the electron density distribution F_0 . We calculate F_0 only to leading order. The simplifying assumptions are (1) the net current giving rise to B_θ has the same Bennett radial profile as the beam; (2) the forward component of delta ray momentum at birth is ignored ($S_z^{\text{dir}} \rightarrow 0$). The result is

$$F_0 = \left[\frac{S^{\text{dir}}}{(2\epsilon/mc^2)S^m} \right] \frac{I_{n0}^2}{I_A^2} \frac{I_{b0}}{\pi a^2 \epsilon c} \frac{1+2u}{2(1+u)^2} + n_g^2 \left[\frac{S^{\text{dir}} S^m}{2\epsilon/3mc^2} \right] \frac{I_{b0}}{4\pi \epsilon c} \ln \left(\frac{1+u}{1-u} \right) \quad (33)$$

This distribution function is readily integrated in y to give the delta ray density. The first term in Eq. (33) is strongly dependent on current and is generally the dominant term for high current beams. In these situations, the steady-state delta ray population enclosed within the walls of radius, r_w , [in units of particles/cm] is given by

$$N_0 = D_2 \left(\frac{I_{b0}}{\epsilon c} \right) \left(\frac{I_{n0}}{I_A} \right)^2 \left[\ln \left(\frac{r_w^2}{a^2} \right) - \frac{1}{2} \right]. \quad (34)$$

The dimensionless atomic parameter, D_2 ,

$$D_2 = \int_{y_c}^{y_{\max}} dy \frac{S^{\text{dir}}}{(2e/3mc^2)S^{\text{m}}} \quad (35)$$

is plotted as a function of y_c in Figure 3c. It is clear that D_2 is a much more sensitive function of y_c than D_0 and D_1 . Our best estimate of D_2 is $D_2 \approx 1$.

Note that the enclosed delta ray population of Eq. (34) has an analogous form to the dominant term in the enclosed delta ray current of Eq. (32'). The delta ray density, however, has a stronger current dependence (by an additional power of I_{n0}/I_A). In addition the dimensionless constant D_2 is larger than D_0 [Eq. (21)]. Hence, in cases in which the delta current is only a small fraction of the beam current, the delta density can be comparable to or larger than the beam density.

III. Steady-State Solution: Finite Energy Loss

In the previous section, the steady-state Boltzmann equations for delta rays was solved exactly in the limit of no energy loss. Within the context of the two-term approximation for the Boltzmann equations, this limit provides a rigorous upper bound for the delta ray currents. In this section, we calculate the reduction of delta ray currents from finite energy loss effects. Several approximations are introduced to make the problem mathematically tractable:

- (1) Delta rays are described by a fluid model.
- (2) The magnetic field B_0 arises from a net current which has the same radial profile as the beam current. The profile is assumed to be Bennett.

(3) High current approximation ($\geq$ few kA). Technically, we assume

$$\frac{I_{n0}}{I_A} \gg \frac{n_g \bar{s}_m a}{2},$$

where $1/(n_g \bar{s}_m)$ is the collision length for the delta rays and a is the beam radius.

The present calculation is based on a fluid description of delta rays. The fluid equations are derived by taking moments of the Boltzmann equations. The mass conservation equation is obtained by integrating Eq. (5) in y space

$$\frac{1}{r} \frac{d}{dr} r \left(\frac{j_r}{ec} \right) = \left(\frac{j_b}{ec} \right) n_g \bar{s}_d - n_\delta n_g \bar{s}_\ell, \quad (36)$$

where the production rate for delta rays is

$$\bar{s}_d \equiv \int_{y_c}^{y_{\max}} dy S^{dir}. \quad (37)$$

The energy loss term in Eq. (36) is an approximation obtained from

$$\begin{aligned} n_\delta \bar{s}_\ell &\equiv - \int_{y_c}^{y_{\max}} dy \frac{\partial}{\partial y} F_0 S^{\text{loss}} \\ &\approx F_0(y_c) S^{\text{loss}}(y_c). \end{aligned} \quad (38)$$

An implicit assumption made here is that the distribution function F_0 at the boundary y_c is proportional to the delta ray density n_δ .

The two equations for momentum conservation, obtained from integrating Eqs. (8) and (11), are

$$\left(\frac{J_b}{ec}\right) n_g \bar{S}_z + \frac{eB_\theta}{mc^2} \frac{J_r}{ec} = n_g \bar{S}_m \left(\frac{J_z}{ec}\right) \quad (39)$$

and

$$- \frac{T}{mc^2} \frac{dn_\delta}{dr} = \frac{eB_\theta}{mc^2} \frac{J_z}{ec} + n_g \bar{S}_m \left(\frac{J_r}{ec}\right), \quad (40)$$

where $\bar{S}_z$ is the production rate for the forward component of delta ray current and Tn_δ and $\bar{S}_m$ are the average energy density and normalized momentum transfer rate of the delta's respectively.

Not surprisingly, the radial structure of the three fluid equations [Eqs. (36), (39) and (40)] is similar to the three Boltzmann equations [Eqs. (5), (8) and (11)] from which they were derived. These three equations determine the three unknowns n_δ , J_z and J_r . Explicit solutions are derived in Appendix 2. The forward delta ray current enclosed within a wall of radius r_w is calculated to be

$$I_{\delta 0} = D_0' I_{b0} \left(\frac{I_{n0}}{I_A}\right) (1 - \lambda) g_w(k_1^2) + D_1' I_{b0}, \quad (41)$$

where the two atomic constants D_0 and D_1 are defined as

$$D_0' = \frac{S_d}{S_m}, \quad (42)$$

$$D_1' = \frac{S_z}{S_m}, \quad (43)$$

which are the fluid-model counterparts of D_0 and D_1 in the Boltzmann solution of Eq. (32). The energy loss effects are characterized by the parameter,

$$k_1^2 = \left(\frac{S_d mc^2}{S_m \tau} \right) \left(\frac{I_{n0}}{I_A} \right)^2, \quad (44)$$

where the dimensionless atomic constant $\left(\frac{S_d mc^2}{S_m \tau} \right)$ is of the order of unity.

The explicit dependence of the delta current on energy loss is given in terms of the function $g_w(k_1^2)$ in Eqs. (A45) and (A49) of Appendix 2. In Figure 5, we plot g_w as a function of k_1^2 . Finally λ is a higher order correction given by

$$\lambda = \left(\frac{S_d}{S_m} \frac{S_z}{S_m} \frac{mc^2}{\tau} \right) \left(\frac{I_{n0}}{I_A} \right). \quad (45)$$

In the limit of no energy loss ($S_d \rightarrow 0$) $\lambda \rightarrow 0$ and $k_1^2 \rightarrow 0$, and

$$\lim_{k_1^2 \rightarrow 0} g_w(k_1^2) = \ln \left(\frac{r_w^2}{a^2} \right) - 1. \quad (46)$$

The forward delta ray current reduces to a form identical to the Boltzmann result [Eq. (32)]. With finite energy loss, deviations from the no-energy-loss limit is strongly dependent on the current (k_1^2 is proportional to I_{n0}^2). In fact, in the limit of very high current, such that $k_1^2 \gg 1$, we obtain the solution

$$\lim_{k_1^2 \gg 1} g_w = \frac{2}{k_1^2}. \quad (47)$$

The delta current then becomes

$$I_{\delta 0} = (I_A) \left(\frac{I_{b0}}{I_{n0}} \right) \left(\frac{2 \bar{S}_d \tau}{\bar{S}_\ell mc^2} \right), \quad (48)$$

which is independent of current to zeroth order; i.e., there is an absolute bound to the delta current which is some fraction of 17 kA. Physically, the bound comes about because in the high current limit, the gyro radii of the deltas continue to decrease, making them less effective in generating forward currents.

The density of deltas can be calculated from mass conservation [Eq. (36)]. The total Delta population enclosed within the walls is obtained by performing a radial integration on Eq. (36). The result is

$$N_0 = \left(\frac{I_{b0}}{ec} \right) \frac{\bar{S}_d}{\bar{S}_\ell} - 2\pi r_w J_r(r_w), \quad (49)$$

where the radial current rJ_r is given explicitly in the appendix [Eq. (A44)]. In the limit of no energy loss [$\bar{S}_\ell \rightarrow 0$], it appears initially that the direct production term [the first term on the RHS of Eq. (49)] would diverge. However, the radial flow term is also divergent and cancels the production term to leading order. We obtain a finite result which is quite analogous to Eq. (34);

$$\lim_{k_1^2 \rightarrow 0} N_0 = D_2 \left(\frac{I_{b0}}{ec} \right) \left(\frac{I_{b0}}{I_A} \right)^2 \ln \left(\frac{r_w^2}{a^2} \right), \quad (50)$$

where

$$D_2 \equiv \frac{\bar{S}_d mc^2}{\bar{S}_m \tau}. \quad (51)$$

Finally, in the limit of very high current, so that $k_1^2 \gg 1$, the radial current flow vanishes, and we are left with the simple expression

$$\lim_{k_1^2 \gg 1} N_0 = \left(\frac{I_{b0}}{ec} \right) \frac{\xi_d}{\xi_l} . \quad (52)$$

IV. Linearly Rising Solution

In the previous two sections, we have calculated the asymptotic delta ray current, which is the value that the forward current would eventually attain for a long pulse. Since an electron beam in a real experiment has a finite pulse length, we need to ask an additional question: how rapidly does the forward delta ray current build up as one moves away from the head of the beam?

The regime of interest for the calculations of this section is behind the beam head where large transient electric fields are present. We therefore include only the magnetic effects on delta rays. Effects of electric fields on the delta ray currents will be illustrated with numerical examples in the next section. In addition, the net current is assumed to have reached quasi-steady state, and B_0 is modeled as arising from a time-independent Bennett net current profile. Although the above assumptions imply that we stay away from the very beam head, there follows a large "front segment" of the beam for which the calculations in this section are valid. Here, as in previous sections, we neglect recombination and cascading effects, but in addition, we ignore both energy loss effects and momentum transfer effects. Hence the regime of interest is defined by $x_{\text{pinch}} < x < x_{\text{collisions}}$.

The Boltzmann equations for the delta rays take on the form⁵

$$\frac{\partial F_0}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} r G_r - \frac{\partial G_z}{\partial x} = \left(\frac{j_b}{ec} \right) n_g S^{dir} , \quad (53)$$

$$\frac{\partial G_r}{\partial x} + \left(\frac{2\epsilon}{3mc^2} \right) \frac{\partial F_0}{\partial r} + \frac{eB_\theta}{mc^2} G_z = 0 , \quad (54)$$

$$\frac{\partial G_z}{\partial x} - \left(\frac{2\epsilon}{3mc^2} \right) \frac{\partial F_0}{\partial x} - \frac{eB_\theta}{mc^2} G_r = \left(\frac{j_b}{ec} \right) n_g S_z^{dir} . \quad (55)$$

An empirical observation from numerical code results (NUTS) is that for ETA type beams, steady-state sets in much earlier for the radial current than for the axial current. There is a long period of time during which G_r stays constant while F_0 and G_z are linearly rising in x . Indeed, an admissible solution to Eqs. (53) to (55) is given by

$$\frac{\partial G_r}{\partial x} = 0 , \quad (56)$$

$$F_0' \equiv \frac{\partial F_0}{\partial x} \quad \text{independent of } x , \quad (57)$$

$$G_z' \equiv \frac{\partial G_z}{\partial x} \quad \text{independent of } x . \quad (58)$$

We will derive the solution to Eqs. (53) to (55) in this case. The Boltzmann equations (53) to (55) can be rewritten in a time-independent form

$$F_0' + \frac{1}{r} \frac{\partial}{\partial r} r G_r - G_z' = \left(\frac{j_b}{ec} \right) n_g S^{dir} , \quad (59)$$

$$\left(\frac{2\epsilon}{3mc^2} \right) \frac{\partial F_0'}{\partial r} + \frac{eB_\theta}{mc^2} G_z' = 0 , \quad (60)$$

$$G_z' - \left(\frac{2\epsilon}{3mc^2} \right) F_0' - \frac{eB_\theta}{mc^2} G_r = \left(\frac{j_b}{ec} \right) n_g S_z^{dir} . \quad (61)$$

This is a system of three equations to be solved for the three unknowns: F_0 , G_z and G_r . The solution is derived in Appendix 3. Both F_0 and G_z are found to have the Bennett radial profile $(1+u)^{-2}$ (where $u = r^2/a^2$) with

$$(1+u)^2 F_0 = \left(\frac{I_{b0}}{\pi a^2 e c} \right) n_g S_0, \quad (62)$$

$$(1+u)^2 G_z = \left(\frac{I_{b0}}{\pi a^2 e c} \right) n_g \left(\frac{I_A}{I_{n0}} \right) \left(\frac{4\epsilon}{3mc^2} \right). \quad (63)$$

Here the energy-dependent coefficient S_0 is given by

$$S_0 \equiv \left(S^{\text{dir}} - \left(\frac{I_A}{I_{n0}} \right) S_z^{\text{dir}} \right) / \left[1 - \left(\frac{2\epsilon}{3mc^2} \right) \left(\frac{I_A}{I_{n0}} \right) \left(1 + \frac{2I_A}{I_{n0}} \right) \right] \quad (64)$$

The physical requirement that F_0 be non-negative implies that

$$\left(\frac{2\epsilon}{3mc^2} \right) \left(\frac{I_A}{I_{n0}} \right) \left(1 + \frac{2I_A}{I_{n0}} \right) < 1 \quad (65)$$

in the entire energy domain. This condition is satisfied only when the net current is sufficiently high. For low net current cases, the scenario of constant G_r and linearly rising F_0 and G_z is never attained. We have implicitly assumed that the numerator of S_0 is always positive since $S_z^{\text{dir}} \ll S^{\text{dir}}$.

The rates of delta ray density and current buildup are obtained by integrating over y . We have

$$\frac{dn}{dx} = \left(\frac{J_b}{ec} \right) n_g S_0, \quad (66)$$

$$\frac{dJ_z}{dx} = J_b \left(\frac{I_A}{I_{n0}} \right) n_g S_2, \quad (67)$$

where the two effective cross-sections are

$$\bar{S}_0 = \int_{y_c}^{y_{\max}} dy S_0 , \quad (68)$$

$$\bar{S}_2 = \int_{y_c}^{y_{\max}} dy \left(\frac{4\epsilon}{3mc^2} \right) S_0 . \quad (69)$$

For high currents $\bar{S}_0$ and $\bar{S}_2$ are weakly dependent on the current. $\bar{S}_0$ approaches the direct production cross section $\bar{S}_d$ in the limit of very high net current. Our best estimate gives $\bar{S}_0 \approx 1.5 \times 10^{-22} \text{ cm}^2$ and $\bar{S}_2 \approx 2.5 \times 10^{-23} \text{ cm}^2$ in the ultrahigh current limit.

Rate of buildup of total delta ray population and delta ray forward current enclosed are then given by

$$\frac{dN_0}{dx} = \left(\frac{I_{b0}}{ec} \right) n_g S_0 \quad (70)$$

and

$$\frac{dI_{\delta 0}}{dx} = I_A \left(\frac{I_{b0}}{I_{n0}} \right) n_g \bar{S}_2 . \quad (71)$$

Note that the delta ray density production rate is proportional to the current while the delta ray current production rate is independent of beam current and beam size to zeroth order. The production rates are linearly proportional to pressure. The dependence on pressure, current and beam size are distinctly different from the asymptotic results of Sections II and III.

V. Numerical Results

The LLNL numerical code NUTS⁵ was constructed to study the evolution of beam-generated plasma channels. The full Boltzmann equations in the two-term approximation are advanced simultaneously with Maxwell's equations. We present results of forward delta ray currents computed by the code.

The mathematical treatment underlying NUTS makes no a priori distinction between high energy and low energy electrons. However, it has been observed that the electron distributions evolve naturally into two distinct populations: a high energy group and a low energy plasma sea. A typical example of the axial current distribution $G_z(y)$ is presented in Figure 6. At the low energy end, $G_z(y)$ is negative, indicating a plasma return current which is moving in a direction opposite to the beam. At the high energy end, there is a smaller positive population which are the forward moving delta rays. A diagnostic that is built into NUTS defines the delta ray current density and current as

$$J_\delta \equiv ec \int_{y_c}^{y_{\max}} dy G_z(y)$$

and

$$I_{\delta 0} \equiv \int_0^{r_w} 2\pi r dr J_\delta .$$

Since $G_z(y) \approx 0$ over a large domain of intermediate y values, the definition of J_δ is quite insensitive to the choice of cutoff value, y_c .

We have typically chosen

$$y_c = \frac{y_{\max}}{2}$$

In Figures 7 and 8, we present $I_{\delta 0}(x)$ computed by NUTS for several choices of beam current, beam size and pressure. In all cases considered, steady-state was not reached in 350 cm of pulse. In fact the delta ray current rises in an approximately linear manner in the body of the pulse ($100 \text{ cm} \lesssim x \lesssim 350 \text{ cm}$). In Figure 7, we show two runs at the same pressure (76 torr) and same rise length (20 cm) but with different beam current and beam radius (10 kA, 0.5 cm and 6.5 kA, 1 cm respectively). The rate of increase of $I_{\delta 0}$ in the body of the pulse is seen to be nearly identical. This is consistent with the analytic formula Eq. (71), which predicts that $\frac{dI_{\delta 0}}{dx}$ should be independent of current and beam size to leading order.

In the beam head ($x < 100 \text{ cm}$), the delta currents show significant deviations from linear rise. Here, electric effects play a dominant role. The 10 kA case exhibits a negative dip in I_{δ} in the first 30 cm. This backward delta current was driven by the large inductive electric field at beam head. The large electric field also serves to heat up low energy electrons, resulting in a large population of high energy electrons. As the inductive electric field subsides, the high energy electrons "hang around," since they are relatively collisionless. The magnetic field then pushes these electrons forward to generate a large forward current. This mechanism is responsible for the steep rise in delta ray current to a plateau value at around $x = 60 \text{ cm}$. The same features can also be seen in the lower current case, although they are less dramatic because of the weaker inductive field.

In Figure 8, we show the results at three different pressures for the same beam (with beam current of 10 kA, beam radius of 0.5 cm and rise length of 20 cm). It is clear that the rate of current buildup is fastest at 500 torr, and decreases with decreasing pressure. In the linearly rising portion, the rate of rise is $\frac{dI_{60}}{dx} = 3.9 \times 10^9$ statamp/cm at 500 torr, 1.9×10^9 statamp/cm at 200 torr, and 1.0×10^9 statamp/cm at 76 torr. This trend is again consistent with the analytic formula of Eq. (71), which predicts a linear dependence on pressure. The rate of current buildup computed numerically is somewhat lower than what one might estimate from the analytic formula. This discrepancy is reasonable since energy loss and momentum transfer processes, which are ignored in the analytic treatment, are significant in the time scale of $x < 350$ cm.

In Figure 9, we show a typical radial profile of the delta ray current density, $J_{\delta}(r)$. The forward delta ray current density peaks off axis and has in general a broader distribution than the beam current. In Figure 10, we display a typical radial profile of $I_n(r)$, which is the total current enclosed within a radius r (beam current, plasma current and delta ray current included). Conceptually we may divide the domain in r into three zones. In the innermost region I, we are inside the beam, and $I_n(r)$ increases as we move towards the beam edge. In region II, we are moving out of the beam, $I_b(r)$ is now leveling off, but $I_n(r)$ continues to rise. The broad distribution of forward delta ray current flowing on the wings of the beam is responsible for the rise in $I_n(r)$ in this region. Finally, in region III, we see a decrease in $I_n(r)$, indicating a backward current flowing outside of the delta ray current. The source of this current is as follows: As the high

energy electrons continue to move away from the beam axis, they gradually lose their energy. In the outer region, the delta rays are moving toward the end of their range. Here they are not sufficiently energetic to produce any significant current, but they are very effective in ionizing the gas. Hence, a diffuse sheath of plasma is produced here. The finite E_z drives this plasma to create an outer layer of backward plasma current. Region III is present in all runs with a finite beam cutoff, but is missing in runs with an infinite Bennett tail. In the latter cases, $I_n(r)$ continues to rise all the way to the edge of the wall.

VI. Summary and Discussions

In this paper, we have presented analytic and numerical calculations of net current enhancement due to magnetic-field-driven delta rays. In our analytic studies, we have addressed two questions: 1) What is the maximal delta ray current that a high current beam can produce and support? or, equivalently, what is the steady-state solution for the delta ray current? 2) How fast does the delta ray current build up? In our numerical studies, we have used the LLNL code NUTS to calculate the evolution of the delta ray current. Results for an approximately 10 ns pulse were presented for several beam parameters and pressures.

Two separate analytic calculations of the steady-state delta ray currents were presented. The calculation of Section II presented a rigorous solution to the Boltzmann equations in the two-term approximation, neglecting energy loss. The result provides an upper bound on the delta ray current. As long

as the beam current is not too high (say, < 10 kA), the approximations in this section are reasonable, and the steady-state solution applies. The key result is given in Eq. (32). The asymptotic delta ray current is predicted to increase strongly with beam current, ($I_{\delta 0} \propto I_{b0}^2$) and to increase (though less strongly) with decreasing beam size ($I_{\delta 0} \propto \ln\left(\frac{r_w^2}{a^2}\right)$). In addition, the asymptotic delta ray current is predicted to be independent of pressure. For a 10 kA beam, the formula gives the current enhancement fraction ($I_{\delta 0}/I_{b0}$) to be slightly less than 20%. This model calculation also predicts a broad radial distribution of delta currents. The asymptotic delta ray density is given by Eq. (34), and predicts a very strong dependence on current ($\propto I_{b0}^3$), weak dependence on beam size ($\propto \ln\left(\frac{r_w^2}{a^2}\right)$), and no dependence on pressure.

The no energy loss model of Section II fails as the current is increased much above 10 kA. In Section III, we present a fluid treatment of the steady-state problem with finite energy loss effects. The main result for delta current is given in Eq. (41). Finite energy loss corrections is given in terms of a single function $g_w(k_1^2)$ which is plotted in Figure 5. The parameter which measures the importance of energy loss effects is

$$k_1^2 = \text{constant} \left(\frac{I_{n0}}{17 \text{ kA}} \right)^2$$

where the constant is proportional to the cross section for energy loss, and is estimated to be of the order of unity for realistic cross sections. When the net current is substantially below 17 kA, $k_1^2 \rightarrow 0$ and Eq. (41) reduces to the result of Section II. But as the net current increases, the deviations from the zero-energy-loss result become increasingly significant. In the limit of very high current, we are led to a value of delta current which is

independent of beam current [Eq. (48)]. In Section III, we also calculate corrections to the asymptotic delta ray density due to finite energy loss.

The rate of delta ray current buildup is calculated analytically in Section IV. The key results are given in Eqs. (70) and (71). The rate of delta ray current buildup turns out to be proportional to pressure, but independent of beam current and beam size. The theory also predicts a time scale for current buildup. This time scale is generally longer than the pulse length for most of the ETA type beams. This means that we do not expect to reach the asymptotic solutions in the ETA experiments. The analytic predictions for current buildup are compared against the results of NUTS in Sec. V. We find qualitative agreement between the analytic and numerical results.

The predicted dependences of net current enhancement effects on beam current and beam size as well as the broad radial spread of forward current are in qualitative agreement with the experiments. Since delta rays are produced in any gas, the theory naturally accounts for the observed universality of the net current enhancement phenomenon. The predicted forward current is a fraction of the beam current as observed. Quantitatively we can get at most 10 to 20% enhancement from the theories, in contrast to $\geq 50\%$ frequently observed in experiments. The large current enhancements at ETA are, however, always associated with large transverse displacements of the beam. A reasonable conjecture is that the discrepancy arises from nonaxisymmetric effects which are not modeled in the present theoretical treatment.

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Throughout the course of this work, we have profited from many discussions with our colleagues at LLNL, both theorists and experimentalists. A discussion with R. J. Briggs forced us to look hard at the energy loss effects and led eventually to the results of Section III. W. M. Sharp first derived with the analytic solution to Eq. (A27) presented in Appendix 2. J. Clark helped to clarify our understanding of the experimental results by reviewing the entire set of data on net current enhancement with us. We have also profited from several discussions with Keith Brueckner.

Appendix 1

In this Appendix, we derive the steady-state solution for the radial profile of delta ray currents. Eq. (26) in the text gives a first order differential equation for $I_\delta(r)$, the delta ray current enclosed in a disk of radius r :

$$\frac{dI_\delta}{dr} = 2 D_0 I_b(r) \left(\frac{I_b + I_\delta}{I_A r} \right) + D_1 \frac{dI_b}{dr} , \quad (A1)$$

where D_0 and D_1 are dimensionless atomic constants. For convenience of analysis, Eq. (A1) can be cast into a dimensionless form by introducing the variable

$$u \equiv \frac{r^2}{a^2} \quad (A2)$$

and

$$F_b(u) \equiv \frac{I_b(r)}{I_{b0}} , \quad (A3)$$

$$F_\delta(u) \equiv \frac{I_\delta(r)}{I_{b0}} , \quad (A4)$$

where I_{b0} is the total beam current. Eq. (A1) then becomes

$$\frac{dF_\delta}{du} = \kappa \left[\frac{F_b^2}{u} + \frac{F_b F_\delta}{u} \right] + D_1 \frac{dF_b}{du} , \quad (A5)$$

where the dimensionless parameter κ is given by

$$\kappa \equiv D_0 \frac{I_{b0}}{I_A} , \quad (A6)$$

The general solution to Eq. (A5) is given by

$$F_{\delta}(u) = \left[\int_0^u du' L(u') \exp \left(- \int_0^{u'} M(u'') du'' \right) \right] \left[\exp \left(\int_0^u M(u') du' \right) \right]. \quad (A7)$$

where

$$L(u) = \kappa \frac{F_b^2}{u} + D_1 \frac{dF_b}{du}, \quad (A8)$$

$$M(u) = \kappa \frac{F_b}{u} \quad (A9)$$

For any beam profile, Equations (A7) to (A9) give the solution for the delta ray current profile. The explicit solution for the two cases of a Bennett with an infinite tail and a Bennett with finite cutoff are presented as follows:

(a) Bennett with no cutoff

For this case, the beam current distribution is given by

$$F_b(u) = \frac{u}{1+u} \quad (A10)$$

and we have

$$L(u) = \kappa \frac{u}{(1+u)^2} + D_1 \frac{1}{(1+u)^2}, \quad (A11)$$

$$M(u) = \kappa \frac{1}{(1+u)}. \quad (A12)$$

The exponential factor in Eq. (A7) is evaluated to be

$$\exp \left(\int_0^u M(u') du' \right) = (1+u)^{\kappa}. \quad (A13)$$

The integral involving $L(u)$ consists of two terms. The first term is given by

$$\kappa \int_0^u du' \frac{u}{(1+u')^{2+\kappa}} = \left[\frac{1}{(1+\kappa)} - \frac{1}{(1+u)^\kappa} + \frac{\kappa}{(1+\kappa)(1+u)^{1+\kappa}} \right] \quad (A14)$$

and the second term is given by

$$D_1 \int_0^u du' \frac{1}{(1+u')^{2+\kappa}} = \frac{D_1}{(1+\kappa)} \left[1 - \frac{1}{(1+u)^{1+\kappa}} \right] \quad (A15)$$

Collecting the results of Eqs. (A13), (A14) and (A15), we obtain

$$F_\delta(u) = \left[\frac{(1+u)^\kappa}{(1+\kappa)} - 1 + \frac{\kappa}{(1+\kappa)(1+u)} \right] + \frac{D_1}{(1+\kappa)} \left[(1+u)^\kappa - \frac{1}{(1+u)} \right] \quad (A16)$$

The delta ray current density is related to the derivative of F_δ , and is given by

$$\frac{J_\delta(r)}{I_{b0}/\pi a^2} = \frac{dF_\delta}{du} = \frac{\kappa}{(1+\kappa)} \left[\frac{1}{(1+u)^{1+\kappa}} - \frac{1}{(1+u)^2} \right] + \frac{D_1}{(1+\kappa)} \left[\frac{\kappa}{(1+u)^{1+\kappa}} + \frac{1}{(1+u)^2} \right] \quad (A17)$$

(b) Bennett with finite cutoff

For this calculation, we will set $D_1 = 0$. (The contribution from the forward current at birth is negligibly small.) Let the beam current density have a sharp cutoff at $u = u_c$. Then

$$F_b = \begin{cases} N_c \frac{u}{(1+u)} & \text{for } u < u_c \\ 1 & \text{for } u \geq u_c \end{cases} \quad (A18)$$

where the normalization constant is

$$N_c = \frac{(1+u_c)}{u_c} \quad (A19)$$

Then, we have

$$L(u) = \begin{cases} \kappa N_c^2 \frac{u}{(1+u)^2} & \text{for } u < u_c, \\ \frac{\kappa}{u} & \text{for } u \geq u_c, \end{cases} \quad (A20)$$

$$M(u) = \begin{cases} \frac{\kappa N_c}{(1+u)} & \text{for } u < u_c, \\ \frac{\kappa}{u} & \text{for } u \geq u_c. \end{cases} \quad (A21)$$

For convenience, define

$$\tilde{\kappa} \equiv \kappa N_c. \quad (A22)$$

Then, the exponential factor is evaluated to be

$$\exp\left(\int_0^u du' N(u')\right) = \begin{cases} (1+u)^{\tilde{\kappa}} & \text{for } u < u_c, \\ (1+u_c)^{\tilde{\kappa}} \frac{u}{u_c} & \text{for } u \geq u_c. \end{cases} \quad (A23)$$

The delta ray current is then computed to be

$$F_\delta(u) = \begin{cases} N_c \left[\frac{(1+u)^{\tilde{\kappa}}}{(1+\tilde{\kappa})} - 1 + \frac{\tilde{\kappa}}{(1+\tilde{\kappa})(1+u)} \right] & \text{for } u < u_c, \\ N_c \left[\frac{(1+u_c)^{\tilde{\kappa}}}{(1+\tilde{\kappa})} - 1 + \frac{\tilde{\kappa}}{(1+\tilde{\kappa})(1+u_c)} + \frac{1}{N_c} \right] \left(\frac{u}{u_c} \right)^{\tilde{\kappa}} - 1 & \text{for } u \geq u_c. \end{cases} \quad (A24)$$

Some numerical examples of $F_\delta(u_\omega) = i_{\delta 0}/i_{b0}$ are given in Table 1. F_δ is calculated for various cutoff values of $u_c = \left(\frac{r_c}{a}\right)^2$. κ is fixed at 0.05 and the wall is at $u_\omega = 100$.

Appendix 2

In this appendix, we derive the steady-state solution in the finite energy loss case. The three fluid equations (36), (39) and (40) determine the three unknowns n_b , $\tilde{J}_z \equiv J_z/ec$ and $\tilde{J}_r \equiv J_r/ec$. We first eliminate n_b and $\tilde{J}_z$, and combine the three equations into a single equation for $\tilde{J}_r$:

$$\begin{aligned} \frac{d}{dr} \frac{1}{r} \frac{d}{dr} r \tilde{J}_r = n_g \left[\frac{dn_b}{dr} \tilde{S}_d + n_b \left(\frac{eB_\theta}{mc^2} \right) \left(\frac{\tilde{S}_z \tilde{S}_z}{\tilde{S}_m T/mc^2} \right) \right] \\ + \left(\frac{\tilde{S}_z}{\tilde{S}_m T/mc^2} \right) \left[\left(\frac{eB_\theta}{mc^2} \right)^2 + (n_g \tilde{S}_m)^2 \right] \tilde{J}_r . \end{aligned} \quad (A25)$$

One additional approximation drastically simplifies the equations with relatively little loss of generality: we assume that for the calculation of the magnetic field B_θ in Eq. (A25), the radial profile of the net current is the same as the radial profile of the beam current. This allows us to rewrite Eq. (A25) in a form where the mathematical structure is much simpler. Introducing scaled variables

$$\begin{aligned} u &\equiv \frac{r^2}{a^2} , \\ R &\equiv r \tilde{J}_r . \end{aligned} \quad (A26)$$

Eq. (A25) can be cast in the form

$$\frac{d^2 R}{du^2} = H(u) + K^2(u) R , \quad (A27)$$

where the two known functions $H(u)$ and $k^2(u)$ are given by

$$H(u) = \frac{a^2 n_g}{2} \left[\frac{dn_b}{du} \bar{S}_d + n_b \left(\frac{1_n(u)}{u I_A} \right) \left(\frac{\bar{S}_d \bar{S}_z}{\bar{S}_m T / mc^2} \right) \right] \quad (A28)$$

and

$$k^2(u) = \left(\frac{\bar{S}_d}{\bar{S}_m T / mc^2} \right) \left[\frac{1_n^2(u)}{u^2 I_A^2} + n_g^2 \bar{S}_m \frac{a^2}{4u} \right] . \quad (A29)$$

Although the method of solution is general, we will restrict ourselves to the case of a Bennett profile, obtaining

$$H(u) = - \frac{h}{(1+u)^3} , \quad (A30)$$

where

$$h = \frac{1}{\pi} \left(\frac{I_{b0}}{ec} \right) n_g \bar{S}_d (1 - \lambda) \quad (A31)$$

with

$$\lambda = \left(\frac{\bar{S}_d \bar{S}_z}{\bar{S}_d \bar{S}_m} \right) \left(\frac{mc^2}{T} \right) \left(\frac{1_{n0}}{2 I_A} \right) \quad (A32)$$

and

$$k^2(u) = \frac{k_1^2}{(1+u)^2} + \frac{1_2^2}{u} . \quad (A33)$$

Here

$$k_1^2 = \left(\frac{\xi_k mc^2}{\xi_m T} \right) \left(\frac{I_{n0}}{I_A} \right)^2, \quad (A34)$$

$$k_2^2 = \left(\frac{\xi_k mc^2}{\xi_m T} \right) \left(\frac{n_g \xi_m a}{2} \right)^2. \quad (A35)$$

Note that k_1 is proportional to net current but independent of pressure, while k_2 depends only on pressure and not on current. Furthermore, for high current beams, it is generally true that

$$\frac{I_{n0}}{I_A} \gg \frac{n_g \xi_m a}{2}. \quad (A36)$$

(The collision length for delta rays is generally much longer than the beam radius.) Hence, we will solve the problem assuming $k_2 = 0$.

Two linearly independent solutions to the homogeneous equation

$$\frac{d^2 y}{du^2} = \frac{k_1^2}{(1+u)^2} y \quad (A37)$$

are given by

$$y_{\pm} = (1+u)^{P_{\pm}} \quad (A38)$$

where

$$P_{\pm} = \frac{1 \pm \sqrt{1 + 4k_1^2}}{2}. \quad (A39)$$

A particular solution to the inhomogeneous equation,

$$\frac{d^2 R}{du^2} = - \frac{h}{(1+u)^3} + \frac{k_1^2}{(1+u)^2} R, \quad (A40)$$

is given by

$$\bar{R} = - \frac{h}{(2 - k_1^2)(1 + u)} \quad (A41)$$

The general solution to the inhomogeneous equation Eq. (A27) consists of a linear combination of the particular solution $\bar{R}$, and the two independent solutions to the homogeneous equation y^+ and y^- . The unique solution to the problem is determined with the addition of the two boundary conditions

$$R(u = 0) = 0 \quad (A42)$$

$$R(u \rightarrow \infty) = 0 \quad (A43)$$

This gives rise to

$$r \tilde{J}_r \equiv R(u) = \frac{h}{(2 - k_1^2)} \left[\frac{1}{(1 + u)^\ell} - \frac{1}{(1 + u)} \right] \quad (A44)$$

where

$$\ell = \frac{\sqrt{1 + 4k_1^2}}{2} - \frac{1}{2} \quad (A45)$$

Having calculated the radial current flow, the axial current can be obtained from axial momentum balance [Eq. (39)]. Combining Eqs. (39), (23'), (A44), (A45) and (A31), we compute the axial current density of deltas to be

$$J_z = \frac{2 I_n(r)}{r^2 I_A} \left(\frac{I_{b0}}{\pi} \right) \left(\frac{\xi_d}{\xi_m} \right) \frac{(1 - \lambda)}{2 - k_1^2} \left[\frac{1}{(1 + u)^\ell} - \frac{1}{(1 + u)} \right] + J_b \left(\frac{\xi_z}{\xi_m} \right) \quad (A46)$$

which is readily integrated to obtain the forward delta current enclosed

$$I_\delta(u) = I_{b0} \left(\frac{I_{n0}}{I_A} \right) \left(\frac{\xi_d}{\xi_m} \right) (1 - \lambda) g(u) + I_b(u) \left(\frac{\xi_z}{\xi_m} \right) \quad (A47)$$

where $g(u)$ is a radial from factor given by

$$g(u) = \frac{2}{(2 - k_1^2)} \left[\frac{1}{\ell} \left(1 - \frac{1}{(1+u)\ell} \right) - \left(\frac{u}{1+u} \right) \right] . \quad (A48)$$

The delta current measured at the wall is given by Eq. (A48) with

$$g_w \approx \frac{2}{(2 - k_1^2)} \left[\frac{1}{\ell} \left(1 - \frac{1}{u_w \ell} \right) - 1 \right] . \quad (A49)$$

Appendix 3

In this Appendix, we derive solutions to the Boltzmann Equations in the linearly rising case. With the explicit form of B_θ and J_b for the Bennett beam, Eqs. (59) to (61) may be recast in the form

$$F_0' + \frac{\partial}{\partial u} \left(\frac{2rG_r}{a^2} \right) - G_z' = \left(\frac{I_{b0}}{\pi a^2 \epsilon c} \right) n_g S_z^{\text{dir}} \frac{1}{(1+u)^2}, \quad (\text{A50})$$

$$\bar{\epsilon} \frac{\partial F_0}{\partial u} + \frac{I_{n0}}{I_A} \frac{G_z'}{(1+u)} = 0, \quad (\text{A51})$$

$$G_z' - \bar{\epsilon} F_0' - \left(\frac{I_{n0}}{I_A} \right) \left(\frac{2rG_r}{a^2} \right) \frac{1}{(1+u)} = \left(\frac{I_{b0}}{\pi a^2 \epsilon c} \right) n_g S_z^{\text{dir}} \frac{1}{(1+u)^2}, \quad (\text{A52})$$

where we have introduced the notation

$$\bar{\epsilon} \equiv \frac{2\epsilon}{3mc^2}.$$

To eliminate G_r , Eq. (A52) can be rearranged to express $\frac{2rG_r}{a^2}$ in terms of G_z' and F_0' . Taking the derivative in u results in the relation

$$\frac{\partial}{\partial u} \left(\frac{2rG_r}{a^2} \right) = \frac{I_A}{I_{n0}} \left\{ \frac{\partial}{\partial u} [(1+u)G_z'] - \bar{\epsilon} \frac{\partial}{\partial u} [(1+u)F_0'] + \left(\frac{I_{b0}}{\pi a^2 \epsilon c} \right) n_g S_z^{\text{dir}} \frac{1}{(1+u)^2} \right\} \quad (\text{A53})$$

Substituting this expression into Eq. (A50), we have an equation involving G_z' and F_0' only. Eq. (A51) is then used to express G_z' in terms of F_0' , resulting finally in an equation that involves F_0' alone.

$$F_0' \left(1 - \bar{\epsilon} \frac{I_A}{I_{n0}} \right) - \bar{\epsilon} \left(\frac{I_A}{I_{n0}} \right)^2 \frac{\partial}{\partial u} \left[(1+u)^2 \frac{\partial F_0'}{\partial u} \right] = \left(\frac{I_{b0}}{\pi a^2 \epsilon c} \right) n_g \left(S_z^{\text{dir}} - \frac{I_A}{I_{n0}} S_z^{\text{dir}} \right) \frac{1}{(1+u)^2}. \quad (\text{A54})$$

The solution to this equation is given by

$$F_0 = \frac{C_0}{(1+u)^2} \quad , \quad (A55)$$

where

$$C_0 = \left(\frac{I_{b0}}{\pi a^2 \epsilon c} \right) n_g \left(S^{dir} - \frac{I_A}{I_{n0}} S_z^{dir} \right) // \left[1 - \bar{\epsilon} \frac{I_A}{I_{n0}} \left(1 + 2 \frac{I_A}{I_{n0}} \right) \right] \quad (A56)$$

Finally, Eq. (A57) is invoked to give

$$G_z = \left(\frac{I_A}{I_{n0}} \right) \frac{2\bar{\epsilon}C_0}{(1+u)^2} \quad . \quad (A57)$$

Table 1

r_c/a	$I_{\delta 0}/I_{b0}$
2	0.24
3	0.22
4	0.21
∞	0.20

The fractional current enhancement, $I_{\delta 0}/I_{b0}$, for various cutoff values of r_c/a for a truncated Bennett beam. $\kappa = D_0 \frac{I_{b0}}{I} = 0.05$ and the wall is at $u_w = 100$, $D_1 = 0$.

Figure Captions

- Figure 1: Schematic of delta ray current as a function of $x = ct - z$ (distance from beam head).
- Figure 2: Atomic data input for delta rays. The four sets of data correspond to (a) S^{dir} -- direct production, (b) S_z^{dir} -- direct production of forward momentum, (c) S^{m} -- momentum transfer, d) S^{loss} -- energy loss. Their precise definitions are given in the text. The independent variable is $y = \ln(\epsilon + \epsilon_s)$ where ϵ is the electron energy in eV, and ϵ_s is set at 1 eV.
- Figure 3: The asymptotic delta current and density are dependent on three dimensionless atomic constants D_0 , D_1 , and D_2 . Sensitivity of these three constants to the low-energy cutoff y_c is shown in Figure 3.
- Figure 4: The asymptotic net current profile $I_n(r) = I_b(r) + I_\delta(r)$ where the delta current $I_\delta(r)$ is given by Eq. (28). The plasma return current is assumed to be zero in this model.
- Figure 5: Effect of energy loss on delta ray current is given in terms of the function g_w . The independent variable k_1^2 is proportional to energy loss S_δ and the square of the net current. In this plot, we set $u_w \equiv \left(\frac{r_w}{a}\right)^2 = 100$. The limit of no energy loss corresponds to $g_w(0) = 3.605$.
- Figure 6: A typical example of the energy dependence of axial current distribution $G_z(y)$.
- Figure 7: NUTS results on the buildup of delta ray current. Two different beams at the same pressure compared.
- Figure 8: NUTS results on the buildup of delta ray current. Fixed beam at three different pressures compared.
- Figure 9: A typical delta ray current profile.
- Figure 10: A typical net current profile for a Bennett beam with finite cutoff.

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Figure 1

Figure 2a

Figure 2b

Figure 2c

Figure 2d

Figure 3a

Figure 3b

Figure 3c

Figure 4

Figure 5

Figure 6

Figure 7

Figure 8

Figure 9

Figure 10