LA-UR -85-1022

TITLE ATTRACTORS FOR THE KURAMOTO-SIVASHINSKY EQUATIONS

LA-UR--85-1022

DE85 009631

AUTHOR(S) B. Nicolaenko
B. Scheurer
R. Temam

SUBMITTED TO AMS-SIAM Series in Applied Mathematics
July 22-August 24, 1984
Santa Fe, NM 87545

Los Alamos National Laboratory
Los Alamos, New Mexico 87545

FORM NO. 836 R4
ST. NO. 2629 5/81

ATTRACTORS FOR THE KURAMOTO-SIVASHINSKY EQUATIONS

B. Nicolaenko[1]

B. Scheurer[2]

R. Temam[3]

[1] Theoretical Division and Center for Nonlinear Studies, MS-B284, Los Alamos National Laboratory, Los Alamos, NM 87545.

[2] Commissariat à l'Energie Atomique, Centre d'Etudes de Limeil-Valenton (Dpt MA) BP 27, 94190 Villeneuve St. Georges, France.

[3] Laboratoire d'Analyse Numerique, CNRS and Universite Paris Sud, Bat 425, 91405 Orsay, France

0. INTRODUCTION

In this paper, we address the question of constructing an upper bound of the Hausdorff and Fractal dimensions $d_H(x)$ and $d_F(x)$ for attractors X of the Kuramoto-Sivanshinsky equation. We investigate the large time behavior of the solution u = u(xib) of:

$$\frac{\partial u}{\partial r} + \nu\Delta^2 u + \Delta u + \frac{1}{2} + |\nabla u|^2 = 0 \quad , \quad (x,r) \in \mathbb{R}^n \times \mathbb{R}_+ \qquad (0.1)$$

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R} \qquad (0.2)$$

$$u(x+Le_i,t) = u(x,t) \qquad 1 \leq i \leq n, \qquad (0.3)$$

where $\nu > 0$ and u_0 is L-periodic. This equation occurs in a large variety of physical situations; it models the formation of cellular patterns whose temporal behavior becomes chaotic, when the typical size L of the cell is large enough. The natural bifunction parameter of the problem is the adimensional number $\frac{L}{2\pi\sqrt{\nu}}$. In the former paper [ ], we considered solutions of (0.1)-(0.2) with either periodic or Neuman boundary conditions. We obtained general upper bounds for $d_H(X)$, $d_F(X)$ in terms of

$$R = \lim_{t \to \infty} ||\nabla u(t)||$$

and

$$Y = \lim_{t\to\infty} \frac{1}{t} \int_0^t ||D^3u(s)||^2 ds \quad \text{if } n=1$$

$$(\text{resp. } Y = \lim_{t\to\infty} \frac{1}{t} \int_0^t ||D^4u(s)||^2 ds \quad \text{if } n= 2,3) \quad .$$

In one dimension, for even solutions of (0.1), (0.2) (including Neuman conditions), we proved the uniform boundedness of orbits. As a consequence:

$$R \le \text{cste } \nu^{-1/4} \tilde{L}^{5/2} \tag{0.4}$$

$$Y \le \text{Cste } \nu^{-5/2} \tilde{L}^{5} \tag{0.5}$$

$$d_H(x) \le d_F(x) \le 1 + \text{Cste } \tilde{L}^{13/8} \quad . \tag{0.6}$$

Here, we significantly sharpen the upper bound (0.6) in the one dimensional context. In [ ], the main step in the derivation of the (0.6), was a classical lower estimate on the trace of same operator associated to linearized flow of (0.1), (0.2). Our main improvement results from extending a remarkable Sobolev type inequality obtained by Lieb and Thirring (see [ ]) in the context of Schrödinger operators. Our extension takes the following form:

$$\sum_{j=1}^{m} \int |\Delta\phi_j|^2 dx \ge \text{Cste} \int (\sum|\phi_j(x)|^2)^{\alpha_n} dx \tag{0.7}$$

where the $\{\phi\}$ are any set of m smooth functions, orthonormal in L^2 and L-periodic; $\alpha = 1 + \frac{4}{n}$ with $n \leq 3$. The constant depends only upon n. Using (0.7), (0.6) now becomes:

$$d_H(x) \leq d_F(x) \leq 1 + \text{Cste}\ \nu^{\frac{9}{40}}\ \tilde{L}^{\frac{3}{2}} \tag{0.8}$$

This paper is organized as follows. In the first part, we review the problem and recall some relevant facts in estimating $d_H(X), d_F(X)$. Then, we show how (0.7) is related to bounds for eigenvalues of $\Delta^2 + q(X)$, where q(X) is a smooth negative function. In the second part, using the Birman-Schwinger principle (see [ ]), we demonstrate the extended Lieb-Thirving inequality (0.7). The methods derived in [ ] and sharpened here are in fact generic to a whole class of operation of the type:

$$\frac{\partial u}{\partial t} + P\left(\frac{\partial}{\partial x}\right) u + f\left(\frac{\partial u}{\partial x}\right) = 0 \tag{0.9}$$

where $P(\frac{\partial}{\partial x})$ is any pseudo-differential operator with even symbol, of order ≥ 2; and

$$f(s) = \frac{s^2}{2} \text{ or } \sqrt{1+s^2} - 1$$

(The latter nonlinearity is a natural one to describe curvature effect in wrinkled flames [ ]. Details will appear elsewhere. Here we just refer to the papers by Kuramoto [ ] and Sivashinsky [ ]; a discussion of these physical situations is given in the introduction of [ ].

1. HAUSDORFF DIMENSION OF AN ATTRACTOR

1.1 Review of the Problem

We consider the Kuramoto-Sivashinsky equation, in space imension $n \leq 3$. That is to find $u = u(x,t)$ solution of:

$$\frac{\partial u}{\partial t} + \nu \Delta^2 u + \Delta u + \tfrac{1}{2}|\nabla u|^2 = 0 \qquad (x,t) \in R^n R_+ \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad x \in n \tag{1.2}$$

$$u(x + Le_i, t) = u(x,t) \qquad 1 < i \leq n \quad , \tag{1.3}$$

where $\nu > 0$ and u_0 is L-periodic. We address the question of an upper bound of the Hausdorff and fractal dimensions of any attractor of (1.1) -(1.3). First, we recall the general setting of [, ,], applied to the Kuramoto-Sivashinsky equation [,].

In what follows, we look at any solution of (1.1)-(1.3) such that:

$$||\nabla u(t)|| \leq k < +\infty \quad , \text{ for } t \geq 0 \quad . \tag{1.4}$$

Such a bound has been proved for even solutions of (1.1)-(1.3), when n = 1 (see [ ]). We define (see [ ]) $\dot{H} = H/R = L^2/R$ the quotient space of $H = L^2$ by R equipped with the natural norm. For any element w of H, $\dot{w}$ will be the corresponding coset in $\dot{H}$. If $S(t): u_0 \to u(.,t)$ is the mapping defined by (1.1)-(1.3), we thus get an associated mapping $\dot{S}(t)$ in $\dot{H}$ (with an appropriate domain). The corresponding limit set:

$$X = \bigcap_{t>0} c\ell \, \{\dot{u}(s) \ , \ s \geq t\} \qquad (1.5)$$

has been shown ([ ]) to be a <u>functional invariant set</u> for the dynamical system (1.1), in the terminology of [, ,]. In particular:

$$\lim_{t\to\infty} \mathrm{dist}\,(\dot{u}(t),X) = \lim_{t\to\infty} \inf_{\dot{w}\in X} \|\dot{u}(t) - \dot{w}\| = 0 \ , \qquad (1.6)$$

for any solution satisfying (1.4). The "differential" $D\dot{S}(t,u_0)$ of $\dot{S}(t)$ around a given solution $u = \dot{S}(t)u_0$ of (1.1)-(1.3) is obtained by introducing the linearized evolution equation:

$$\frac{\partial U}{\partial t} + \nu\Delta^2 U + \Delta U + \nabla u . \nabla U = 0 \qquad (1.7)$$

$$U(x,0) = \xi(x) \qquad (1.8)$$

$$U(x + Le_i, t) = U(x,t) \qquad 1 \leq i \leq n \ . \qquad (1.9)$$

Precisely, $D\dot{S}(t,u_0): \xi \to U(t)$ where $U(t)$ is the value at time t of the solution to (1.7) (1.8)[(1)]; this is a linear compact operator in $\dot{H}$. Now for $m \in \mathbb{N}$ given, to $\xi_1, \ldots, \xi_m$ given in $\dot{H}$ it corresponds $U_1(t), \ldots, U_m(t)$ solutions of (1.7) (1.8). Then

(1) The solution U exists for $0 < t < \xi_1(u)$.

a direct calculation shows that:[1]

$$\frac{d}{dt} \log \|\bigwedge_{j=1}^{m} U_j(t)\| + \text{Trace}\,(A(u) \circ P_m) = 0 \quad , \qquad (1.10)$$

for $u = \dot{S}(t)u_0$ and where by definition:

$$A(u) = \nu\Delta^2 + \Delta + \nabla u \nabla$$
$$(1.11)$$

P_m orthogonal projection of L^2 onto span $\{U_1(t), \ldots, U_m(t)\}$.

Note that the projector P_m depends smoothly on $\xi_1, \ldots, \xi_m$, u, t and m. The relation (1.10) is equivalent to:

$$\|\bigwedge_{j=1}^{m} U_j(t)\| = \|\bigwedge_{j=1}^{m} \xi_j\| \exp \int_0^t \text{Trace}\,(A(u) \circ P_m)ds. \quad (1.12)$$

Now, we introduce the norm of $\wedge^m D\dot{S}(t,u_0)$ in $\wedge^m \dot{H}$:

$$\omega_m(D\dot{S}(t,u_0)) = \sup\{\|\bigwedge_{j=1}^{m} U_j(t)\| : \xi_j \in \dot{H} \;,\; \|\xi_j\| \le 1 \;,$$
$$1 \le j \le m\} \quad . \qquad (1.13)$$

This norm measures the volume variation of a "m-dimensional cube" $\bigwedge_{j=1}^{m} \xi_j$ convected along the orbit $\dot{S}(t)u_0$. From (1.12), (1.13),

(1) Hereafter $\| \; \|$ is the natural norm on $\wedge^m \dot{H} = \underbrace{\dot{H}\wedge \ldots \wedge\dot{H}}_{m}$, where $\wedge$ denotes the exterior product.

for u_0 running over X (defined by (1.5)), we then obtain easily (P is any projector of rank m):

$$\bar{\omega}_m(t) \equiv \sup_{u_0 \in X} \omega_m(D\dot{S}(t,u_0))$$

$$\leq \sup_{u_0 \in X} \exp\left(-\int_0^t \operatorname{Inf}_P \operatorname{Trace}\,(A(u)_\circ P)ds\right) \quad . \qquad (1.14)$$

A general result of [,] relates the fractal and Hausdorff dimensions of X, $d_F(X)$ and $d_H(X)$, to the first integer m_0 such that $\bar{\omega}_{m_0}(t) < 1$ (for t large) i.e., $\dot{S}(t)$ contracts all the m_0-dimensional cubes at every point u_0 of X. Precisely, if there exists $m_0 \in \mathbb{N}$ for which:

$$\lim_{t \to +\infty} \frac{1}{t} \log \bar{\omega}_{m_0}(t) < 0 \qquad (1.15)$$

then the following upper bound holds:

$$d_H(X) \leq d_F(X) \leq m_0 + 1 \quad . \qquad (1.16)$$

We are thus led to find the first $m_0 \in \mathbb{N}$ such that:

$$\lim_{t \to \infty} \left\{ \operatorname{Inf}_{u_0 \in X} \operatorname{Inf}_P \frac{1}{t} \int_0^t \operatorname{Trace}\,(A(u)_\circ P)\,ds \right\} > 0 \qquad (1.17)$$

As a consequence, to find lower bounds for the expression

$\frac{1}{t}\int_0^t$ Trace $(A(u)_\circ P)$ ds will be of crucial importance. This is achieved in the next section.

1.2 The Main Result

We state and prove the main result of this paper; the proof relies on a Sobolev inequality for the operator $\bigwedge_{j=1}^{m} \Delta^2$. As a consequence, we compute $m_0 \in \mathbb{N}$ such that (1.15) holds. Recall that X is defined by (1.5), A(u) by (1.11).

Theorem 1.1. Let u be any solution of (1.1)-(1.3) such that (1.4) holds. Denote by $E_1, \ldots, E_m$ the first m eigenvalues of the biharmonic operator Δ^2 (with periodic boundary conditions) and by P any projector of rank m. Then:

$$\lim_{t\to\infty}\{\sup_{u_0\in X}\inf_{P}\frac{1}{t}\int_0^t \text{Trace}\,(A(u)_\circ P)\,ds\}$$

$$\geq m(L)^{-\frac{n}{\alpha_n}}\{c_0(n)\frac{\nu}{2}m^{\alpha_n-1}(L)^{-\frac{(\alpha_n-1)^2}{\alpha_n}}$$

$$-\frac{1}{2}\lim_{t\to\infty}\frac{1}{t}\int_0^t \|\Delta u+\frac{1}{\nu}\|_{\alpha'_n}\,ds\}\,, \qquad (1.18)$$

where $\alpha_n = 1 + \frac{4}{n}$, $\alpha'_n = \frac{\alpha_n}{\alpha_n - 1}$.

<u>Proof</u>. Let us introduce $\{\phi_j\}_{j \in \mathbb{N}}$ an orthonormal basis of smooth functions of $\dot{H}$ such that $\{\phi_1, \ldots, \phi_m\}$ is an <u>orthonormal</u> basis of $P\dot{H}$. By definition:

$$\text{Trace } (A(u)_o P) = \text{Trace } ((\nu\Delta^2 + \Delta + \nabla u.\nabla)_o P$$

$$= \sum_{j=1}^{m} \nu||\Delta\phi_j||^2 - ||\nabla\phi_j||^2 + \quad \nabla u.\nabla\phi_j \ \phi_j \ dx$$

$$\geq \frac{\nu}{2} \sum_{j=1}^{m} ||\Delta\phi_j||^2 - \tfrac{1}{2} \quad (\Delta u + \frac{1}{\nu}) \sum_{j=1}^{m} |\phi_j|^2 \ dx,$$

where we used a classical interpolation inequality. We now claim that there exists a constant $c_0(n)$, independent of m and L such that:

$$\sum_{j=1}^{m} ||\Delta\phi_j||^2 \geq c_0(n) \quad (\sum_{j=1}^{m} |\phi_j|^2)^{\alpha_n} dx \quad , \alpha_n = 1+\frac{4}{n} \quad . \tag{1.20}$$

This key inequality will be discussed and proved in the next sections. From (1.19), (1.20), we get, using Hölder inequality:(1)

$$\text{Trace } (A(u)_o P) \geq \{c_0(n) \frac{\nu}{2} ||\rho||_{\alpha_n}^{\alpha_n - 1} - \frac{1}{2} ||\Delta u + \frac{1}{\nu}||_{\alpha_n'}\} ||\rho||_{\alpha_n} \tag{1.21}$$

(1) $|| \ ||_\alpha$ is the norm in L^α.

where $\alpha'_n = \frac{\alpha_n}{\alpha_n - 1}$ and $\rho(x) \equiv \sum^m |\phi_j(x)|^2$. Since the basis $\{\phi_1,$ $\ldots, \phi_m\}$ is orthonormal, we have $m < (L)^{\frac{n}{\alpha'_n}} ||\rho||_{\alpha_n}$; therefore after taking the time average of (1.21) we obtain:

$$\frac{1}{t}\int_0^t \text{Trace } (A(u)_o P)\, ds > m(L)^{-\frac{n}{\alpha'_n}} \{c_0(n)\, \frac{\nu}{2}\, m^{\alpha_n - 1} (L)^{-\frac{(\alpha_n-1)^2}{\alpha_n}}$$

$$- \frac{1}{2}\frac{1}{t}\int_0^t ||\Delta u + \frac{1}{\nu}||_{\alpha'_n}\, ds\} \quad . \quad (1.22)$$

The conclusion of the theorem is immediate from (1.22). □

It is now an easy matter to find the smallest $m_0 \in N$ such that the right hand side of (1.21) is strictly positive. As in [ ], we will restrict ourselves to the case where we proved assumption (1.4) (see Corollary 2.2 of [ ]).

Corollary 1.2. Let X (defined [illegible] (1.5)) be the functional invariant set associated to (1.1) (1.3) where n [illegible] 1 and n is even. Then, the Hausdorff and fractal dimensions $d_H(X)$ and

$d_F(X)$ are finite. Precisely:

$$d_H(X) \le d_F(X) \le m_0+1 \tag{1.23}$$

where, m_0 is defined by

$$m_0 = c_0(1)^{-\frac{1}{4}}(2\pi)^{\frac{4}{5}}\,\nu^{-\frac{3}{20}}\,\tilde{L}^{\frac{4}{5}}\,(2\pi\sqrt{\nu}\,\tilde{L})^{-\frac{3}{10}}\,R^{\frac{1}{2}}Y^{\frac{1}{4}} + (2\pi)^{\frac{4}{5}}\,\nu\,L^{\frac{3}{5}}\;{}^{\frac{1}{4}} \tag{1.24}$$

for $\tilde{L} = \dfrac{L}{2\pi\sqrt{\nu}}$ and:

$$R = \lim_{t\to\infty} ||Du(t)||\ ,\quad Y = \lim_{t\to\infty}\frac{1}{t}\int_0^t ||D^3u(s)||^2\,ds\ . \tag{1.25}$$

Proof. We use (1.18) in the case where $n = 1$, i.e., $\alpha_1 = 5$, $\alpha_1' = 5/4$. The right hand of (1.18) will be strictly positive as soon as:

$$m > c_0(1)^{-\frac{1}{4}}\,(2\pi)^{\frac{4}{5}}\,\nu^{\frac{3}{20}}(\tilde{L})^{\frac{4}{5}}\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t ||D^2u + \frac{1}{\nu}||_{\frac{5}{4}}\,ds\right)^{\frac{1}{4}}\ . \tag{1.26}$$

The following inequalities are easy consequences of Hölder inequality and standard interpolation:

$$\frac{1}{t}\int_0^t \|D^2u\|_{\frac{5}{4}}\,ds \le L^{\frac{3}{10}} \left(\frac{1}{t}\int_0^t \|Du\|\,ds\right)^{\frac{1}{2}} \left(\frac{1}{t}\int_0^t \|D^3u\|^2 ds\right)^{\frac{1}{2}}$$

$$\le L^{\frac{3}{10}} \left(\sup_{0<s<t} \|Du(s)\|\right)^{\frac{1}{2}} \left(\frac{1}{t}\int_0^t \|D^3u\|^2\,ds\right)^{\frac{1}{2}} \tag{1.27}$$

Therefore, with the definitions (1.25):

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|D^2u + \frac{1}{\nu}\|_{\frac{5}{4}}\,ds \le L^{\frac{3}{10}} R^{\frac{1}{2}} Y^{\frac{1}{2}} + \nu^{-1} L^{\frac{4}{5}} \tag{1.28}$$

and (1.26) is a fortiori satisfied if $m \geq m_0$ where m_0 is defined by (1.24). The proof is complete. □

From [ ], we know upper bounds, in terms of ν and L, for R and Y. (See (2.18), (2.19) in [ ].) A direct computation gives

Corollary 1.3. Under the above assumptions, if $L \geq L_0 \geq 2.44$

$$m_0 \le c_1 \nu^{\frac{9}{40}} L^{\frac{3}{2}} + c_2 L \tag{1.29}$$

where c_1, c_2 are absolute constants.

Remark 1.4. The reason for the condition $\hat{L} \geq \tilde{L}_0$ has been discussed in [ ]. □

1.3 Generalization of a Sobolev inequality of Lieb and Thirring

In this section we prove the inequality (1.20) used during the proof of the Theorem 1.1. The main step in the proof consists of showing, using orthonomality of the ϕ_j's, the equivalence of (1.20) with a bound for the moments of certain eigenvalues. Precisely, these are the eigenvalues of the generalized Schrödinger operator $\Lambda^2 + q(x)$, for an appropriate potential q. Here we make precise this equivalence; the bound for moments is discussed in the next paragraph.

Theorem 1.5. Let $\{\phi_1, \ldots, \phi_m\}$ any set of m (nonconstants) smooth functions, who are orthonomals in L^2 and 1-periodic. Then, there exists a constant $c_0(n)$ depending on the space dimension n but not on m such that:

$$\sum_{j=1}^{m} \|\Lambda\phi_j\|^2 \geq c_0(n) \int \{\sum_{j=1}^{m} |\phi_j(x)|^2\}^{\alpha_n} dx \qquad (1.30)$$

where $\alpha_n = 1 + \frac{4}{n}$ and $n \geq 3$.

Proof. Let us introduce the operator $A = \Lambda^2 + q(x)$ on the n-dimensional torus $\Omega = [-\frac{1}{2}, \frac{1}{2}]^n$; we will assume $q(x) \le 0$, defined and smooth for simplicity although it is not necessary. The operator Λ, unbounded in $H = L^2(M)$, admit $H^4_{per}(M)$ for domain and is self adjoint; its spectrum is discrete and denumerable. Let us denote by λ_j, $j \geq 1$, its eigenvalues counted with multiplicities. Finally, we set by definition:

$$N_\alpha(q) = \sum_{\lambda_j < \alpha} 1 \quad . \tag{1.31}$$

The following sequence of inequalities is then straightforward:

$$\begin{aligned}\sum_{j=1}^{m} \|\Lambda\phi_j\|^2 + \int q(x) \sum_{j=1}^{m} |\phi_j(x)|^2 \, dx &= \sum_{j=1}^{m} \lambda_j \\ &\geq - \sum_{j=1}^{m} |\lambda_j| \\ &\geq - \sum_{\lambda_j < 0} |\lambda_j| \\ &= - \int_0^\infty N_{-\alpha}(q) d\alpha \quad .\end{aligned} \tag{1.32}$$

We thus have to estimate the sum $\sum_{\lambda_j < 0} |\lambda_j| = \int_0^\infty N_{-\alpha}(q) d\alpha$ from above. This will be done in the next section; the result is (recall that we assumed $q(x) \leq 0$):

$$\int_0^\infty N_{-\alpha}(q) \, d\alpha \leq c(n) \int |q(x)|^{1+\frac{n}{4}} \, dx \tag{1.33}$$

From (1.32), (1.33) we get:

$$\sum_{j=1}^{m} ||\Delta\phi_j||^2 \geq \int -q(x) \sum_{j=1}^{m} |\phi_j(x)|^2 \, dx - c(n) \int |q(x)|^{1+\frac{n}{4}} \, dx \; . \tag{1.34}$$

Now we choose $q(x) = -c_3\left(\sum_{j=1}^{m} |\phi_j(x)|^2\right)^{c_4}$, where the constants c_3, c_4 satisfy $c_4+1 = c_4(1+\frac{n}{r})$, $c_3 = 2c(n)\, c_3^{1+n/4}$, i.e., $c_4 = \frac{4}{n}$, $c_3 = (2c(n))^{-n/4}$. Therefore, from (1.34) we deduce (1.30), with $\alpha_n = c_4+1$ and $c_0(n) = c_3$. The proof is complete. □

2. BOUNDS FOR MOMENTS OF THE EIGENVALUES

Here we prove the bounds (1.33). The method is general and relies on the Birman-Schwinger principle (as it has been quoted in the introduction) and the computation of the trace for an integral operator. In particular, it could be applied for Δ^2 replaced by any elliptic operator. We recall the result.

Theorem 2.1. Let q be a smooth function defined on the torus $M = [-\frac{1}{2}, \frac{1}{2}]^n$, $n \geq 3$. Then, if $q_-(x) = \text{Min}\{q(x), 0\}$:

$$\sum_{\lambda_j < 0} |\lambda_j| = \int_0^\infty N_\sigma(\ldots)\, d\sigma \leq c(n) \int |q_-(x)|^{1+\frac{n}{4}} \, dx \; , \tag{1.33}$$

where the λ_j's are eigenvalues of $\Delta^2 + q(x)$ defined on M and $N_{-\alpha}(q) = \sum_{\lambda_j < 0} 1$. The expression of the constant $\varepsilon(n)$ appear in (2.10) below.

<u>Proof</u>. Due to the min-max principle, we first remark that for $\alpha > 0$

$$N_{-\alpha}(q) \le N_0(q+\alpha) = N_0(t\alpha + q + (1-t)\alpha)$$

$$\le N_0(t\alpha + \text{Min}\,\{q+(1-t)\alpha, 0\}) \quad , \quad 0 < t < 1 \quad . \qquad (2.1)$$

In what follows, we shall use the shorter notation $(q + (1-t)\alpha)_-$ for Min $\{q + (1-t)\alpha, 0\}$. Thus, we are led to estimate the number of negative eigenvalues for $\Delta^2 + t\alpha + (q+(1-t)\alpha)_-$. The Birman-Schwinger principle (see the Appendix) state that the number of negative eigenvalues of $\Delta^2 + t\alpha + (q+(1-t)\alpha)_-$ is equal to the number of eigenvalues greater than one of $|(q+(1-t)\alpha_-|^{1/2}$ $(\Delta^2 + t\alpha)^{-1}$ $|q+(1-t)\alpha)_-|^{1/2} = K_{\alpha,t}$. The latter is clearly bounded from above by the number of eigenvalues greater than one of $(K_{\alpha,t})^m$, for $m \in \mathbb{N}^*$, and a fortiori by [(1)] Trace $(K_{\alpha,t})^m$. Therefore, from (2.1), we obtain:

$$N_{-\alpha}(q) \le \text{Trace}\,(K_{\alpha,t})^m \quad , \qquad (2.2)$$

where the positive integer m will be chosen below. From the definition of $K_{\alpha,t}$ and classical commutativity property of the

(1) One check (see the Appendix) that $K_{\alpha,t}$ is trace class for $m > 3$.

trace, we still get:

$$N_{-\alpha}(q) \le \mathrm{Trace}\,(|(q + (1-t)\alpha)_-|^m (\Lambda^2 + t\alpha)^{-m}) \quad . \qquad (2.3)$$

By the definition of the trace, we now obtain:

$$N_{-\alpha}(q) \le L^{-n} \sum_{k \in Z^n} \int |(q(x) + (1-t)\alpha)_-|^m \sigma_{\alpha,t}(k)\, dx \quad , \qquad (2.4)$$

where $\sigma_{\alpha,t}(k) = ((\frac{2\pi}{L})^4 |k|^4 + t\alpha)^{-n}$ is the symbol of $(\Lambda^2 + t\alpha)^{-m}$.

Finally, (2.4) is equivalent to ($\alpha > 0$, $t \in [0,1]$):

$$N_{-\alpha}(q) \le \{L^{-n} \sum_{k \in Z^n} \sigma_{\alpha,t}(k)\} \int |(q(x) + (1-t)\alpha)_-|^m\, dx \qquad (2.5)$$

In (2.5), the term is brackets is bounded from above by

$$I_{m,n}(t\alpha) = L^{-n} \int_{R^n} ((\tfrac{2\pi}{L})^4 |\xi|^4 + t\alpha)^m\, d\xi$$

$$= (2\pi)^{-n} (t\alpha)^{\frac{n}{4}-m} \int_{S^{n-1}} d\omega_{n-1} \int_0^\infty (\rho^4 + 1)^{-m} \rho^{n-1}\, d\rho \qquad (2.6)$$

where $d\omega_{n-1}$ is the measure on the sphere $S^{n-1} = \{\xi \in R^n,$ $|\xi| = 1\}$. The last integral is finite iff $m > \frac{n}{4}$ and (2.5), (2.6) gives the desired bound on $N_a(q)$:

$$N_\alpha(q) \le \Gamma_m(t\alpha) \int |(q(x) + (1-t)\alpha)_-|^m \, dx$$

$$m > \frac{n}{4}, \quad \alpha > 0, \quad t \in [0,1] \qquad (2.7)$$

We are now in position to bound from above $\int_0^\infty N_{-\alpha}(q)\, d\alpha$ (and more generally $\int_0^\infty N_{-\alpha}(q)\, \alpha^\ell \, d\alpha$). From (2.7) and Fubini's theorem, we have:

$$\int_0^\infty N_{-\alpha}(q)\, d\alpha \le \int dx \int_0^\infty |(q(x) + (1-t)\alpha)_-|^m \, \Gamma_{m,n}(t\alpha)\, d\alpha$$

$$= \int dx \int_0^{\frac{1}{1-t}|q(x)_-|} (|q(x)_-| - (1-t)\alpha)^m \, \Gamma_{m,n}(t\alpha)\, d\alpha$$

$$= \int dx \int_0^1 |q(x)_-|^m (1-\beta)^m \, \Gamma_{m,n}\left(\frac{t}{1-t}|q(x)_-|\beta\right) \frac{|q(x)_-|}{1-t}\, d\beta \qquad (2.8)$$

Using the value (2.6) of $\Gamma_{m,n}$, we still obtain:

$$\int_0^\infty N_{-\alpha}(q)\, d\alpha \le \gamma_{m,n}\, t^{\frac{n}{4}-m} (1-t)^{m-(1+\frac{n}{4})} \int_0^1 (1-\beta)^m \beta^{\frac{n}{4}-m}\, d\beta$$

$$\cdot \int |q(x)_-|^{1+\frac{n}{4}}\, dx \,, \qquad (2.9)$$

where

$$\gamma_{m,n} = (2\rho)^{-n} \int_{S^{n-1}} d\omega_{n-1} \int_0^\infty (\rho^4+1)^{-m} \rho^{n-1} d\rho \text{ and } \frac{n}{4} < m < 1+\frac{n}{4}$$

(to have a finite integral in β). It remains to find the best $t \in [0,1]$ and $m \in]\frac{n}{4}, 1+\frac{n}{4}[$ that is t_0, m_0 giving the smallest $t^{\frac{n}{4}-m}(1-t)^{m-(1+\frac{n}{4})}$. The proof of (1.33) is complete; the precise value of $\varepsilon(n)$ in (1.33) is:

$$\varepsilon(n) = \gamma_{m_0,n}\, t_0^{\frac{n}{4}-m_0} (1-t_0)^{m_0-(1+\frac{n}{4})} \int_0^1 (1-\beta)^{m_0} \beta^{\frac{n}{4}-m_0} d\beta$$

with $\gamma_{m_0,n} = (2\pi)^{-n} \int_{S^{n-1}} d\omega_{n-1} \int_0^\infty (\rho^4+1)^{-m_0} \rho^{n-1} d\rho$.

APPENDIX

Here we recall, for the sake of completeness, some facts concerning the Birman-Schwinger principle (see [,]). Let $V(x)$ be a bounded potential, assumed strictly negative and let $\lambda_0 > 0$. On the n-dimensional torus $M = [-\frac{1}{2},\frac{1}{2}]^n$, we consider the following eigenvalue problem:

$$(\Delta^2 - \lambda_0 + V)\phi = \lambda\phi \quad , \tag{A.1}$$

where the biharmonic operator Δ^2 could be replaced by any elliptic operator. Associated to (A.1) in another eigenvalue problem:

$$(\Delta^2 - \lambda_0)\chi = -\mu V \chi \quad . \tag{A.2}$$

Since $\lambda_0 < 0$, $(\Delta^2 - \lambda_0)^{-1}$ is well defined as an operator on M; setting $\psi = |V|^{\frac{1}{2}} \chi$, (A.2) is therefore equivalent ($V < 0$ and smooth), by a direct computation to:

$$S\psi = |V|^{\frac{1}{2}} (\Delta^2 - \lambda_0)^{-1} |V|^{\frac{1}{2}} \psi = \frac{1}{\mu}\psi \quad . \tag{A.3}$$

that is an eigenvalue problem for the operator S.

Theorem A.1: Birman-Schwinger principle.

For $V < 0$, $\lambda_0 < 0$ and if λ_j (resp. ν_j) are the eigenvalues of $\Delta^2 - \lambda_0 + V$ (resp. $|V|^{\frac{1}{2}} (\Delta^2 - \lambda_0)^{-1} |V|^{\frac{1}{2}}$), we have:

$$\sum_{\lambda_j < 0} 1 \qquad \sum_{\nu_j > 1} 1 \tag{A.4}$$

<u>Proof</u>. From (A.1), we deduce:

$$\lambda = ||\phi||^{-2} \left(||\Delta\phi||^2 - \lambda_0 \; ||\phi||^2 + \quad V(x)|\phi|^2 \, dx\right)$$

$$= ||\phi||^{-2} \left(||\Delta\phi||^2 - \lambda_0 \; ||\phi||^2 - \quad |V(x)| \; |\phi|^2 \, dx\right)$$

$$= ||\phi||^{-2} \quad |V(x)| \; |\phi|^2 \, dx \left(\frac{||\Delta\phi||^2 - \lambda_0 \; ||\phi||^2}{|V(x)| \; |\phi|^2 \, dx} - 1\right)$$

therefore $\lambda < 0$ is equivalent to $\left(\frac{||\Delta\phi||^2 - \lambda_0 \; ||\phi||^2}{|V(x)| \; |\phi|^2 \, dx} - \mu < 1\right)$

that is (since $V < 0$) to $\Delta^2\phi + \mu V\phi = \lambda_0\phi$. The equivalence of (A.2) and (A 3) complete the proof. □

<u>Remark A.1</u>. In the proof of Theorem 2.1, we take $\lambda_0 = -t\alpha$, $V = \text{Min}\,\{q + (1-t)\alpha, 0\}$ $t \in [0,1]$. □

<u>Remark A.2</u>. It is worthwhile to notice some properties of the operator $S = |V|^{1/2}(\Delta^2 - \lambda_0)^{-1} \; |V|^{1/2}$. First of all $S > 0$, by direct computation using Parseval identity. Second S is a compact operator in $H = L^2(M)$, because $(\Delta^2 - \lambda_0)^{-1}$ is compact and $|V|^{1/2}$ is bounded. Finally, for $n < 3$, $(\Delta^2 - \lambda_0)^{-1}$ and hence S is trace classe, i.e., the serie of its eigenvalues is absolutely convergence. □