251
EW
PPP 1871
UC-20 F, G

I-1596

D

2h.285

(25)

MASTER

RADIATIVE ELECTRON CAPTURE IN NONEQUILIBRIUM PLASMAS

BY

H.M. Milchberg and J.C. Weisheit*

JANUARY 19, 1982

PLASMA
PHYSICS
LABORATORY

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

PREPARED FOR THE U.S. DEPARTMENT OF ENERGY,
UNDER CONTRACT DE-AC22-76-CO-3073.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

Radiative Electron Capture in Nonequilibrium Plasmas

H. M. Milchberg and J. C. Weisheit*

Plasma Physics Laboratory, Princeton University

Princeton, New Jersey 08544

PPPL--1871

DE82 011308

ABSTRACT

Formulae have been obtained for the degree of linear polarization of recombination radiation from a homogeneous plasma having an anisotropic electron velocity distribution, $f(\vec{v})$, characterized by an axis of symmetry. Polarization measurements are described which utilize these formulae to determine aspects of the anisotropy such as the symmetry axis direction and the lowest order even angular moments of $f(\vec{v})$. As a special case, if the plasma conforms to a distribution such as a "bi-Maxwellian with drift," one can determine the quantities $u_D/T_\perp$ and $(1/T_\perp - 1/T_\parallel)$ which involve the electron drift speed, and the "perpendicular" and "parallel" electron temperatures.

Also, the radiative recombination rate has been calculated for ions whose speeds are comparable to or greater than the electron thermal speed. The change in the rate is small for thermonuclear products in fusion plasmas, but large for cosmic rays in interstellar plasma.

* Present address: Theoretical Physics Division, Lawrence Livermore National Laboratory, P. O. Box 808, Livermore, CA 94550 USA.

DISCLAIMER

This document contains information which may be classified as "Secret" or "Confidential" under the provisions of the Atomic Energy Act of 1954, as amended, and the regulations thereunder. It is the policy of the United States Government to make available to the public information which is not classified as "Secret" or "Confidential" under the provisions of the Atomic Energy Act of 1954, as amended, and the regulations thereunder. This document is being made available to the public in accordance with this policy. It is the responsibility of the user of this document to determine whether or not the information contained herein is classified as "Secret" or "Confidential" under the provisions of the Atomic Energy Act of 1954, as amended, and the regulations thereunder. If the information is classified as "Secret" or "Confidential", it should be handled accordingly. If the information is not classified as "Secret" or "Confidential", it may be freely disseminated.

DISTRIBUTION

100

leg

I. INTRODUCTION

In plasmas of sufficiently low density and high temperature, radiative electron capture, or radiative recombination, is an important process. The basic quantal process, which is summarized in the Appendix, is well understood, and there exist numerous tables of thermal recombination rate coefficients. In Secs. II and III of this paper two nonequilibrium recombination phenomena are investigated. The first is the net polarization of recombination radiation that is observed when the relative velocity distribution function is anisotropic; the second is the change in the radiative recombination rate coefficient that occurs when the ions are suprathermal with respect to the electrons. Part of our analysis is restricted to electron capture by fully stripped ions, but the extension to more complex systems is straightforward.

II. POLARIZATION OF RECOMBINATION RADIATION

Two often used plasma heating methods--electron cyclotron resonance radiation,¹ and lower hybrid frequency radiation²--direct energy largely into electron motions transverse to a confining magnetic field, and thereby give rise to anisotropies in the electron velocity distribution $f(\vec{v})$. Laser heated plasmas are also commonly characterized by non-Maxwellian distributions.³ Further, solar flares evidently contain streams of accelerated electrons^{4,5} and recently it has been suggested that astrophysical plasma beams carry current.⁶

It has been known for some time that anisotropies in $f(\vec{v})$ result in net linearly polarized free-free (bremsstrahlung) radiation.^{7,8,9} In fact, most of the anisotropies referred to above were identified by means of polarization measurements of free-free continuum radiation. Unfortunately,

such measurements provide limited information about $f(\vec{v})$ because a range of electron velocities contributes to free-free radiation at given frequency ω . In addition, the free-free intensity $J_{ff}(\omega)$ may be less than the intensity $J_{fb}(\omega)$ of free-bound radiation from electron recombinations with highly charged ions ($Z \gg 1$). This is the case, for example, in spectra recorded from the Princeton Large Torus¹⁰ and from some laser heated plasmas.^{11,12}

Except for a brief note several years ago by Paranjape,¹³ it seems not to have been realized that recombination emission also has net linear polarization when $f(\vec{v})$ is anisotropic. Moreover, in the recombination process, only electrons of specific relative energy E contribute to the emission at a given frequency ω . For capture into a level Γ with binding energy χ_Γ , the nonrelativistic, center of mass energy relationship is

$$\hbar\omega = E + \chi_\Gamma . \quad (1)$$

It is clear that, at least in principle, recombination radiation can be a more sensitive diagnostic than bremsstrahlung for plasmas in which $f(\vec{v})$ does not have a simple analytic form. This idea is developed here in some detail.

A. A General Polarization Formula

We assume that the electrons are recombining in a (homogeneous) plasma with stationary ions so that the collision energy is $E = (1/2)m_e v^2$; effects due to ion motions are discussed later. We also assume that a single spectral series dominates the continuum radiation in the observed frequency region. Then, it follows that the number of photons of frequency ω and

polarization $\hat{e}$ emitted per unit time per unit solid angle as a result of recombinations (to level Γ) involving electrons of speed v is

$$\frac{dN(\hat{e}, \omega)}{d\Omega} \propto \int d\Omega(\vec{v}) f(\vec{v}) v \frac{d\sigma_{\Gamma}(\hat{e}, v)}{d\Omega} , \quad (2)$$

where $d\Omega(\vec{v})$ is an element of solid angle in velocity space, and $d\sigma_{\Gamma}/d\Omega$ is the differential cross section for recombination to level Γ . As discussed in the Appendix, it has the general form

$$\frac{d\sigma_{\Gamma}(\hat{e}, v)}{d\Omega} = \frac{\sigma_{\Gamma}(\hat{e}, v)}{4\pi} [1 + \beta_{\Gamma} P_2(\vec{v} \cdot \hat{e}/v)] . \quad (3)$$

For recombination to the K-shell the asymmetry parameter satisfies $\beta_K = 2$ for all ionic potentials. For higher shells, β_{Γ} is given by the formula (A.7). In the Appendix it is shown that, for hydrogenic ions, the L-shell asymmetry parameter, β_L , increases from 8/9 at the recombination edge to 2 as $\omega \rightarrow \infty$ (cf. Table A.1).

Hereafter, we restrict our analysis to situations in which $f(\vec{v})$ has cylindrical symmetry, the axis of which is used to define the z-direction. Then $f(\vec{v})$ can be expanded in a series of Legendre polynomials,

$$f(\vec{v}) = \sum_{n=0}^{\infty} f_n(v) P_n(\cos\theta) , \quad (4)$$

and Eq. (2) reduces to

$$\begin{aligned} \frac{dN}{d\Omega}(\hat{e}, \omega) \propto & (2 - \beta_{\Gamma}) K_1 + 3\beta_{\Gamma} (\sin^2 \psi + \cos^2 \psi \cos^2 \gamma) K_2 \\ & + 3\beta_{\Gamma} \cos^2 \psi \sin^2 \gamma K_3 , \end{aligned} \quad (5)$$

where the angles ψ and γ are identified in Fig. 1 and where the integrals K_i are defined as

$$K_1 = 2\pi \int_{-1}^1 d(\cos\theta) f(\vec{v}) = 4\pi f_0(v) \quad , \quad (6)$$

$$K_2 = 2\pi \int_{-1}^1 d(\cos\theta) f(\vec{v}) \left(\frac{1}{2} \sin^2\theta \right) = 4\pi \left(\frac{1}{3} f_0(v) - \frac{1}{15} f_2(v) \right) \quad , \quad (7)$$

$$K_3 = 2\pi \int_{-1}^1 d(\cos\theta) f(\vec{v}) (\cos^2\theta) = 4\pi \left(\frac{1}{3} f_0(v) + \frac{2}{15} f_2(v) \right) \quad . \quad (8)$$

Now consider a detector placed so as to measure radiation intensities $J_{x''}(\omega)$ and $J_{z''}(\omega)$ with polarization resolved along the x'' and z'' axes shown in Fig. 1. Then

$$J_{z''}(\omega) \propto \left. \frac{dN(\hat{e}, \omega)}{d\Omega} \right|_{\psi=\rho} \quad J_{x''}(\omega) \propto \left. \frac{dN(\hat{e}, \omega)}{d\Omega} \right|_{\psi=\rho+\pi/2} \quad . \quad (9)$$

Upon defining

$$\Pi(\omega) = \frac{J_{z''} - J_{x''}}{J_{z''} + J_{x''}} \quad , \quad (10)$$

as the degree of linear polarization of the recombination radiation, evaluation of Eq. (9) yields the general expression

$$\Pi(\omega) = \frac{3\beta_{\Gamma} \cos 2\rho \sin^2 \gamma (K_3 - K_2)}{2(2 - \beta_{\Gamma})K_1 + 3\beta_{\Gamma}[(1 + \cos^2 \gamma)K_2 + \sin^2 \gamma K_3]}$$

$$\Pi_0 = \frac{3\beta_T \cos 2\rho \sin^2 \gamma f_2}{20f_0 + \beta_T f_2(3 \sin^2 \gamma - 2)} \quad (11)$$

In order for Eq. (11) to be useful, the orientation of the symmetry axis with respect to the detector's line of sight must be known. If the angles ρ and γ are unknown, they can be determined in the laboratory as follows (see Fig. 1). Let Π_0 denote the degree of polarization recorded when the detector is aligned with the coordinates (x'' y'' z''); Π_0 satisfies Eq. (11). Then, let Π_1 denote the result of a measurement made after the (x'' z'') axes have been rotated through an angle Δ_1 counterclockwise about y'' . Then Π_1 satisfies Eq. (11), but with $\rho \rightarrow \rho + \Delta_1$. The equations for Π_0 and Π_1 can be combined to give

$$\tan 2\rho = \frac{\Pi_0 \cos 2\Delta_1 - \Pi_1}{\Pi_0 \sin 2\Delta_1} \quad (12)$$

This is an unambiguous determination of ρ since $-\pi/2 \leq \rho \leq \pi/2$.

In order to determine the angle γ , another measurement Π_2 must be made after (1) the original detector orientation (x'' y'' z'') has been rotated clockwise about y'' through the calculated angle ρ to bring the z'' axis into the (y'' z) plane, and (2) this orientation has been rotated through an angle Δ_2 clockwise about the x axis. Then Π_2 satisfies Eq. (11) but with $\gamma \rightarrow \gamma + \Delta_2$ and $\rho = 0$. The equations for Π_0 and Π_2 can be combined to give

$$\frac{\sin^2 (\gamma + \Delta_2)}{\sin^2 \gamma} = \frac{\Pi_2 (\Pi_0 - \cos 2\rho)}{\Pi_0 (\Pi_2 - 1)} \quad (13)$$

Only the positive root of this quadratic equation is consistent with the combination $\Pi_0(\gamma = 0) = 0$ and $0 < \Delta_2 < \pi$. Thus

$$\cot \gamma = -\cot \Delta_2 + \frac{1}{\sin \Delta_2} \left(\frac{\Pi_2(\Pi_0 - \cos 2\rho)}{\Pi_0(\Pi_2 - 1)} \right)^{1/2}. \quad (14)$$

This is an unambiguous determination of γ since $0 \leq \gamma \leq \pi$.

B. Effect of Ion Motions, Ion Charge, and Net Plasma Motion

The effect of ion motions is to introduce some ambiguity in the inferred speeds of recombining electrons that give rise to radiation of a certain measured frequency. For electrons of a given speed v_e , one would observe a distribution of frequencies about ω_0 , the photon frequency for recombination with stationary ions. To calculate this frequency spreading, use is made of the nonrelativistic, lab frame momentum and energy conservation equations

$$m_e \vec{v}_e + m_i \vec{v}_i = m \vec{v} + \hbar \vec{k}, \quad (15)$$

$$\frac{1}{2} m_e v_e^2 + \frac{1}{2} m_i v_i^2 + \chi_T = \hbar \omega + \frac{1}{2} m v^2, \quad (16)$$

along with the conditions $\hbar \omega / (m_i c^2)$, v_i / c , v_e / c and $m_e / m_i \ll 1$. Here $m = m_e + m_i$ is the mass of the resulting ion or atom and $\vec{v}$ is its velocity. Defining $\hbar \omega_0 = (1/2) m_e v_e^2 + \chi_T$, solving Eqs. (15) and (16) for $(\omega - \omega_0)$ and then averaging $(\omega - \omega_0)^2$ over a thermal ion velocity distribution yields for the spectral broadening,

$$\frac{\langle (\omega - \omega_o)^2 \rangle^{1/2}}{\omega_o} \lesssim \frac{1/\sqrt{3} m_e v_{eth}}{1/2 m_e v_e^2 + \chi_T}, \quad (17)$$

where $v_{eth}^2 = 3kT_1/m_1$. It is readily seen that for sufficiently low ion temperature, high ion mass and high χ_T , the frequency broadening is negligible, and especially so near the recombination edge ($v_e \gtrsim 0$). Moreover, this expression for the spreading is an overestimate since the slowest ions have the largest recombination cross sections, and here it has been assumed that recombination occurs with equal weight throughout the ion distribution.

High ion charges are favorable for measurements of recombination radiation polarization. In plasmas, large Z ions are generally more massive, so for a given ion temperature, the thermal velocity is smaller, decreasing the frequency spreading of Eq. (17). In addition, recombination edge frequencies scale roughly as Z^2 , further reducing this broadening and enhancing the ratio of recombination intensity to bremsstrahlung intensity, $J_{fb}(\omega)/J_{ff}(\omega) \sim \hbar\omega/kT_e \exp(\hbar\omega/kT_e)$, near the recombination edge of a highly stripped ion.

When the center of mass of the plasma is not at rest with respect to the observer (e.g., a cosmic plasma), the radiation field components at the source and those at the detector are related by a Lorentz transformation. However, the net effect of the transformation is to leave the degree of polarization Π invariant.¹⁴

C. Specific Polarization Formulae

The coefficients f_0 and f_2 will now be evaluated for some specific velocity distributions. Two frequently used analytic forms are a "bi-Maxwellian" distribution, with different temperatures $T_{\parallel}$ and $T_{\perp}$

characterizing electron thermal motions parallel and perpendicular to a symmetry axis; and a Maxwellian with a net drift $\vec{u}_D$ which defines the symmetry axis. These two forms can be combined into a "bi-Maxwellian plus axial drift" distribution,

$$f(\vec{v}) = \left(\frac{m_e}{2\pi\kappa T_e} \right)^{3/2} \frac{T_{\parallel}}{T_{\perp}} \exp \left(- \frac{m_e v^2 \sin^2 \theta}{2\kappa T_{\perp}} - \frac{m_e (v \cos \theta - u_D)^2}{2\kappa T_{\parallel}} \right) , \quad (18)$$

and the resulting expressions for f_0 and f_2 are, to within the same proportionality constant,

$$f_0 \propto S(\lambda, \xi) , \quad (19)$$

$$f_2 \propto - \frac{5}{2} S(\lambda, \xi) - \frac{15}{2} \frac{\partial S(\lambda, \xi)}{\partial \lambda} , \quad (20)$$

where

$$\lambda = [2m_e(\hbar\omega - \chi_{\Gamma})]^{1/2} \frac{u_D}{\kappa T_{\parallel}} , \quad \xi = \left(\frac{\hbar\omega - \chi_{\Gamma}}{\kappa T_{\parallel}} \right) \left(1 - \frac{T_{\parallel}}{T_{\perp}} \right) , \quad (21)$$

are two dimensionless variables, and

$$S(\lambda, \xi) = \int_{-1}^1 dx \exp(-\xi x^2 + \lambda x) = 2 \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} \frac{d^{2n}}{d\lambda^{2n}} \left(\frac{\sinh \lambda}{\lambda} \right) . \quad (22)$$

In what follows, it is assumed that the orientation angles (ρ , γ) have been determined so that it is necessary to consider only the plasma's intrinsic polarization

$$\tilde{\Pi} = \Pi(\rho = 0, \gamma = \frac{\pi}{2}) = \frac{3\beta_{\Gamma} f_2}{20 f_0 + \beta_{\Gamma} f_2} . \quad (23)$$

If $T_{\parallel} \approx T_{\perp}$ and/or $\hbar\omega \approx \chi_{\Gamma}$, then ξ is small and the coefficients f_n may be taken to first order in ξ , resulting in the expression

$$\tilde{\Pi} = \frac{3\beta_{\Gamma}[a_1 - (\xi + 3)a_2 + 3\xi a_3]}{4(\beta_{\Gamma} - 2)(a_1 - \xi a_2) + 3\beta_{\Gamma}[-a_1 + (\xi - 1)a_2 + \xi a_3]} \quad (24)$$

where $a_1 = \sinh\lambda/\lambda$, $a_2 = d^2 a_1/d\lambda^2$, and $a_3 = d^4 a_1/d\lambda^4$.

In the limit $\xi \rightarrow 0$, Eq. (24) reduces to an expression for the degree of linear polarization due to a net drift of plasma electrons with respect to the ions,

$$\tilde{\Pi}_D = \frac{3\beta_{\Gamma}[3\lambda \cosh\lambda - (3 + \lambda^2) \sinh\lambda]}{2(\beta_{\Gamma} - 2)\lambda^2 \sinh\lambda + 3\beta_{\Gamma}[\lambda \cosh\lambda - (1 + \lambda^2) \sinh\lambda]} \quad (25)$$

The asymptotic behaviors of $\tilde{\Pi}_D$ are

$$\tilde{\Pi}_D \underset{\lambda \rightarrow \infty}{\sim} \frac{3\beta_{\Gamma}}{\beta_{\Gamma} + 4}, \quad \tilde{\Pi}_D \underset{\lambda \rightarrow 0}{\sim} \frac{\beta_{\Gamma}\lambda^2}{20} \quad (26)$$

The expression for $\tilde{\Pi}_D$ for the special case $\beta_{\Gamma} = 2$ (K-shell capture) is plotted in Fig. 2.

If the drift velocity is small, then the coefficients f_n may be taken to lowest order in λ , using the expression

$$S(\lambda \ll 1, \xi) \approx 2 \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} \left(\frac{1}{2n+1} + \frac{\lambda^2}{2(2n+3)} \right) \quad (27)$$

The limiting case $\lambda \rightarrow 0$ does not yield a closed form expression for $\tilde{\Pi}_{2T}$, the degree of linear polarization of recombination radiation from a bi-Maxwellian plasma. However, the series in

$$\tilde{\Pi}_{2T} = \frac{3\beta_T \sum_{n=0}^{\infty} (-\xi)^n / (2n+3)(2n+1)n!}{\sum_{n=0}^{\infty} [2 + \beta_T n / (2n+3)] [(-\xi)^n / (2n+1)n!]} , \quad (28)$$

converge rapidly. The asymptotic behaviors of $\tilde{\Pi}_{2T}$ are

$$\tilde{\Pi}_{2T} \underset{\xi \rightarrow \pm \infty}{\sim} \pm \frac{3\beta_T}{\beta_T - 8} , \quad \tilde{\Pi}_{2T} \underset{\xi \rightarrow 0}{\sim} -\frac{\beta_T \xi}{10} . \quad (29)$$

Figure 3 is a plot of $\tilde{\Pi}_{2T}$ vs ξ for K-shell capture ($\beta_T = 2$).

D. Comments on Polarization Measurements

It has been assumed that a single spectral series dominates plasma emission in the observed frequency range. Often this is not the case. But, unless the characteristics of the background radiation change appreciably across the recombination edge of interest, measurements of intensity made above and below the edge can be used to subtract out the background so that Π can be determined for that particular recombination transition.

In general, if polarization measurements were consistent with predictions of a particular model distribution, then the parameters λ and/or ξ could be determined. If the measured polarization values were not in accord with a known distribution, Eq. (11) indicates that a plot of $\Pi(\omega)$ vs ω should yield f_2/f_0 as a function of ω , and hence of v , through Eq. (1). The lowest order even angular moments of $f(\vec{v})$ are thus determinable. Further, since the formulae which give the orientation of the symmetry axis are independent of the form of $f(\vec{v})$, a set of polarization measurements should reveal that orientation. A particular application of this would be the determination of magnetic field line direction.

It is interesting to compare the polarization of bremsstrahlung with that of recombination radiation. One example is provided by the calculation

of Raimbault and Shohet,¹⁵ in which bremsstrahlung polarization was determined for a bi-Maxwellian plasma with $kT_{\parallel} = 1.5$ KeV and $kT_{\perp} = 15$ KeV, conditions existing in their magnetic mirror device. At $\hbar\omega = kT_{\perp}$, they found that the polarization of free-free radiation was about 45 percent. For these same plasma conditions, it is seen from Fig. 3 that, at the same frequency, the polarization of (K-shell) recombination radiation exceeds this amount as long as the threshold energy $\chi_T \lesssim 11$ KeV. This inequality is satisfied for most ions of interest.

Finally, with regard to the existence of velocity anisotropies, two interesting astrophysical applications of polarization measurements are noted. (1) Davidson and McCray¹⁶ commented that there may be substantial electric currents in the jets of the bizarre galactic object SS 433. There is strong hydrogen Balmer emission from these luminous jets, and determination of its polarization could confirm their suggestion. (2) Hinata¹⁷ has proposed a difficult spectroscopic observation of a combined Zeeman-Stark pattern in HeI lines to determine if an energetic electron beam (instead of a thermal wave) transports energy to the chromosphere at the beginning of a solar flare. Polarization measurements of the strong recombination continuum radiation (cf. Ref. 18) may also prove useful for inferring this beam's existence.

III. RADIATIVE RECOMBINATION OF SUPRATHERMAL IONS

In thermal equilibrium, plasma ions have a mean speed much less than that of the electrons, and it usually suffices to treat an electron-ion interaction as if the ion was at rest. However, this simplification is not correct in fusion plasmas, wherein the fusion products are "born" with speeds comparable to or much greater than the electron thermal speed. The

same problem arises whenever highly energetic ions (e.g., stripped neutral beam particles, cosmic rays) interact with a relatively cool background plasma. In this section it is determined to what extent recombination, which favors a low relative speed $v = |\vec{v}_i - \vec{v}_e|$ of the colliding particles, is inhibited by large ion speeds.

It is assumed that the electron velocity distribution $f(\vec{v}_e)$ is Maxwellian at a temperature T_e . Also, Eq. (A.5), the cross section for bare ion recombination to level n , is approximated by Kramers' formula,

$$\sigma_n(v) = \sum_l \sigma_{nl}(v) \approx \frac{2^7 \pi \alpha_0^2 (n \hbar \omega_n)^3}{3\sqrt{3} \hbar \omega (Z m_e v c)^2} . \quad (30)$$

This approximation is valid for $\omega < 10 \omega_1$. Because of the assumed isotropy of the electrons, the rate coefficient for recombination to level n ,

$$B_n(v_i, T_e) = \int d^3 v_e f(\vec{v}_e) |\vec{v}_i - \vec{v}_e| \sigma_n(|\vec{v}_i - \vec{v}_e|) , \quad (31)$$

is independent of the direction of $\vec{v}_i$.

The angular parts of the integral Eq. (31) can be performed immediately to yield an expression in terms of the dimensionless variables

$$x_i^2 = \frac{m_e v_i^2}{2 \kappa T_e} , \quad y_n^2 = \frac{\hbar \omega_n}{\kappa T_e} , \quad (32)$$

namely,

$$B_n(x_i, T_e) = \frac{2^9 \pi^2 \alpha_0^2 (n \hbar \omega_n)^3}{3\sqrt{3} (Zc)^2} \left(\frac{1}{2 m_e \kappa T_e} \right)^{3/2} I(x_i, y_n) . \quad (33)$$

The remaining integral,

$$I(a, b) = \frac{1}{ab} \int_0^{\infty} dx x \exp(-x^2) \{ \arctan[(x+a)^2/b^2]^{1/2} - \arctan[(x-a)^2/b^2]^{1/2} \}$$

$$= \frac{\exp(-a^2)}{a} \int_0^{\infty} d\tau \frac{\exp(-\tau^2) \sinh(2a\tau)}{(\tau^2 + b^2)}, \quad (34)$$

can be evaluated term by term:

$$I(a, b) = \exp(-a^2) \left(\frac{\sin(2ab)}{2ab} I(0, b) + \sum_{m=1}^{\infty} \frac{(-4a^2 b^2)^m}{(2m+1)!} \sum_{p=1}^m \frac{(p-1)!}{(-b^2)^p} \right). \quad (35)$$

In this expression

$$I(0, b) \equiv \lim_{a \rightarrow 0} I(a, b) = \exp(b^2) E_1(b^2), \quad (36)$$

where E_1 is the first exponential integral function.

Two limiting forms of Eq. (35) are of interest. When $b \ll 1$, $E_1(b^2) \sim -\ln b^2$ so that

$$I(a, b \ll 1) \approx \exp(-a^2) \ln(1/b^2). \quad (37)$$

When b is large and the $p = 1$ term dominates,

$$I(a, b \gg 1) \approx \exp(-a^2) \left[\frac{\sin(2ab)}{2ab} I(0, b) + \frac{1}{b^2} \left(1 - \frac{\sin(2ab)}{2ab} \right) \right]$$

$$\approx \exp(-a^2)/b^2 \quad (38)$$

since $I(0, b \gg 1) \approx b^{-2}$. Also, it is easily verified that in the

limit $v_i \rightarrow 0$, the usual hydrogenic formula is regained:

$$B_n(0, T_e) = \frac{2^6 \alpha^3 a_0^2 c n}{3} \left(\frac{\hbar \omega_n}{\kappa T_e} \right)^2 \left(\frac{\pi \kappa T_e}{3 m c^2} \right)^{1/2} \exp\left(\frac{\hbar \omega_n}{\kappa T_e}\right) E_1\left(\frac{\hbar \omega_n}{\kappa T_e}\right). \quad (39)$$

For α particles produced in D-T fusion plasmas, the parameter $y_n \ll 1$ for all n . For $(1/2) m_\alpha v_\alpha^2 = 4$ MeV, $x_\alpha \approx (0.5 \text{ KeV}/\kappa T_e)^{1/2} < 1$ and a satisfactory approximation is

$$\frac{B_n(x_\alpha, T_e)}{B_n(0, T_e)} \approx \exp(-x_\alpha^2) \left(1 - \frac{2x_\alpha^2}{3 \ln(y_n^2)} \right) \quad (40)$$

It is interesting to note that this x_α dependence is the same as that for the ratio of the number of electrons in each case ($v_\alpha \neq 0$ and $v_\alpha = 0$) which, relative to the ion, have speeds smaller than some maximum value $\hat{v} \approx (2\kappa T_e/m_e)^{1/2} / \ln(\kappa T_e/\hbar \omega_n)$ for efficient electron capture. Because x_α typically is $\lesssim 0.1$, the α -particle's speed is not high enough to reduce its recombination rate coefficient very much.

On the other hand, the rate coefficient for free electron capture by a cosmic ray particle of energy E_{CR} traversing interstellar plasma ($\kappa T_e \gtrsim 1$ eV) is reduced significantly by its motion. In this case, relevant to low energy cosmic rays, the parameter y_n^2 is large for the lower states ($n \lesssim 4$) and $x_{CR}^2 \approx 5 E_{CR} (\text{MeV/amu}) > 1.21$. Therefore, from Eqs. (38) and (39) it follows that

$$\frac{B_n(x_{CR}, T_e)}{B_n(0, T_e)} \approx \exp(-x_{CR}^2) \ll 1. \quad (41)$$

Here, the x_α dependence corresponds simply to the Boltzmann factor for electrons having speeds $v_e \approx v_{CR}$.

ACKNOWLEDGMENTS

The topic of Sec. II arose in conversations one of the authors (J. C. W.) had with Dr. Eric Silver. Both he and Dr. Doug Post are thanked for useful discussions.

This research was supported by the United States Department of Energy under Contracts DE-AC02-76-CHO3073 and W-7405-ENG-48. H. Milchberg acknowledges the support of the Natural Sciences and Engineering Research Council of Canada.

APPENDIX: THE RECOMBINATION CROSS SECTION

In the dipole approximation, the differential cross section for an electron and an ion with reduced mass μ and relative momentum $\hbar\vec{q} = \mu\vec{v}$ to recombine and emit a photon of polarization $\hat{e}$ and momentum $\hbar\vec{k}$ within the differential solid angle $d\Omega$ about $\vec{k}$ is

$$d\sigma(\hat{e}, q) = \frac{k^3}{2\pi\hbar} |\langle \hat{e}, \text{ion} | \hat{e} \cdot \vec{d} | \text{atom} \rangle|^2 d\Omega, \quad (\text{A.1})$$

where $\vec{d}$ is the instantaneous dipole moment of the electron-ion system, and where the continuum wave function is normalized to unit flux.

When the bound state is specified by hydrogenic quantum numbers, e.g., $|\text{atom}\rangle = |nlm\rangle$ and the continuum state is expanded in partial waves,

$$|\hat{e}, \text{ion}\rangle = \frac{4\pi}{q} \left(\frac{\pi\mu}{2\hbar q} \right)^{1/2} \sum_{\lambda, \nu} i^\lambda \exp(i\eta_\lambda) Y_{\lambda\nu}^*(\hat{q}) Y_{\lambda\nu}(\hat{r}) R_{q\lambda}(r). \quad (\text{A.2})$$

Equation (A.1) becomes

$$d\sigma_{nl}(\hat{e}, q) = \left(\frac{16\pi^3 e^2 k^3 \mu}{3\hbar^2 q^3} \right) \sum_m \left| \sum_{\lambda=\pm 1} (-i)^\lambda \exp(-i\eta_\lambda) [\max(\lambda, l)]^{1/2} \rho_\lambda \right. \\ \left. \times \sum_{\nu M} Y_{\lambda\nu}(\hat{q}) Y_{lM}(\hat{e}) \begin{pmatrix} \lambda & 1 & l \\ \nu & M & m \end{pmatrix} \right|^2 d\Omega. \quad (\text{A.3})$$

In this equation $\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is a 3-j symbol, $Y_{\lambda\nu}$ is a spherical harmonic, $\rho_\lambda = \langle q\lambda | r | nl \rangle$ is a radial matrix element, and

$$\eta_\lambda = \arg \Gamma(1 + \lambda - i \frac{Z\alpha c}{v}), \quad (\text{A.4})$$

is the coulomb phase shift. Note that the cross section has been summed

with respect to degenerate atomic states having different azimuthal quantum numbers m . Integration with respect to solid angle yields the cross section for recombination into level $n\ell$, with the emission of a photon of specified frequency and polarization,

$$\sigma_{n\ell}(\hat{e}, q) = \left(\frac{4\pi^2 e^2 \hbar^3}{3n^2 q^3} \right) \sum_{\lambda=\ell\pm 1} \max(\lambda, \ell) |\rho_\lambda|^2. \quad (A.5)$$

The total cross section $\sigma_{n\ell}(q)$, which involves a summation with respect to polarization states, is simply twice this expression.

Cooper and Zare¹⁹ showed that the equation for the differential cross section Eq. (A.3) can be simplified to yield

$$d\sigma_{n\ell}(\hat{e}, q) = \sigma_{n\ell}(\hat{e}, q) [1 + \beta_{n\ell} P_2(\hat{q} \cdot \hat{e}/q)] d\Omega/4\pi, \quad (A.6)$$

where $P_2(x)$ is the second order Legendre polynomial and where the asymmetry parameter

$$\beta_{n\ell} = \frac{\ell(\ell-1)\rho_{\ell-1}^2 + (\ell+1)(\ell+2)\rho_{\ell+1}^2 - 6\ell(\ell+1)\rho_{\ell-1}\rho_{\ell+1} \cos(\eta_{\ell+1} - \eta_{\ell-1})}{(2\ell+1)[\ell\rho_{\ell-1}^2 + (\ell+1)\rho_{\ell+1}^2]}, \quad (A.7)$$

is defined in terms of continuum phase shifts η and radial matrix elements ρ . In general, $\beta_{n\ell}$ varies between -1 and 2, but for s states ($\ell = 0$) $\beta_{ns} = 2$, independent of η , ρ , and q . Thus for the K-shell

$$\frac{d\sigma_K}{d\Omega}(\hat{e}, q) = \frac{\sigma_{1s}(\hat{e}, q)}{4\pi} [1 + 2P_2(\hat{q} \cdot \hat{e}/q)], \quad (A.8)$$

while for the L-shell

$$\begin{aligned} \frac{d\sigma_L(\hat{e}, q)}{d\Omega} &= \frac{\sigma_{2s}(\hat{e}, q)}{4\pi} [1 + 2P_2(\hat{q} \cdot \hat{e}/q)] \\ &\quad + \frac{\sigma_{2p}(\hat{e}, q)}{4\pi} [1 + \beta_{2p} P_2(\hat{q} \cdot \hat{e}/q)] \\ &= \frac{\sigma_L(\hat{e}, q)}{4\pi} [1 + \beta_L P_2(\hat{q} \cdot \hat{e}/q)] \quad . \end{aligned} \quad (A.9)$$

The total L-shell recombination cross section is $\sigma_L = \sigma_{2s} + \sigma_{2p}$, and the effective asymmetry parameter for the L-shell is

$$\beta_L = 2 + (\beta_{2p} - 2) \frac{\sigma_{2p}}{\sigma_L} \quad . \quad (A.10)$$

In order to calculate β_{2p} we use the general result

$$\eta_{l+1} - \eta_{l-1} = \arctan \left(\frac{(2l+1)\tau_n}{\tau_n^2 - l(l+1)} \right) \quad , \quad (A.11)$$

where the parameter

$$\tau_n = \frac{Zac}{v} = n \left(\frac{\omega_n}{\omega - \omega_n} \right)^{1/2} \left(\frac{\mu}{m_e} \right) \quad (A.12)$$

is defined in terms of the photon energy $\hbar\omega$ and the binding energy $\hbar\omega_n$ of the n^{th} level. In addition, Burgess' formulae²⁰ for the radial matrix elements ρ_λ are used to obtain for the 2p state

$$\frac{\rho_2}{\rho_0} = 2 \left(\frac{\omega + 3\omega_2}{\omega} \right)^{1/2} \quad , \quad (A.13)$$

and

$$\frac{\sigma_{2p}}{\sigma_L} = \frac{(\omega_2/\omega)(3 + 8\omega_2/\omega)}{(1 + 3\omega_2/\omega) + (\omega_2/\omega)(3 + 8\omega_2/\omega)} \quad (A.14)$$

Altogether, Eqs. (A.7) and (A.10) - (A.14) yield the values of β_{2p} and β_L listed in Table A.I.

For simplicity, only K and L-shell capture of an electron by a bare nucleus has been considered. More complicated ions and higher shells can be treated by generalization of the equations given here.

TABLE A.1.

HYDROGENIC ASYMMETRY PARAMETERS

ω/ω_2	β_{2p}	β_L
1.00	0.48485	0.88889
1.02	0.49186	0.89977
1.04	0.49880	0.91047
1.06	0.50566	0.92101
1.08	0.51246	0.93138
1.10	0.51917	0.94160
1.15	0.53566	0.96646
1.20	0.55172	0.99038
1.25	0.56738	1.0134
1.30	0.58263	1.0356
1.40	0.61202	1.0777
1.50	0.64000	1.1169
1.75	0.70440	1.2039
2.00	0.76190	1.2778
2.50	0.86022	1.3960
2.90	0.92615	1.4696
3.00	0.94118	1.4857
3.10	0.92486	1.4863
3.50	0.86486	1.4909
4.00	0.80000	1.5000

REFERENCES

- ¹D. G. S. Greene and J. L. Shohet, *Plasma Phys.* 15, 5 (1973).
- ²S. B. Litvin and V. T. Tikhonchuk, *Sov. J. Plasma Phys.* 6, 183 (1980).
- ³G. Bekefi, C. Deutsch, and B. Yaakobi, in *Principles of Laser Plasmas*, ed. G. Bekefi, Wiley-Interscience, New York (1976), pp 596-602.
- ⁴G. A. Dulk, D. B. Melrose, and S. F. Smerd, *Proc. Astron. Soc. Australia* 3, 243 (1978).
- ⁵A. G. Emslie and J. C. Brown, *Astrophys. J.* 237, 1015 (1980).
- ⁶G. Benford, *Mon. Not. Roy. Astron. Soc. (London)* 183, 29 (1978).
- ⁷R. L. Gluckstern and M. H. Hull, *Phys. Rev.* 90, 1030 (1953).
- ⁸A. A. Korchak, *Sov. Phys. Doklady* 12, 192 (1967).
- ⁹K. Eidmann, *Plasma Phys.* 17, 121 (1975).
- ¹⁰K. Brau, S. Von Goeler, M. Bitter, R. D. Cowan, D. Eames, K. Hill, N. Sauthoff, E. Silver, and W. Stodiek, *Phys. Rev. A.* 22 (6), 2769 (1980).
- ¹¹M. H. Key, J. G. Lunney, J. M. Ward, R. G. Evans, and P. T. Rumsby, *J. Phys. B.* 12, L213 (1979).
- ¹²G. Charatis, D. C. Slater, F. J. Mayer, J. A. Tarvin, G. E. Busch, D. L. Matthews, L. Koppel, in *Low Energy X-ray Diagnostics - 1981*, ed. D. Attwood, B. Henke, A.I.P., New York (1981).
- ¹³V. Paranjape, *Proc. Roy. Soc.* 69B, 765 (1956).
- ¹⁴L. D. Landau and E. M. Lifschitz, *Classical Theory of Fields*, 3rd. ed., Pergamon Press, Oxford (1971), pp. 119-123.
- ¹⁵P. A. Raimbault and J. L. Shohet, *Phys. Fluids* 15, 1477 (1972).
- ¹⁶K. Davidson and R. McRay, *Astrophys. J.* 241, 1082 (1980).
- ¹⁷S. Hinata, *Astrophys. J.* 234, 363 (1979).
- ¹⁸J. T. Jeffries and F. Q. Orrall, *Astrophys. J.* 133, 946 (1961).

¹⁹J. Cooper and R. N. Zare, in Atomic Collision Processes, ed.

S. Geltman, K. T. Manthappa, W. E. Brittin, Gordon and Breach, New York
(1969) pp. 317-335.

²⁰A. Burgess, Mem. Roy. Astron. Soc. (London) 69, 1 (1964).

²¹V. L. Ginzburg, Elementary Processes for Cosmic Ray Astrophysics,
Gordon and Breach, New York (1969), Section I.1.

Figure Captions

Fig. 1. Geometry of observation of recombination radiation. The line of sight lies in the yz plane.

Fig. 2. Degree of linear polarization vs. λ for the drift case.

Fig. 3. Degree of linear polarization vs. ξ for the two temperature case.

#81T0371

Fig. 1

Fig. 2

#81T0372

Fig. 3