BETATRON OSCILLATIONS IN A FREE ELECTRON LASER *

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Abstract

A dispersion relation is used to study the effect of betatron oscillations on the gain of a guided mode in a free electron laser operating in the high-gain regime before saturation. We assume the electron beam to be matched to the natural focussing of a helical wiggler, and our treatment includes the effects of the finite energy spread and emittance of the electron beam, the transverse gradient in the wiggler, and the variation of the electromagnetic wave transverse phase experienced by the electrons due to betatron oscillations.

1. Introduction

This work is motivated by interest in the design of a single pass free electron laser operating in the wavelength region below $1000\text{\AA}$. First estimates of the gain can be made using an idealized one-dimensional theory as developed, for example, by Bonifacio et al. [1], and three-dimensional diffraction effects can be included following Moore's [2] discussion of guided modes. The start-up from shot noise in the electron beam has been treated within the one-dimensional theory by Kim [3] and Wang and Yu, [4] and including three-dimensional diffraction effects by Kim [5] and Krinsky and Yu [6].

An important constraint on how short a radiation wavelength is practical results from the gain reduction due to the transverse emittance of the electron beam. This problem has been studied by Goldstein et al. [7] using numerical simulation techniques. Here, we consider the complementary approach of

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using Vlasov-Maxwell equations to derive a dispersion relation determining the gain in the exponential regime before saturation. We consider a helical wiggler with no external focussing and assume the electron beam to be matched to the natural focussing of this magnet. It is found that the transverse electromagnetic phase variation experienced by the electrons due to the betatron motion will not have a significant effect, if the phase variation is small in one gain length. Also, the condition for negligible gain reduction due to the transverse emittance is that the rms emittance be less than the radiation wavelength divided by 2π times the betatron phase advance in one gain length.

2. Dispersion Relation

In the absence of external focussing, the vector potential in a helical wiggler is well-approximated for small transverse displacements from the wiggler axis by [8]

$$\vec{A} = A_w \left[\hat{x} \left(1 + \frac{k_w^2 y^2}{2} \right) \cos k_w z + \hat{y} \left(1 + \frac{k_w^2 x^2}{2} \right) \sin k_w z \right]. \quad (1)$$

After averaging over the fast wiggler oscillations [9], the transverse electron motion can be described by harmonic betatron oscillations in the transverse displacement $\vec{R}$:

$$\frac{d^2 \vec{R}}{dz^2} = -k_\beta^2 \vec{R}, \quad (2)$$

where $k_\beta = K k_w / \gamma \sqrt{2}$ is the betatron wave number and $K = e A_w / mc$ is the field strength parameter. The electron energy in units of its rest mass is denoted γ , and we consider $\gamma \gg 1$ so space charge effects are negligible.

When the radiation field is ignored, the unperturbed electron distribution f_0 is determined by the Vlasov equation:

$$\frac{\partial f_0}{\partial t} + v_{||} \frac{\partial f_0}{\partial z} + \frac{\vec{P}}{m\gamma} \cdot \frac{\partial f_0}{\partial \vec{R}} - mc^2 \gamma k_\beta^2 \vec{R} \cdot \frac{\partial f_0}{\partial \vec{P}} = 0, \quad (3)$$

where $\vec{P} = \gamma m \vec{v} - e \vec{A}$ is the canonical momentum, and the longitudinal velocity is approximately given by

$$v_{||}/c \approx 1 - \frac{1+K^2}{2\gamma^2} - \frac{1}{2} \left[k_\beta^2 R^2 + P^2/(\gamma mc)^2 \right]. \quad (4)$$

The matched electron beam is specified by the condition that f_0 be independent of z and t , hence

$$f_0 = f_0(k_\beta^2 x^2 + P_x^2/(\gamma mc)^2, k_\beta^2 y^2 + P_y^2/(\gamma mc)^2). \quad (5)$$

We now introduce the circularly polarized radiation field with electric field

$$\vec{E} = \frac{1}{\sqrt{2}} E e^{ik_0 z - i\omega_0 t} \left(\frac{\hat{x} + i\hat{y}}{\sqrt{2}} \right) + \text{c.c.}, \quad (6)$$

where E is a slowly varying complex amplitude and $k_0 = \omega_0/c = 2\gamma_0^2 k_w/(1+K^2)$ is the resonant radiation wavenumber. It can be demonstrated that if $v_{||}/c$, K/γ and $k_w R$ are small, then to lowest order in the electric field amplitude E , the electron distribution in the presence of the radiation field approximately satisfies the Vlasov equation

$$\frac{\partial f}{\partial t} + v_{||} \frac{\partial f}{\partial z} + \frac{\vec{p}}{m\gamma} \cdot \frac{\partial f}{\partial \vec{R}} - \gamma mc^2 k_\beta^2 \vec{R} \cdot \frac{\partial f}{\partial \vec{P}} + \dot{\gamma} \frac{\partial f}{\partial \gamma} = 0, \quad (7)$$

where

$$\dot{\gamma} = - \frac{e^2 A_w}{2m^2 c^2 \gamma} \left[e^{i(k_0 + k_w)(z - v_0 t)} E + \text{c.c.} \right], \quad (8)$$

and $v_0 = v_{||}(\gamma = \gamma_0, R=0, P=0)$. It is useful to transform to the dimensionless variables [6]

$$\tau = k_w z, \quad \zeta = (k_0 + k_w)(z - v_0 t), \quad \vec{r} = \sqrt{2k_0 k_w} \vec{R}, \quad \vec{p} = d\vec{r}/d\tau, \\ F = e^{-i\zeta} f, \quad k = k_\beta/k_w.$$

The wave equation in the paraxial approximation and the linearized Vlasov equation can now be written as

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \zeta} - i\nabla_\perp^2 \right) E = D_1 \int \frac{d\gamma}{\gamma} d^2 p F, \quad (9)$$

$$\frac{\partial F}{\partial \tau} + \eta \left(i + \frac{\partial}{\partial \zeta} \right) F + \vec{p} \cdot \frac{\partial F}{\partial \vec{r}} - k^2 \vec{r} \cdot \frac{\partial F}{\partial \vec{p}} = \frac{D_2}{\gamma} \frac{\partial f_0}{\partial \gamma} E, \quad (10)$$

with $D_1 = n_0 \mu_0 e^2 c^2 A_w / 2m\omega_w$, $D_2 = e^2 A_w / 2m^2 c^2 \omega_w$, and

$$\eta = \frac{v_{||} - v_0}{c - v_0} \approx 2\left(\frac{\gamma - \gamma_0}{\gamma_0}\right) - \frac{1}{4} (p^2 + k^2 r^2) . \quad (11)$$

The Laplacian $\nabla_{\perp}^2$ corresponds to the dimensionless vector $\vec{r}$, and the equilibrium distribution f_0 is taken such that the spatial density $g_0(r) = \int dy d^2p f_0$ is maximum at $r=0$ with $g_0(0) = 1$, hence, n_0 appearing in the expression for D_1 is the peak electron density.

We consider the equilibrium distribution to be of the form

$$f_0 = h(\gamma) u(p^2 + k^2 r^2) . \quad (12)$$

The Fourier transforms, $\partial/\partial r \rightarrow -i\Omega$ and $\partial/\partial \zeta \rightarrow iq_{||}$, are employed in Eqs. (9) and (10). Then using Eq. (10) we solve for F in terms of E and use the result in Eq. (9) to derive the dispersion relation [10]:

$$\begin{aligned} (-i\Omega + iq_{||} - i\nabla_{\perp}^2) E = D_1 D_2 \int \frac{d\gamma}{\gamma^2} h'(\gamma) \int d^2p u(p^2 + k^2 r^2) \\ * \int_{-\infty}^0 ds e^{-i\alpha s} E(\vec{r} \cos ks + \frac{\vec{p}}{k} \sin ks) , \end{aligned} \quad (13)$$

where $\alpha = \Omega - \eta (1 + q_{||})$. For simplicity, we use the same symbol E for the function and its Fourier transform.

This integrodifferential equation is an eigenvalue problem determining coherent frequencies Ω and eigenfunctions E . As a consequence of rotational invariance about the z -axis, the eigenfunctions can be seen to be of the form $E = H(r) e^{im\theta}$, where r, θ are polar coordinates specifying $\vec{r}$. The appearance of $E(\vec{r} \cos ks + \frac{\vec{p}}{k} \sin ks)$ rather than $E(\vec{r})$ on the right-hand side of Eq. (13) describes the variation of the transverse electromagnetic phase seen by the electrons due to the betatron motion. When the transverse phase change seen by an electron in one gain length l_G is small, the variation of the transverse phase does not significantly reduce the gain and the dispersion relation simplifies. There are two cases to consider. If $k_{\beta} l_G < 1$, i.e. the betatron phase advance in one gain length is small, the condition for small transverse electromagnetic phase change is

$$(k_{\beta G}^1) \frac{r_e}{r_{EM}} \ll 1, \quad (14)$$

where r_e characterizes the transverse size of the electron beam and r_{EM} that of the guided mode. If $k_{\beta G}^1 > 1$, the condition for small phase change is

$$\frac{r_e}{r_{EM}} \ll 1, \quad (15)$$

which assures that even at the maximum of the betatron oscillation, the electrons are still well inside the guided mode. In either case, we can use the series expansion:

$$E(\vec{r} + \vec{\delta}(s)) = E(\vec{r}) + \vec{\delta} \cdot \frac{\partial E(\vec{r})}{\partial \vec{r}} + \dots \quad (16)$$

on the right-hand side of Eq. (13), yielding

$$\begin{aligned} (-i\Omega + iq_{||} - i\nabla_{\perp}^2) E = D_1 D_2 \int \frac{d\gamma}{\gamma^2} h'(\gamma) \int d^2p u(p^2 + k^2 r^2) \\ * \frac{i}{\alpha} \left[E - \frac{k^2}{k^2 - \alpha^2} r \frac{\partial E}{\partial r} + \frac{3k^4}{(k^2 - \alpha^2)(4k^2 - \alpha^2)} r^2 \frac{\partial^2 E}{\partial r^2} \right. \\ \left. + \frac{1}{4k^2 - \alpha^2} \frac{p^2}{2} \nabla_{\perp}^2 E + \dots \right]. \end{aligned} \quad (17)$$

When the inequalities of Eq. (14) or (15) hold, the series in Eq. (17) can be truncated, and the effect of the transverse phase variation on the gain can be determined by perturbation theory applied to the resulting differential equation.

If we neglect the effect of the transverse phase variation, the dispersion relation reduces to

$$[\Omega - q_{||} + \nabla_{\perp}^2 + U(\Omega, q_{||}; r)] E = 0 \quad (18)$$

where

$$U = D_1 D_2 \int \frac{d\gamma}{\gamma^2} h'(\gamma) \int \frac{d^2p u(p^2 + k^2 r^2)}{\Omega - (1 + q_{||}) [2(\frac{\gamma - \gamma_0}{\gamma_0}) - \frac{1}{4}(p^2 + k^2 r^2)]} \quad (19)$$

For a given electron, $p^2 + k^2 r^2$ is a constant of the motion [9]. Electrons with larger values of this invariant have lower longitudinal velocities,

therefore, electrons on the outer edge of the electron beam have on average lower longitudinal velocity than those near the beam center. This variation of the average longitudinal velocity with transverse position modifies the spatial dependence of the guiding potential U , so the effect of the transverse emittance is not simply an effective energy spread.

From Eq. (19), we see that the reduction of the longitudinal velocity due to the transverse emittance will have a negligible effect on the gain, if

$$\text{Im } \Omega \gg (p^2 + k^2 r^2)/4 . \quad (20)$$

Noting that $p^2 + k^2 r^2 = (2k_o k_\beta / k_w)(k_\beta R^2 + k_\beta^{-1} (dR/dz)^2)$ and estimating $k_\beta R^2 + k_\beta^{-1} (dR/dz)^2 = 2\epsilon$, twice the rms emittance ϵ , we can rewrite the inequality of Eq. (20) as $(\text{Im}\Omega = 1/k_w l_G)$:

$$\epsilon \ll \frac{1}{k_\beta l_G} \frac{\lambda_o}{2\pi} , \quad (21)$$

where $\lambda_o = 2\pi/k_o$ is the radiation wavelength. This restriction on the transverse emittance ϵ becomes very difficult to satisfy for short wavelength λ_o . The condition is somewhat relaxed when $k_\beta l_G$ is small, i.e. when there is only a small fraction of a betatron oscillation in a gain length.

3. Mode Degeneracy

When the field gradient in the wiggler is neglected, and the electron beam is assumed to have vanishing energy and angular spread, the dispersion relation becomes that originally discussed by Moore [2]:

$$[\nabla_\perp^2 + \Omega - q_{||} - \frac{(2\rho)^3}{\Omega^2} u(r/a)] E(\vec{r}) = 0 \quad (22)$$

with $[1] (2\rho)^3 = 2D_1 D_2 / \gamma_o^3$, and a being a constant characterizing the electron beam radius. Transforming to the new variables $\vec{w} = \vec{r}/a$, $\Omega = 2\rho\Lambda$ and $q_{||} = 2\rho\Delta$, Eq. (22) can be rewritten:

$$\left[\frac{1}{a^2} \nabla_w^2 + \Lambda - \Delta - \frac{1}{\Lambda^2} u(w) \right] \psi(\vec{w}) = 0 , \quad (23)$$

where $E(\vec{r}) = \psi(\vec{r}/a)$, and

$$\tilde{a}^2 = 2\rho a^2 = 2\rho \frac{2k_0 k_w R_e^2}{\omega^2}, \quad (24)$$

with R_e being the electron beam radius. Note that $\tilde{a}^2 \propto k_0 R_e^2 / l_G^{1D}$, where $k_0 R_e^2$ is the Rayleigh range associated with the electron beam radius and $l_G^{1D} \propto 1/2\rho k_w$ is the gain length given by one-dimensional theory.

Eq. (23) shows that the growth rates of the guided modes become degenerate for $\tilde{a} \rightarrow \infty$. For large $\tilde{a}$, the optical mode is small compared to the electron beam and we can use the expansion $u(w) = 1 - b_2 w^2 + \frac{1}{2} b_4 w^4 \dots$. The terms with w^k ($k \geq 4$) are treated as perturbations [11] off the soluble problem (of the two-dimensional quantum harmonic oscillator):

$$\left[\nabla_w^2 + \tilde{a}^2 (\Lambda - \Delta - \frac{1}{\Lambda^2}) + b_2 \frac{\tilde{a}^2}{\Lambda^2} w^2 \right] \psi = 0. \quad (25)$$

The unperturbed eigenfunctions are $\psi = w^m L_n^m(\beta w^2) \exp(-\beta w^2/2) \exp(im\theta)$, with $\beta = \pm i\sqrt{b_2} \tilde{a}/\Lambda$. The eigenvalues Λ of Eq. (25) are determined by

$$\Lambda^2(\Lambda - \Delta) = 1 \pm 2i \frac{\Lambda}{a} (2n+m+1) \sqrt{b_2}. \quad (26)$$

This result is correct including $O(\Lambda/\tilde{a})$ for the true problem of Eq. (23), and the higher-order terms can be determined from perturbation theory. For the fundamental mode $m=n=0$ of Eq. (23):

$$\Lambda^2(\Lambda - \Delta) = 1 \pm 2i \frac{\Lambda}{a} \sqrt{b_2} - \frac{b_4}{b_2} \frac{\Lambda^2}{a^2} + O\left(\frac{\Lambda^3}{a^3}\right). \quad (27)$$

In the case of a Gaussian profile $b_2 = b_4 = 1$, Eq. (27) including $O(\Lambda^2/\tilde{a}^2)$ agrees with the variational calculation of Xie and Deacon [12].

When $\tilde{a} \gg 1$, diffraction is negligible and the gain is given by one-dimensional theory; many modes are degenerate and the transverse size of the mode $R_{EM} < R_e$ [See discussion following Eq. (25)]:

$$\frac{R_{EM}}{R_e} = \beta^{-1/2} \propto \tilde{a}^{-1/2} \ll 1. \quad (28)$$

When $\tilde{a} \leq 1$, diffraction reduces the gain, one mode dominates and $R_{EM} \geq R_e$.

4. An Example

We present an illustrative set of parameters relevant to the design of a single pass free electron laser operating at 330Å wavelength. The electron energy is 750 MeV ($\gamma_0=1500$) and the energy spread $\sigma_\gamma/\gamma < 10^{-3}$. The wiggler period $\lambda_w = 3$ cm, and we choose $K = 2$. Electron beam emittance ϵ is expressed in terms of the rms size σ_R and angular spread σ_θ by

$$\epsilon = k_\beta \sigma_R^2 = k_\beta^{-1} \sigma_\theta^2, \quad (29)$$

and we assume Gaussian distributions. The betatron wavenumber

$k_\beta = Kk_w/\gamma\sqrt{2} = 0.2 \text{ m}^{-1}$, and the electron emittance is taken to be

$$4\pi\epsilon = 3 \times 10^{-8} \text{ mrad}, \quad (30)$$

hence $\sigma_R = 0.11 \text{ mm}$. The Pierce parameter ρ is expressed in practical units according to

$$(2\rho)^3 = 3 \times 10^{-6} \frac{I(\text{amp})}{\gamma_0} \frac{\lambda_0 \lambda_w}{\sigma_R^2} \frac{K^2}{1+K^2}, \quad (31)$$

so assuming a peak current of 100 Amp, we find $\rho = 1.2 \times 10^{-3}$. Also,

$\tilde{a} = \sqrt{2\rho 2k_0 k_w} \sigma_R = 1.6$, which indicates the electron beam and radiation mode are roughly of the same size. The gain length will be close to the one-dimensional value $l_G^{1D} = 1/2\rho k_w = 2\text{m}$, and $k_\beta l_G^{1D} = 0.4$. It then follows that $k_\beta l_G^{1D} R_e / R_{EM} \approx 0.4 < 1$, so the transverse phase variation in one gain length is not too large, but still not negligible. The condition of Eq. (21) on the emittance is satisfied:

$$\epsilon = 2.4 \times 10^{-9} \text{ m-rad} < \frac{1}{k_\beta l_G^{1D}} \frac{\lambda_0}{2\pi} = 1.3 \times 10^{-8} \text{ m-rad}, \quad (32)$$

hence the reduction of the gain due to the electron beam emittance should not be severe.

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