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GRAVITY THEORIES IN MORE THAN FOUR DIMENSIONS[1]

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ABSTRACT

String theories suggest particular forms for gravity interactions in higher dimensions. We consider an interesting class of gravity theories in more than four dimensions, clarify their geometric meaning and discuss their special properties.

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1. Introduction

The study of gravity in more than 4 dimensions is motivated by the approach of Kaluza [1] and Klein [2] to the problem of unification of gravity with electromagnetism and the other elementary interactions. The Kaluza-Klein point of view has been revived recently [3] by the study of supergravity. The most promising approach to unification seems to be that based on string theories [4-6].

The 10 dimensional gravity which emerges from supersymmetric string theories in the low energy limit contains in its action terms quadratic in the Riemann curvature tensor. Warren Siegel has emphasized that these terms give rise to ghosts and violate unitarity, while the string theory is unitary. This puzzling contradiction has been resolved by Barton Zwiebach [7] who has pointed out that the n-dimensional action

$$\int \sqrt{g}\, d^n x \left(\mathcal{R}_{ab,cd}\mathcal{R}^{ab,cd} - 4\mathcal{R}_{ab}\mathcal{R}^{ab} + \mathcal{R}^2 \right) , \tag{1.1}$$

leads to ghost-free nontrivial gravitational interactions for $n > 4$. By explicit computation Zwiebach has shown that, if one expands (1.1) about Minkowski space, the terms quadratic in the gravitational field combine to a total derivative and integrate to zero, so that (1.1), added to the usual Einstein-Hilbert action, introduces no propagator corrections. In 4 dimensions the entire expression (1.1) is a total derivative and is recognized as proportional to the Euler topological invariant.

Halpern and Zwiebach have observed that, similarly, the Einstein action in 4 dimensions, $\int \sqrt{g}\, d^4x \mathcal{R}$, has exactly the form of the Euler invariant in 2 dimensions except that the indices run over 4 values instead of 2. This has led them to believe that dimensionally continued Euler densities may play a role in the low energy limit of string theories. They have also conjectured that, in each case, the leading term in an expansion around Minkowski space integrates to zero.

We shall see that this conjecture is rather easy to prove, once the geometric meaning of the dimensionally continued Euler densities is understood. As we show below, they form a particular class of gravity Lagrangians in higher dimensions, which are constructed in terms of the vielbein and curvature forms, without the use of the Hodge dual of the curvature 2-form and are, therefore, a natural generalization of the Einstein action and of the cosmological term.

2. Basic differential geometry

We recall here some basic notions of differential geometry [2]. We shall use the vielbein 1-forms

$$e^a = e^a{}_m(x)dx^m, \tag{2.1}$$

and the connection 1-forms

$$\omega_a{}^b = \omega_a{}^b{}_m(x)dx^m. \tag{2.2}$$

Latin (tangent space) indices are raised and lowered by means of the constant Minkowski metric η_{ab} and

$$\omega_{ab} = -\omega_{ba}. \tag{2.3}$$

Torsion and curvature are 2-forms defined respectively as [3]

$$T^a = De^a \equiv de^a + \omega^a{}_b e^b, \tag{2.4}$$

and

$$R_a{}^b = d\omega_a{}^b + \omega_a{}^c\omega_c{}^b, \quad R_{ab} = -R_{ba}. \tag{2.5}$$

They satisfy the Bianchi identities

$$DT^a \equiv dT^a + \omega^a{}_b T^b = R^a{}_b e^b, \tag{2.6}$$

and

$$(DR)_a{}^b = (dR + \omega R - R\omega)_a{}^b = 0, \tag{2.7}$$

where matrix multiplication in implied. It is well know, and easy to verify, that (2.6) and (2.7) follow from the basic property of the exterior differential d

[2] A very useful review is given by Eguchi, Gilkey and Hanson [8].

[3] We omit the wedge sign $\wedge$ in the product of exterior differential forms. This convention causes no confusion and simplifies the formulas.

$$d^2 = 0. \tag{2.8}$$

If one applies d to a Lorentz invariant polynomial, the result can be worked out according to the Leibnitz rule of differentiation with d replaced by the covariant differential D on the various terms of a product and suitable signs corresponding to the fact the d is an antiderivative.

A small variation $\delta\omega$ of the connection forms induces, by (2.5), a variation of R given by

$$\delta R = D\delta\omega = d\delta\omega + \omega\delta\omega + \delta\omega\omega. \tag{2.9}$$

The last two terms both have the plus sign because ω and $\delta\omega$ are both odd.

Finally, let us recall that, for a (pseudo-) Riemannian manifold the torsion vanishes

$$T^a = 0. \tag{2.10}$$

3. Candidate Lagrangians

In n dimensions the Lagrangian is given by an n-form which, integrated over the manifold, gives the action. A particularly interesting class of Lagrangians invariant under local Lorentz transformations is given by

$$R_{ab}R_{cd}\cdots e_f e_g \cdots \epsilon^{abcd\cdots fg\cdots}. \tag{3.1}$$

The totally antisymmetric epsilon tensor has n indices. If the number of R's is $k \leq \left[\frac{n}{2}\right]$, the number of vielbein forms e is $r = n - 2k$ so that (3.1) is an n-form, which we denote by $L_{k,r}$. For instance, in 2 dimensions we could take either

$$L_{0,2} = e_a e_b \epsilon^{ab}, \tag{3.2}$$

or

$$L_{1,0} = R_{ab}\epsilon^{ab}. \tag{3.3}$$

The latter is proportional to the Euler invariant. Because of its topological nature the corresponding equations of motion are trivial, there is no Einstein-like gravity in 2-dimensions.

In 4 dimensions we can take a linear combination of

$$L_{0,4} = e_a\, e_b\, e_c\, e_d\, \epsilon^{abcd}, \tag{3.4}$$

$$L_{1,2} = R_{ab}\, e_c\, e_d\, \epsilon^{abcd}, \tag{3.5}$$

and

$$L_{2,0} = R_{ab}\, R_{cd}\, \epsilon^{abcd}. \tag{3.6}$$

The first is a cosmological term, the second is proportional to the Einstein-Hilbert Lagrangian and the third to the Euler invariant.

In 6 dimensions we can take a linear combination of

$$L_{0,6} = e_a\, e_b\, e_c\, e_d\, e_f\, e_g\, \epsilon^{abcdfg}, \tag{3.7}$$

$$L_{1,4} = R_{ab}\, e_c\, e_d\, e_f\, e_g\, \epsilon^{abcdfg}, \tag{3.8}$$

$$L_{2,2} = R_{ab}\, R_{cd}\, e_f\, e_g\, \epsilon^{abcdfg}, \tag{3.9}$$

and

$$L_{3,0} = R_{ab}\, R_{cd}\, R_{fg}\, \epsilon^{abcdfg}. \tag{3.10}$$

Again, the first is a cosmological term, the second is proportional to the Einstein-Hilbert action and the last to the Euler invariant. Now we have the new possibility (3.9). Similarly for higher dimensions. Odd numbers of dimensions can be considered as well, but in this case the Euler invariant is absent, of course.

To be concrete, let us stay with 6 dimensions. Can one really have a term like (3.9) in the Lagrangian? At first sight one may think that such a term, which is quadratic in the Riemann tensor, will contribute to the bilinear part of the Lagrangian for the field h which describes the deviation from Minkowski space

$$e^a{}_m = \delta^a{}_m + h^a{}_m \tag{3.11}$$

and thus spoil the particle interpretation by introducing ghosts [9]. However, one can see that this is not the case.

Let us consider an infinitesimal variation of the connections and vielbein forms. The corresponding variation of $L_{2,2}$ is

$$\begin{aligned} \delta L_{2,2} &= 2\delta R_{ab} R_{cd}\, e_f\, e_g\, \epsilon^{abcdfg} \\ &\quad + 2 R_{ab} R_{cd}\, e_f\, \delta e_g\, \epsilon^{abcdfg} \quad . \end{aligned} \tag{3.12}$$

Using (2.9), the first term on the right hand side is

$$2\,(D\delta\omega_{ab})\, R_{cd}\, e_f\, e_g\, \epsilon^{abcdfg}. \tag{3.13}$$

On the other hand, using the Bianchi identity (2.7) and the definition (2.4) of the torsion we have

$$\begin{aligned} 2d\,(\delta\omega_{ab} R_{cd}\, e_f\, e_g)\,\epsilon^{abcdfg} &= 2\,(D\delta\omega_{ab})\, R_{cd}\, e_f\, e_g\, \epsilon^{abcdfg} \\ &\quad + 4\delta\omega_{ab} R_{cd}\, e_f\, T_g\, \epsilon^{abcdfg} \quad . \end{aligned} \tag{3.14}$$

Therefore, if the torsion vanishes, (3.12) can be written

$$\begin{aligned} \delta L_{2,2} &= 2d\left(\delta\omega_{ab} R_{cd}\, e_f\, e_g\, \epsilon^{abcdfg}\right) \\ &\quad + 2R_{ab} R_{cd}\, e_f\, \delta e_g\, \epsilon^{abcdfg} \quad . \end{aligned} \tag{3.15}$$

This equation tells us that, if we consider a power series expansion in h starting from flat space, the terms in $L_{2,2}$ which are quadratic in h appear under a derivative sign (first term on the right hand side in (3.15)); for a compact manifold or with suitable conditions at infinity, they drop out after integration. The first non trivial term in the integrated action is cubic; it comes from the second term on the right hand side of (3.15) and can be immediately obtained from it. Clearly, the same result is true for $L_{2,r}$ in $4+r$ dimensions. These Lagrangians with 2 Riemann tensors and r vielbeins, do not introduce propagator corrections. Instead they generate interaction terms which are at least cubic.

A perfectly analogous argument can be carried out when there are $k > 2$ Riemann tensors. For $L_{k,r}$, the terms of order k in the field h appears under a total derivative sign and the leading term in the action is actually of order $k+1$. For $r > 0$, $L_{k,r}$ gives interaction vertices with $2k$ derivatives (momenta) and at least $k+1$ graviton lines.

The argument given in this section is a simple generalization of the argument one uses to derive the equations of motion from the Einstein-Hilbert Lagrangian (3.5), and to show that the corresponding action does not contain a term linear in h (which integrates to zero) and starts instead with a kinetic term of order h^2. In the case of the Euler invariants $L_{k,0}$ one finds of course that the entire variation is a total derivative

$$\delta L_{k,0} = kd \left(\delta\omega_{ab} R_{cd} R_{ef} \cdots \epsilon^{abcdef\cdots} \right) , \tag{3.16}$$

corresponding to the fact that the integral of $L_{k,0}$ is a topological invariant.

4. Conclusion

The Lagrangians described in the previous section are the natural generalization to more than 4 dimensions of the Einstein action and of the cosmological term. It is easy to rewrite them in terms of the four-index Riemann tensor which is related to the curvature 2-form by

$$\begin{aligned} R_{ab} &= \frac{1}{2} R_{ab,mn}\, dx^m\, dx^n \\ &= \frac{1}{2} R_{ab}{}^{cd}\, e_c\, e_d \quad . \end{aligned} \tag{4.1}$$

For instance (3.9) in 6 dimensions becomes

$$\begin{aligned} L_{2,2} &= \frac{1}{4} R_{ab}{}^{a'b'}\, R_{cd}{}^{c'd'}\, e_{a'}\, e_{b'}\, e_{c'}\, e_{d'}\, e_f\, e_g \epsilon^{abcdfg} \\ &= \frac{1}{4} R_{ab}{}^{a'b'}\, R_{cd}{}^{c'd'}\, e\, \epsilon_{a'b'c'd'fg}\, \epsilon^{abcdfg}\, d^6 x \quad , \end{aligned} \tag{4.2}$$

where

$$e = \det e^a{}_m, \tag{4.3}$$

is the determinant of the vielbein field. On the other hand, the Euler invariant (3.6) in 4-dimensions becomes

$$L_{2,0} = \frac{1}{4} R_{ab}{}^{a'b'} R_{cd}{}^{c'd'} e\,\epsilon_{a'b'c'd'}\,\epsilon^{abcd} d^4x. \tag{4.4}$$

Expressing the product of epsilon tensors in (4.4) in terms of Kronecker delta's one finds the well known expression (1.1) for the Euler invariant in 4-dimensions

$$L_{2,0} \propto R_{ab,cd}R^{ab,cd} - 4R_{ab}R^{ab} + R^2, \tag{4.5}$$

in terms of the Riemann tensor, the Ricci tensor

$$R_{ab} = R_{ac,b}{}^{c}, \tag{4.6}$$

and the scalar curvature

$$R = R_a{}^a. \tag{4.7}$$

Now it is obvious that the product of epsilon tensors in the 6 dimensional expression (4.2) will generate a formula for $L_{2,2}$ exactly like (4.5) (up to an overall factor) except that now the indices run over 6 values instead of 4. The relative coefficients 1, -4 and 1 of the three terms are the same. The same remark applies to all other Lagrangians with 2 Riemann tensors in higher numbers of dimensions. One can say that $L_{2,r}$, for $r \geq 1$ is obtained from the 4 dimensional Euler invariant $L_{2,0}$ in the form (4.5) by letting the indices run over n values instead of 4 and by integrating over n dimensions, instead of 4. Because of the antisymmetry of the epsilon tensors, it is also clear that, if in (4.5) one lets the indices run over fewer than 4 values, one obtains identically zero.

In a similar way, the Lagrangians $L_{k,r}$, written in a form analogous to (4.5), can be interpreted as the extension to a higher number of dimensions of the Euler number $L_{k,0}$ in $2k$ dimensions also written by working out the product of epsilon tensors. Again, one obtains zero if one limits the range of the indices to fewer than $2k$ values. Therefore, as it is also obvious from Sec. 3, in a given dimension the number of Lagrangians of the class considered here is finite. It would be truly remarkable if they were the only ones emerging from string theories in the low energy limit. Indeed, Zwiebach [7] has pointed out that the 3-graviton on-shell vertex obtained from the bosonic closed string theory contains a term with the momentums to the 6^{th} power. On the other hand a term with 6 derivatives can come only from $L_{3,r}$ and must involve at least 4 gravitons. To obtain an interaction with 6 derivatives and 3 gravitons one must go beyond the class of Lagrangians given by (3.1) and use explicitly the Hodge dual of the curvature 2-form.

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References

[1] T. Kaluza, P. A. W. 1921, p. 966.

[2] O. Klein, Z. Phys. 37 (1926) 895; Nature 118 (1926) 516.

[3] E. Witten, Nucl. Phys. B186 (1981) 412.

[4] M. B. Green and J. H. Schwarz, Phys. Lett. 149B (1984) 117.

[5] D. J. Gross, J. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 6 (1985) 502.

[6] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, Santa Barbara preprint NSF-ITP-84-170.

[7] B. Zwiebach, Berkeley preprint UCB-PTH-85/10.

[8] T. Eguchi, P. B. Gilkey and A. J. Hanson, Phys. Reports 66 (1980) 213.

[9] K.Stelle, Phys. Rev. D16 (1977) 953.

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