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BEAMS IN FREE ELECTRON LASERS**

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Small Signal Gain for Parabolic Profile Beams in Free Electron Lasers

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ABSTRACT

Three new types of free electrons lasers (FEL) that are being examined in new ranges of parameter design space are: compact systems, XUV systems, and high power devices. Shorter wiggler wavelengths, shorter or longer lasers, higher currents, and higher quality electron beams are a few of the trends in the FEL community. The primary predictor of FEL oscillation is the small signal gain. We present 3-D small-signal calculations for more realistic parabolic-profile electron beams in the limit of moderately wide to wide electron beams. This limit compliments the thin electron beam limit and needs be included in any analytical approximation that encompasses all 3-D effects. The system of equations for the optical modes are of Hamiltonian form and are solved as the analytical eigenmodes of the stationary quantum mechanical harmonic oscillator. We show the complete solution to the initial value problem in the special case of a cold, resonant electron beam, including the damped modes heretofore neglected. From this we derive the asymptotic solution as a superposition of Hermite [square symmetry] or Laguerre (circular symmetry) modes. We give expressions for the mode size, the spatial growth rate and injection fraction for the dominant mode.

1. INTRODUCTION

Although the governing equations for optical guiding in FELs are well understood¹, there is practical interest in obtaining simple formulas for the gain of such systems. Doing so transforms the problem of general system design to an algebraic level and can represent and codify experience in running computer codes. The case of a thin electron beam has been treated² elsewhere and here we treat the opposite limit of a wide electron beam. The full initial value problem has been treated numerically, but not analytically, however, asymptotic solutions giving what corresponds to the fastest growing eigenmode of this treatment have been found for a radial top hat electron beam. The completeness of such modes and the injection fraction has not been examined analytically. Our treatment handles a special case of an on-resonant parabolic electron beam in the standard radial-lane approximation. It has the advantage of serving as a prototype for doing the full initial value problem. As such, it complements the initial value treatment available in the thin beam case.

The approach here may be considered a generalization that adds diffraction to the well known resonant 1-D small signal cubic equation usually appearing in dispersion³ relation form:

$$\frac{d^3}{d\tau^3} A_{1D} = \frac{ij}{2} A_{1D}, \quad (1)$$

where $A_{1D}(0) = 1$, $dA_{1D}/d\tau(0) = 0$, $d^2 A_{1D}(0)/d\tau^2 = 0$. Here j is the dimensionless current and the optical field spatial growth rate is $\sqrt{3}(j/2)^{1/3}/2L$. The parameter τ may be thought of as a time parameter measuring the progress of the pulse through the wiggler and becomes unity at the wiggler end. To solve the initial value problem, we substitute $\exp(\Gamma\tau)$ for A_{1D} and solve for the three values of Γ . Then we use a linear superposition and three boundary conditions. The three-dimensional treatment, by analogy, would also use an exponential time dependence. Generalizing this idea, we use a laplace transform technique to obtain the complete solution of the asymptotic field. We give the spatial growth rate, mode size, and the field injection ratio. These are reduced to simple algebraic formulas in terms of a single parameter that may be thought of as the ratio of the 3D gain length to the rayleigh range of the excited wave.

2. MATHEMATICAL ANALYSIS

Adding diffraction to the cubic equation is straight forward and well known^{1,4}. The partial differential equation for the propagation of the optical field A is third order in τ and second order in the two dimensional space coordinates x and y :

$$\frac{d^2}{d\tau^2} \left[\frac{dA}{d\tau} + \frac{L\nabla_{\perp}^2 A}{2ik} \right] = \frac{ij}{2} A(1 - \alpha r^2), \quad (2)$$

here $r^2 = x^2 + y^2$, k is the optical wavenumber and L is the length of the wiggler. The electron beam density profile is proportional to $1 - \alpha r^2$ and the zero density occurs at $r = r_0 = 1/\sqrt{\alpha}$. This equation can be cast in a more compact and familiar form⁴:

$$\frac{d}{d\tau} \begin{pmatrix} P \\ Q \\ A \end{pmatrix} = M \cdot \begin{pmatrix} P \\ Q \\ A \end{pmatrix} = M \cdot \vec{V}, \quad (3)$$

where P is a measure of the sinusoidal velocity modulation amplitude and phase and Q is a measure of bunching and

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & ij(1 - \alpha r^2)/2 & -L\nabla_{\perp}^2/2ik \end{pmatrix}. \quad (4)$$

The solution to the eigenvalue problem [Eq.(5) with $V_0 = 0$] gives (P_m, Q_m, A_m) for eigenvalue $s = s_m$. With the orthonormal⁴ condition rewritten as

$$\int d^2\vec{r} [A_m A_n + \frac{ij}{2}(1 - \alpha r^2)(P_m Q_n + P_n Q_m)] = I_n \delta_{mn}. \quad (4b)$$

Because of the question of the incompleteness of the eigenfunctions, we solve the equation, with initial conditions, $\vec{V}_0 = (0, 0, A_0)^T$ by a laplace transform technique. The transform of a function F is written $\tilde{F}(s) = \int_0^\infty \exp(-s\tau) F(\tau) d\tau$, thus,

$$(M - s) \cdot \tilde{\vec{V}}(s, \vec{r}) + \tilde{\vec{V}}_0(\vec{r}) = 0. \quad (5)$$

In principle we will invert the expression for $\tilde{\vec{V}}(s, \vec{r})$ by the bromwich integral (in practice we identify known pieces):

$$\vec{V}(t, \vec{r}) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \exp(st) \tilde{\vec{V}}(s, \vec{r}) ds, \quad (6)$$

where $\Re(\gamma) > 0$ and is also greater than any singularity in $\tilde{\vec{V}}(s, \vec{r})$. Therefore, we require the solution for $\Re(s) > 0$, only.

Eliminating the first two components of the vector gives

$$\frac{ij}{2s^2} \tilde{A}(s, \vec{r})(1 - \alpha r^2) - \frac{L\nabla_{\perp}^2 \tilde{A}(s, \vec{r})}{2ik} = s\tilde{A}(s, \vec{r}) - A_0(\vec{r}). \quad (7)$$

Transforming to dimensionless coordinates $\vec{r}/a = \vec{r}$ and $a\nabla_{\perp} = \nabla_{\perp}$, where complex a is determined below, and using $\tilde{A}_-(s, \vec{r}) = \tilde{A}(s, \vec{r})$, the spatially driven two dimensional quantum mechanical harmonic oscillator equation is obtained

$$\nabla_{\perp}^2 \tilde{A}_-(s, \vec{r}) + (\Lambda(s) - r^2) \tilde{A}_-(s, \vec{r}) = 2iku^2 A_0(a\vec{r})/L, \quad (8)$$

where

$$\Lambda(s) = \frac{a^2 k}{L} \left(\frac{j}{s^2} + 2is \right), \quad (9)$$

and

$$a^2 = \sqrt{\frac{L}{k\alpha j}} s. \quad (10)$$

The associated homogeneous equation ($A_0 = 0$ and $\Lambda(s) \rightarrow \lambda_{m,n}$) has a simple ground state solution, $\exp(-\bar{r}^2/2)$ with corresponding eigenvalue $\lambda_{00} = 2$.

The eigenfunctions $u_{m,n}(\bar{r})$ can be expressed in either cartesian or polar coordinates and satisfy the eigenvalue equation

$$\bar{\nabla}_\perp^2 u_{m,n}(\bar{r}) - \bar{r}^2 u_{m,n}(\bar{r}) = -\lambda_{m,n} u_{m,n}(\bar{r}). \quad (11)$$

These solutions are known to be complete and are found by separation of variables product function technique. The completeness of the two dimensional functions is guaranteed by the completeness of the one dimensional functions through the construction of $\delta(\bar{x} - \bar{x}_0)\delta(\bar{y} - \bar{y}_0)$. The one dimensional completeness is guaranteed by the sturm-liouville nature of the one-dimensional second order differential equations. The solutions are given by

$$u_{m,n}(\bar{x}, \bar{y}) = N_{nm} H_m(\bar{x}) H_n(\bar{y}) \exp(-\bar{r}^2/2), \quad (12)$$

in cartesian coordinates with $\lambda_{m,n} = 2m + 2n + 2$, $N_{nm} = 1/\sqrt{2^{m+n} \pi n! m!}$ and in polar coordinates

$$u_{p,q}(\bar{r}, \theta) = \frac{L_p^{(q)}(\bar{r}^2) \bar{r}^q \exp(-\bar{r}^2/2) \exp(iq\theta)}{\sqrt{\pi(q+p)!/p!}}, \quad (13)$$

with $\lambda_{p,q} = 4p + 2q + 2$. The rest of the paper applies to both representations but utilizes the cartesian representation for detailed calculations. The value of s determines whether or not $\Lambda(s)$ takes on one of the eigenvalues $\lambda_{m,n}$. In the off-resonant case, where the value of s does not result in $\Lambda(s) = \lambda_{m,n}$, the drive term can be expanded in a series of the on-resonant eigenfunctions whose width depends on the value of s . Here we examine the solutions of $\Lambda(s) = \lambda_{m,n}$. This condition can be written in the scaled variables as

$$\mathcal{L} = \lambda_{mn} \sqrt{\frac{\alpha L}{2k(j/2)^{1/3}}} = \frac{1}{\bar{s}} + i\bar{s}^2. \quad (14)$$

where $\bar{s} = s/(j/2)^{1/3}$. A great deal of physical insight can be garnered from the solution of this cubic. We solve the cubic by using a polar form for the scaled s value that lies within the complex $\bar{s}$ plane. In Fig. (1) we show the scaled eigenvalue $\mathcal{L}$ plotted against the angle θ . The three branches correspond to the three cubic roots for all possible positive values of the scaled eigenvalue at a half-degree and degree increments in θ .

In Fig. (2), we plot the root locus of scaled s values, $\bar{s}$, as a function of the scaled eigenvalue $\mathcal{L}$. With the scaled eigenvalue set to zero, the two right hand curves begin at their uppermost points. The left hand curve, which we call branch number 3 or the loss branch, begins at its lowest point, and this beginning point is a reflection in the y-axis of the uppermost point of the right-hand side curves. The uppermost of the right-hand side curves, which we call FEL gain mode or branch 1, contains the physics of high-gain and optical guiding. As the scaled eigenvalue is increased (by means of a smaller radius electron beam or higher mode numbers), we expect diffraction to increase and the losses to be higher. Indeed, the growth rate, which corresponds to the real part of $\bar{s}$, decreases. The lower branch of the right-hand side, which we call negative density gain (NDG) mode or branch 2, shows the highest, unbounded growth rates as the scaled eigenvalue approaches infinity. This is as it should be. The parabolic shaped electron beam has a negative particle density at high radii. Were negative density particles available, we could think of this region as giving rise to a negative-density FEL. Mathematically, the 1-D version of such a system requires $j \rightarrow -j$ in Eq.(1). When the value of $\mathcal{L}$ of this NDG branch is equal to about unity, the mode width extends beyond the electron zero density boundary. If we look at the average negative density versus the scaled eigenvalue, we find it increases and makes this system a laser with gain that increases with optical beam width. For practical purposes, we may think of this curve as bending over much quicker and perhaps even bending backwards when no negative density electrons are present to maintain the parabolic density profile. The left hand branch corresponds to decaying solutions and it as well as all the other branches are necessary to meet the boundary conditions at $r = 0$. At low values of $\mathcal{L}$, the branch 3 mode appears to be associated with the FEL because $s_3 = -s_1^*$, whereas as $\mathcal{L} \rightarrow \infty$, it appears associated with the NDG mode where $s_3 = -s_2$. Without the negative density (ND) part of the electron beam, the branch 3 curve will also be modified far away from the origin when $\mathcal{L} \gg 1$.

Fig.(1) The scaled eigenvalues $\mathcal{L}$ versus the angle θ for each of the three roots $s_i(\mathcal{L})$, $i = 1, 2, 3$.

Fig. 2 The root locus of the three eigenvalues of spatial growth rates $s_i(\mathcal{L})$, for $i = 1, 2, 3$. The branches are numbered clockwise by 1, 2 and 3 corresponding to 2, 4, and 10 o'clock.

The solution to the 2-D driven harmonic oscillator equation for non-resonant s is then

$$\hat{A}(s, \vec{r}) = \frac{2ik}{L} a^2(s) \sum \frac{u_{m,n}(\vec{r})}{\Lambda(s) - \lambda_{m,n}} \int d^2 \vec{r}' u_{m,n}(\vec{r}') A_0[a(s) \vec{r}'] \quad (15)$$

Letting $A_0(\vec{r}) = \exp(-\beta r^2)$, we have in the cartesian representation,

$$\begin{aligned} \int d^2 \vec{r}' A_0[a(s) \vec{r}'] u_{mn}(\vec{r}') &= 1/\sqrt{\pi} N_{nm} \int d^2 \vec{r}' \exp[-(\beta a^2(s) + 1/2) r'^2] H_m(x') H_n(y') \\ &= N_{nm} \frac{2\sqrt{\pi}}{2\beta a^2 + 1} \left[\frac{\beta a^2 - 1/2}{\beta a^2 + 1/2} \right]^{(m+n)/2} H_m(0) H_n(0) = \frac{2\sqrt{\pi} s_0}{s + s_0} \left[\frac{s - s_0}{s + s_0} \right]^{(m+n)/2} N_{nm} H_m(0) H_n(0), \end{aligned} \quad (16)$$

where $s_0 = \sqrt{k\alpha j/L}/(2\beta)$. This then gives

$$\begin{aligned} \hat{A}(s, \vec{r}) &= 2s_0 \sum_{mn} \frac{\sqrt{\pi} u_{m,n}[\vec{r}/a(s)] s^2}{(s + s_0)(s - s_{1mn})(s - s_{2mn})(s - s_{3mn})} \left[\frac{s - s_0}{s + s_0} \right]^{(m+n)/2} N_{nm} H_m(0) H_n(0) \\ &= \sum_{mn} \hat{A}_{mn}(s, \vec{r}). \end{aligned} \quad (17)$$

The selected component that gives rise to the asymptotic behavior is represented by

$$\hat{A}_{00}(s, \vec{r}) = \frac{2}{\sqrt{\pi}} \frac{s_0 s^2}{(s + s_0)(s - s_1)(s - s_2)(s - s_3)} \exp(-\rho/s), \quad (18)$$

where $\rho = r^2 \sqrt{k\alpha j/L}/2$ independent of mode and we have defined $s_1 = s_{100}$, $s_2 = s_{200}$, $s_3 = s_{300}$. Pulling out the bessel function form,

$$A_{00}(r, \vec{r}) = \frac{2}{\sqrt{\pi}} s_0 L T^{-1} \left[\frac{s^3}{(s + s_0)(s - s_1)(s - s_2)(s - s_3)} \right] \otimes J_0(2\sqrt{\rho r}), \quad (19)$$

here LT^{-1} indicates the inverse laplace transform, and $\otimes$ indicates a convolution of the indicated functions detailed below. Alternately, the complete cartesian representation for the solution can be written

$$A(\tau, \vec{r}) = \sum \sqrt{\pi} H_m(0) H_n(0) LT^{-1} \left[H_m \left(\frac{x}{a_0} \sqrt{\frac{-\partial}{\partial \rho}} \right) H_n \left(\frac{y}{a_0} \sqrt{\frac{-\partial}{\partial \rho}} \right) N_{mn}^2 \exp(-\rho/s)/s \right] \otimes \frac{\partial}{\partial \tau} \left\{ LT^{-1} \frac{s^2}{(s-s_{1mn})(s-s_{2mn})(s-s_{3mn})} \otimes LT^{-1} \left[\frac{2s_0}{s+s_0} \left(\frac{s-s_0}{s+s_0} \right)^{(m+n)/2} \right] \right\}, \quad (19b)$$

where $a_0^4 = L/k\alpha j$. The first term is a bessel function, the second a sum of exponentials. The last inverse transform in Eq.(19b) is a Laguerre polynomial of degree $(m+n)/2$ multiplied by $\exp(-s_0\tau)$. By defining the four constants c_i , $i = 0, 3$ as

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{s_0^3}{(s_0-s_1)(s_0-s_2)(s_0-s_3)} \\ \frac{s_1^3}{(s_1+s_0)(s_1-s_2)(s_1-s_3)} \\ \frac{s_2^3}{(s_2+s_0)(s_2-s_1)(s_2-s_3)} \\ \frac{s_3^3}{(s_3+s_0)(s_3-s_1)(s_3-s_2)} \end{pmatrix}, \quad (20)$$

we can write the LT^{-1} bessel function term (an analogous approach holds for the complete solution) as

$$-[c_0 \exp(-s_0\tau) + c_1 \exp(s_1\tau) + c_2 \exp(s_2\tau) + c_3 \exp(s_3\tau)].$$

Thus the superposition of the lowest harmonic oscillator mode of different widths can be written as

$$A_{00}(\tau, \vec{r}) = \frac{2}{\pi} s_0 \left[c_0 \exp(-s_0\tau) \int_0^\tau \exp(s_0 t) J_0(2\sqrt{\rho t}) dt + \sum_{i=1}^3 c_i \exp(s_i\tau) \int_0^\tau \exp(-s_i t) J_0(2\sqrt{\rho t}) dt \right]. \quad (21)$$

The late time dependence for positive real parts of s_i gives the laplace transform of the bessel function times an exponentially increasing function in time. Thus, the eigenfunction $\pi u_{00}(\vec{r}/a(s_1))/s_1$ returns, multiplied by with a coefficient. The pole with the largest positive real piece for the FEL gain mode is thus comes from branch 1 and is the mode with the least diffraction, the ground state mode, when $\lambda_{00} = 2$, with pole at $s_1 = s_{100}$. This root contributes to asymptotic form:

$$A(\vec{r}, \tau) \propto A_{00}(\vec{r}, \tau) \propto 2s_0 c_1 \exp(s_1\tau) \sqrt{\pi} u_{00}(\vec{r}/a(s_1))/s_1. \quad (22)$$

A necessary requirement for the establishment of the asymptotic form is that at $\tau = 1$

$$|2s_0 c_1 \exp(s_1\tau)/s_1| > 1. \quad (23)$$

This is the condition for the gain to overcome the injection loss of the mode.

3. Numerical Evaluation

The spot size, w_g of the time independent dominant gaussian mode depends on $\Re[1/2a^2(s)]$ and is given by

$$w_g^2 = \frac{2L_g}{k} \frac{|s|^2}{\mathcal{L}_0} = w_c^2 \mathcal{L}_0 \frac{|s_1|^2}{|s_{1r}|}, \quad (24)$$

where $\mathcal{L}_0 = \mathcal{L}(m = n = 0)$. This parameter $\mathcal{L}_0$ can be thought of as a measure of diffraction relative to gain in the system. If we designate the rayleigh range of the excited wave as $z_{re} = kw_e^2/2$, and the 1-D or planar gain length as $L_{pg} = 3^{-1/2}2^{4/3}j^{-1/3}L$, then

$$\mathcal{L}_0^2 = \frac{\sqrt{3}}{2} \frac{L_{pg}}{z_{re}}. \quad (25)$$

Given $\mathcal{L}_0$, the 3-D gain length L_g is easily computable as $L_g = L/\Re(s_1) = L(j/2)^{-1/3}/\Re(\bar{s}_1)$ where $\Re(\bar{s}_1) = r_1 \cos(\theta_1)$, and where $|\bar{s}_1| = r_1 = [\sin(\theta_1)/\cos(2\theta_1)]^{1/3}$, and to 2-1/2% on $[0,4]$,

$$\text{ctn}\theta_1 = 1.6889 + 1.4003\mathcal{L}_0 - 1.3012\mathcal{L}_0^2 + 1.2223\mathcal{L}_0^3 \pm 2.5\%, \quad (26)$$

and for $\mathcal{L}_0 > 4$, $\text{ctn}\theta_1 = \mathcal{L}_0^3 \pm 0.3\%$. In this formalism $\mathcal{L}_0$ is the independent variable and the quantities $\text{ctn}\theta_1$, $\Re(\bar{s}_1)$, $\Im(\bar{s}_1)$, w_g^2/w_e^2 , L_{pg}/z_{re} , L_g/z_{re} , L_{pg}/L_g are functions of $\mathcal{L}_0$ only. The ratio of the equilibrium spot size to the spot size that would result from a gaussian free space beam with rayleigh range equal to the gain length is defined as $W(\mathcal{L}_0) = |\bar{s}_1|/\sqrt{\mathcal{L}_0}$. This ratio can be very large for small $\mathcal{L}_0$, and as $\mathcal{L}_0 \rightarrow 0$ gives rise to the 1-D dispersion equation. The only requirement to obtain the 1-D result then is that 1-D gain dominates over diffraction of the excited wave. Even though the gain may be large, the optical beam leaks out laterally from diffraction and is in this case larger than the electron beam. The validity of the parabolic profile e-beam dispersion equation is thereby verified in the case where the electron beam is small compared to the optical beam mode size. In this case, however, the part of the optical beam that interacts with the electron beam is primarily that at the high gain center of the electron beam, and it appears that the ND portion does not contribute by comparison. The opposite limit of $\mathcal{L}_0$ becoming very large compared to unity gives rise to a spot size w_g that becomes equal w_e on branch 1; branch 2 and 3 run into the ND portion of the electron beam, and, therefore, if the electron beam is truncated at the zero density point, it is not clear whether the expression for the temporal injection ratio is valid in the region $\mathcal{L}_0 > 1$, but the mode size and gain, which depend on branch 1 properties, appear to be valid for all $\mathcal{L}_0$. Thus, what we mean by wide electron beams is that their excited waves have rayleigh ranges that are larger than the 1-D gain length. When electron beams are not wide, the temporal injection ratio is probably underestimated by this theory.

The physical quantity called the injection fraction is the ratio of the asymptotic mode amplitude, extrapolated to zero time, to the injected field amplitude on axis $s_0 c_1/s_1$. It is the product of three factors. The first is $I_{om} = 2s_0/(s_0 + s_1) = 2\beta_g/(\beta_g + \beta)$, where $\beta_g = 1/2a^2(s)$. It gives unity when $\beta = \beta_g$.

$$I_{om}(\alpha/\beta, \mathcal{L}_0) = \frac{\alpha/\beta}{\alpha/\beta + \mathcal{L}_0 \bar{s}_1/2} \quad (27)$$

This can be evaluated approximately by the ctn expression, $s/\bar{s} = (j/2)^{1/3}$, and $\bar{s}_1 = r_1 \exp(i\theta_1)$. This first factor is the mode content of the incident optical mode. The second factor is $I_{tm} = |s_1^2/[(s_1 - s_2)(s_1 - s_3)]|$. When $\mathcal{L}_0 = 0$, $I_{tm} = 1/3$. For $\mathcal{L}_0 > 10.0$, $I_{tm} = \mathcal{L}_0^{-3}$ to 2.2%. For $\mathcal{L}_0 \in [0, 2]$, $I_{tm} = 0.335 - 0.449\mathcal{L}_0 - 0.0295\mathcal{L}_0^2 \pm 1.7\%$; on $[2, 10]$ $I_{tm} = 1.05\mathcal{L}_0^{-3} \pm 7\%$; or alternately on $[2, 10]$,

$$I_{tm} = (0. - 357 + 1.3551\mathcal{L}_0 - 0.4746\mathcal{L}_0^2 + 0.07809\mathcal{L}_0^3 - 0.006142\mathcal{L}_0^4 + 0.0001863\mathcal{L}_0^5)/\mathcal{L}_0^3 \pm 1.7\%. \quad (28)$$

This is the injection fraction into the fastest growing FEL branch of the three temporal modes. The temporal injection fraction depends on the three temporal growth rates that in turn depend on the value of $\mathcal{L}_0$. Because $\bar{s}_1 \rightarrow 0$ as $\mathcal{L}_0 \rightarrow \infty$, irrespective of the distortion of branch 2 and 3, $I_{tm} \rightarrow 0$ as $\mathcal{L}_0 \rightarrow \infty$. The third factor is $\sqrt{\pi}$. It combines with u_{00} to give unit intensity at $r = 0$. The requirement of gain overcoming injection loss is

$$\bar{s}_1(\mathcal{L}_0) > (j/2)^{-1/3} \{ \ln[I_{om}(\mathcal{L}_0, j)] + \ln[I_{tm}(\mathcal{L}_0)] \}. \quad (29)$$

The factor I_{tm} can be thought of as being derived from an equivalent 1-D problem with the same boundary conditions as Eq.(1) but with a term to describe diffraction

$$\frac{d^3}{d\tau^3} A_{1D} + i\sqrt{\frac{\alpha j L}{k}} \frac{d}{d\tau} A_{1D} = \frac{ij}{2}. \quad (30)$$

Here $A_{1D} \propto I_{im} \exp -s_1 \tau$.

4. CONCLUSIONS

We have presented calculation of the asymptotic mode for wide ($\mathcal{L}_0 \ll 1$) parabolic profile electron beams but the growth rate and mode size may be useful at all values $\mathcal{L}_0$. We have also addressed the situation of parabolic beam profile truncated at zero density. The asymptotic behavior can be expressed as a superposition of gauss hermite or gauss laguerre modes, the lowest order dominant mode is a stationary gaussian. The growth rates, widths and injection fractions of parametric variations of these modes are found analytically and with convenient numerical approximations. The theory loses its accuracy at large values of the parameter $\mathcal{L}_0$.

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