Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

LBL--16723

DE84 002909

To be published as a chapter in FLUIDS AND PLASMAS:
GEOMETRY AND DYNAMICS, J. Marsden, Ed., American
Mathematical Society, Providence, RI

CANONICAL DERIVATION OF THE VLASOV-COULOMB
NONCANONICAL POISSON STRUCTURE

A.N. Kaufman and R.L. Dewar

September 1983

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

CANONICAL DERIVATION OF THE VLASOV-COULOMB
NONCANONICAL POISSON STRUCTURE*

Allan N. Kaufman
and
Robert L. Dewar[†]

Accelerator Fusion and Research Division
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

September 1983

*This work was supported by the Director, Office of Energy Research of the U.S. Department of Energy under Contract Number DE-AC03-76SF00098.

[†]Institutional Affiliation: Australian National University, Canberra, Australia

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

A handwritten signature, possibly "E. J. B.", is written in the bottom right corner of the page.

CANONICAL DERIVATION OF THE VLASOV-COULOMB NONCANONICAL POISSON STRUCTURE

Allan N. Kaufman* and Robert L. Dewar**

ABSTRACT. Starting from a Lagrangian formulation of the Vlasov-Coulomb system, canonical methods are used to define a Poisson structure for this system. Successive changes of representation then lead systematically to the noncanonical Lie-Poisson structure for functionals of the Vlasov distribution.

The Vlasov system with Coulomb interaction is not only the standard model for plasma theory, but it is also a paradigm for the understanding of noncanonical Poisson structures. The Vlasov distribution $f(\underline{r}, \underline{p})$ is its dynamical field variable and has no canonical conjugate. Yet it possesses a noncanonical Lie-Poisson structure for observables $A(f)$, functionals of f :

$$\{A_1, A_2\} = \int d^3r d^3p f(\underline{r}, \underline{p}) [a_1(\underline{r}, \underline{p}), a_2(\underline{r}, \underline{p})], \quad (1)$$

where $a(\underline{r}, \underline{p}) = \delta A / \delta f(\underline{r}, \underline{p})$ is the functional derivative, and

$$[a_1, a_2] = (a_1 / a \underline{r}) (a_2 / a \underline{p}) - (a_1 / a \underline{p}) (a_2 / a \underline{r})$$

is the canonical bracket on functions of $(\underline{r}, \underline{p})$.

Since a knowledge of the Poisson structure is essential for a Hamiltonian field theory, it is highly desirable to understand the derivation of this structure. In this paper, we present such a derivation, based on old-fashioned canonical field theory. This derivation is rigorous, and is accessible to readers ignorant of modern mathematics.

Before beginning, we survey the earlier derivations of the Poisson structure (1). In 1980, this structure was discovered by Morrison and Kaufman by manipulating Morrison's results for the Vlasov-Maxwell system [9]; the latter had been obtained by inspired guesswork. (The guesswork method was systematically reviewed by Morrison in [10]. It has been extended to wave systems by McDonald and Kaufman [8].)

Independently, the structure (1) was obtained by Gibbons [3], using

1980 Mathematics Subject Classification 76X05, 70H05, 58F05.

This work was supported by the Director, Office of Energy Research, of the U.S. Department of Energy under Contract Number DE-AC03-76SF00098.

Lie-group concepts. This approach was extensively investigated by Marsden and Weinstein [7], who have applied these concepts to other systems as well.

A third approach to deriving the structure (1) was introduced by Bialynicki-Birula and Hubbard [1]. They began with the canonical structure for a system of particles, derived from it the noncanonical structure for the Klimontovich distribution, and then postulated the identical structure for the Vlasov distribution. The success of this approach led Kaufman to apply it to multi-fluid electrodynamics [4] and to waves [5].

The method adopted here is to begin with a field Lagrangian for the Vlasov-Coulomb system. Our form is a modification of the Low Lagrangian due to Dewar [2]. With an eye on the application to gyrokinetics [6], we consider one species (for simplicity) of particle, moving non-relativistically in a specified (possibly nonuniform) static magnetic field $\underline{B}(\underline{x})$, with only Coulomb interaction. At some reference time ($t = 0$), we specify the smooth Vlasov distribution f_0 of initial conditions $(\underline{r}_0, \underline{v}_0)$ in six-dimensional phase space. The dynamical field in a Lagrangian description is $\underline{R}(t; \underline{r}_0, \underline{v}_0)$, the corresponding position at any time t .

The Coulomb interaction energy C is thus $[z_0 = (\underline{r}_0, \underline{v}_0)]$

$$C = (e^2/2) \int d^6 z_0 \int d^6 z'_0 f_0(z_0) f_0(z'_0) |\underline{R}(t; z_0) - \underline{R}(t; z'_0)|^{-1}, \quad (2)$$

while the noninteractive field Lagrangian L_0 is

$$L_0 = \int d^6 z_0 f_0(z_0) (m \dot{\underline{R}}^2/2 + (e/c) \dot{\underline{R}} \cdot \underline{A}(\underline{R}(t; z_0))), \quad (3)$$

where $\dot{\underline{R}}(t; z_0) = \partial \underline{R}(t; z_0) / \partial t$, and $\underline{B}(\underline{x}) = \nabla \times \underline{A}(\underline{x})$. We consider the total Lagrangian $L = L_0 - C$, and apply the standard canonical procedure.

The first step is to define the momentum field $\underline{\pi}(z_0)$ canonically conjugate to the position field $\underline{R}(z_0)$:

$$\underline{\pi}(z_0) = \delta L / \delta \dot{\underline{R}}(z_0) = f_0(z_0) \underline{P}(z_0), \quad (4)$$

where $\underline{P} = m \dot{\underline{R}} + \frac{e}{c} \underline{A}(\underline{R})$ is the usual canonical particle momentum.

The next step is to express the canonical Poisson Bracket relation for the position field and its conjugate (at equal times):

$$\{ \underline{R}(z_0), \underline{\pi}(z'_0) \} = \underline{1} \delta^6(z_0 - z'_0),$$

with other brackets vanishing. Inserting (4), we obtain the interesting relation:

$$\{ \underline{R}(z_0), \underline{P}(z'_0) \} = \underline{1} \delta^6(z_0 - z'_0) / f_0(z_0), \quad (5)$$

which requires that $f_0(z_0) > 0$ everywhere. The final conventional step is to form the field Hamiltonian:

$$H = \int d^6 z_0 \underline{z}(z_0) \cdot \underline{R}(z_0) - L$$

$$= \int d^6 z_0 f_0(z_0) (\underline{P}(z_0) - (e/c) \underline{A}(\underline{R}(z_0)))^2 / 2m + C. \quad (6)$$

We now wish to transform our representation to the Vlasov field $f(\underline{r}, \underline{p}; t)$:

$$f(\underline{r}, \underline{p}; t) = \int d^6 z_0 f_0(z_0) \delta^3(\underline{r} - \underline{R}(\underline{z}; z_0)) \delta^3(\underline{p} - \underline{P}(\underline{z}; z_0)). \quad (7)$$

Thus f is a function on $(\underline{r}, \underline{p})$ -space, and simultaneously is a functional of the fields $\underline{R}(z_0)$, $\underline{P}(z_0)$. Setting aside the former property, let us consider general functionals $G(\underline{R}(z_0), \underline{P}(z_0))$. The Poisson Bracket on such functionals is obtained by using its property of differentiation, and the chain rule:

$$\begin{aligned} \{G_1, G_2\} &= \int d^6 z_0 \delta^6 z'_0 \left(\delta G_1 / \delta \underline{R}(z_0) \right) \left(\delta G_2 / \delta \underline{P}(z'_0) \right) : \{ \underline{R}(z_0), \underline{P}(z'_0) \} \\ &\quad + \left(\delta G_1 / \delta \underline{P}(z_0) \right) \left(\delta G_2 / \delta \underline{R}(z'_0) \right) : \{ \underline{P}(z_0), \underline{R}(z'_0) \} \\ &= \int d^6 z_0 \left((\delta G_1 / \delta \underline{R}(z_0)) \cdot (\delta G_2 / \delta \underline{P}(z_0)) \right. \\ &\quad \left. - (\delta G_1 / \delta \underline{P}(z_0)) \cdot (\delta G_2 / \delta \underline{R}(z_0)) \right) / f_0(z_0), \end{aligned} \quad (8)$$

where we have used the fundamental Poisson Bracket (5).

We now may immediately apply (8) to the special functional (7), choosing $G_1 = f(z_1)$, $G_2 = f(z_2)$, where $z_j = (\underline{r}_j, \underline{p}_j)$ is a parameter for purposes of the functional differentiation in (8).

After a few straightforward steps, we obtain

$$\begin{aligned} \{f(z_1), f(z_2)\} &= \int d^6 z_0 f_0(z_0) (\partial^2 / \partial \underline{r}_1 \cdot \partial \underline{p}_2 - \partial^2 / \partial \underline{r}_2 \cdot \partial \underline{p}_1) \\ &\quad \times \delta^3(\underline{r}_1 - \underline{R}(z_0)) \delta^3(\underline{p}_1 - \underline{P}(z_0)) \delta^3(\underline{r}_2 - \underline{R}(z_0)) \delta^3(\underline{p}_2 - \underline{P}(z_0)) \\ &= (\partial^2 / \partial \underline{r}_1 \cdot \partial \underline{p}_2 - \partial^2 / \partial \underline{r}_2 \cdot \partial \underline{p}_1) f(z_1) \delta^6(z_1 - z_2) \end{aligned} \quad (9)$$

Finally, we consider functionals $A(f)$, and their Poisson Bracket. We have

$$\{A_1, A_2\} = \int d^6 z_1 \int d^6 z_2 \delta A_1 / \delta f(z_1) \delta A_2 / \delta f(z_2) \{f(z_1), f(z_2)\},$$

substitute (9), and integrate by parts, obtaining (1), the desired result.

The Hamiltonian (6), (2) reads, in terms of f ,

$$H(f) = \int d^6z f(z) \left(p - (e/c) A(\underline{r}) \right)^2 / 2m + C,$$

$$C(f) = (e^2/2) \int d^6z \int d^6z' f(z) f(z') |\underline{r} - \underline{r}'|^{-1}.$$

This completes the canonical derivation of the Hamiltonian structure of the Vlasov-Coulomb system.

BIBLIOGRAPHY

1. I. Bialynicki-Birula and J. C. Hubbard, "Gauge-independent Canonical Formulation of Relativistic Plasma Theory," *Phys. Rev. A* (in press).
2. R. L. Dewar, "A Lagrangian Theory for Nonlinear Wave Packets in a Collisionless Plasma", *J. Plasma Physics* 7 (1972) 267-284.
3. J. Gibbons, "Collisionless Boltzmann Equations and Integrable Moment Equations," *Physica* 3D (1981) 503-511.
4. A. N. Kaufman, "Elementary Derivation of Poisson Structures for Fluid Dynamics and Electrodynamics," *Phys. Fluids* 25 (1982) 1993-1994.
5. A. N. Kaufman, "Natural Poisson Structures of Nonlinear Plasma Dynamics," *Physica Scripta* T2/2 (1982) 517-521.
6. A. N. Kaufman and B. M. Boghosian, "Lie-Transform Derivation of the Gyrokinetic Hamiltonian System," this volume.
7. J. Marsden and A. Weinstein, "The Hamiltonian Structure of the Maxwell-Vlasov Equations," *Physica* 4D (1982) 394-406.
8. S. W. McDonald and A. N. Kaufman, "Hamiltonian Kinetic Theory of Plasma Ponderomotive Processes," in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, M. Tabor and Y. M. Treve, eds. (AIP Conf. Proc. 88, 1982), 117-120.
9. P. J. Morrison, "The Maxwell-Vlasov Equations as a Continuous Hamiltonian System", *Phys. Lett.* 80A (1980) 383-386.
10. P. J. Morrison, "Poisson Brackets for Fluids and Plasmas," in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, M. Tabor and Y. M. Treve, eds. (AIP Conf. Proc. 88, 1982), 13-46.

* PHYSICS DEPARTMENT AND
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA, BERKELEY
BERKELEY, CA 94720

** DEPARTMENT OF THEORETICAL PHYSICS
RESEARCH SCHOOL OF PHYSICAL SCIENCES
AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA ACT 2600, AUSTRALIA

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.