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COMPUTATIONAL PROBLEMS
IN
MAGNETIC FUSION RESEARCH

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COMPUTATIONAL PROBLEMS IN MAGNETIC FUSION RESEARCH

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ABSTRACT

Numerical calculations have had an important role in fusion research since its beginning, but the application of computers to plasma physics has advanced rapidly in the last few years. One reason for this is the increasing sophistication of the mathematical models of plasma behavior, and another is the increased speed and memory of the computers which made it reasonable to consider numerical simulation of fusion devices. The behavior of a plasma is simulated by a variety of numerical models. Some models used for short times give detailed knowledge of the plasma on a microscopic scale, while other models used for much longer times compute macroscopic properties of the plasma dynamics.

The computer models used in fusion research are surveyed. One of the most active areas of research is in time-dependent, three-dimensional, resistive magnetohydrodynamic models. These codes are reviewed briefly.

1. INTRODUCTION

For the purpose of discussing the physical models, and also the numerical methods, it is convenient to consider the following categories of computer codes used to model the physics of fusion devices,

1. Time-dependent magnetohydrodynamics
2. Plasma transport in a magnetic field
3. MHD and guiding-center equilibria
4. MHD stability of confinement systems
5. Vlasov and particle models
6. Multi-species Fokker-Planck codes
7. Hybrid codes

In a short paper it is impossible to review all of the above topics, so a brief description of the models is given, followed by a review of time-dependent, three-dimensional, resistive magnetohydrodynamic codes.

1.1 Time-dependent magnetohydrodynamics

Detailed comparison of experimental data from pulsed high-beta devices with theory, taking due account of experimental complications depends on the

application of 2-D and 3-D (two- and three-dimensional) versions of codes analogous to the 1-D Hain-Roberts code [1]. Most of the physical phenomena important here lie in the fast MHD time scale (nanoseconds to microseconds). For Tokamak configurations, the corresponding effects occur on longer time scales-milliseconds; however, the questions of stability of Tokamak discharges toward MHD modes are very important. One example of great interest is the study of the early stages of a Tokamak discharge and the formation and destruction of magnetic surfaces.

Fast time scale MHD codes are typically used to investigate the time dependent behavior of instabilities. The main question to be answered is whether or not a particular MHD mode will be unstable, and if so, how fast will it grow and what is its structure. Although linear MHD stability calculations have made a significant contribution to our understanding of plasma phenomena, nonlinear MHD problems, including the effects of resistivity, rely totally on computers. In order to analyze nonlinear resistive instabilities, the time dependent MHD equations of motion must be solved. The most advanced resistive MHD stability codes are nonlinear and three-dimensional. This degree of generality is necessary in order to study the coupling of modes.

There are a great variety of MHD codes being developed. Within the fluid theory various degrees of complexity are considered. The so-called ideal MHD is an infinite conductivity approximation. The more realistic models include the transport coefficients, e.g., thermal conductivity and electrical resistivity, and these can be scalars or tensors [1,2]. Two-dimensional codes are now fairly standard and there are several three-dimensional codes. In some cases perturbation theory is used and the equations are then linearized and Fourier analyzed in one or two coordinates.

The choice of coordinate system varies among these codes. A fixed Eulerian grid is the usual choice, but Lagrangian descriptions, particularly using magnetic flux surfaces as coordinate surfaces [3], are proving useful in certain problems as are particle-in-cell methods [4]. In the work of Brackbill [5] a moving grid is used which is not a Lagrangian grid. A variety of difference schemes are being used, ranging from fully explicit using a Lax-Wendroff or a leap-frog scheme, to implicit methods employing the ADI scheme or "splitting" (the method of fractional time steps). In a later section of this paper we shall describe the variety of numerical methods used in three-dimensional codes.

1.2 Plasma transport in a magnetic field

In order to simulate the transport of a plasma in a magnetic confinement device

over most of its lifetime--from tens to hundreds of milliseconds--a set of partial differential equations of the diffusion type must be solved. Typical dependent variables are the number densities and temperatures of each particle species, current densities, and magnetic fields. The transport coefficients such as thermal conductivity, electrical resistivity, and diffusion coefficients are obtained from the best available theories, but the codes also have the capability of easily changing the form of the coefficients in order to develop phenomenological models. In the past years, a considerable effort has been devoted to the numerical solution of these equations for toroidal plasmas which provides an excellent means of comparing theory with experiment. Reviews of these models and their application to Tokamaks are given in Refs. [6,7]. In all of these codes implicit difference methods are used for the solution of the coupled diffusion equations.

1.3 MHD and guiding-center equilibria

It is necessary to develop time-independent codes to support the design and operation of each major fusion experiment. These include codes used to compute and study prospective equilibrium plasma configurations. Experimental devices incorporating the idea of axial symmetry in a torus appear to be capable of plasma confinement for times which are of great interest. One reason for this result is the assurance of equilibria in such devices, as predicted by MHD and guiding-center theories. The computation of these two-dimensional equilibria involves the solution of an elliptic partial differential equation. Iterative methods such as SOR, ADI, and ICCG, and direct methods including cyclic reduction and FFT have all been used in these codes. Equilibrium computational methods are reviewed in Refs. [8, 9].

1.4 MHD stability of confinement systems

Another important class of time-independent code is used to determine the ideal MHD stability of equilibrium configurations by calculating eigenvalues of the linearized perturbed equations. Variational methods are used, and finite elements and Fourier series are used to represent eigenmodes. Reviews of these computations have been given by Grimm et al [10] and Gruber et al [11].

1.5 Vlasov and particle models

Particle codes are fundamental in that they compute in detail the motion of particles under the influence of their self-consistent electric and magnetic fields, as well as any externally imposed fields. These codes give phase-space distribution functions, fluctuation and wave spectra, and orbits of individual particles. They are ideal for providing detailed information on the growth and

saturation of strong instabilities and the effects of turbulence.

Particle codes are usually classified as either "electrostatic" or "electromagnetic". In the first type only the self-consistent electric field is computed via Poisson's equation and the magnetic field is either absent or constant in time. Recent methods and results are reviewed in Ref. [12]. In the last five years there has been a considerable development in electromagnetic codes. They are either relativistic and fully electromagnetic, i.e., the particle equations of motion are relativistic and the electric and magnetic fields are obtained from the full Maxwell equations (wave equations) as in Ref. [13] or they are in the nonradiative limit where the equations are nonrelativistic and displacement currents are neglected as in Ref. [14]. Recent research [15-19] in particle codes has concentrated on developing implicit, orbit-averaging and moment algorithms for moving the particles and computing their charge and current densities. These results are of great importance to the fusion program, as they will allow particle codes to be used to simulate plasma phenomena on time scales much longer than previously considered.

1.6. Multi-species Fokker-Planck codes

In the simulation of magnetically confined plasmas where the ions are not Maxwellian and where a knowledge of the distribution functions is important, kinetic equations must be solved. At number densities and energies typical of mirror machines, the end losses are due primarily to the scattering of charged particles into the loss cones in velocity space by classical Coulomb collisions. The kinetic equation describing this process is the Boltzmann equation with Fokker-Planck collision terms [20]. The use of this equation is not restricted to mirror systems. The heating of plasmas by energetic neutral beams, the thermalization of α -particles in DT plasmas, the study of runaway electrons and ions in Tokamaks, and the performance of two-energy component fusion reactors are other examples where the solution of the Fokker-Planck equation is required [21].

The first injection of neutral beams into Tokamak plasmas took place at the Culham, Princeton, and Oak Ridge laboratories in 1972-73. The injected ions were studied with linearized Fokker-Planck models [22,23,24] and the expected plasma heating was observed experimentally.

With the advent of much more powerful neutral beams, it is now possible to consider neutral-beam-driven Tokamak fusion reactors. For such devices, three operating regimes [25] can be considered: (1) the beam-driven thermonuclear reactor, (2) the two-energy component torus (TCT), and (3) the energetic-ion-reactor e.g., the counterstreaming ion torus (CIT). In order to study reactors

in regimes (2) or (3), a non-linear Fokker-Planck model must be used because most of the fusion energy is produced by beam-beam or beam-plasma reactions. Furthermore, when co and counter injection are used, or major radius compression is employed, a two velocity-space dimensional Fokker-Planck operator is required. A non-linear, two-dimensional, multi-species Fokker-Planck model [21] developed for the mirror program was applied successfully to several scenarios of TCT operation [26,27,28]. The successful application of the two-dimensional Fokker-Planck model to the energy multiplication studies of TCT led to the formulation of a more complete model of beam-driven Tokamak behavior [29,30].

Neutral beam heated tokamaks [25] are characterized by a warm Maxwellian background plasma, whose evolution can be described by a set of macroscopic transport equations, and one or more energetic species which are quite non-Maxwellian, whose evolution should be represented by Fokker-Planck equations. The coupling of these systems is by means of particle and energy sources in the multispecies transport equations and a Maxwellian target plasma in the multi-species Fokker-Planck equation.

The Fokker-Planck/Transport (FPT) Code [29,30] models the time-dependent behavior of such a system. The model assumes the existence of an arbitrary number of Maxwellian warm ion species which are described by their individual densities $n_i(r, t)$ and by a common temperature profile $T_i(r, t)$ where r is the average radius of a flux surface. The electrons are described by a separate temperature profile $T_e(r, t)$ and their density is determined by quasineutrality. The energetic species are represented by velocity space distribution functions $f_p(v, \mu, t)$, where v is speed and μ is pitch angle. In addition to one-dimensional radial transport equations for the bulk plasma densities and temperatures, one nonlinear Fokker-Planck equations in two-dimensional velocity space for the energetic ion distribution functions; neutral beam deposition and neutral transport are modeled using Monte Carlo codes [31,32].

1.7 Hybrid codes

There is a need for codes which can best be described as Hybrid Codes; these are codes which combine the good features of fluid codes with the good features of particle codes. The advantage of a particle code is that it contains the most complete treatment of the physics. Its disadvantage also stems from this feature because it is forced to follow the development of the plasma on the fastest time scale and shortest space scale at which significant plasma phenomena occur. These scales are typically much shorter than the time and size scale in fusion devices. The feature of fluid codes which is attractive is that they treat the plasma on a coarser scale and hence need many fewer time

steps and spatial points; however, the motions of certain classes of particles are often crucial. It is clear that proper treatment of such phenomena requires an accurate description of the important class of particles. On the other hand, it is possible to treat the rest of the plasma by means of fluid equations.

Another class of hybrid code is the coupling of a Fokker-Planck code to a plasma transport (diffusion) code which was described in the last subsection.

2. TIME-DEPENDENT, THREE-DIMENSIONAL, RESISTIVE MAGNETOHYDRODYNAMICS

In order to achieve the high densities and temperatures required for a successful fusion reactor, a plasma must be confined by a magnetic field for a sufficiently long time. In the attempts to achieve this confinement, the problem of stability has emerged as one of the most important. The most dangerous types of instabilities are the magnetohydrodynamic (MHD) instabilities in which the plasma is assumed to behave as a conducting fluid and the instabilities involve displacement of macroscopic portions of the plasma. It is a particular MHD instability, the resistive instability which is considered in this section.

Resistivity can destroy the stabilization achieved by the shearing of the lines of force. In the case of a magnetic field which has shear or which changes direction, the magnetic energy can be reduced by allowing the fields to mix and annihilate. This is prevented in a perfectly conducting plasma, but with finite conductivity an instability can develop in which the magnetic lines of force are torn into "islands". This type of resistive instability is known as a resistive tearing mode [33].

There are three types of resistive modes: (1) the rippling mode, which is driven by a gradient in the resistivity and is usually not important when large temperature gradients are unlikely; (2) the gravitational mode (g-mode) which is the resistive equivalent of the interchange instability and is important in sheared systems; and (3) the tearing mode, which is the resistive equivalent of the kink mode and involves displacement of the whole plasma.

The modes grow on a time scale intermediate between the resistive diffusion time $\tau_R = 4a^2/\eta c^2$ and the hydromagnetic transit time $\tau_H = a(4\pi/\mu)^{1/2} B^{-1}$ where 'a' is a characteristic dimension of the plasma layer, η is the resistivity, μ is the mass density of the plasma, B is the magnetic field, and c is the speed of light.

Due to the many possible equilibrium configurations and the many approximations necessary to make the problem analytically tractable, it is usually not possible to analytically describe the general parameter dependence of the growth rates. In order to obtain results for specific and wide choices of equilibrium magnetic fields and boundary conditions, numerical models have been developed to study these resistive instabilities. Some codes employ a set of "reduced" MHD equations which are appropriate for large aspect ratio, low β , Tokamak plasmas.

The development of three-dimensional, initial-value codes for the solution of MHD equations has evolved from a few exploratory studies in the 1970's to several production type codes, which at the present time are making important contributions to the understanding of resistive instabilities in toroidal plasmas. This evolution is examined from several points of view. It is useful to consider the following options for initial-value MHD code development:

- primitive MHD equations, reduced MHD equations
- compressible, incompressible
- ideal MHD, dissipative MHD
- one, two, three dimensions
- linear, nonlinear
- Eulerian, Lagrangian, ALE, dynamical grid
- finite differences, finite elements, expansions
- explicit difference equations, implicit difference equations

The codes considered in this review will be categorized according to the above options. In addition the applications of the codes will be discussed, as these usually explain the choice of options.

3.1 Primitive, incompressible, linear, resistive MHD codes

The numerical study of linear MHD stability has followed two paths. In the first, the linearized MHD equations (either resistive or ideal) are treated as an initial value problem [34-43]. The equilibrium state of the system is specified and is given a perturbation. The MHD equations are advanced forward in time to trace the evolution of the plasma. The fastest growing instability eventually dominates over all other motions. The second approach which has been used very successfully is to utilize the energy principle [10,11] to gain information on the full spectrum of ideal MHD instabilities.

The resistive instability of an incompressible plasma was first analytically investigated by Furth, Killeen, and Rosenbluth [33]. They used the plane slab model, in which the equilibrium depends only on y , the magnetic field is $\hat{x} B_{x0} + \hat{z} B_{z0}$, and $\hat{y} \cdot \nabla = 0$. In that paper perturbations of the form

$f_1(y) \exp [i(k_x x + k_z z) + \omega t]$ are assumed, and the problem is to solve an eigenvalue problem for ω , the growth rate of the instability. In order to solve the problem the plasma is divided into two regions, a narrow inner region about the plane for which the wave vector is perpendicular to the zero-order magnetic field ($\mathbf{k} \cdot \mathbf{B}_0 = 0$) and an outer region where the infinite conductivity equations hold. By matching the solutions within the resistive layer to the outer ideal MHD solutions, FKR found resistive tearing modes with growth rates, $\mu \approx \frac{1}{2} R$, proportional to $S^{2/5}$, where $S \approx \tau_R / \tau_H$.

We can also assume an arbitrary time-dependence and the problem becomes an initial-value problem. Two regions are not used, i.e., the same equations hold throughout the plasma. The initial-value problem is then solved numerically. This method of solution was developed [34] for the linear model simultaneously with the analytic technique and is described in Ref. [35]. The initial-value codes, MPPLE, use the same basic equations and assumptions as FKR and are capable of finding tearing, rippling, and gravitational modes as well as mixed modes.

We assume that the hydromagnetic approximation is valid, and the ion pressure and inertia terms are neglected in Ohm's law. An isotropic resistivity is assumed, the fluid is assumed to be incompressible, and perturbations in resistivity result only from convection. The basic equations are:

$$\frac{\partial \mathbf{B}}{\partial t} = \text{curl} (\hat{\mathbf{v}} \times \mathbf{B}) = \text{curl} \left(\frac{1}{4\pi} \text{curl} \mathbf{B} \right), \quad (2.1)$$

$$\text{div} \mathbf{B} = 0, \quad \text{div} \hat{\mathbf{v}} = 0, \quad (2.2)$$

$$\text{curl} \left(\rho \frac{d\hat{\mathbf{v}}}{dt} \right) = \text{curl} \left(\frac{1}{4\pi} \text{curl} \mathbf{B} \times \mathbf{B} \right), \quad (2.3)$$

$$\frac{d}{dt} (\hat{\mathbf{v}} \cdot \mathbf{B}) = 0. \quad (2.4)$$

In Eqs. (2.1) - (2.3) we consider $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ and $\hat{\mathbf{v}} = \hat{\mathbf{v}}_0 + \hat{\mathbf{v}}_1$, where $\mathbf{B}_0$ and $\hat{\mathbf{v}}_0$ are given and the subscript 1 denotes perturbed quantities. We obtain, to first order, the following set of linearized equations:

$$\frac{\partial \mathbf{B}_1}{\partial t} = \text{curl} (\hat{\mathbf{v}}_0 \times \mathbf{B}_1 + \hat{\mathbf{v}}_1 \times \mathbf{B}_0) = \frac{1}{4\pi} \text{curl} \left(\hat{\mathbf{v}}_0 \text{curl} \mathbf{B}_1 + \hat{\mathbf{v}}_1 \text{curl} \mathbf{B}_0 \right) \quad (2.5)$$

$$\text{div} \mathbf{B}_1 = 0, \quad \text{div} \hat{\mathbf{v}}_1 = 0, \quad (2.6)$$

$$\rho_0 \text{curl} \left[\frac{\partial \hat{\mathbf{v}}_1}{\partial t} + (\hat{\mathbf{v}}_0 \cdot \nabla) \hat{\mathbf{v}}_1 \right] = \frac{1}{4\pi} \text{curl} \left[(\hat{\mathbf{B}}_0 \cdot \nabla) \mathbf{B}_1 + (\mathbf{B}_1 \cdot \nabla) \hat{\mathbf{B}}_0 \right], \quad (2.7)$$

with ρ_0 determined from the first-order version of Eq. (2.4).

In order to consider more realistic equilibrium magnetic fields, a cylindrical model, RIPPLE IV, was developed [35], which also used the above equations of incompressible magnetohydrodynamics. This model has been extensively applied to the study of tearing modes in reversed field pinch [36,37], and tokamak [37] equilibria. DiBiase developed a new cylindrical model [37,39] which includes the effects of compressibility, viscosity, and thermal conductivity along with finite resistivity. This model has also been applied to the RFP [37-39].

The linear model given by Eqs. (2.5) - (2.7) is applied in RIPPLE IV in order to study specific diffuse pinch configurations. The equilibrium is given by $\vec{B}_0(r) = \hat{r} B_{\theta 0}(r) + \hat{z} B_{z0}(r)$, $\vec{v}_0 = 0$, and $n_0 = n_0(r)$. These functions are chosen to describe a particular experiment, and the stabilizing effect of the location of the conducting walls (R_w) with reference to the singular surface can be determined. We assume perturbations of the form $f_1(r,t) \exp[i(m\theta + k_z z)]$. We can find a consistent system of four equations involving the components $B_{\theta 1}$, B_{z1} , v_{r1} , $v_{\theta 1}$ and an equation for n_1 . These equations are solved by implicit difference methods in the r direction for each mode (m, k_z) .

Recently there has been interest in the effect of equilibrium flow on the tearing mode [40,45]. We have developed a new linear initial-value code, RIPPLE V, [40] to study this problem. For this work we have gone back to the plane slab model using the incompressible MHD equations. We have also applied the linear slab model to the double tearing mode [46]. In this case there are two neighboring singular surfaces, i.e., surfaces for which $k \cdot \vec{B}_0 = 0$. If these surfaces lie close to one another, the modes at each singular surface may interact leading to an enhanced growth rate.

In all of the above linear models the initial-value problems solved are one-dimensional, i.e., the zero-order fields are given by $\vec{B}_0(y)$ or $\vec{B}_0(r)$ and the perturbed variables take the form

$$f_1(y,t) \exp[i(k_y y + k_z z)] \text{ or } f_1(r,t) \exp[i(m\theta + k_z z)]$$

In many toroidal confinement devices it is not possible to specify the equilibrium fields as functions of one variable. In tokamaks and compact torii the zero-order field can be specified by $\vec{B}_0(r,z)$. To study tearing modes in such configurations we have developed a new two-dimensional, linear code, RIPPLE VI, [41] in which the perturbations are of the form $f_1(r,z,t) \exp[i(n\theta + k_z z)]$. We use the incompressible MHD equations to derive a set of eight coupled linear partial differential equations. For the case, $n = 0$, we have developed a 2D axisymmetric linear code (ALIMO) [47], which makes use of field and velocity stream functions, resulting in a system of four equations.

These linear codes are used for extensive parameter studies of prospective equilibria. Stable and unstable regions of wave number space, growth rates of exponentially growing modes and their mode structure are calculated. In order to study the longtime, large amplitude behavior of these modes, and to simulate experimental devices in controlled fusion research, the non-linear fluid equations must be solved. In general, such a calculation requires the simultaneous advancement in time of eight non-linear partial differential equations in several spatial dimensions.

2.2 Primitive, compressible, nonlinear, resistive MHD codes

After the linear modes have been determined, the resultant eigenmodes may be used to start a nonlinear calculation which can then follow their growth to some final state. These nonlinear calculations have all been with initial-value codes in two and three space dimensions. The nonlinear codes are not limited to stability studies. They are also used to simulate total experiments. By incorporating time dependent boundary conditions, such as circuit equations, a simulation can start with a neutral gas, and model its ionization, pinching, and subsequent relaxation.

The primitive resistive magnetohydrodynamic (MHD) equations relate the electromagnetic fields E and B to the fluid velocity v and the thermodynamic variables (the pressure, P , and mass density, ρ). In terms of the nondimensional variables

$$\begin{aligned} x/a &\rightarrow x & t/t_H &\rightarrow t & B/B_0 &\rightarrow B \\ v/v_A &\rightarrow v & \rho/\rho_0 &\rightarrow \rho & P/P_0 &\rightarrow P \\ \mu_0 J_0 &\rightarrow j \end{aligned}$$

these equations take the form

$$\frac{\partial B}{\partial t} - \nabla \times (v \times B - \frac{1}{S} \nabla \times B) = 0 \quad (2.8a)$$

$$\frac{\partial v}{\partial t} + v \cdot \nabla v + \frac{1}{2} \nabla P + B \times (\nabla \times B) = 0 \quad (2.8b)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \quad (2.8c)$$

$$\frac{\partial P}{\partial t} + \nabla \cdot (v P) - (\gamma - 1) \left\{ \frac{1}{2} \nabla \cdot P + \frac{2\eta}{S} (\nabla \times B)^2 \right\} = 0 \quad (2.8d)$$

where η is the electrical resistivity of the fluid, and we have assumed Ohm's law in the form

$$\underline{E} + \underline{v} \times \underline{B} = \frac{1}{c} \underline{j} \quad (2.3)$$

We assume the perfect gas law is valid so that $e = P/(\epsilon - 1)$, where e is the specific internal energy and $\epsilon = C_p/C_v$ is the ratio of specific heats for the fluid.

In the normalization described above subscripts $\{ \}_0$ refer to characteristic values of various quantities, a is a characteristic length for changes in the magnetic field, $v_A = B_0/(4\pi n_0)^{1/2}$ is the Alfvén velocity, and $t_H = a/v_A$ is the Alfvén transit time. The normalization of the thermodynamic quantities is chosen such that $\rho_0 = B_0^2/8\pi = \rho_A^2/2$. The quantity $S = t_R/t_H = 4\pi a v_A/c^2$ is the magnetic Reynolds number, and is the ratio of the two characteristic time scale, of Eq. (1.8): the resistive diffusion time $t_R = 4\pi a^2/c^2$, and the Alfvén transit time t_H .

Equations (2.4c) are a set of eight nonlinear equations in the eight unknowns B, v, ρ, ϵ and P . The above set of equations are not all in conservation form. Equation (2.4c) is in conservation form, and Eq. (2.4a) is in the form of a pseudovector conservation law. Equation (2.4b) could be converted to conservation form by adding $B \cdot \nabla B = 0$ to the equation. This addition has not been done because on a finite difference mesh $\nabla \cdot B$ is not necessarily zero. The pressure equation (2.4d) is also not in conservation form. If we define in our normalized variables a total energy density w as

$$w = \frac{1}{2} v^2 + \frac{B^2}{8\pi} + P/(\epsilon - 1) \quad (2.1e)$$

an equation in conservation form for w can be obtained [4d]. When conservation form is used the pressure becomes a subsidiary variable. When the plasma $\beta = (P/B^2)$ is small, the error in the pressure calculation becomes equal to or greater than the calculated pressure. In this form, then, low β plasmas cannot be simulated.

At this point we should consider the question of why use the primitive MHD model, which leads to eight coupled equations, when the reduced MHD model is available, leading to only two equations. The reduced equations employ Tokamak ordering, i.e. ϵ is of order ϵ^2 , where $\epsilon = a/R \ll 1$, is the aspect ratio of the torus, and the toroidal field is much larger than the poloidal field. The toroidal field is assumed constant in the reduced model. If we wish to consider high β Tokamaks, or the reversed field pinch (RFP), where the toroidal and poloidal field are comparable, or a compact torus (small aspect ratio) then the reduced equations are not applicable and we must use the primitive MHD model.

One of the earliest 3D resistive MHD codes was TRINITY developed by Roberts and Boris [49]. It was an Eulerian code using an explicit leap-frog difference scheme with the resistive diffusion terms employing the Dufort and Frankel algorithm. Similar 3D codes subsequently developed were those of Wooten et al (ideal) [50], Cochran et al [51], and Pritchett et al (ideal) [52]. These codes have been used to study internal kink and interchange instabilities in simple geometries.

Sykes and Wesson have also developed a 3D Eulerian code which includes a tensor resistivity and thermal conductivity [53-55]. Their code employs a lax-Wendroff scheme and the diffusion terms are also treated by an explicit method. This code has been used to study internal sawtooth oscillations in Tokamaks [53], spontaneous toroidal field reversal in the RFP [54], and high beta internal sawtooth oscillations in toroidal geometry [55].

All of the above codes must satisfy a Courant time-step condition. Generally this is much smaller than one imposed by the time scale of the physical instability being studied. To avoid this limitation Finan [56,57] developed a 3D Eulerian code which uses the Douglas-Gunn algorithm for Alternating-Direction Implicit temporal advancement. The eight equations are solved simultaneously to avoid synchronization errors. The resulting finite difference equations are a coupled system of nonlinear algebraic equations which are solved by the Newton-Raphson iteration technique. Time steps 10-50 times the Courant condition are used in long time simulations. The Finan code, IMP, generalizes to three dimensional techniques that have proved successful in two dimensional resistive MHD calculations [2,46,48,58]. The code has been used to model the tearing mode and internal kink instability in cartesian geometry and the internal kink in toroidal geometry [56,57]. The code has also successfully modeled spontaneous field reversal in the RFP [56].

All of the above 3D nonlinear codes use finite differences in all three directions. In order to obtain the required spatial resolution for high S resistive instabilities, and simultaneously to decrease the computer time for the simulation of these instabilities, recent 3D MHD code developments have used expansions in one or two directions with finite differences in the remaining direction(s).

Hender [59] has developed a 3D Eulerian code to study the coupling of resistive "q" modes in reversed field pinches. It assumes cylindrical geometry and starts with an equilibrium which is a function of r only. The linear codes RIPPLE 4a [35,36] or RESTAB [39] are used to calculate unstable modes which are parametrized by m and k_z . The 3D nonlinear code uses finite differences

in the r direction and a Fourier expansion in θ and z . The finite difference method is a mixture of implicit and explicit algorithms.

Mirin [60] has developed a 3D Eulerian code to study the nonlinear growth of MHD instabilities in axisymmetric toroidal equilibria which are functions of r and z . The linear code RIPPLE VI [41] calculates unstable modes, periodic in ϕ , which can be used as initial values for the 3D nonlinear code. The nonlinear code uses an explicit Lax-Wendroff method in r and z with a Fourier expansion in ϕ .

Turner and Wesson are developing a similar code [61] for 3D MHD simulations of JET.

2.3 Reduced, nonlinear, resistive MHD codes

In order to study kink modes in Tokamaks, a set of reduced equations was derived [62] and solved in two dimensions (cylindrical coordinates with helical symmetry) by several groups. Strauss [63,64] has extended the two dimensional model [62] to three dimensions, and solves the equations on an Eulerian mesh with a leap-frog time advancement algorithm. Strauss [63] uses this 3D code to study internal kinks in a square pipe.

The resistive MHD equations in three dimensions can be similarly simplified [65] by the application of Tokamak ordering. The first assumption is $B_z/B_\theta \sim \epsilon$, where B_θ is the poloidal field, B_z is the toroidal field, and $\epsilon = a/R_0 \ll 1$ is the aspect ratio. Here a is the minor radius and R_0 is the major radius of the torus. A cylindrical coordinate system (r, θ, z) is employed, where r ($0 \leq r \leq a$) is the minor radius, θ ($0 \leq \theta \leq 2\pi$) is the poloidal angle, and z ($0 \leq z \leq 2\pi$) is the toroidal angle. This ordering has the important effect that the time variation of the toroidal magnetic field, B_z , can be ignored and, consequently, the fastest time scale of the equations, the time for the propagation of Alfvén waves across the magnetic field, is removed from the dynamics. The fastest time-scale remaining in the equations is the time for Alfvén waves to propagate along the magnetic field. The slow timescale is the resistive diffusion time. The second assumption is to consider only a low β plasma, where β is the ratio of plasma pressure to magnetic field pressure. Specifically, $\beta \sim \epsilon^2$ is assumed.

The above assumptions yield two scalar partial differential equations which in dimensionless form are [65]

$$\frac{\partial \psi}{\partial t} = -\mathbf{J} \cdot \nabla \psi - \frac{\partial \psi}{\partial t} = E_{\parallel}^W \quad (2.11a)$$

and

$$\frac{DU}{Dt} = -S^2 \left[\hat{e}_z \cdot (\nabla_\perp \times \nabla \psi) + \frac{\partial J_z}{\partial t} \right] \quad (2.11b)$$

where

ψ is the poloidal flux function (normalized to $a^2 B_{z0}$) defined by

$$\underline{B} = (-\nabla \psi \times \hat{e}_z + \hat{e}_z) B_{z0},$$

$\frac{D}{Dt}$ is the convective derivative $\frac{\partial}{\partial t} + \underline{v}_\perp \cdot \nabla$,

$\underline{v}$ is the fluid velocity in units of a/r_0 ,

$\perp$ denotes perpendicular to $\hat{e}_z$,

S is the resistivity normalized to unity at the magnetic axis,

$J_z = \frac{1}{2} \nabla_\perp^2 \psi$ is the toroidal component of the plasma current density
 normalized to $B_{z0} R_0 / B_{z0}$,

ψ is the velocity stream function, $\underline{v}_\perp = \nabla \psi \times \hat{e}_z$,

$E_\perp$ is the equilibrium toroidal electric field at the wall, and

$\omega = \frac{1}{2} \nabla_\perp^2 \psi$ is the toroidal vorticity.

The Oak Ridge code RS3 [66] solves Eq. (2.11) using differences in all three directions with a mixed explicit-implicit scheme. This code has been superseded by RSF [67] which uses finite differences in r and expansions in the two periodic directions, and runs considerably faster for the same accuracy. In order for a code such as RSF to model three dimensional behavior, care must be taken to insure that a sufficient number of modes are chosen for possible coupling and excitation. RSF typically uses 35 modes, but can handle 80.

The code has been applied to the feedback stabilization of the 2/1 tearing mode [68], the study of Mirmov oscillations [69], and an extensive study of major disruptions under varying conditions [70-73]. The explanation of disruptions found in these investigations involves a coupling of tearing modes, in particular 2/1 and 3/2 modes. Such studies are only possible with three dimensional codes.

Another 3D reduced MHD code which makes use of an expansion in the two periodic directions is that of Edery et al [74,75]. Similar codes are also in operation at the Princeton Plasma Physics Laboratory [76] and in Japan at JAERI [77].

2.4 Summary

A number of codes which solve either the primitive or reduced equations of resistive magnetohydrodynamics in three dimensions, as initial-value problems, have been examined. Because of the nature of resistive instabilities e.g., tearing and reconnection, the Eulerian representation is generally used. Recent developments favor the use of expansions in one or two directions because of increased speed and accuracy, as long as the number of modes considered is not too large. The codes which employ expansions are also more compatible with the linear codes [34-43] which solve each mode separately. These modes can be combined to start the nonlinear problems.

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