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Stochastic Propagation of an Array of Parallel Cracks: Exploratory Work on Matrix Fatigue Damage in Composite Laminates

R. E. Williford

September 1989

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Transverse cracking of polymeric matrix materials is an important fatigue damage mechanism in continuous-fiber composite laminates. The propagation of an array of these cracks is a stochastic problem usually treated by Monte Carlo methods. However, this exploratory work proposes an alternative approach wherein the Monte Carlo method is replaced by a more closed-form recursion relation based on "fractional Brownian motion." A fractal scaling equation is also proposed as a substitute for the more empirical Paris equation describing individual crack growth in this approach. Preliminary calculations indicate that the new recursion relation is capable of reproducing the primary features of transverse matrix fatigue cracking behavior. Although not yet fully tested or verified, this recursion relation may eventually be useful for real-time applications such as monitoring damage in aircraft structures.

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1.0 SUMMARY

Transverse fatigue cracking of polymeric matrix materials is important in laminated, continuous-fiber composite structures because it determines (in part) the rate at which damage accumulates toward failure of the composite. The phenomenon can be described as the propagation of an array of parallel cracks that effectively "shield" each other by shear lag effects. Such a problem is stochastic, and is usually approached via Monte Carlo methods. Individual crack propagation within the array is usually addressed with an empirical Paris-type equation.

The purpose of this report is to begin development of an alternative and more closed-form mathematical solution that should be more useful for real-time simulations, such as monitoring damage in aircraft structures. In this regard, the Monte Carlo method is replaced by a stochastic modeling superstructure based on "fractional Brownian motion." The Paris equation is replaced by a scaling equation proposed to relate fatigue cycle number to individual crack growth. Both of these two model elements are based on fractals.

A recursive relationship is developed to describe the dependence of the numbers of cracks on fatigue cycles and stress. Very limited preliminary calculations indicate that this new model is capable of reproducing the major features of transverse matrix fatigue cracking, including the dependence on laminate thicknesses, the delay in cycles until the first few cracks form, and the accumulation of transverse cracks toward a saturation value. Additional calculations would be expected to verify the model. However, the reader should note that this brief work is very exploratory in nature, and can make no claim of rigor. There are many rather tenuous assumptions that must be justified in future work.

Finally, it should be noted that this report presumes the reader to have prior knowledge of the basic essentials of composite fatigue and fractals.

2.0 INTRODUCTION

Fatigue concepts for composite laminates have been reasonably well developed from a continuum mechanics perspective. For example, Talreja (1981) has described fatigue-life diagrams for composite laminates with polymeric matrix materials in terms of three basic damage regimes. In a simplified form that ignores all recognized delamination phenomena, these regimes are dominated by fiber breakage, matrix cracking, and the fatigue limit of the matrix material. Each mechanism is confined to a particular strain or loading regime (Figure 1).

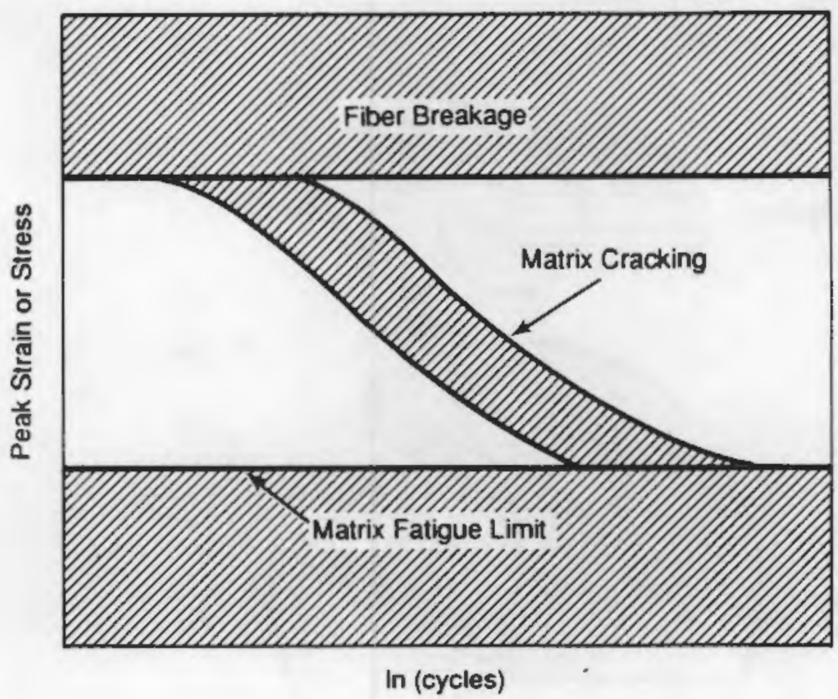

Figure 1. Fatigue-life diagram for composites.

It is evident that matrix cracking is an important damage evolution mechanism because it determines the allowable strain or loading as a function of accumulated fatigue cycles. In comparison, the other mechanisms may appear relatively constant with cycle number.

Much data has been collected to understand composite fatigue mechanisms, and observations have been generalized. For example, Wang, et al. (1984), argue that "for polymer-based, continuous fiber systems, the fundamental mode of damage usually involves matrix cracking as the first sign of failure." They further state that "clearly, the ability to analytically describe the crack formation and distribution characteristics by a general methodology is an essential first step towards assessing the various effects [that matrix cracking can have] on laminate properties," such as strength and stiffness. A particular case studied by Wang, et al. (1984), was transverse matrix cracking of $[0_2/90_m]_s$ continuous-fiber laminates (m = 2,3) in the AS-3501-06 graphite-epoxy system. A schematic of these transverse matrix fatigue cracks is shown in Figure 2. As fatigue cycles advance, the number of matrix cracks accumulate as in Figure 3. These data are representative of the phenomenon to be modeled in the present work.

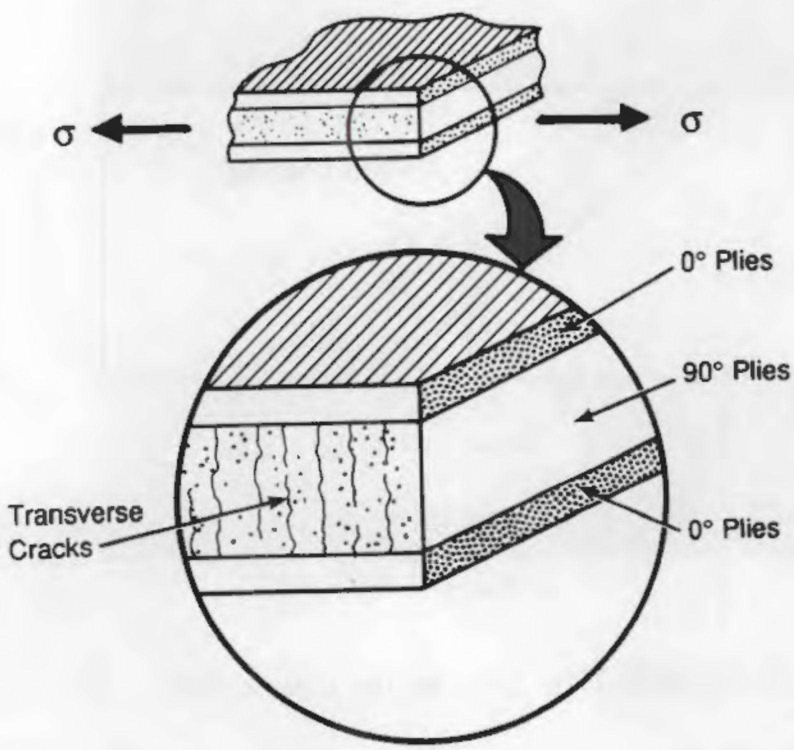

Figure 2. Schematic view of multiple transverse cracks.

The mechanics of the growth of such an array of cracks appears uncertain. Wang, et al. (1983), noted that partially developed cracks (crack length "2a" less than laminate thickness "2h") were rarely observed in post-test examinations. This would appear consistent with the brittle nature of the epoxy matrix, resulting in unstable crack growth from small flaws of sizes less than the fiber diameters. These flaws were not measured by Wang, et al. (1982, 1983, 1984), because they could not be observed by optical microscopy. They chose to describe this array of microflaws with an "effective flaw" concept where crack lengths (a) and spacings (s) both assumed normal distributions, as shown in Figure 4. This, in turn, permitted the use of linear elastic fracture mechanics (LEFM) to describe the growth of the "effective flaws" into transverse matrix cracks of length a=h.

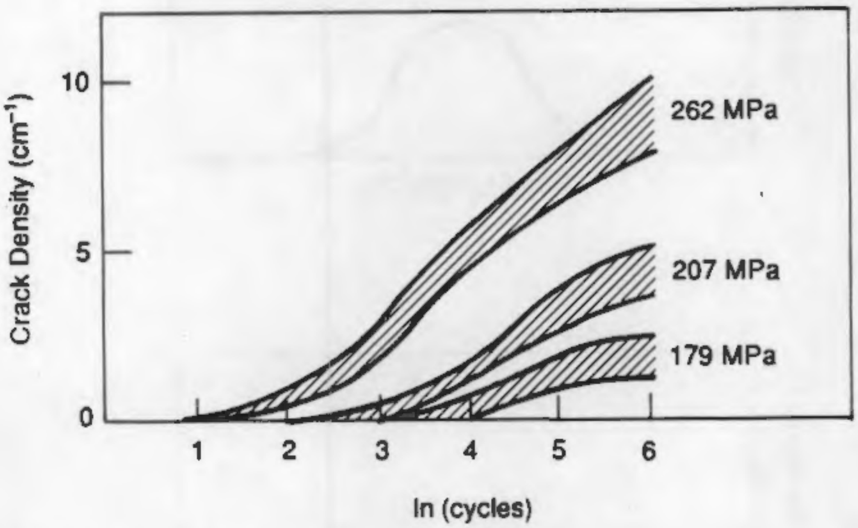

Figure 3. Experimental transverse crack density versus fatigue cycle N for $[0_2/90_3]_s$ composites.

Individual crack growth rates were computed with a form of the Paris Law, specialized for brittle materials:

$$\frac{da}{dN} = \alpha \left[\frac{G(\sigma_x, a)}{G_c}\right]^p \tag{1}$$

where N is cycle number, G_c is the critical energy release rate for the material, G is the available energy release rate for a crack of length a and depends on the local "shielded" stress σ_x, and α and p are empirical

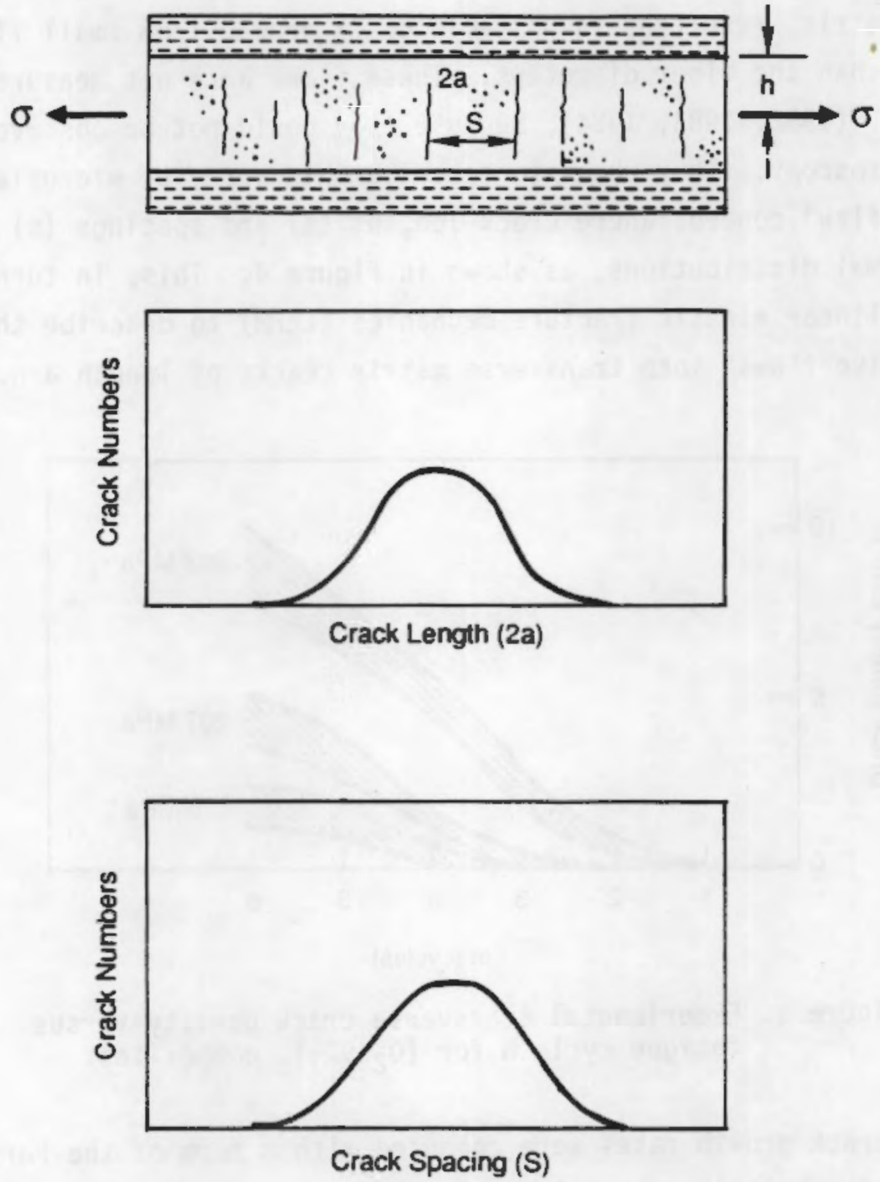

Figure 4. Hypothetical effective flaw distributions in the 90° layer of a $[0/90]_s$ composite.

constants. Shielding is caused by shear lag effects from adjacent cracks, and is briefly described in the next section.

Wang, et al. (1984), simulated the evolution of matrix crack density (Figure 3) using a Monte Carlo approach that applied Equation (1) to each propagating "effective flaw" in turn. However, the initial "effective flaw" distributions for a and s were specified at the outset of each simulation to ensure that the model reproduced the behavior in Figure 3. For example, the mean and standard deviation for the initial "flaw" size a_0 were 0.64h and 0.16h, respectively, for the $[0_2/90_2]_s$ composite. Flaws of such sizes would seem observable, and thus contradict their statement of non-observability.

The result was a rather complex empirical approach that is valid only for the data base used to fit the empiricisms. This approach seems inconsistent with the statement by Wang, et al. (1984), that "...for a given laminate under a given class of loading, the matrix cracking pattern is essentially mechanically reproducible, with perhaps a certain statistical variation only." If the problem is that simple, perhaps a simpler model may thus be justified. For field applications such as real-time simulation of damage accumulation in an aircraft structure, a more useful and simpler model would be a set of equally empirical curve fits. A more desirable alternative would be to incorporate some measure of predictive capability for other loading and cyclic conditions. This exploratory paper attempts to begin development of such a "simpler, more predictive" model, with fewer empiricisms.

propagating "effective flaw" in turn. However, the initial "effective flaw" distributions for a and c were specified at the onset of each simulation to ensure that the model reproduced the behavior in Figure 1. For example, the mean and standard deviation for the initial "flaw" size a_o were 0.064 and 0.16h, respectively, for the $[0_2/90_4]_s$ composite. Flaws of such sizes would seem observable, and thus contradict their statement of non-observability.

The result was a rather complex empirical approach that is valid only for the data base used to fit the empiricisms. This approach seems inconsistent with the statement by Wang, et al. (1985), that "...for a given laminate under a given class of loading, the matrix cracking pattern is essentially mechanically reproducible, with perhaps a certain statistical variation only." If the problem is that simple, perhaps a simpler model may thus be justified. For field applications such as real-time simulation of damage accumulation in an aircraft structure, a more useful and simpler model would be a set of equally empirical curve fits. A more desirable alternative would be to incorporate some measure of predictive capability for other loading and cyclic conditions. This exploratory paper attempts to begin development of such a "simpler, more predictive" model, with fewer empiricisms.

3.0 MODEL DEVELOPMENT

3.1 ELEMENTS OF THE MODEL

The five basic elements of the proposed new model are:

(1) a submodel for fractional Brownian motion, which establishes the new model "superstructure" or overall mathematical form,

(2) a similarity relationship between a single crack's length, its growth increment, and the fatigue cycle number, and which replaces the empirical Paris equation,

(3) an approximation for the "available strain energy release rate" (G_0) for an individual isolated crack,

(4) an equation to account for the reduction in G_0 caused by "shielding," or shear lag effects from neighboring cracks, and

(5) a relationship between the crack growth increment (Δa) and the far field strain (ϵ) for single cracks.

Each of these elements is described in more detail in the following paragraphs.

3.1.1 Fractional Brownian Motion (FBM)

This model can most simply be viewed as an extension of the probability arguments used to develop the basic diffusion equation. However, instead of the usual exponent 1/2, as in $(Dt)^{1/2}$ where D is the diffusion coefficient, we have the Hurst exponent H, where $0 \leq H \leq 1$. The significance of H is as follows. For H = 1/2, the successive steps of a random walk are uncorrelated or independent (truly random). For other values of H, successive steps have long range correlations. $1/2 < H \leq 1$ implies "persistence," where an increasing (decreasing) process will continue to increase (decrease). $0 \leq H < 1/2$ implies "antipersistence," where an increasing (decreasing) process will tend to decrease (increase) with time. Most natural phenomena are persistent, with $H \simeq 0.72$. This is described in more detail in Feder (1988), Chapters 8-10. It is interesting to note that this model was first developed in 1968.

Equation (9.22) of Feder (1988) expresses the form of this model that is most useful for the present analysis:

$$\frac{B(bt) - B(0)}{B(t) - B(0)} = b^H \tag{2}$$

where B is the vertical height and t is the horizontal position on a two dimensional plot of stochastic data, such as Figure 5. b is a scale factor for the horizontal axis. This figure shows a striking visual resemblance to the crack array in Figure 4 above. This resemblance can be pursued further to construct a fractional Brownian motion (FBM) model for the crack array by letting B be the crack length (a), and t be the axial position along the composite (X). The positions bt, t, and 0 can then be designated by the crack locations X_3, X_2, and X_1, where X_1 is an arbitrary reference. It can then be shown that $b = X_3 - X_2 = \Delta X$, so that the "geometric" FBM takes the form

$$\frac{a(X_3) - a(X_1)}{a(X_2) - a(X_1)} = \Delta X^H . \tag{3}$$

This equation expresses the relationship that the ratio of crack lengths at two axial positions (X_2 and X_3) is correlated with their spacing to the power H. Equation (3) thus provides motivation for the application of FBM to the problem of an array of propagating parallel cracks. This equation was investigated in the earlier stages of this work, but was abandoned because of numerous difficulties described below. Realization that FBM applied to many types of two dimensional plots of stochastic data led to more useful forms that will be described in later sections.

It should be noted that the aforementioned general applicability of FBM has a price: it is valid only "in distribution", or in the mean. That is, individual values for B chosen for the left side of Equation (2) usually do not equal b^H, where H is the mean value for the entire data set. Only the average of many such B choices will equal b^H. Consequently, B values must be chosen carefully to be useful. It will be seen that the method of choice is not well defined.

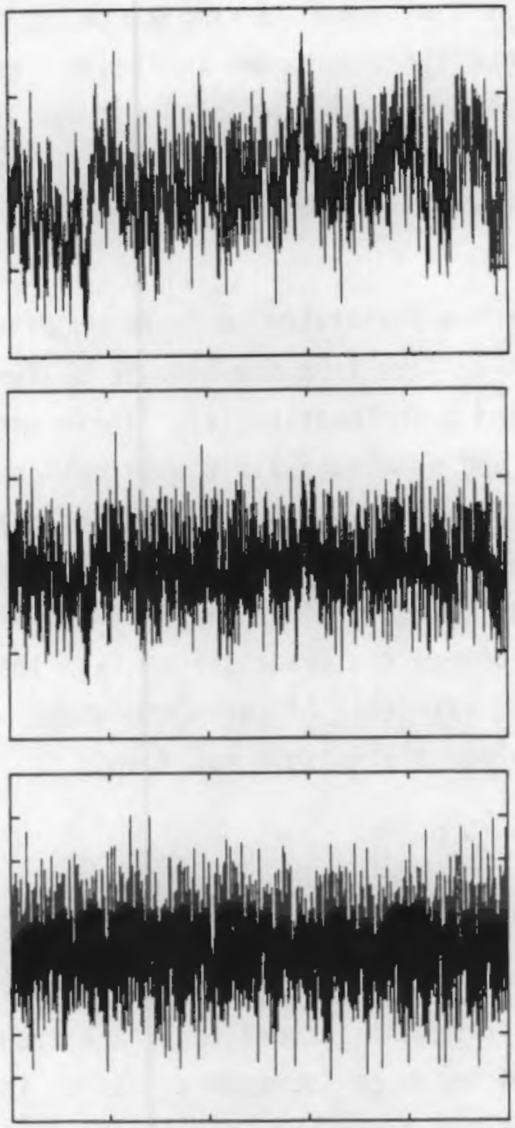

Figure 5. Plots of the fractional Brownian motion function in Equation (2) for H = 0.9 (top), H = 0.7 (middle), and H = 0.5 (bottom). Vertical axis is B (or a), horizontal axis is t (or X).

3.1.2 Proposed Replacement for the Paris Law

The Paris law in Equation (1) expresses the concept that fatigue cracks appear to continue growth even when the strain energy release rate (G) is less than its critical value (G_c) for the initiation of crack growth from static conditions. This concept may be visualized as the accumulation of microdamage at the crack tip, and may be consistent with the viscoelastic nature of the epoxy matrix material in Wang's studies. In this regard, a truly flat matrix fatigue limit as shown in Figure 1 may not actually exist because the polymer chains undergo some finite "rearrangement" on each fatigue loading cycle. Such "rearrangements" can be interpreted as microdamage, but verification of these arguments is a subject for future work.

Although the Paris law has historically been quite useful in the study of fatigue fracture, it suffers from the need to calibrate empirical parameters, such as α and p in Equation (1). These empiricisms require a significant data base, and sometimes limit applications to the same or similar materials and conditions. However, an alternative that may reduce the number of empiricisms may be available. This alternative can be developed from Equations (6)-(8) in the paper by Williford (1988). Based on the self-similarities between the crack length (a), its growth increment (Δa), the characteristic size (L_0) of the microdamage, and the total damage length (L_t), the following relationship was found:

$$\frac{L_t}{L_0} = \left[\frac{a}{\Delta a}\right]^D \quad . \tag{4}$$

D is the fractal dimension of the damage, and is described below. L_t is the sum of all microdamage increments across the crack front, $L_t = NL_0$, where N is the number of such microdamage increments. If it is assumed that one L_0 is incurred for each fatigue cycle, as in a linear damage accumulation rule, then Equation (4) describes the number of fatigue cycles to cause a crack of length a to advance by Δa.

If it is further assumed that the microdamage increments L_0 accumulate in a more or less random manner across the crack front (see Figure 6), the distribution of microdamage will form a fractal of dimension D, where $2 \lesssim D \lesssim 3$. A plan view of the crack tip damage will have dimension $1 \lesssim D \lesssim 2$ (see Williford 1989). The propagating damage of such a plan view may be quite similar to the two dimensional FBM plot for the crack array described in the last section. Assuming such similarity, the FBM for the crack tip damage will have the same H as the crack array. Further, the relationship between D and H is known for such two-dimensional representations: D = 2-H

(Feder 1988). This proposal is depicted in Figure 6, and results in the following equation.

$$\Delta N = \left[\frac{a}{\Delta a}\right]^{2-H} \tag{5}$$

where ΔN is the number of elapsed fatigue cycles to cause the crack to advance by Δa, and $0 \leq H \leq 1$.

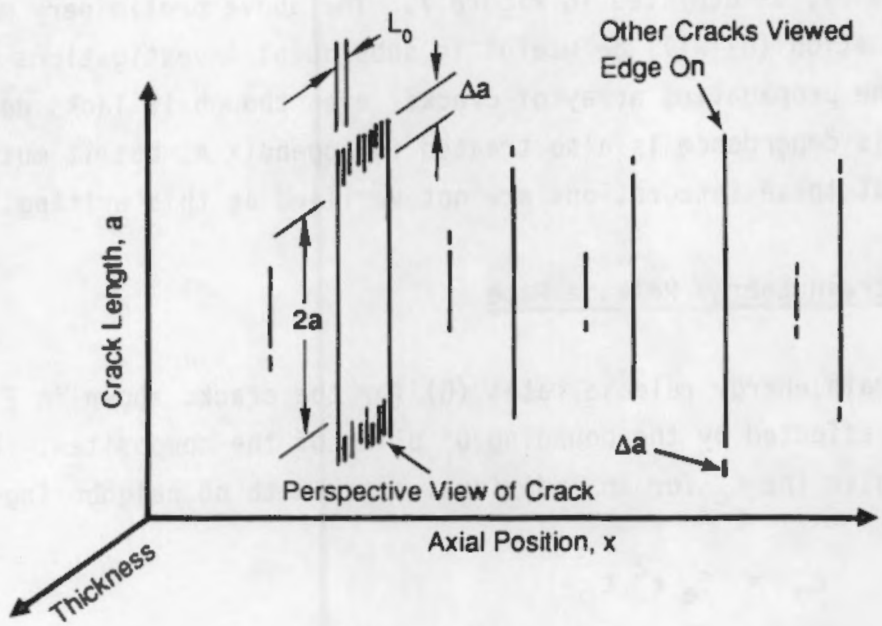

Figure 6. Similarities between crack tip microdamage and the crack array: $D_{crack\ tip} = D_{array} = 2-H$.

Equation (5) appears to yield reasonable results when integrated to investigate the behavior of a versus N, as follows. Rearranging and assuming we can let the Δ go to differentials, we have

$$\frac{da}{a} = (dN)^q \tag{6}$$

where $$q = -\frac{1}{2-H} \quad . \tag{7}$$

Integration of the left side of Equation (6) is straightforward, but the right side requires the fractional calculus (Oldham and Spanier 1974). This integration is performed in Appendix A. The result is:

$$\frac{a}{a_o} = \exp\left[1.1(N-N_o)^{0.22}\right] \quad , \tag{8}$$

for the expected value $H \approx 0.72$. The exponential behavior of crack length with fatigue cycle number seems consistent with observation (e.g., see Williford 1987), as depicted in Figure 7. The above preliminary mathematical form for Equation (8) will be useful in subsequent investigations of the FBM model for the propagating array of cracks, even though it lacks dependence on stress. This dependence is also treated in Appendix A, but it must be stressed that these integrations are not verified at this writing.

3.1.3 Strain Energy Release Rate

The strain energy release rates (G) for the cracks shown in Figures (2) and (4) are affected by the bounding 0° plies of the composites. Wang, et al. (1984) give the G_o for an individual crack with no neighboring cracks as

$$G_o = C_e \, \epsilon^2 \, t \tag{9}$$

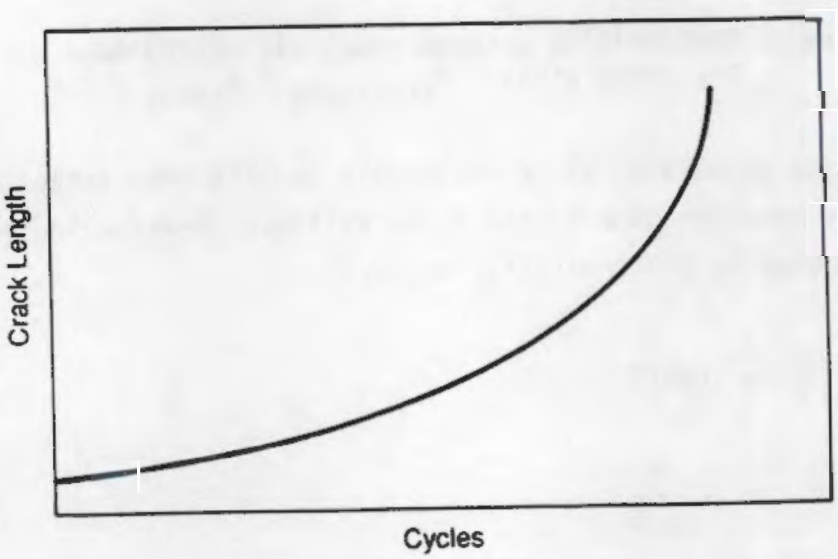

Figure 7. Exponential behavior of Equation (8), typical of fatigue data.

where ϵ is the mean or far field strain, t is the ply thickness (2h = mt where m is the number of 90° plies), and C_e is shown in Figure 8 (Wang, et al. 1982). The reduction in G_0 as the crack length approaches the 90° layer thickness is caused by transfer of the axial load to the bounding 0° plies. C_e, and thus G_0, are approximately linear up to a/h values of about 2/3. Wang, et al. (1982) found the a/h values corresponding to

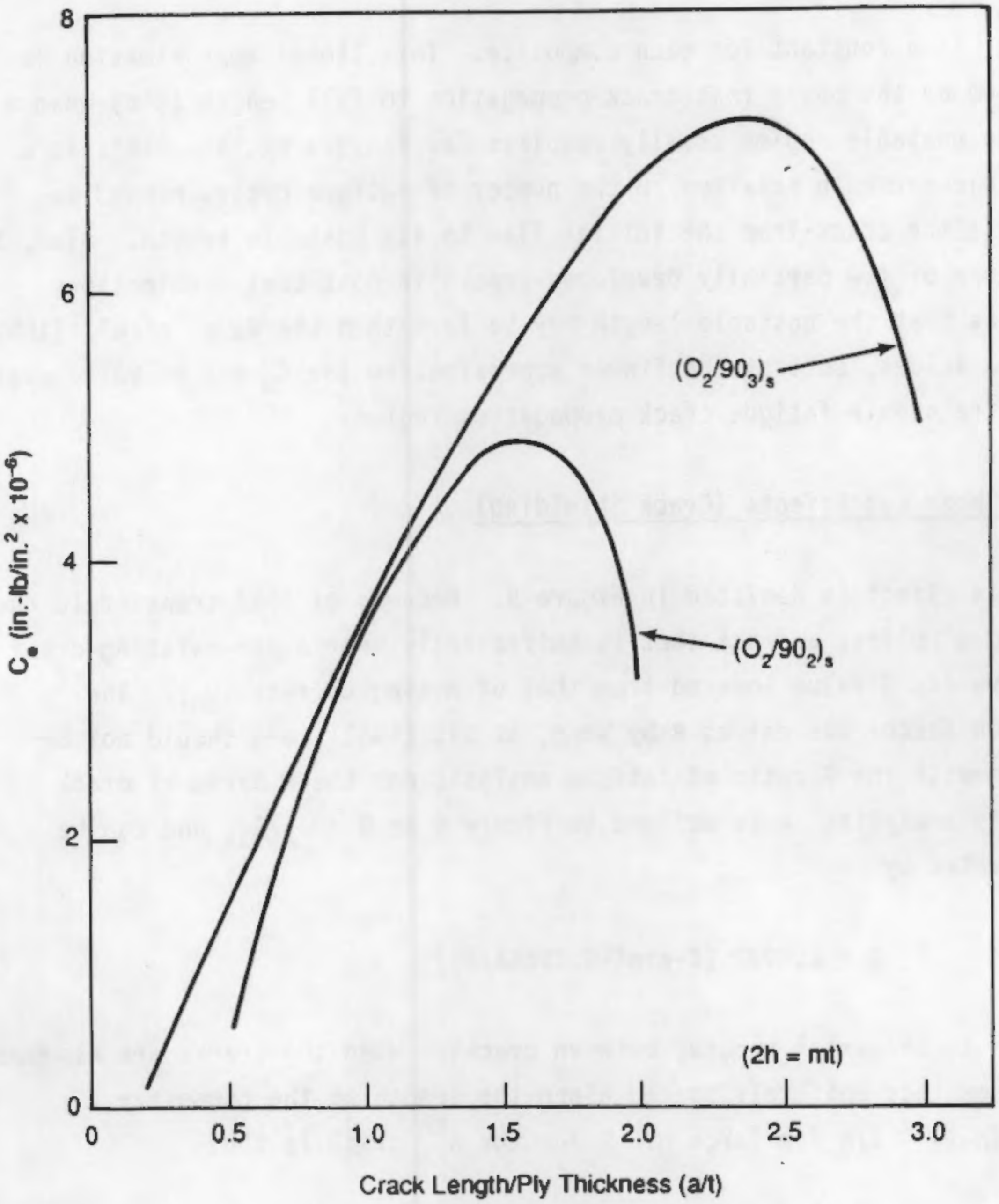

Figure 8. Mechanical load shape functions (C_e) plotted versus relative crack length (a/t), where t is ply thickness (2h = mt).

unstable crack growth to be 2/3 to 1 for the fatigue load ranges of interest: 36-53 ksi for $[0_2/90_2]_s$ and 26-38 ksi for $[0_2/90_3]_s$. Although these crack lengths are slightly larger than the linear range for C_e, a linear relationship will be assumed for G_o:

$$G_o = C_1 \epsilon^2 a \tag{10}$$

where C_1 is a constant for each composite. This linear approximation is justified on the basis that crack propagation to full length (a=h) when a is near the unstable regime usually requires few fatigue cycles. This is a negligible error in relation to the number of fatigue cycles needed to propagate the crack from the initial flaw to its unstable length. Also, the observance of few partially developed cracks in post-test examinations indicates that the unstable length may be less than the Wang, et al. (1982) computed values, so that the linear approximation for G_o may be valid over the entire stable fatigue crack propagation regime.

3.1.4 Shear Lag Effects (Crack Shielding)

This effect is depicted in Figure 9. Because of load transfer to the bounding 0° plies, a crack that is sufficiently near a pre-existing crack will have its G value lowered from that of a single crack (G_o). The reduction factor was called R by Wang, et al. (1982), and should not be confused with the R ratio of fatigue analysis nor the R curve of crack stability analysis. R is defined in Figure 9 as $R = G_2/G_1$, and can be approximated by

$$R = 1.0092\ [1-\exp(-0.396\Delta X/h)] \tag{11}$$

where ΔX is the axial spacing between cracks. When the cracks are assumed to be more or less uniformly spaced along the length of the composite $\Delta X = 1/(n-1) \approx 1/n$ for large n. R for the n^{th} crack is thus

$$R = 1.0092\ [1-\exp(-0.396/nh)] \tag{12}$$

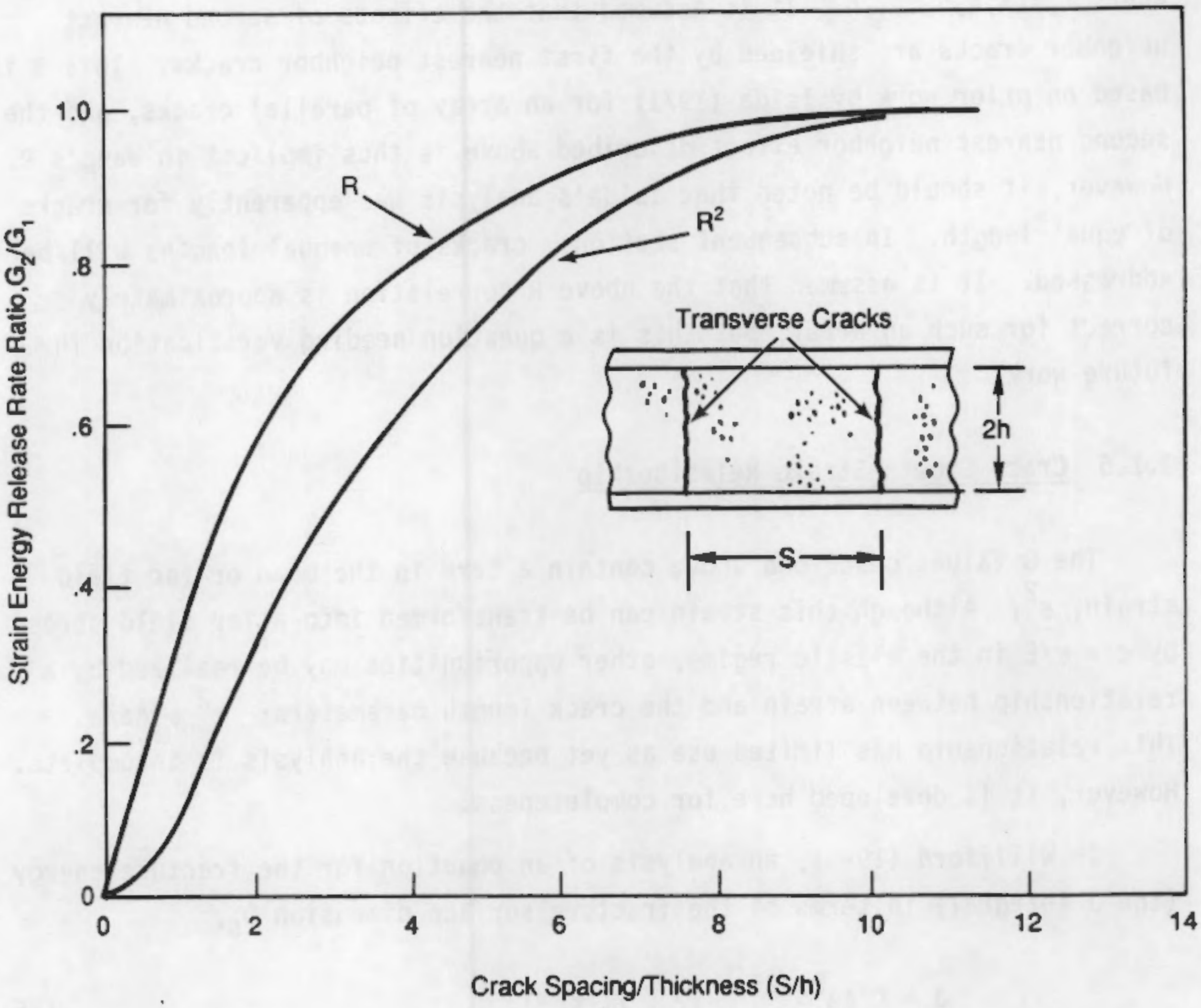

Figure 9. Ratio of strain energy release rates for a crack occurring a distance ΔX from a prior crack.

When there are only two cracks, the second crack has a G value described by

$$G_2 = RG_1 = RG_o \quad . \tag{13}$$

The third and subsequent cracks will have neighbors on each side, and a G value described by

$$G_i = R^2G_o \tag{14}$$

where $i = 3,4,5,\ldots,n$. It is assumed that the effects of second nearest neighbor cracks are shielded by the first nearest neighbor cracks. This R is based on prior work by Isida (1971) for an array of parallel cracks, and the second nearest neighbor effect described above is thus implicit in Wang's R. However, it should be noted that Isida's analysis was apparently for cracks of equal length. In subsequent sections, cracks of unequal lengths will be addressed. It is assumed that the above R correlation is approximately correct for such an array, but this is a question needing verification in future work.

3.1.5 Crack Growth-Strain Relationship

The G values described above contain a term in the mean or far field strain, ϵ^2. Although this strain can be transformed into a far field stress by $\epsilon = \sigma/E$ in the elastic regime, other opportunities may be realized by a relationship between strain and the crack length parameters: $\epsilon^2 \propto \Delta a/a$. This relationship has limited use as yet because the analysis is incomplete. However, it is developed here for completeness.

In Williford (1989), an analysis of an equation for the fracture energy (the J integral) in terms of the fracture surface dimension D_s,

$$J = C\,\Delta a^{D_s - 2} \qquad (2 \leq D_s \leq 3) \quad , \tag{15}$$

revealed that for the average case $D_s = 2.5$, the material constant C has units of toughness or stress intensity, K. Equation (15) thus reduces to

$$J = K\,\Delta a^{1/2} \quad . \tag{16}$$

From fracture mechanics, we have for plane stress

$$K = Y\sigma\sqrt{\pi a} \tag{17}$$

and

$$J = K^2/E \quad . \tag{18}$$

Combining Equations (16) and (18) gives

$$K\Delta a^{1/2} = K^2/E \tag{19}$$

and with $\sigma = E\epsilon$, then

$$\Delta a^{1/2} = K\epsilon/\sigma \tag{20}$$

Substituting Equation (17) for K gives

$$\Delta a^{1/2} = Y\epsilon\sqrt{\pi a} \quad , \quad \text{or}$$

$$\frac{\Delta a}{a} = \pi Y^2 \epsilon^2 \tag{21}$$

which is the desired relationship.

It is interesting to note that substitution of Equation (21) into Equation (5) yields a version of the Coffin-Manson relation: $\epsilon \Delta N^t = C_2$, where $t = 1/2(2-H) = 0.39$ and $C_2 = 1/Y\sqrt{\pi}$. The exponent t is smaller than Coffin-Manson's for plastic strain ($t = 0.5$), and larger than Morrow's for elastic strain ($t = 0.1$), so the above fractal approach appears to yield an "average" exponent. Also note that ΔN above is not the cycles to failure, ΔN_f, as in the Coffin-Manson formulation. See Meyers and Chawla (1984, p. 697) for further details.

3.1.6 Closure

The above Equations (2), (8), (10), (12), (14), and (21) are assembled in the next section to explore the possibilities of an improved model for fatigue crack accumulation in composite laminates. Results are as yet incomplete, so only preliminary calculations are available for analysis. The lack of rigor in this early stage of development should also be noted again to maintain the proper perspective for these results.

3.2 ASSEMBLY OF A CANDIDATE MODEL

Some motivation for considering use of the FBM correlation in Equation (2) as a model superstructure will probably be of benefit for the reader. The original proposal was to develop a crack array propagation model from Equation (4) alone. Approximately one-third of the project effort was expended trying to do this, without success. It was the introduction of FBM concepts that finally permitted definition of the exponent D in Equation (4), and subsequent development of Equations (5) and (8). However, Equation (5) alone still does not adequately describe the crack array, as seen from the following simplified analysis. Rearranging Equation (5) and dividing by ΔN gives

$$\frac{\Delta a}{\Delta N} = a(\Delta N)^{-p}$$

where $p = (3-H)/(2-H) = 1.78$ for the anticipated H of 0.72. The approximation $\Delta a = h\Delta n$ relates the crack growth increment to the increment in the number of cracks in the array (Δn). With a approximated as the average value h/2, the above equation becomes

$$\frac{\Delta n}{\Delta N} \simeq 0.5\ (\Delta N)^{-1.78} \quad .$$

From Figure 3, the value of $\Delta n/\Delta N$ can be estimated for given increments in the fatigue cycle numbers (ΔN). This value is approximately $\Delta n/\Delta N \simeq 10/\Delta N$, or $\Delta n/\Delta N \propto \Delta N^{-1}$. The above equation gives $\Delta n/\Delta N \propto \Delta N^{-1.78}$, and results in a discrepancy of about four orders of magnitude for $\Delta N = 1000$. Even very small values for H result in a discrepancy of about three orders of magnitude. Although the proposed Paris law replacement may be adequate for propagation of a single crack, this does not appear so for an array of cracks. This was the motivation that led to consideration of fractional Brownian motion (FBM) as a model superstructure in which to insert the proposed Paris law replacement.

3.2.1 Selection of Model Superstructure

Recalling that FBM is applicable to a wide range of two-dimensional plots of stochastic data, it seems evident that a number of possible model superstructures may be chosen. Designating the left side of Equation (2) as $B[Y(Z)]$, and the right side as $(\Delta Z)^H$, some possible FBM model superstructures for the present crack array are

$$B[a(X)] = \Delta X^H \tag{22}$$

$$B[\Delta a(X)] = \Delta X^H \tag{23}$$

$$B[G(N)] = \Delta N^H \tag{24}$$

$$B[\Delta U(N)] = \Delta N^H \tag{25}$$

where $\Delta U = G\Delta a$ in Equation (25). Although more candidates are possible, only the above four were investigated in this brief exploratory work. A summary of the results of this investigation follows.

An FBM for Equation (22) can be obtained by substituting Equation (10) into Equation (14), solving for the $a_i(X_i)$, and substituting these into Equation (2). The result is

$$\frac{\frac{G_3}{G_1}\left[\frac{R_1\epsilon_1}{R_3\epsilon_3}\right]^2 - 1}{\frac{G_2}{G_1}\left[\frac{R_1\epsilon_1}{R_2\epsilon_2}\right]^2 - 1} = \Delta X_{23}{}^H \tag{26}$$

where ΔX_{23} is the spacing between the second and third successive cracks under consideration. Recalling that $\Delta X \cong 1/n$, where n is the number of cracks per unit length, and that R is an exponential function of $1/n$, solution of this implicit equation would appear difficult. Also, although stress dependence can be included via $\epsilon = \sigma/E$, there is no means to introduce the dependence on fatigue cycle N. Most importantly, the unknown G values for each crack are retained in the model. Elimination of these G values requires either that they be known or that they simply be equal so that their ratio is unity. The only condition available for a unity G ratio is if both cracks are at the critical state $G = G_c$. However, the Paris-type approaches discussed in Section 3.1.2 permit crack propagation by microdamage accumulation when $G < G_c$. The condition $G = G_c$ is thus considered overly restrictive, and possibly self-contradictory. Equation (23) can be developed into a similar model with G ratios by substituting Equation (21) into Equation (10) and proceeding as above. In either case the troublesome G ratios remain, even if the right sides of Equations (22) and (23) are defined as in Equations (24) and (25). The direct geometric approach for constructing the FBM superstructure [described for Equation (3)] was thus abandoned.

Equations (24) and (25) yield more useful results. FBM superstructures for these equations can be constructed as above, and give for Equation (24)

$$\frac{\left[\frac{R_3}{R_1}\frac{\epsilon_3}{\epsilon_1}\right]^2 \frac{a_3}{a_1} - 1}{\left[\frac{R_2}{R_1}\frac{\epsilon_2}{\epsilon_1}\right]^2 \frac{a_2}{a_1} - 1} = \Delta N_{23}{}^{H} \tag{27}$$

and for Equation (25)

$$\frac{\left[\frac{R_3}{R_1}\frac{\epsilon_3}{\epsilon_1}\right]^2 \frac{a_3 \Delta a_3}{a_1 \Delta a_1} - 1}{\left[\frac{R_2}{R_1}\frac{\epsilon_2}{\epsilon_1}\right]^2 \frac{a_2 \Delta a_2}{a_1 \Delta a_1} - 1} = \Delta N_{23}{}^{H} \tag{28}$$

The relative merits of these two equations are discussed next.

Both Equations (27) and (28) assume that there exist three successively developing cracks somewhere in the array that satisfy the FBM assumptions for the mean H value for the array. Recall that this is actually a probabilistic statement, with presently unknown probability and tolerance for H. This is a subject for investigation in future work.

Both Equations (27) and (28) designate the scale factor b of Equation (2) as ΔN_{23}, the number of elapsed fatigue cycles between the completion of crack 2 and the completion of crack 3. This designation seems reasonable at present, but better alternatives may be found in future work, such as the cycles between completions of Δa_3 and Δa_2.

Both Equations (27) and (28) are implicit in the number of cracks (n), but both contain n on only one side of the equation. This may facilitate solutions. Both presently contain the cycle increment ΔN_{23} in explicit form on the right side of the equation, which permits use of the equation as a tool for predicting when the next crack will occur. In this sense, Equations (27) and (28) are recursive. Note that $R_3 = f(n)$, $R_2 = f(n-1)$, and $R_1 = f(n-2)$. However, this explicitness in ΔN_{23} will be lost in subsequent analyses described below.

Both equations appear to relate an energy parameter (G or ΔU) to an independent variable (N), and both permit introduction of stress dependence via $\epsilon = \sigma/E$. The incremental formulation of both sides of Equation (28) seems intuitively appealing, and may have acceptable fractal foundations according to Voss' method of "successive random additions" (Feder, 1988, p. 180). With a_i given by Equation (8), the unknown products $a_i \Delta a_i$ in Equation (28) could be defined by $a_i(da_i/dN_i)\Delta N_i$, but this introduces an additional complexity that is presently not justified by the preliminary nature of Equation (8). For this reason, Equation (27) was selected for this exploratory investigation because of its relative simplicity. However, future work may address Equation (28) as an alternative formulation of the model.

3.2.2 Exploratory Calculations

Having selected Equation (27) for further investigation, it seemed prudent to briefly explore its behavior numerically before proceeding. Substituting in the R equations and converting strains to stresses, Equation (27) takes the form

$$\frac{\left[\dfrac{1-\exp(-C_1/n)}{1-\exp(-C_1/(n-2))}\right]^2 \left[\dfrac{\sigma_3}{\sigma_1}\right]^2 \dfrac{a_3}{a_1} - 1}{\left[\dfrac{1-\exp(-C_1/(n-1))}{1-\exp(-C_1/(n-2))}\right]^2 \left[\dfrac{\sigma_2}{\sigma_1}\right]^2 \dfrac{a_2}{a_1} - 1} = \Delta N_{23}{}^{H} \tag{29}$$

where $C_1 = 79.2$ for the $[0_2/90_2]_s$ composite, and $C_1 = 52.8$ for the $[0_2/90_3]_s$ composite. The subscripts i = 3,2,1 on the a_i and σ_i correspond to the present, the last, and the next to the last cracks respectively: $n_2 = n-1$, $n_1 = n-2$. It has also been assumed that the Young's Moduli (E_i) are approximately equal because the cracks are successive, and thus cancel out of Equation (29). This should be verified in future work.

The first numerical exploration concerned estimation of the number of cycles, $N = \Sigma\Delta N_{23}$, for the beginning of matrix cracking. Because of singularities in the n_2 and n_1 terms, this was estimated for the third (or later) crack that formed. Calculations were performed on a programmable hand calculator. The limited accuracy of this calculator frequently resulted in the "onset of crack array development" being defined for the fourth or fifth crack.

Because the $\sigma_i^2 a_i$ terms were unknown, it was assumed that all far field stresses were constant, as in constant loading fatigue, and that the a_i/a_1 ratios were unity to represent incipient completion of the "last" crack. A constant value of $H = 0.72$ was assumed because this has been observed for many natural phenomena (Feder, 1988). The smallest number of cracks that could be solved for the $[0_2/90_3]_s$ composite was 4.4. For consistency with the assumption $a_i/a_1 = 1$, this limited definition of the "onset" condition as n=5. For this composite, the number of cycles to reach five cracks was computed to be 5208, compared to about 100 cycles estimated from the plotted fatigue data of Wang, et al. (1984) for 38 ksi loading. For the $[0_2/90_2]_s$ composite, the earliest solution was at 3.3 cracks, and five cracks were computed at 35131 cycles, compared to about 2100 cycles at 38 ksi from the plotted data of Wang, et al. (1984). These are the only two fatigue data sets at the same stress, and there is no stress calibration between the above computations and data. It is therefore impossible to determine with certainty if the proposed model is biased toward overprediction. All that can be said is that the dependence of cracking on the 90° layer thickness (2h) seems to be correctly represented by the model: a thicker 90° layer cracks earlier because it can store more strain energy than the thinner design.

The apparent overprediction of cycles to reach a given number of cracks may be caused by the rather crude assumption that the ratios $(\sigma_i/\sigma_1)^2$ and (a_i/a_1) were unity. Defining these ratios as C_{i1} for $i = 2,3$, additional exploratory calculations were performed for the condition $C_{31} = C_{21}^2$, for reasons which will become apparent later in this writing. Results were that the above data for cycles to five cracks could be reproduced with $C_{21} = 12$ for the $[0_2/90_3]_s$ composite, and $C_{21} = 110$ for the $[0_2/90_2]_s$ composite. This corresponds to stress ratios between two successive cracks of 12 to 110, and does not appear reasonable. This effect must therefore originate primarily from the a_i/a_1 ratios. A physical interpretation is that later cracks must reach appreciably longer lengths before unstable growth occurs. The longer lengths required are caused by the shielding effect as crack numbers (n) increase. It should be noted that if this is true, the linear approximation for G_0 in Equation (10) is probably inadequate unless the initial microflaws are less than 1/110 of the 90° layer thickness. Although this may be reasonable, it has not been checked in the present work because of limited project resources and time. Also note that the above results for the "onset condition" could be checked with other FBM models such as Equation (26), because widely separated cracks at small n may permit cancellation of the nearly equal G_i values. These are questions to be addressed in future work.

A final series of calculations were performed to investigate the behavior of Equation (29) at larger n. For $C_{21} = C_{31} = 1$, the number of cycles between cracks approaches unity at large n. This is too rapid a crack accumulation rate to match the data in Wang, et al. (1984). However, the above arguments using other values for C_{21}, with $C_{31} = C_{21}^2$, indicated that crack accumulation rates may slow substantially as n increases. Although these calculations are very preliminary, the approach to an asymptotic number of cracks in the array may thus be possible to model if a suitable equation for the a_i/a_1 ratios can be found in terms of the elapsed number of cycles, ΔN_{23}. The search for this equation continues in the next subsection.

3.2.3 Inclusion of Proposed Paris Law Replacement in the Model

It should first be recalled that the form of Equation (8) is uncertain and has not been verified at this writing, so that subsequent discussion is tenuous at best.

The purpose of including Equation (8) in the model is to eliminate the unknown crack length ratios a_i/a_1, $i = 2,3$. This equation is repeated here for convenience:

$$\frac{a}{a_0} \propto \exp\,[(\Delta N)^r] \tag{30}$$

where $\Delta N = N - N_0$ is the number of fatigue cycles to propagate the crack from an initial length a_0 to a final length a, and $r = 0.22$ for the assumed value of $H = 0.72$. The exponential form reproduces typical fatigue behavior, while the small exponent r provides a slightly asymptotic or damped characteristic at large ΔN. The latter condition effectively prevents the crack growth rate from becoming prematurely infinite and producing another mathematical singularity. Another possible value for the exponent $r<1$ is discussed in Appendix A for a stress-dependent version of Equation (30).

There is an Equation (30) for each crack that appears in the FBM, so insertion of this equation into Equation (29) requires the subscript $i = 1,2,3$ for each a and ΔN. Assuming that all three cracks began from the same initial flaw size a_0, substitution of Equation (30) into Equation (29) results in

$$\frac{\left[\frac{1-\exp(-C_1/n)}{1-\exp(-C_1/(n-2))}\right]^2 \left[\frac{\sigma_3}{\sigma_1}\right]^2 \exp\,[\Delta N_3{}^r - \Delta N_1{}^r] - 1}{\left[\frac{1-\exp(-C_1/(n-1))}{1-\exp(-C_1/(n-2))}\right]^2 \left[\frac{\sigma_2}{\sigma_1}\right]^2 \exp\,[\Delta N_2{}^r - \Delta N_1{}^r] - 1} = \Delta N_3{}^H \tag{31}$$

where a constant is omitted from the exponential terms containing the ΔN_i, for clarity. Also note that ΔN_3 on the right side of Equation (31) replaces ΔN_{23}, because they are equal if the third crack begins propagation immediately after the second crack reaches $a_2 = h$.

Although Equation (31) is no longer explicit in ΔN_3, it appears to retain a predictive capability by virtue of its recursive nature. However, it is clear that further work is needed to determine ΔN_1 and ΔN_2, and thus to better define the "onset condition," so that this predictive capability can be verified and put into practice in the field. One possible approach is to develop relationships between the ΔN_i, such as the simple linear extrapolation $\Delta N_3{}^r - \Delta N_1{}^r = 2\,(\Delta N_2{}^r - \Delta N_1{}^r)$, which leads to the assumption $C_{31} = C_{21}{}^2$ employed in the last subsection for numerical explorations.

Future work will probably also require the approximations $n_2 = n_3-1+\delta$ and $n_1 = n_3-2+\delta$, where δ is a small but finite number that prevents numerical singularities in the R ratios.

In any event, the use of the proposed replacement for the Paris law in the $B(G) = \Delta N^H$ version of the FBM model for the propagating crack array introduces $\exp(\Delta N^r)$ terms that appear quite capable of attaining values of 12 to 110, which were found necessary to match the data in the last subsection.

3.2.4 Closure

Equation (31) would thus appear to have a good chance of reproducing all major features of the data of Wang, et al. (1982, 1984), including the dependence on the 90° layer thickness, the delay in fatigue cycles until the first few cracks form, crack accumulation toward a saturation value, and dependence on the stress level. However, testing and verification of Equation (31) will require access to more extensive computing facilities, and is precluded by the limited resources of this project.

4.0 DISCUSSION AND CONCLUSIONS

The foregoing analysis can make no pretense of rigor. The words "assume" and "approximate" were those most frequently used to develop Equation (31). This was necessary in order to propose a workable candidate for a new model that departs substantially from accepted practice. Although such an approach is sometimes necessary and often useful in an exploratory analysis, it leaves many questions open for further scrutiny. Some of these questions of rigor are outlined below.

(a) The FBM model is only valid "in the mean", so that the choice of three successive cracks that satisfy the model for the mean Hurst exponent (H) requires at least a probabilistic qualification, and eventually a proof that this is in fact possible.

(b) The value of $H = 0.72$ was assumed because this value is exhibited by many natural phenomena, but it is unknown if fatigue fracture of an engineered material such as a composite can be considered "natural." The value of H needs experimental verification, with the hope that it will at least remain a constant rather than vary with fatigue cycles.

(c) The assumption that $\Delta N = L_t/L_o$ in the proposed Paris law replacement seems tenuous if each cycle produces only one increment of microdamage L_o, as in a linear damage rule. Probabilistically, occasions of two, three, or even zero increments in a given cycle seem as likely, and may eventually require a more generalized interpretation of the ratio L_t/L_o. This is probably related to the behavior of the fatigue limit for the polymer matrix material, which also needs clarification.

(d) The assumption of self-similarity between individual crack tip damage and the array of crack growth increments in the proposed Paris replacement should be experimentally verified.

(e) Due to time and resource constraints, integration of the proposed Paris replacement in Equation (8) was not verified. This is a crucial relationship that must be proven rigorously before any further progress can be made. Also see Appendix A for the inclusion of stress dependence in Equation (8).

(f) The linearized approximation for the strain energy release rate equation needs further investigation because of the large a_i/a_I ratios found in Section 3.2.2.

(g) The crack shielding effects described by the R value in Equation (12) should be verified for neighboring cracks of unequal lengths.

(h) Other FBM approaches, such as Equation (25), should be attempted to determine if the model superstructure is as general as claimed. It should be possible to achieve the same predictive capability and results with such alternative approaches. In particular, prediction of the "onset condition" should be verified as described in Section 3.2.2.

(i) The proposed model in Equation (31) assumes that the Young's Moduli are nearly equal for successive cracks, and thus cancel out. This should be verified. The proposed model also contains no explicit dependence on other fatigue loading characteristics, such as the stress cycle amplitude, and this should be included in future work.

(j) Other hidden or less obvious assumptions that will surface in future investigations should be addressed. An example is that a clearer definition of ΔN_3 may be needed. Most mysterious is the lack of critical material properties, such as σ_c or G_c, in the model. This is unusual for a fracture problem.

Future work should address the above questions in order of priority, as follows. Items (e), (j), (b), and (a) are highest priority, and their resolution will permit numerical testing of Equation (31). Items (d), (h), and (c) are second priority, while items (f), (g), and (i) are lowest priority.

In spite of the above difficulties, Equation (31) appears to show promise for providing the desired simplified model in a recursive form. Preliminary calculations indicated that it has the capability of reproducing all major features of the transverse matrix fatigue cracking data base that was analyzed. It is therefore recommended that work continue in the future so that this model can be better understood. A long-term goal is to develop similar predictive capabilities for other composite damage forms, such as delamination.

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APPENDIX A

FURTHER NOTES ON THE PROPOSED PARIS LAW REPLACEMENT

Integration of Equation (6) via the fractional calculus is as follows. Equation (6) can be expressed in the form

$$\frac{da}{a} = \frac{1}{(dN)^q} \tag{A.1}$$

where $q = 1/(2-H)$. (A.2)

If it is assumed that the numerator 1 in Equation (A.1) is actually the qth order fractional differential of the Heaviside unit step function (HS), then in the notation of the fractional calculus, we have the following differential form:

$$d\ln a = \frac{d^q(HS)}{(dN)^q} \quad , \tag{A.3}$$

where $q = 0.78 > 0$ for $H = 0.72$. Integrating both sides of this equation to order unity gives

$$\ln\left(\frac{a}{a_o}\right) = \frac{d^s(HS)}{(dN)^s} \tag{A.4}$$

where $s = q-1 = -0.22$, and negative orders signify integration in the notation of the fractional calculus. Because $s < 0$, an integral thus remains on the right side of Equation (A.4). Performing this integration gives (Oldham and Spanier, 1974, p. 105):

$$\frac{a}{a_o} = \exp\left[\frac{(N-N_o)^{-s}}{\Gamma(1-s)}\right] \tag{A.5}$$

where $\Gamma(1-s) = \Gamma(1.22) = 0.91$. The final form is thus

$$\frac{a}{a_o} = \exp\left[1.1\ (N-N_o)^{0.22}\right] \tag{A.6}$$

Another problem with Equation (8) is that there is no dependence on stress. This can be included in an approximate fashion by the use of Equation (21), which is actually valid only for H = 0.5, as follows. From Equation (5), we have

$$\frac{\Delta a}{a} = \left[\frac{\Delta a}{a}\right]^{3/2} \left[\frac{a}{\Delta a}\right]^{1/2} = \Delta N^{-q} \qquad (A.7)$$

From Equation (21) and $\epsilon = \sigma/E$, Equation (A.7) becomes

$$\frac{\Delta a}{a} = \left[\frac{\sigma Y\sqrt{\pi}}{E}\,\Delta N^{-q}\right]^{2/3} \qquad (A.8)$$

Again letting the Δ go to differentials, we have

$$\frac{da}{a} = \left[\frac{\sigma Y\sqrt{\pi}}{E}\right]^{2/3} \frac{1}{(dN)^{r}} \qquad (A.9)$$

where $r = 2q/3 = 0.52$. Invoking the above Heaviside argument and integrating as before, with the term in brackets taken as constant, gives

$$\frac{a}{a_o} = \exp\left[\left(\frac{\sigma Y\sqrt{\pi}}{E}\right)^{2/3} \frac{(N-N_o)^{-s}}{\Gamma(1-s)}\right] \qquad (A.10)$$

where $s = r-1 = -0.48$ for $H = 0.72$. The generally small values of σ/E are compensated by a larger exponent for N.

It should be noted again that these methods of "integration" are not verified at this writing. However, it does seem possible to incorporate the stress dependence in the proposed Paris law replacement.

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