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**LINKING THE GAUSS-BONNET-CHERN THEOREM,
ESSENTIAL HOPF MAPS AND MEMBRANE SOLITONS
WITH EXOTIC SPIN AND STATISTICS ***

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ABSTRACT

By way of the Gauss-Bonnet-Chern theorem, we present a higher dimensional extension of Polyakov's regularization of Wilson loops of point solitons. Spacetime paths of extended objects become hyper-ribbons with self-linking, twisting and writhing numbers. Specifically we discuss the exotic spin and statistical phase entanglements of geometric n -membrane solitons of D -dimensional KP_1 σ -models with an added Hopf-Chern-Simons term where $(n, D, K) = (0, 3, \mathbb{C}), (2, 7, \mathbb{H}), (6, 15, \mathbb{O})$. They are uniquely linked to the complex ($\mathbb{C}$) and quaternion ($\mathbb{H}$) and octonion ($\mathbb{O}$) division algebras.

Two overlapping research areas have recently attracted much attention. They are the topological quantum field theories (TQFT) in $D \geq 3$ spacetime dimensions [1, 2], inspired by the works of Donaldson and Jones [3], and theories of particles bearing *any* spin and statistics or *anyons* [4], fuelled by the quest for a mechanism underlying the fractional quantum Hall effect or high temperature superconductivity.

By relating quantum $D=2$ conformal and $D=3$ Chern-Simons field theories, Witten [5] showed a correspondence between the expectation values of the Wilson loops traced by 'colored' *point* sources in spacetime and Jones's polynomials for knots. Notably, to so obtain the fundamental Skein relations, he had to regularize or *frame* the Wilson loops, in addition to doing the standard regularization. It is here that an interesting overlap occurs with the theory of anyons. In fact, such a regularization had been discovered by Polyakov

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[6] in his proof of the $D=3$ fermi-bose transmutation of baby Skyrmions in their *point-like limit*. Our talk takes off at this intersection, a meeting point of topology, geometry and physics. It is based on work done with Soonkeon Nam.[7]. We defer to this longer article for greater details and a comprehensive list of references as time and space limitations only permit a quick review of some of the highlights here.

We start from the well-studied CP_1 σ - model with a Chern-Simons term [6]

$$A = \int d^3x \left[|D_\mu Z|^2 + \frac{\theta}{8\pi^2} \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} + A_\mu J^\mu \right]. \quad (1)$$

$0 \leq \theta \leq \pi$. The two component complex spinor field $Z^T = (Z_1, Z_2)$ with $|Z|^2 = 1$ lives on S^3 . The unit normed field n is given by the Hopf projection map taking $Z \in S^3$ to $n = Z^\dagger \sigma Z \in S^2$. D_μ is the covariant derivative with its holonomic $U(1)$ gauge field $A_\mu = i Z^\dagger \partial_\mu Z$, $F_{\mu\nu}$ being the field strenght. This model admits exact S^2 - solitons. While the third term is the Aharonov-Bohm term, the second term also reads as $S_H = \frac{1}{2} \int d^3x A_\mu J^\mu$ (for $\theta = \pi$) i.e. as an interaction between the field A_μ and the conserved topological current $J_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\lambda} \epsilon_{abc} n^a \partial^\nu n^b \partial^\lambda n^c$. The latter is so normalized that $Q = \int d^2x J_0$, the soliton (electric) charge is integral and labels the elements of $\pi_2(S^2) = \mathbb{Z}$. The field boundary condition is such that spacetime is $\mathbb{R}^3 \cup (\infty) = S^3$; the Chern-Simons action is the Hopf invariant for the maps $n : S^3 \rightarrow CP_1 = S^2$, classified by the generators of $\pi_3(S^2) = \mathbb{Z}$, the additive group of the integers.

Wilczek and Zee [4] showed that interchanging two $Q=1$ -solitons or equivalently rotating one of them around the other by 2π gives raise to a statistical, alias spin phase factor $e^{i\theta}$ to the wave function. Hence the soliton is an anyon with exotic spin $s = \frac{\theta}{2\pi}$ and intermediate statistics. This phase originates from the gauge interactions among anyons and corresponds to a mapping with Hopf invariant 1. The latter feature is a key ingredient in our higher dimensional generalizations of the θ -spin and statistics connection.

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Polyakov opted for a tractable Wilson loop approach to the large distance behavior of soliton Green functions of system (1). To study the effects induced by the long range Chern-Simons interactions, he approximated the partition function Z by

$$Z = \sum_{(P)}^{\text{all closed paths}} e^{-m L(P)} \langle \exp\left(i \int_P dx^\mu A_\mu\right) \rangle. \quad (2)$$

P denotes a Feynman path of a soliton seen as a curve in spacetime R^3 , $L(P)$ is the total path length.

The first exponential factor in (2) is the Schwinger action of a free relativistic *point-like* soliton of mass m . The other factor $\Phi(P) = \left\langle \exp\left(i \oint_P A^\mu dx_\mu\right) \right\rangle$ where the functional averaging $\langle \dots \rangle$ is w.r.t. the Hopf action embodies the Aharonov-Bohm effect, typical of topologically massive gauge theories. The Chern-Simons-Hopf action induces magnetic flux on electric charges and vice versa. Being Gaussian (and for $\theta = \pi$), this phase is exactly calculable, hence the analytic appeal of the Polyakov approximation. By direct integration of the equation of motion [8], $\Phi(P)$ is given by exponentiating the effective action :

$$\Phi(P) = \frac{1}{N} \exp \left[i S_0 + i \int d^3x \left(\frac{\theta}{4\pi^2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu \right) \right]. \quad (3)$$

S_0 is the free point particle action and N is a suitable normalization. The conserved current of a $Q=1$ point source is now $J_\mu(x) = \int d\tau \delta^3(x-y(\tau)) \frac{dy_\mu(\tau)}{d\tau}$. From (1) the key equation for J_μ reads

$$J_\mu(x) = - \frac{\theta}{2\pi^2} \epsilon_{\mu\nu\rho} \partial^\nu A^\rho \quad (4)$$

Upon substitution in (3) with $\theta = \pi$ it yields $\Phi(P) = \exp \left(\frac{i}{2} \int d^3x A_\mu J^\mu \right)$. So for the above given point current and, in the Lorentz gauge $\partial^\alpha A_\alpha = 0$, A_μ can be solved to give Polyakov's result :

$$\Phi(P) = \exp \{ i\pi I_G(P) \} \quad (5)$$

where

$$I_G(C_\alpha \rightarrow C_\beta) = \frac{1}{4\pi} \oint_{C_\alpha} dx^\mu \oint_{C_\beta} dy^\nu \frac{\epsilon_{\mu\nu\lambda} (x^\lambda - y^\lambda)}{|x - y|^3} \quad (6)$$

in the limit where the two smooth closed 3-space curves C_α and C_β coincide, namely $C_\alpha = C_\beta = P$, the soliton worldline.

Were C_1 and C_2 in R^3 (or S^3) *disjoint* curves , (6) would just be their Gauss linking coefficient. If $\Omega (M_2)$ is the solid angle subtended by C_1 at the point M_2 of C_2 , then Stoke's theorem yields $I_G = \frac{1}{4\pi} \int_{C_2} d\Omega (M_2)$, which measures the variation of this solid angle divided 4π as M_2 runs along C_2 ; it is the algebraic number of loops of one curve around the other.

There is however an indeterminacy in the analytic result (6) . While its integrand is that of Gauss' invariant, the integration is over one and the same curve. $I_G(P)$, which represents the net action of the Aharonov-Bohm and the Chern-Simons terms in (1) on a charged particle, is thus *undetermined*. Clearly this artifact of the geometric point-limit approximation must be corrected for by a proper definition or regularization of $I_G(P)$.

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Polyakov's curing prescription is to trade the delta function in the alternative expression of $\int_P dx^\mu \iint d^2y_\mu \delta(x-y)$ of $4\pi I_G$ for the Gaussian $\delta_\varepsilon(x-y) = (2\pi\varepsilon)^{-\frac{1}{2}} \exp\left[-\frac{(x-y)^2}{\varepsilon}\right]$.

He found $I_G(P)_{\text{Reg}}$ to be $-T(P)$, the total torsion or twist of the curve P in spacetime with $T(P) = \frac{1}{2\pi} \int_P d\mathbf{x} \cdot \left(\mathbf{n} \times \frac{\partial \mathbf{n}}{\partial s} \right) \equiv \frac{1}{2\pi} \int_P \tau(s) ds$. s and $\mathbf{n}$ denote the arc length and the principal normal vector to P at the point $\mathbf{x}(s)$. What then are the geometric underpinnings of this regularization procedure?

By substituting the Gaussian the dominant contribution of the surface integral comes from an infinitesimal strip Σ_P ; Polyakov [6] effectively turns a spacetime curve into a ribbon. Indeed, in 1959, Calugareanu [9] discovered that $I_G(C_\alpha \rightarrow C_\beta)$ is in fact a perfectly well defined entity, a new topological invariant SL , the self-linking number for a simple closed ribbon[9]. SL is in fact the linking number of C_β with a twin curve C_α moved an infinitesimally small distance ε along the principle normal vector field to C_β . Being disjoint these two curves can be linked and unlinked, like the strands of a circular supercoiled DNA molecule [10]. Nowadays in knot theory this construction is called the *framing* of a curve C_β . Most significant to our considerations is the existence of the "conservation law": $SL = T + W$ whereby SL , is the algebraic sum of two *differential geometric* characteristics of a closed ribbon, its total torsion or twisting number T and its *writhing number* or *writhe* W . While both T and W , which are metrical properties of the ribbon and its "axis" respectively, can take a continuum of values, their sum SL must be an integer. The writhing number W , better known in topology [11] as the Gauss integral for the map $\phi : S^1 \times S^1 \rightarrow S^2$, is the element solid angle or the pullback volume 2-form $d\Omega_2$ of S^2 under ϕ . So Calugareanu's formula reads

$$SL(P) = \frac{1}{4\pi} \int_{P \times P} d\Omega_2 + \frac{1}{2\pi} \int_P \tau ds. \quad (7)$$

Using the invariance of W under dilations and let ϕ be the map, $e(s,u)$ ($e^2 = 1$), a local Frenet-Serret frame vector attached to the curve, we get

$$W = \frac{1}{4\pi} \int_0^L ds \int_0^1 du \epsilon_{abc} e^a \partial_s e^b \partial_u e^c, \quad a,b,c = (1,2,3) \quad \text{and} \quad \partial_s = \partial/\partial s, \quad \partial_u = \partial/\partial u.$$

A conformally invariant action for the frame field e , W is manifestly a Wess-Zumino-Novikov-Witten term and readily identified as a Berry phase upon exponentiation. It is precisely Polyakov's [6] double integral representation (modulo an integer) for the torsion $T(P)$. From our vantage point, this equivalence, whose importance will soon be clear, is just the equality $W = -T \pmod{Z}$. A mundane illustration of the relation $W + T = SL$ for a ribbon is a coiled phone cord. When unstressed with its axis curling like a helix in space, its writhe is large while its twist is small. When stretched with its axis almost straight, its twist is large while its writhe is small.

By now many people [12] have verified that the regularized Hopf (statistical) phase factor is given by the writhe, $\Phi(P) = \exp(i\pi W(P))$. Its alternate form $\Phi(P) = \exp(-i\pi T(P)) \exp(+i\pi n)$ by way of the relation $W = -T \pmod{Z}$ is the "spin" phase factor essential to Polyakov's proof that the 1-solitons in (1) are fermions by obeying a Dirac equation in their point-like limit. Had θ been kept arbitrary, we would have the more general theory of pointlike anyons carrying fractional spin and intermediate statistics [4]. In other words the relation $W = -T + SL$ is the mathematical expression of the connection between statistics and spin in the geometric point soliton limit.

In the geometry of 2-surfaces, a form of the celebrated Gauss-Bonnet theorem is $K = 2\pi\chi$. Its telling property, unique in the whole of differential geometry, is the following: Like the formula $SL = T + W$, it relates entities defined solely in terms of topology such as the Euler characteristics χ of a closed surface M and metrical entities defined purely in terms of distances and angles such as total Gaussian curvature K for M . It is perhaps not too surprising that Fuller [10] showed that the relation $SL = T + W$ is in fact a consequence of the Gauss-Bonnet formula. So we see an interesting role that the latter theorem plays in a fundamental physics principle - the relation between spin and statistics!

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Furthermore since White [11] obtained the higher dimensional version of Calugareanu's formula from his formulation of Gauss-Bonnet-Chern theorem for Riemannian manifolds [13], it is natural for us to extend Polyakov's analysis to the $D>3$ counterparts of the Wilczek-Zee model (1). We first define the generalized Gauss linking number for manifolds.

Generalizing the procedure for linking 3-space curves, consider [11] two continuous maps $f(M)$ and $g(N)$ from two smooth, oriented, non intersecting manifolds M and N , $\dim(M) = m$ and $\dim(N) = n$, into R^{m+n+1} . Let S^{m+n} be a unit $(m+n)$ -sphere centered at the origin of R^{m+n+1} and $d\Omega_m$ be the pull-back volume form of S^{m+n} under the map $e : M \times N \rightarrow S^{n+m}$ where to each pair of points $(m,n) \in M \times N$ we associate the unit vector e in R^{m+n+1} : $e(m,n) = \frac{g(n)-f(m)}{|g(n)-f(m)|}$. The degree of this map is then the generalized Gauss linking number of M and N , namely

$$L(f(M),g(N)) \equiv L(M,N) = \frac{1}{\Omega_{n+m}} \int_{M \times N} d\Omega_{n+m}, \quad (8)$$

$\Omega_n (=2\pi^{(n+1)/2}/\Gamma((n+1)/2))$ is the volume of S^n . We first note that the linking number for even dimensional submanifolds M and N is zero due to the non-commutativity property $L(M, N) = (-1)^{(m-1)(n-1)} L(N, M)$.

Next we specialize White's main theorem : Let $f : M^n \rightarrow R^{D=2n+1}$ be an smooth embedding of a closed oriented differentiable manifold into Euclidean $(2n+1)$ space. Let v be an unit vector along the mean curvature vector of M^n . If n is odd (i.e. $D=3, 7, 11, 15$ etc...) then

$$SL(f, f_\theta) = \frac{1}{\Omega_{2n}} \int_{M \times M} d\Omega_{2n} + \frac{1}{\Omega_n} \int_M \tau^* dV \quad (9)$$

is the self-linking number of a hyper-ribbon made up of M^n and the same manifold deformed a small distance ϵ along v , and where the terms on the RHS of (9) are respectively the writhing and twisting numbers, W and T , of the hyper-ribbon. For even ($D=1,5,9,\dots$), both W and T are zero and hence also $SL=0$, the situation is uninteresting for the physics of nonintegrable phases.

The universality of the formula $SL = W + T$ mirrors that of Gauss-Bonnet-Chern theorem. It is only natural to expect that for higher dimensional solitons in suitable models White's formula $T = -W \pmod{Z}$ similarly links their spin and statistical phases. As applied to physics, it could define and relate the twisting and writhing of odd dimensional closed S^3 -, S^5 -, S^7 -... hyper-ribbons, the world volumes of topological S^2 -, S^4 -, S^6 -membranes solitons in $D=7, 11, 15$... dimensional spacetime respectively. How to cut down this infinity of choices? What are the natural $D > 3$ σ -model counterparts of (1) which may admit such solitons with exotic spin and statistics?

One approach suggests itself. These models should have the three essential ingredients of the $CP(1)$ model [4]: 1) The existence of topological membrane solitons; 2) the presence in the action of an Abelian Chern-Simons term, the Hopf invariant; 3) the associated Hopf mappings $S^{2n-1} \rightarrow S^n$ include ones with Hopf invariant 1. The first two conditions are embodied in the time component of the key equation (4). Its integration over all of space yields the topological charge-magnetic flux coupling which is the very basis of the statistics phenomenon in $(2+1)$ dimensions. As to the third condition, a crucial element in the Wilczek-Zee proof of the fractional spin and statistics for one soliton [4], we have the following property of Hopf mappings in higher dimensions. While for any n even there always exists a map $f: S^{2n-1} \rightarrow S^n$ with only even integer Hopf invariant $\gamma(f)$, but as for Hopf maps of invariant 1, the celebrated Adams' theorem [14] tells us: If there exists a Hopf map $f: S^D \rightarrow S^{(D+1)/2}$ of Hopf invariant $\gamma(f) = 1$, indeed one with any integer $\gamma(f)$, then D must equal 1, 3, 7 and 15 i.e. $m = (D+1)/2 = 1, 2, 4$ and 8.

From it follow other relevant theorems[15]. They are

1) R^n admit the structure of a real division algebra if and only if $n=1, 2, 4$ and 8. The

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1) R^n admit the structure of a real division algebra if and only if $n = 1, 2, 4$ and 8. The

algebras are the real numbers (**R**), the complex numbers (**C**), the quaternions (**H**) and the octonions (**Ω**).

2) The only parallelizable spheres are S^1 , S^3 and S^7

3) There exists a vector product on R^n if and only if $n = 3$ or 7 . These products are the familiar Gibbs-Hamilton product of vectors or imaginary quaternions and the Cayley product of imaginary octonions.

4) Among the spheres S^n only S^2 or S^6 admit an almost complex structure.

So Adams' theorem singles out 4 unique sets of spacetime and field topologies for the sought for field theories. These unique four families of Hopf maps, the $K = R, C, H, \Omega$ Hopf bundles and their various properties are made manifest by the tell-all diagram

$$\begin{array}{c}
 U(1) = SO(2) \\
 \parallel \\
 Z_2 = O(1) = S^0 \rightarrow S^1 \rightarrow S^1/Z_2 = \mathbf{RP}(1) \approx SO(2)/Z_2 \\
 \parallel \\
 SO(2) = U(1) = S^1 \rightarrow S^3 \rightarrow S^2 = \mathbf{CP}(1) \approx SU(2)/U(1) \\
 \parallel \\
 SU(2) = Sp(1) = S^3 \rightarrow S^7 \rightarrow S^4 = \mathbf{HP}(1) \approx Sp(2)/Sp(1) \times Sp(1) \\
 \parallel \\
 Spin(8)/Spin(7) = S^7 \rightarrow S^{15} (\approx Spin(9)/Spin(7)) \rightarrow S^8 = \mathbf{\Omega P}(1) \approx Spin(9)/Spin(8).
 \end{array}$$

The Hopf maps $f : S^{2n-1} \rightarrow S^n$, $n=1, 2, 4, 8$, with Hopf invariant one have found important applications in condensed matter physics and in quantum field theory. Particularly the connection with exotic spin and statistics had long been lurking in the background. Thus, in the $D=2$ ϕ^4 field theory, the $n=1$ real Hopf map realizes the 1-kink soliton bearing intermediate spin and obeying exotic statistics [16]. The $n=2$ complex Hopf map underlies the θ spin and statistics of $D=3$ $\mathbf{CP}(1)$ model with Hopf term. We have shown this pattern to persist in models built on the remaining two Hopf fiberings, namely the $D=7$ quaternionic $\mathbf{HP}(1)$ ($\approx S^4$) and the $D=15$ octonionic $\mathbf{\Omega P}(1)$ ($\approx S^8$) σ -models augmented with their respective Hopf invariant term. Modulo possible Skyrme

terms needed for soliton stability, their action generalize (1) as

$$S_{(n)} = \int_M \partial_\mu \vec{N} \cdot \partial^\mu \vec{N} d^{2n-1}x + \frac{\theta}{a} \int_M A_{n-1} \wedge dA_{n-1} \quad (10)$$

$n = 4, 8, M = S^7, S^{15}$

where $\vec{N} \cdot \vec{N} = 1$, the $(n+1)$ -vector parametrizing S^n . If $K^T = (K_1, K_2)$ such that $K^\dagger K = 1$ with $K_1, K_2 \in H, \Omega$, is a K -valued 2-spinor parametrizing S^{2n-1} , the Hopf map is $\vec{N} = Sc(K^\dagger \vec{\gamma} K)$ with $K^\dagger = (\bar{K}_1, \bar{K}_2)$, $\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ \bar{e}_\mu & 0 \end{pmatrix}$, $\mu = 0, 1, \dots, m-1$ and $\gamma_m = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $m = 4$ and 8 . The γ 's are just the Dirac matrices of $SL(2, K)$ with γ_m being the analog of γ_5 of the standard $D=4$ formalism.

The A_{n-1} are composite rank $(n-1)$ Kalb-Ramond fields. They are nonlocal in N but local in K . The second term in () is up a multiplicative constant θa^{-1} the Whitehead-Chern-Simons form of the Hopf invariant $\chi(\Phi)$ classifying the mappings $\Phi : S^{2n-1} \rightarrow S^n$. It also reads

$$S_I = \frac{1}{(n-1)!} \int d^{2n-1}x J^{\mu_1 \dots \mu_{n-1}} A_{\mu_1 \dots \mu_{n-1}} \quad (11)$$

i.e. as an interaction of the field A_{n-1} with the conserved topological current $J_{n-1} = -\frac{(n-1)! \theta}{4\pi^2} F_n$, the $*$ means Hodge duality. The sources of J_{n-1} are charged solitonic 2-membranes and 6-membranes respectively, in accordance with the Kleman-Toulouse classification of topological defects [17].

Since these membrane solitons can be readily shown to have a thin London limit, Polyakov's approximation translates into the Chern-Simons-Kalb-Ramond electrodynamics of Nambu-Goto membranes. To get the statistical phase, we consider [12] the propagation of two pairs of membranes-antimembrane and seek the phase upon adiabatically exchanging the membranes. We then get

terms needed for soliton stability, their action generalize (1) as

$$S_{(n)} = \int_M \partial_\mu \bar{N} \cdot \partial^\mu \bar{N} d^{2n-1}x + \frac{\theta}{a} \int_M A_{n-1} \wedge dA_{n-1} \quad (10)$$

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$$\left\langle \exp\left(\frac{i}{3!} \oint_{P_1} A^{(\mu\nu\lambda)} dx^\mu \wedge dx^\nu \wedge dx^\lambda\right) \exp\left(\frac{i}{3!} \oint_{P_2} A^{(\mu\nu\lambda)} dx^\mu \wedge dx^\nu \wedge dx^\lambda\right) \right\rangle \quad (12)$$

P_1 and P_2 are S^3 hyper-curves. The functional average $\langle \dots \rangle$ is taken over the Hopf action (9). As in $D=3$ case, the resulting phase here is the sum of three phases. The first contribution yields the phase factor $\exp(2i (\pi^2/\theta) L)$ with L being the generalized Gauss' linking coefficient () for two S^3 -loops. We get π^2/θ for the statistical phase. The other two phase factors $\Phi(P_i)$ are given by the expectation value of one loop:

$$\Phi(P) = \left\langle \exp\left(\frac{i}{3!} \oint_P A^{(\mu\nu\lambda)} dx^\mu \wedge dx^\nu \wedge dx^\lambda\right) \right\rangle. \quad (13)$$

In the London-Nielsen-Olesen limit we use the effective action

$$S = S_0 + \frac{\theta}{(3!)^2 a} \int_{S^7} d^7x \epsilon_{\mu\nu\lambda\alpha\beta\gamma\delta} A^{\mu\nu\lambda} \partial^\alpha A^{\beta\gamma\delta} + \frac{1}{3!} \int_{S^7} d^7x J_{\mu\nu\lambda} A^{\mu\nu\lambda}. \quad (14)$$

S_0 is the free Nambu-Goto action for a relativistic 2-membrane, $0 \leq \theta \leq \pi$ and the constant a is yet to be chosen. By direct integration of the equation of motion

$$J_{\mu\nu\lambda} + 2 \frac{\theta}{3!a} \epsilon_{\mu\nu\lambda\alpha\beta\gamma\delta} \partial^\alpha A^{\beta\gamma\delta} = 0 \quad (15)$$

with the given current $J^{\mu\nu\lambda}(y) = \int d^3x \delta^7(x-y) \frac{\partial(x^\mu, x^\nu, x^\lambda)}{\partial(\tau, \sigma_1, \sigma_2)}$, in the Lorentz gauge

$$\partial^\alpha A_{\alpha\beta\gamma} = 0,$$

$$i \frac{1}{2 \cdot 3!} \int d^7x J_{\beta\gamma\delta} A^{\beta\gamma\delta}(x) = i \frac{a}{4\theta \Omega_6} \int_{S^3} d\Sigma_{\beta\gamma\delta} \int_{S^3} d\Sigma_{\mu\nu\lambda} \frac{\epsilon^{\mu\nu\lambda\alpha\beta\gamma\delta} (x-y)_\alpha}{|x-y|^7} \quad (16)$$

However just like in the $D=3$ case, the double integration is over one and the same hypercurve S^3 so the phase (16) is *undetermined*. We thus apply White's result to get the

regularized phase

$$\Phi(P \approx S^3) = \exp\left\{i \frac{a}{4\theta} W(P)\right\} \quad (17)$$

where $W(P) = \frac{1}{\Omega_6} \int_{S^3 \times S^3} d\Omega_6$ is the writhe of the Feynman path P of the Nambu-Goto S^2 -membrane, a S^3 hyper-ribbon in 7-spacetimes. A similar computation gives the same result for the octonionic case of $P \approx S^7$ in S^{15} -spacetime.

For the value of $a = 4\pi^2$, $\Phi(P) = \exp(\pi^2 i W/\theta)$. By setting $\theta = \pi$, we get the S^3 - (S^7 -) counterpart of Polyakov's phase factor $\Phi(P) = \exp(-\pi i T(P)) \exp(\pi i n)$, T being the generalized torsion for an S^3 - (S^7) ribbon P . We similarly expect this phase factor to embody the thin membrane's spin in a functional integral formalism. Provided that this reasonable conjecture is verified by a construction à la Polyakov of the spin factor for membranes, we will have a 7- (15-) dimensional analog of the $D=3$ Fermi-Bose transmutation. Since the value of θ is not fixed by the $U(1)$ gauge invariance of the Kalb-Ramond field, for arbitrary values of θ we thus have the possibility of fractional statistics and spin via the relation $W = -T \pmod{Z}$ for the membranes.

Without probing the complexity of model dependent soliton structure or performing a detailed canonical quantization of the above KP_1 σ -models, the evidence of θ -spin and statistics among the membranes is further supported by the topology of their cobfiguration space of fields. In the Schrödinger picture, the configuration space Γ of finite energy static solutions of our $KP(1)$ σ -model is the *mapping space* of all based preserving smooth soliton maps $\vec{N}(x) : x \in S^n \rightarrow \vec{N}(x) \in S^n$, $n = 2, 4, 8$. The space Γ is an infinite Lie group with connectivity given by

$$\pi_0(\Gamma = \{\vec{N} : S^n \rightarrow S^n\}) = \pi_n(S^n) = Z. \quad (18)$$

So it is divided into an infinite set of pathwise-connected components Γ_α , $\alpha \in Z$, corresponding to the various soliton sectors labelled by the charge Q . Moreover for our membranes, as with Skyrmions and Yang-Mills instantons, each Γ_α has further

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topological obstructions. According to G.W. Whitehead [18] all Γ_α in Γ have the *same* homotopy type i.e. $\pi_i(\Gamma_\alpha) \approx \pi_i(\Gamma_\beta)$. Of relevance to the question of exotic spin and statistics, for the 1-soliton sector are the key relations

$$\pi_i(\Gamma_1) \approx \pi_i(\Gamma_0) \approx \pi_{i+n}(S^n) = \mathbb{Z} \text{ for } (i, n) = (1, 2), (3, 4), (7, 8). \quad (19)$$

by the Whitehead and Hurewicz [19] isomorphisms, the latter stating $\pi_i(\Gamma_\alpha) \approx \pi_{i+n}(S^n)$.

The connections (18) reflect the *multi-valuedness* of Γ_i and imply the possibility of adding to the KP(1) σ -model action a Hopf invariant $\chi(\vec{N})$, the generator of the torsion free part of $\pi_{i+n}(S^n)$ ($\pi_3(S^2) = \mathbb{Z}$, $\pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}_{12}$ and $\pi_{15}(S^7) = \mathbb{Z} \oplus \mathbb{Z}_{120}$). Generalizing the CP(1) model $\{(i, n) = (1, 2)\}$, the nontriviality of these $\pi_i(\Gamma_1)$ implies the possibilities of Aharonov-Bohm effects of a multiply connected configuration space Γ and signals the existence for the membrane solitons of a higher dimensional analog of an θ spin and statistics connection.

In the CP(1) case, the Hopf term induces, upon a 2π rotation P of the Skyrmion or an interchange of two Skyrmions, a projective spin phase factor $\Phi(P) = \exp(i\theta) = \exp(i2\pi s)$, s being the soliton spin. The equality $\theta = 2\pi s$ for this process of rotation is a physical realization of the homomorphism:

$$\pi_1(\text{SO}(2)) \approx \pi_3(S^2) \approx \pi_1(\Gamma_1) = \mathbb{Z}. \quad (20)$$

It establishes the equality of the kinematically allowed exotic spin to the dynamically induced θ -spin by way of the Hopf term. Yet (20) is but a special case of the Hopf-Whitehead J-homomorphism [20] $\pi_i(\text{SO}(n)) \approx \pi_{i+n}(S^n)$. Indeed we have the following chain of homomorphisms :

$$\pi_i(\Gamma_1) \approx \pi_i(\Gamma_0) \approx \pi_{i+n}(S^n) \approx \pi_i(\text{SO}(n)) = \mathbb{Z} \quad (21)$$

with $(i, n) = (1, 2), (3, 4), (7, 8)$. $\pi_3(SO(4)) \approx \pi_7(S^4) \approx \mathbb{Z}$, $\pi_7(SO(8)) \approx \pi_{15}(S^8) \approx \mathbb{Z}$

It suffices to say the physical interpretation of these topological relations implies a dynamically induced exotic spin and statistics connection for the 2- and 6-membranes.

In conclusion, our work may cast new light on the uncharted problem of the spin and statistics connection for higher dimensional extended objects. It is also a first step both in the bosonic functional integral formulation for spinning extended objects [6,21] and in the study of the θ -vacuum phenomenon in Kaluza-Klein compactification. As their historical developments [22] testify, division algebras stand at the crossroad of several frontier areas of mathematics and physics. Over the last decade, ever more linkages have been discovered between division algebras and basic aspects of unified theories. Examples are field theories with rigid supersymmetries, superstrings and supermembranes in critical dimensions. Here we have uncovered their special relevance in yet another connection, a generalized θ -spin and statistics connection.

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