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A study of nonlinear waves described by the cubic Schrodinger equation

**Arthur Eugene Walstead
(Ph.D. Thesis)**

March 12, 1980

Lawrence
Livermore
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A Study of Nonlinear Waves Described by the Cubic Schrödinger Equation

by

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Abstract

The cubic Schrödinger equation is ubiquitous as a model equation for the long-time evolution of finite amplitude near-monochromatic dispersive waves. It incorporates the effects of the radiation field pressure on the constitutive properties of the supporting medium in a self-consistent manner. We study the properties of the uniformly translating periodic wave solutions of the one-dimensional CSE. These (so-called *cnoidal*) waves are characterized by the values of four parameters. We use Whitham's¹ averaged variational principle to derive a system of quasilinear evolution equations (the *modulational equations*) for the values of these parameters when they are slowly varying in space and time. Explicit expressions for the characteristic velocities of the modulational equations are obtained for the full set of cnoidal waves. The system is subsequently determined to be hyperbolic or elliptic, stable or unstable, according to the sign of a certain coefficient pq . Riemann invariants are obtained for several limits for the stable case ($pq < 0$) and growth rates are obtained for several limits, including the solitary wave chain, for the unstable case ($pq > 0$). The results for

several nontrivial limiting cases agree with those obtained by independent methods by other authors.

The dynamics of the CSE generalized to two spatial dimensions are studied for the case $p q > 0$. A large class of similarity solutions with cylindrical symmetry are obtained systematically using infinitesimal transformation group techniques. The methods are adapted to obtain the symmetries of the action functional of the CSE and to deduce nine integral invariants. A numerical study of the self-similar solutions reveals that they are modulationally unstable and that singularities dominate the dynamics of the CSE in two dimensions.

The CSE is derived using perturbation theory for a specific problem in plasma physics - the evolution of the envelope of a near-monochromatic electromagnetic wave in a cold magnetized plasma.

- 1 G. B. Whitham, Linear and Nonlinear Waves, (Wiley, New York, 1974), Chaps. 14-16.

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The observant reader may note that this thesis has been composed by computer. I used the RED dialect of the TRIx interpreter available on the MPECC CDC 7600 to compose the text and equations. The final copy was output on the high resolution hardcopy camera. I am indebted to Hank Moll, who wrote TRIx, for his assistance with the subroutines I used to insert and label equations automatically in the text.

Several of the results in Chapter 5 were obtained using the MIT MACSYMA symbolic interpreter maintained by the Mathlab group and supported by USERDA contract E(11-1)-3070 and NASA grant NSG 1323. I hope to make further use of this exciting tool in the future.

I am also indebted to the Fannie and John Hertz Foundation for generous support during the early years of my graduate study.

Chapter 1

A. The Cubic Schrödinger Equation.

Broadly speaking, this thesis is concerned with the qualitatively new phenomena associated with dispersive wave motion when the finite amplitude of the waves is taken into account. More specifically, we shall study the Cubic Schrödinger Equation (hereafter the CSE) which has appeared time and again in many areas of physics¹⁻⁵ as a model equation for an important special class of problems - the long time evolution of nearly monochromatic and finite amplitude wave trains. The universal nature of the CSE is suggested by the heuristic derivation presented in the following paragraphs.

Our understanding of nonlinear dispersive waves is advanced by contrast with the known properties of linear dispersive waves. For our purposes a linear dispersive wave is a propagating field composed by superposition of plane waves satisfying a dispersion relation $\omega = \omega(k)$ with real values of the frequency ω for real values of the wave number k with the additional condition $\omega_{kk} \neq 0$. Such disturbances can be advanced from initial conditions using the Fourier integral theorem

relating a field $E(x, t)$ to its spatial Fourier decomposition $\tilde{E}(k, t)$ at $t = 0$:

$$E(x, t) = \frac{1}{\sqrt{2\pi}} \int dk \tilde{E}(k, 0) e^{ikx - i\omega(k)t} \quad 1.1$$

To represent a localized near-monochromatic wave packet, we multiply a plane wave function $e^{ik_0 x}$ by an envelope function $e^{-\varepsilon^2 x^2/4}$ with ε a small real number. The Fourier transform of $E(x, 0)$ is then a gaussian:

$$\tilde{E}(k, 0) = \frac{\sqrt{2}}{\varepsilon} e^{-(k-k_0)^2/\varepsilon^2} \quad 1.2$$

The spread in wavenumbers is thus localized about k_0 with variance $\langle (k-k_0)^2 \rangle \sim O(\varepsilon^2)$. The dominant contribution to (1.1) comes from a small region of wavenumbers about k_0 . Within this region the dispersion relation $\omega(k)$ is accurately approximated by a Taylor series in the variable $k' = k - k_0$. We expand $\omega(k)$ about k_0 and keep terms of $O(\varepsilon^2)$. The integral in (1.1) is then approximated by

$$E(x, t) = \frac{e^{ik_0 x - i\omega_0 t}}{\sqrt{1 + i\varepsilon^2 \omega_{kk}^0 t/2}} \exp\left(-\frac{\varepsilon^2}{4} \frac{(x - \omega_{kk}^0 t)^2}{1 + i\varepsilon^2 \omega_{kk}^0 t/2}\right) \quad 1.3$$

We note:

- i) The wave packet spreads or disperses provided $\omega_{kk}^0 \neq 0$ with $\langle x^2 \rangle - \langle x \rangle^2 \sim t$ as $t \rightarrow \infty$.

ii) The wave packet propagates at the group velocity $\omega_k^0 = \omega_k(k_0)$.

To derive the CSE we use the integral representation (1.1), only we rewrite $E(x,t)$ in the *phasor* representation

$$E(x,t) = A(x,t)e^{ik_0x - i\omega_0 t} \quad 1.4$$

where $A(x,t)$ is slowly varying in x,t . We obtain

$$A = \frac{1}{\sqrt{2\pi}} \int dk \tilde{E}(k,0) e^{i(k-k_0)x - i[\omega(k)-\omega_0]t} \quad 1.5a$$

$$iA_t = \frac{1}{\sqrt{2\pi}} \int dk \tilde{E}(k,0) [\omega(k)-\omega_0] e^{i(k-k_0)x - i[\omega(k)-\omega_0]t} \quad 1.5b$$

If the group velocity ω_k^0 is slow enough so that the material medium supporting the wave responds quasistatically to the radiation pressure of the field, then the constitutive properties of the medium will be modified for a particular wavenumber so that we may let $\omega(k) \rightarrow \omega(k, |A|^2)$ in (1.5b). Expanding $\omega(k, |A|^2)$ about $\omega_0 = \omega(k_0, 0)$ in the term in braces in the integrand of (1.5b), we obtain

$$iA_t \sim \frac{1}{\sqrt{2\pi}} \int dk \tilde{E}(k,0) \left[(k-k_0)\omega_k^0 + \frac{1}{2}(k-k_0)^2\omega_{kk}^0 + \omega_{|A|^2}^0 |A|^2 \right] e^{i(k-k_0)x - i[\omega(k, |A|^2) - \omega_0]t} \quad 1.6$$

Using the operator replacement

$$k-k_0 \rightarrow -i \frac{\partial}{\partial x} \quad 1.7$$

and (1.5a) we obtain

$$i(A_t + \omega_k^0 A_x) + \frac{1}{2} \omega_{kk}^0 A_{xx} - \omega_A^0 |A|^2 A \sim 0 \quad 1.8$$

Equation (1.8) is the Schrödinger equation of quantum mechanics with the potential replaced by a self-consistent term proportional to the field intensity. If we ignore the nonlinear term in (1.8) we obtain

$$i(A_t + \omega_k^0 A_x) + \frac{1}{2} \omega_{kk}^0 A_{xx} = 0 \quad 1.9$$

which has a particular solution identical to (1.3) (provided we restore the exponential factor $e^{ik_0 x - i\omega_0 t}$ to the phasor). For our purposes it is convenient to write (1.8) in a frame where $\omega_k^0=0$ and to define constitutive constants p and q characteristic of the medium. Thus the CSE will hereafter be written

$$iA_t + pA_{xx} + q|A|^2 A = 0 \quad 1.10$$

Equations (1.10) or (1.8) suggest an ordering scheme that will be necessary to obtain the CSE by systematic perturbation theory. Each term in (1.10) must be $O(\epsilon^3)$ dictated by the nonlinear term. Thus the

time derivative of the envelope function or phasor must be $O(\varepsilon^2)$ and the spatial gradient of the phasor must be $O(\varepsilon)$.

What are the qualitatively new phenomena inherent in the CSE?

i). The CSE (1.10) has nonspreading, localized, propagating solutions. If we write the envelope function in the form $\sqrt{I}e^{i\varphi}$, these new wave types can be written:

$$pq > 0: \quad I = e_1 \operatorname{sech}^2 \left[\left(e_1 \frac{q}{2p} \right)^{1/2} (x - ut) \right] \quad 1.11$$

$$pq < 0: \quad I = e_2 - e_3 \operatorname{sech}^2 \left[\left(e_2 \frac{q}{2p} \right)^{1/2} (x - ut) \right] \quad 1.12$$

with a phase structure φ that is unimportant for our present discussion.

We immediately note that these new solutions are qualitatively different from linear wave packets - *they do not decay asymptotically with time.*

More significantly, for the case $pq > 0$ the localized packets (1.7a) are *solitons*⁶⁻⁹ - i.e. two or more such solitons preserve their identities

during collisions, emerging asymptotically unchanged apart from slight displacements in position and phase. Even more remarkable is the fact

that these solitons form a discrete basis set for the complete solution of the initial value problem of the CSE¹⁰ ($pq > 0$) for packets

that are initially localized in space. This fact was proven analytically in a profound development in applied mathematics - the

*Inverse Scattering Transform*¹⁰⁻¹³ (hereafter IST). The CSE has been shown to belong to a class of nonlinear equations - most of physical

significance - that are solvable by the IST. The IST theory is a generalization of Fourier analysis¹³ to certain nonlinear equations whereby initial conditions are mapped into a dual space where they are represented by the solution of a linear problem that can be mapped forward in time in a simple manner. The inverse mapping of the time advanced representation back to the original space then gives the original packet at the advanced time. However, we shall not be concerned with this powerful and beautiful theory in this work. Rather, our study will be more closely related to the following phenomena inherent in the CSE.

ii). A second qualitatively new phenomenon introduced by the CSE is a class of instability - *modulational instability*¹⁴. We can illustrate this class of instabilities by an example. We expand the envelope function A about a uniform state using the following substitution:

$$A = \sqrt{\epsilon} e^{i\beta x - i\gamma t} \{ 1 + \epsilon [u \cos(kx - \omega t) + i v \sin(kx - \omega t)] \} \quad 1.13$$

and collect terms of $O(\epsilon)$ and $O(\epsilon^2)$:

$$O(1): \quad \gamma = p\beta^2 - qI \quad 1.14a$$

$$O(\epsilon): \quad \omega = (2\beta p \pm p\sqrt{k^2 - 2qI/p})k \quad 1.14b$$

The dispersion relation (1.14b) shows that the case $pq > 0$ is unstable to the formation of modulations in the envelope. In line with the

discussion of the previous paragraphs, the nonlinear stage of the instability evolves towards a sequence of solitons plus a residual of decaying oscillations.

iii). A third feature of the CSE is a class of uniform periodic wave trains that are the analogues of the plane waves of the linear theory. If we again write $A = \sqrt{I} e^{i\varphi}$ then the uniform waves are given by the following expressions in terms of the Jacobian elliptic function $\text{cn}(\theta|m)$.

$$\begin{aligned} pq > 0: \quad I &= e_2 + (e_1 - e_2) \text{cn}^2 \left[(e_1 - e_2)^{1/2} \left(\frac{q}{2p} \right)^{1/2} (x - ut) | m \right] \\ m &= \frac{e_1 - e_2}{e_1 - e_3} \end{aligned} \quad 1.15$$

$$\begin{aligned} pq < 0: \quad I &= e_3 - (e_2 - e_3) \text{cn}^2 \left[(e_1 - e_3)^{1/2} \left(\frac{q}{2p} \right)^{1/2} (x - ut) | m \right] \\ m &= \frac{e_2 - e_3}{e_1 - e_3} \end{aligned} \quad 1.16$$

These waves are periodic in the intensity I with a period given by

$$l = 2 \left(\frac{2p}{q} \right)^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} K(m) \quad 1.17$$

where $K(m)$ is the complete elliptic integral of the first kind. In the limit $m \rightarrow 1$, $l \rightarrow \infty$ and (1.15, 16) approach (1.11, 12).

In this work we shall treat these (so called *cnoidal*) waves as nonlinear carriers. The appearance of the scale length l will allow us to develop an approximate set of evolution equations for the parameters e_1, e_2, e_3 and u appearing in (1.15) and (1.16) when they are slowly varying in space and time. The evolution equations shall be referred to as the *modulational equations* and will have the following form:

$$\left[\begin{array}{c} \text{matrix function} \\ \text{of } e_1, e_2, e_3 \\ \text{and } u \end{array} \right] \frac{\partial}{\partial t} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ u \end{bmatrix} + \left[\begin{array}{c} \text{matrix function} \\ \text{of } e_1, e_2, e_3 \\ \text{and } u \end{array} \right] \frac{\partial}{\partial x} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ u \end{bmatrix} = 0 \quad 1.18$$

We shall determine the *modulational stability* of the cnoidal carriers by determining the elliptic (unstable) or hyperbolic (stable) character of the system (1.18). In the case of modulationally stable cnoidal waves, we shall determine the characteristic velocities at which long wavelength perturbations propagate and the Riemann invariants associated with wave motion in certain limits. The derivation and analysis of the modulational equations forms the content of Chapters 2 through 5.

We shall derive our theory using a powerful technique due to G. B. Whitham^{15,16,17}. This method is based on a variational principle closely related to the theory of adiabatic invariants for a mechanical system exhibiting nearly periodic motion. It is well known¹⁸ that a

mechanical system with nearly periodic motion for a coordinate q satisfies a relation of the form

$$\frac{d}{dt} \oint dt q L_{, \dot{q}} = \frac{d}{dt} \oint p dq \sim 0 \quad 1.19$$

where L is the Lagrangian and p the momentum conjugate to q . Weakly nonuniform cnoidal waves are also nearly periodic in a phase variable θ and will be shown to satisfy analogous expressions of the form

$$\frac{\partial}{\partial t} \oint d\theta u_{\theta} \mathcal{L}_{u_t} - \frac{\partial}{\partial x} \oint d\theta u_{\theta} \mathcal{L}_{u_x} \sim 0 \quad 1.20$$

where $\mathcal{L}$ is the Lagrangian density of the CSE. Relations such as (1.20) will allow us to write a closed set of equations for the slowly varying parameters of the cnoidal waves.

One of the advantages of Whitham's approach is that the closed system of evolution equations for the cnoidal wave parameters are always quasilinear in lowest order and can therefore be classified as elliptic or hyperbolic following standard procedures. For the hyperbolic (modulational stable) case, the theory provides a means to determine Riemann invariants of the wave motion - information that would be lost in a strictly linear theory of perturbed motion about a cnoidal wave and that allows the modulational equations to be replaced by an equivalent set of ordinary differential equations.

B. An example of the application of the CSE from Plasma Physics.

In this section we follow the heuristic derivation of the CSE and construct a specific model of the evolution of an electromagnetic wave in a cold magnetized plasma taking into account the higher order effects of the radiation pressure. In Appendix I we derive the CSE for this same problem using perturbation theory - the resulting CSE is the same.

We consider an electromagnetic wave in a cold plasma ($T_e, T_i \sim 0$) propagating at right angles to a zero order uniform magnetic field $B_0 = B_0 \hat{z}$ produced by an external current source. The electric field vector E is assumed to be parallel to the external magnetic field vector B_0 so that the lowest order currents induced are parallel and therefore decouples from B_0 . The wave is assumed to propagate in the $\hat{x}$ direction and to depend only on the independent variable x . Under these circumstances the electromagnetic wave satisfies the following dispersion relation¹⁹ which is independent of B_0 :

$$\omega^2 = \omega_0^2 + k^2 c^2, \quad \omega_0^2 = \omega_{pe}^2 + \omega_{pi}^2$$

$$\omega_{pa}^2 = \frac{4\pi e^2 n}{m_a} \quad 1.21$$

The collective motion of the charged particles contributes to the total field of the electromagnetic wave in a constructive manner for $\omega^2 > \omega_0^2$. Destructive interference ensues when $\omega^2 < \omega_0^2$ yielding exponential decaying waves away from any source or boundary. We easily compute the group velocity ω_k and dispersion term ω_{kk} in the long wavelength limit obtaining

$$\omega_k^0 = \pm \frac{c^2 k}{\omega} \Big|_{k=0} = 0 \quad 1.22$$

$$\omega_{kk}^0 = \pm \frac{c^2}{\omega_0} \quad 1.23$$

Our problem is to deduce the modification of the dispersion relation (1.21) to lowest order in the intensity of the electromagnetic wave. As $\omega_k^0 = 0$, we can be confident that the response of the plasma to the radiation pressure of a nearly monochromatic packet of long wavelength can be quasistatic.

Clearly the nonlinear modification of the dispersion relation (1.20) will appear parametrically through the density n . For equilibrium in the quasistatic response of the plasma to the radiation pressure of a packet, we demand that forces remain in balance at all times. To balance the forces we need the radiation pressure and the magnetic pressure acting on the plasma. The radiation pressure is easily derived from an energy argument. We assume that the work done on the plasma during a virtual compression of a segment of plasma of length l be compensated by a decrease in the total energy content of the wave

$$-P_r \Delta l = \Delta(Ul) \quad 1.24$$

where P_r is the radiation pressure and U is the wave energy density. To compute U , we construct Poynting's vector and equate it to the product of the directed group velocity $\mathbf{k} \omega_k$ and the energy density U :

$$S = \frac{1}{8\pi} c \mathbf{E}^* \times \mathbf{B} = \frac{c}{8\pi} k |\mathbf{E}|^2 = \hbar \omega_k U \quad 1.25$$

Using (1.22) we obtain the following expression for U:

$$U = \frac{|\mathbf{E}|^2}{8\pi} \quad 1.26$$

An electromagnetic wave with frequency ω executes simple harmonic motion for which there is a well known adiabatic invariant conserved for slow changes in a parameter, specifically the quantity *total energy/frequency* or *wave action* is a constant so that

$$\Delta\left(\frac{U}{\omega}\right) = 0 \quad 1.27$$

Thus we have

$$\Delta(U) = U \frac{\Delta\omega}{\omega} = U \frac{\Delta n}{2n_0} = -U \frac{\Delta l}{2} \quad 1.28$$

and in combination with (1.24) and (1.25) we have

$$P_r = \frac{-\Delta(U)}{\Delta l} = \frac{U}{2} = \frac{|\mathbf{E}|^2}{16\pi} \quad 1.29$$

Now the magnetic pressure is given by

$$P_m = \frac{(B_0 + B_2)^2}{8\pi} \quad 1.30$$

where B_2 is a second order small correction to B_0 . We demand that the forces balance, so that the pressure is everywhere uniform yielding

$$P_m + P_r = \frac{B_0^2}{8\pi} \quad 1.31$$

Using (1.29), (1.30) and (1.31) and the fact that $|E|^2$ and B_2 are second order small, we obtain

$$\frac{B_2}{B_0} = -\frac{|E|^2}{4B_0^2} \quad 1.32$$

We are clearly in a regime where the magnetic flux and plasma density are frozen together, which allows us to write

$$\frac{n_2}{n_0} = \frac{j_2}{B_0} \quad 1.33$$

Thus the nonlinear dispersion relation can be written

$$\omega^2 = \omega_0^2 \left(1 - \frac{|E|^2}{4B_0^2}\right) + k^2 c^2 \quad 1.34$$

yielding the following coefficient for the nonlinear term in the CSE:

$$\omega_0^2 |E|^2 = \pm \left(-\frac{\omega_0^2}{8B_0^2} \right) \quad 1.35$$

We have all of the necessary elements from which to assemble our heuristically derived CSE (1.8):

$$iE_t \pm \frac{c^2}{2\omega_0} E_{xx} \pm \frac{\omega_0}{8B_0^2} |E|^2 E = 0 \quad 1.36$$

For this CSE $pq > 0$ and we conclude that plane waves are modulationally unstable. It is instructive to compare (1.36) with the results of a systematic application of perturbation theory to the same problem in Appendix I. There we obtain the following set of equations valid for $\partial/\partial t \ll \Omega_i$,

$$iE_t + \frac{c^2}{2\omega_0} E_{xx} - \frac{\omega_0 n_2}{2n_0} E = 0 \quad 1.37$$

$$(1 + \frac{a_0^2}{c^2}) \frac{\partial^2 n_2}{\partial t^2 n_0} - a_0^2 \frac{\partial^2 n_2}{\partial x^2 n_0} = a_0^2 \frac{\partial^2 |E|^2}{\partial x^2 4B_0^2} \quad 1.38$$

where Ω_i is the ion cyclotron frequency $eB_0/m_i c$ and a_0 is the *Alphén* or *magnetosonic velocity* ($a_0^2 = B_0^2/4\pi n_0(m_e + m_i)$). If $\partial/\partial t \ll a_0 \partial/\partial x$ we obtain

$$\frac{n_2}{n_0} = -\frac{|E|^2}{4B_0^2} \quad 1.39$$

and the resulting CSE is identical to (1.36) (provided we choose the + sign). However, equations (1.37) and (1.38) reveal the physics and the approximations implicit in the adiabatic hypothesis. Equation (1.37) describes the slow evolution of the electromagnetic wave in a weakly modulated plasma while the second equation describes the slow evolution of the plasma as a driven magnetosonic wave with a driving term proportional to the radiation pressure. Thus the adiabatic assumption is valid provided the packet velocity is much smaller than the magnetic sound speed.

The diamagnetic field B_2 that subtracts from the external field B_0 is produced by electron and ion drifts (*precessional currents*) that are driven by the radiation pressure force and that flow at right angles to both the radiation force and magnetic field B_0 .

In the sequel we shall study the CSE *per se* and not discuss specific applications further.

C. Symmetry considerations and the CSE in two spatial dimensions

The second part of this thesis is an overview of the dynamics of the CSE in two spatial dimensions. Whereas the first study concerned the details of evolution of specific motions (weakly nonuniform cnoidal waves) in one spatial dimension, we shall be content to uncover information that is less specific to a particular motion and common to all motions - qualities that are 'typical' or 'universal'. Our perspective is therefore radically different and so the means of the analysis.

This broader perspective is appropriate when the motions of a system are largely unknown or where customary means of analysis - such as those applied to linearized problems - are not applicable. For example, the IST, which solves a large class of initial value problems for the CSE in one dimension, does not apply to fully two dimensional problems. What is known about the CSE in two dimensions is gleaned from numerical experiments performed on the computer, particular analytic solutions and conservation laws - and extrapolations from these sources.

Of particular importance are so called similarity solutions that are customarily obtained with the aid of dimensional analysis. Similarity solutions are important because they are associated with the asymptotic form of solutions 'far' from boundaries or initial conditions²⁰. By 'far' we usually mean that some natural scale characteristic of the disturbance has evolved through a change of over an order of magnitude so that the initial or boundary conditions appear singular relative to this natural scale. Similarity solutions are commonly singular in space and/or time and are always highly symmetric. Thus it is reasonable that they should emerge as asymptotic forms at times when initial data appear relatively singular.

In Chapter 6 of the thesis we shall use a specific mathematization of symmetry notions²² to systematically deduce a large class of similarity solutions of the CSE in two dimensions. We shall also adapt the techniques used to obtain similarity solutions to deduce symmetries of the action functional of the CSE. Noether's theorem²² is then used to obtain a conservation law corresponding to each symmetry. As

symmetries must be found before they can be used in the theorem, our procedure is the natural complement to Noether's theorem.

As mentioned, the perspective gained from knowledge of similarity solutions and conservation laws is augmented by numerical experiments. Thus we shall also present select numerical solutions of the CSE in two dimensions. We shall study several of the similarity solutions and illuminate their role in the asymptotics of more general motions described by the CSE in two dimensions. The numerical experiments with the similarity solutions of the CSE form the content of Chapter 7.

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Chapter 2

In this chapter we shall introduce and develop Whitham's principle to a form that will be most useful for us. The interested reader should consult Whitham's book¹ and fundamental papers^{2,3,4} for an in-depth discussion of the principle in the context of applications to linear and nonlinear waves. The full power of the viewpoint can only be appreciated after an examination of the variety of its adaptations and applications. In this chapter we present a brief development of the essence of Whitham's ideas.

There is an underlying concept that we will encounter that involves the extension⁵ of the number of independent variables. We represent the situation symbolically by writing $u=f(x,t)=F(\theta(x,t),x,t)$. The dependent variable u solves a partial differential equation in x and t that yields wave motion. In the second function F the dependence on x and t is mediated in part by a variable θ . For our purposes θ parameterizes the oscillatory behavior of f . The functional form of θ is usually determined implicitly once the periodicity requirement for wave motion is specified; i.e. $F(\theta,x,t)=F(\theta+2\pi,x,t)$. Let us say that F is

"synchronized" when for the variable θ in $F(\theta, x, t)$ we substitute the function value $\theta(x, t)$ for the given x, t . When F is synchronized, little may be gained by substituting the representation F into the differential equation. However, there is an alternative viewpoint from which the problem can be posed. If we obtain the functional form of F as a function of the three independent variables θ, x and t and then obtain the functional form of θ as a separate problem, we can synchronize F by using $\theta(x, t)$ in F . From this viewpoint, F is temporarily regarded as a function of three independent variables. We have effectively transformed our original problem into two new problems; the solution for F as a function of θ, x and t and the solution for θ as a function of x and t . The reduction of the number of independent variables from three to two involves the solution for θ as a function of x and t . Whitham has succeeded in formulating this problem on an equal footing with the determination of F . The conditions for the solution of both F and θ can be stated in compact and universal terms via a variational principle.

A. Whitham's averaged variational principle.

1. The principle as an exact statement for nonuniform waves.

To begin, we assume that we have a system of partial differential equations that are the Euler equations corresponding to a Lagrangian $\mathcal{L}$. For simplicity we shall assume that we have only one dependent field variable $u(x, t)$, and that our equations admit uniform wave solutions. By this we mean that we have a class of solutions that depend on space

and time only through a phase variable $\theta=kx-\omega t$, with k and ω constants, and that the solutions are periodic in θ with period 2π . We can write $u=U(\theta)$ with $U(\theta)=U(\theta+2\pi)$. Isolated pulses are included as possible wave forms as limiting cases where $1/k$ is infinite. These waves are to be considered to have finite amplitudes and will usually not be simple circular functions and will be the solution of nonlinear equations in general. The two constants ω and k have their usual significance for linear waves. In the general case of nonlinear waves, the ratio ω/k will still correspond to a phase velocity, or frame velocity wherein periodic wave forms remain stationary in time. We write our fundamental variational requirement:

$$\frac{\delta}{\delta u} \iint dx dt \mathcal{L}(u_t, u_x, u) = 0 \quad 2.1$$

which yields the Euler equation:

$$\frac{\partial}{\partial t} \mathcal{L}_{u_t} + \frac{\partial}{\partial x} \mathcal{L}_{u_x} - \mathcal{L}_u = 0 \quad 2.2$$

Uniform wave solutions to (2.2) are obtained by integration with respect to the variable θ . Recall that $u=U(\theta)$ and $\theta=kx-\omega t$ so that differentiation with respect to x and t can be replaced by differentiation with respect to the single variable θ . We use the new symbol L for $\mathcal{L}$ restricted to problems depending on space and time through a single independent variable θ .

$$L = L(-\omega u_\theta, k u_\theta, u) \quad 2.3$$

We denote the derivative of L with respect to u_t or equivalently $-\omega u_\theta$ by the symbol L_1 ; similarly we define $L_{u_x} = L_{k u_\theta} = L_2$. The Euler equation (2.2) is then written:

$$\frac{d}{d\theta}(-\omega L_1 + k L_2) - L_u = 0 \quad 2.4$$

As $L_{u_\theta} = -\omega L_1 + k L_2$, (2.4) further reduces to:

$$\frac{d}{d\theta} L_{u_\theta} - L_u = 0 \quad 2.5$$

This equation is identical to the Euler equation for a mechanical system if θ is thought of as a variable analogous to time. This system possesses an energy integral:

$$u_\theta L_{u_\theta} - L = H \quad 2.6$$

Equation (2.6) can be solved for u_θ then integrated to give u as a function of $\theta + \theta_0$, where θ_0 is an additive integration constant. This formally solves the problem and yields the explicit form of the function $u = U(\theta)$. Note that the constant θ_0 is of little consequence as it represents a displacement in phase, whereas the constant H affects the qualitative form of the wave. We write our uniform solution in the form $U(\theta; H, \omega, k)$ in order to indicate the dependence of the solution on the

parameters H , ω and k . Intuitively, we would expect weakly nonuniform waves to be similar to the uniform waves. These solutions should be adequately described by the same functional form $U(\theta; H, \omega, k)$ as our uniform waves, only with the constants H, ω and k now weakly dependent on space and time. Guided by this intuitive notion, we now generalize our assumed functional form to include nonuniformly periodic waves. We write $u=U(\theta, x, t)$ with $U(\theta, x, t)=U(\theta+2\pi, x, t)$; that is, all of the periodic variation of U occurs as a function of x and t through the variable θ . This assumed form results in no loss of generality. Note that the time and space derivatives of U will depend on θ . Derivatives with respect to x and t are modified:

$$\frac{\partial}{\partial x} \rightarrow k \frac{\partial}{\partial \theta} + \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} \rightarrow -\omega \frac{\partial}{\partial \theta} + \frac{\partial}{\partial t} \quad 2.7$$

with $k=\theta_x$ and $-\omega=\theta_t$ so that our Euler equation (2.2) becomes

$$\frac{\partial}{\partial \theta}(-\omega L_1 + k L_2) - L_3 + \frac{\partial}{\partial t} L_1 + \frac{\partial}{\partial x} L_2 = 0 \quad 2.8$$

where the partial derivatives $\partial/\partial t$ and $\partial/\partial x$ are taken now for fixed θ . What Whitham noticed was that (2.8) is the Euler equation for a new variational principle where θ, x and t are regarded as fully independent variables. That is, in the variation of u ; the variables θ, x and t are treated as fully independent. Accordingly the new variational requirement is:

$$\frac{\delta}{\delta u} \iiint dx dt d\theta L(u_t - \omega u_\theta, u_x + k u_\theta, u) = 0 \quad 2.9$$

The relationship between the variables θ and x and t can be recovered from (2.8) and the fact that u is periodic in θ . Alternatively, we note that the action integral (2.9) is now a functional of θ as well as u and it is natural to inquire what the variation of (2.9) with respect to θ will yield. We require that the relation of θ to x and t (our "synchronization" condition) be specified by the variational requirement:

$$\frac{\delta}{\delta \theta} \iiint dx dt d\theta L(u_t - \omega u_\theta, u_x + k u_\theta, u) = 0 \quad 2.10a$$

or

$$\frac{\delta}{\delta \theta} \iint dx dt \langle L \rangle = 0 \quad 2.10b$$

where

$$\langle L \rangle = \oint d\theta L \quad 2.10c$$

yielding the Euler equation

$$\frac{\partial}{\partial t} \langle L_\omega \rangle - \frac{\partial}{\partial x} \langle L_k \rangle = 0 \quad 2.11$$

It is a remarkable fact that (2.11) is precisely the condition that emerges as a consistency requirement in a consideration of (2.8). To see this we first write (2.8) in an equivalent conservation law form. We write, following Whitham:

$$\frac{\partial}{\partial \theta} (u_\theta L_{u_\theta} - L) + \frac{\partial}{\partial t} (u_\theta L_{u_t}) + \frac{\partial}{\partial x} (u_\theta L_{u_x}) = 0 \quad 2.12$$

To verify that (2.12) is equivalent to (2.8) we expand (2.12):

$$\begin{aligned} u_{\theta\theta} L_{u_\theta} + u_\theta \frac{\partial}{\partial \theta} L_{u_\theta} + u_{\theta t} L_{u_t} + u_\theta \frac{\partial}{\partial t} L_{u_t} + u_{\theta x} L_{u_x} + u_\theta \frac{\partial}{\partial x} L_{u_x} \\ - L_{u_\theta} u_\theta - L_{u_\theta} u_{\theta\theta} - L_{u_t} u_\theta - L_{u_x} u_{x\theta} = 0 \end{aligned} \quad 2.13$$

Cancelling terms we obtain:

$$u_\theta \left(\frac{\partial}{\partial \theta} L_{u_\theta} - L_{u_\theta} + \frac{\partial}{\partial t} L_{u_t} + \frac{\partial}{\partial x} L_{u_x} \right) = 0 \quad 2.14$$

or

$$\frac{\partial}{\partial \theta} (-\omega L_1 + k L_2) - L_3 + \frac{\partial}{\partial t} L_1 + \frac{\partial}{\partial x} L_2 = 0 \quad 2.15$$

We also note that (2.12) can be obtained by general considerations of the invariance of the action integral under infinitesimal transformations of the independent and dependent variables using Noether's⁸ theorem. If we have a general variation of the action integral, that is, if all of the independent and dependent variables are extended as continuous functions of some parameter that is varied, and if the action integral remains unchanged by the variation, then Noether's theorem asserts:

$$\frac{\partial}{\partial \theta}(L_{u_\theta} \delta u + L \Delta \theta) + \frac{\partial}{\partial t}(L_{u_t} \delta u + L \Delta t) + \frac{\partial}{\partial x}(L_{u_x} \delta u + L \Delta x) = 0 \quad 2.16a$$

$$\delta = \Delta - \Delta x u_x - \Delta t u_t - \Delta \theta u_\theta \quad 2.16b$$

Comparing (2.12) and (2.16a,b) it is clear that the associated symmetry is a simple displacement in phase $\Delta \theta = \epsilon$ with $\Delta t = \Delta x = \Delta u = 0$. This reflects the fact that our solution admits a simple additive phase constant of the motion. We now return to a consideration of equation (2.12). Having assumed that u is a periodic function of θ , by necessity the left most term of (2.12) in brackets must be periodic in θ , thus we have:

$$\oint d\theta \frac{\partial}{\partial \theta} (u_\theta L_{u_\theta} - L) = \oint d(u_\theta L_{u_\theta} - L) = 0 \quad 2.17$$

thus

$$\frac{\partial}{\partial t} \oint d\theta L_{u_t} u_\theta + \frac{\partial}{\partial x} \oint d\theta L_{u_x} u_\theta = 0 \quad 2.18a$$

or

$$-\frac{\partial}{\partial t} \oint du L_\omega + \frac{\partial}{\partial x} \oint du L_k = 0 \quad 2.18b$$

or

$$\frac{\partial}{\partial t} \langle L_\omega \rangle - \frac{\partial}{\partial x} \langle L_k \rangle = 0 \quad 2.18c$$

This verifies that (2.11) is indeed a consistency requirement.

2. *Whitham's optimal algorithm.*

It can be seen at this point that the variational formulation is the most compact statement of the consistency requirement. However, its advantages are even greater when used to their full potential. This point will be emphasized as we develop the averaged variational principle to its optimal form. Before continuing we should mention that the expressions that we have derived are exact and that no reference has been made to disparate temporal or spatial scales. In order to cast the variational principle in its optimal form, we must revert to an assumed ordering scheme. This is also the first step in a perturbation series solution of the Euler equation.

The fundamental length in the problem is l , the local period of the nonlinear waves. This is the distance to which the scale lengths of gradients are to be compared. We choose l to be $O(\epsilon^0)$ so that $\partial_x = k = 2\pi/l \sim O(\epsilon^0)$. We require that $\omega = -\theta_t$ be of the same order as k . Thus $\partial/\partial t$ and $\partial/\partial x$ must yield $O(\epsilon^0)$ quantities when they operate on θ . The scale over which k and ω (and the other parameters that are constant for uniform waves) vary is assumed to be $O(\epsilon^{-1})$, i.e. the modulations of the parameters are to have long wavelengths. This ordering is implemented by the requirement that $\partial/\partial t$ and $\partial/\partial x$ yield $O(\epsilon)$ quantities when they operate on the explicit x and t arguments of a dependent variable. The operator $\partial/\partial \theta$ is assumed to yield quantities of $O(\epsilon^0)$. When the space of the independent variables is enlarged as before, this ordering scheme can be denoted in a convenient manner using the operator replacements

$$\frac{\partial}{\partial x} \rightarrow k \frac{\partial}{\partial \theta} + \epsilon \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} \rightarrow -\omega \frac{\partial}{\partial \theta} + \epsilon \frac{\partial}{\partial t} \quad 2.19$$

With these replacements, the Euler equation (2.8) becomes:

$$\frac{\partial}{\partial \theta} (-\omega L_1 + k L_2) - L_3 + \epsilon \frac{\partial}{\partial t} L_1 + \epsilon \frac{\partial}{\partial x} L_2 = 0 \quad 2.20$$

and the corresponding variational requirement is:

$$\frac{\delta}{\delta u} \iint \oint du L(\epsilon u_t - \omega u_\theta, \epsilon u_x + k u_\theta, u) \quad 2.21$$

To lowest order we have;

$$\frac{d}{d\theta} (-\omega L_1^0 + k L_2^0) - L_3^0 = 0 \quad 2.22a$$

$$L^0 = [L]_{\epsilon=0} = L(-\omega u_\theta, k u_\theta, u) \quad 2.22b$$

We recognize (2.22a) as the Euler equation (2.4) satisfied by our homogeneous wave. Although H , ω and k are now slowly varying functions of space and time, our formulation has allowed us to treat them as independent variables, and they are effectively constants. Following the procedure used to reduce the homogeneous wave equation, we perform a Legendre transformation from old variables u , u_θ to new coordinate u and conjugate momentum Π where

$$\Pi = L_{u_\theta}^0 = -\omega L_1^0 + k L_2^0 \quad 2.23$$

so that the energy integral is

$$\Pi u_\theta - L^0 = H \quad 2.24$$

The explicit form of H as a function of its canonical coordinates Π and u can be obtained by replacing u_θ by the appropriate function of Π

in the left hand side of equation (2.24). Also, if (2.24) is solved for u_0 and integrated, we obtain u as a function of θ ; in fact, we obtain the same functional form as for uniform waves, as in this case L and L^0 are identical. Thus we see that our intuitive notion concerning the form of the weakly inhomogeneous waves is substantiated. The averaged Lagrangian now takes the form:

$$\langle L^0 \rangle = \frac{1}{2\pi} \oint du \Pi - H = \frac{1}{2\pi} \oint du \Pi(H, u) - H \quad 2.25$$

Equation (2.25) is not a purely formal expression, for the averaging process can be carried out explicitly, as Π can be found as a function of H and u through inversion of $H = H(\Pi, u)$. We observe that as a consequence of an assumed ordering we were able to solve for u to first order and that this yielded an integration constant H . The addition of a new variable H on the slow time scale requires a new condition to close our system. Again we anticipate the power and compactness of the variational approach by requiring

$$\frac{\delta \langle L^0 \rangle}{\delta H} = \frac{\partial \langle L^0 \rangle}{\partial H} = 0 \quad 2.26$$

This condition closes the system of equations for the slowly varying parameters H , ω and k . Clearly (2.26) will give $\omega = \omega(k, H)$ which is a nonlinear dispersion relation. The other variational requirement is of course

$$\frac{\partial}{\partial \theta} \langle L^0 \rangle = \frac{\partial}{\partial t} \langle L^0_\omega \rangle - \frac{\partial}{\partial x} \langle L^0_k \rangle = 0 \quad 2.27$$

If we prefer to use k and ω as variables rather than θ , we also have the consistency requirements

$$\theta_{xt} - \theta_{tx} = \frac{\partial}{\partial t} k + \frac{\partial}{\partial x} \omega = 0 \quad 2.28$$

The equations (2.28), (2.27) and (2.28) represent the averaged variational principle in its optimal form.

B. Averaging in the optimal principle as a canonical transformation.

The preceding sections summarize Whitham's ideas leading to the optimal form of the variational principle. We now present a derivation of the averaged Lagrangian $\langle L^0 \rangle$ occurring in the optimal form of the principle that illuminates its significance from another viewpoint. Our notion is this: if the averaged Lagrangian $\langle L^0 \rangle$ contains all of the information implicit in L , it should be possible to construct a canonical transformation relating $\langle L^0 \rangle$ directly to L^0 .

We construct a canonical transformation⁷ from original coordinate u and momentum Π to new coordinate q' and momentum p' :

$$\begin{aligned} q &= u & q' &= ? \\ &\Rightarrow & & \\ p &= \Pi & p' &= \frac{1}{2\pi} \oint du \Pi = J \end{aligned} \quad 2.29$$

That is, our new momentum variable p' is the action integral corresponding to our old coordinate and momentum. The transformation is completely specified by a generating function that is a function of the old coordinate and new momentum. Our generating function F must satisfy:

$$q' = \frac{\partial F}{\partial p'} = \frac{\partial F}{\partial J} \quad 2.30a$$

$$p = \Pi = \frac{\partial F}{\partial q} = \frac{\partial F}{\partial u} \quad 2.30b$$

Using (2.30b) we write for F :

$$F = \int du \Pi \quad 2.31$$

We note that $\oint \Pi du / 2\pi = J$ can be inverted to give H as a function of J . Our generating function is to be considered as a function of old coordinate u and new momentum J , so we write:

$$F = \int du \Pi[H(J), u] \quad 2.32$$

Our new Hamiltonian H' is given by

$$H'(J) = H + \frac{\partial F}{\partial \theta} = H \quad 2.33$$

and the angle variable conjugate to J by

$$q' = \frac{\partial F}{\partial J} = \frac{\partial H}{\partial J} \int du \frac{\partial \Pi}{\partial H} \quad 2.34$$

But Hamilton's equations reveal that

$$u_\theta = \frac{\partial H}{\partial \Pi} \quad , \quad \Pi_\theta = -\frac{\partial H}{\partial u} \quad 2.35$$

so that

$$q' = \frac{\partial H}{\partial J} \int d\theta = \frac{\partial H}{\partial J} (\theta + \theta_0) = \theta + \theta_0 \quad 2.36a$$

as

$$\frac{\partial J}{\partial H} = \frac{1}{2\pi} \oint d\theta = 1 \quad 2.36b$$

In summary, we have demonstrated a canonical transformation from old generalized coordinates to new generalized coordinates:

$$\begin{aligned} q &= u & q' &= \theta + \theta_0 \\ p &= \Pi & p' &= \frac{1}{2\pi} \oint du \Pi = J \end{aligned} \quad 2.37$$

with generator

$$F = \int du \Pi [H(J), u] \quad 2.38$$

and transformed Hamiltonian:

$$H'(J) = H(u, \Pi) \quad 2.39$$

Let us now compute the new Lagrangian corresponding to H' :

$$L^{0'} = p' q'_\theta - H' = J - H = \langle L^0 \rangle \quad 2.40$$

That is, $\langle L^0 \rangle$ is in actuality a transformed equivalent of the Lagrangian L^0 in angle-action variables. We have no difficulty writing the total action integral for L^0 as a function of the independent variables x and t , we just integrate over x and t :

$$I = \iint dx dt L^{0'} = \iint dx dt \langle L^0 \rangle = \iint \oint dx dt d\theta L^0 \quad 2.41$$

C. The variational principle for a Lagrangian with a pseudo phase variable.

In the next chapter we will see that the Lagrangian for the cubic nonlinear Schrödinger equation can be written in the form

$$L(u_t, u_x, u; \varphi_t, \varphi_x) \quad 2.42$$

Thus we need a generalization of the preceding formalism to cover the addition of a new dependent variable. It turns out that the needed generalization is nontrivial in the case where only the derivatives of the dependent variable occur in the Lagrangian. First, let us consider what happens to L when we substitute $\varphi = \beta x - \gamma t$ into it. We obtain

$$L(u_t, u_x, u; -\gamma, \beta) \quad 2.43$$

This substitution has reduced the number of dependent variables and will allow us to obtain a class of solutions that will depend on the two constants β and γ . Note that the substitution $\varphi = \beta x - \gamma t$ was successful because φ did not appear in L and so no explicit x, t dependence is introduced into L from the substitution. Now we consider the more usual procedure for treating (2.42). We assume $u = U(\theta)$ and $\varphi = \psi(\theta)$ with $u(\theta) = u(\theta + 2\pi)$ and $\psi(\theta) = \psi(\theta + 2\pi)$ and $\theta = kx - \omega t$. Clearly this substitution is not the most general, as the substitution $u = U(\theta)$, $\varphi = \beta x - \gamma t + \psi(\theta)$ adds no explicit x, t dependence and provides two additional wave parameters. Thus for full generality we write:

$$L(-\omega_{\theta}, k u_{\theta}, u; -\gamma - \omega \Phi_{\theta}, \beta + k \Phi_{\theta}) \quad 2.44$$

The homogeneous waves will satisfy the Euler equations

$$\frac{\partial}{\partial \theta} (-\omega L_1 + k L_2) - L_3 = 0 \quad 2.45a$$

$$\frac{\partial}{\partial \theta} (-\gamma L_4 + \beta L_5) = 0 \quad 2.45b$$

With solutions U^0 and ψ^0 . In analogy with the preceding theory, we can derive equations for the slowly varying parameters of U^0 , ψ^0 which describe the enlarged class of weakly inhomogeneous waves close to U^0 , ψ^0 . We define $k=\theta_x$, $\omega=\theta_t$ and $\beta=\Gamma_x$, $\gamma=-\Gamma_t$ with θ and Γ slowly varying in time and space; furthermore, we have $u=U(\theta, x, t)$ and $\varphi=\Gamma+\psi(\theta, x, t)$ with U and ψ periodic in θ with period 2π . Clearly k, ω, γ and β satisfy the consistency requirements

$$\frac{\partial}{\partial t} k + \frac{\partial}{\partial x} \omega = 0 \quad 2.46$$

$$\frac{\partial}{\partial t} \beta + \frac{\partial}{\partial x} \gamma = 0 \quad 2.47$$

Using the ordering scheme of Section A, the lowest order Lagrangian

is

$$L^0(u, -\omega u_\theta, k u_\theta; -\gamma - \omega \psi_\theta, \beta + k \psi_\theta) \quad 2.48$$

which is identical to L for homogeneous waves where ω, k, γ and β are strictly constants. The variational requirements are

$$\frac{\delta \langle L^0 \rangle}{\delta \Gamma} = \frac{\partial}{\partial t} \langle L_\gamma^0 \rangle - \frac{\partial}{\partial x} \langle L_\beta^0 \rangle = 0 \quad 2.49$$

$$\frac{\delta \langle L^0 \rangle}{\delta \theta} = \frac{\partial}{\partial t} \langle L_\omega^0 \rangle - \frac{\partial}{\partial x} \langle L_k^0 \rangle = 0 \quad 2.50$$

Although the variable Γ enters the formalism in a manner similar to the phase θ , it is not a true phase variable. In the language of the canonical formalism, the system is degenerate with only one true frequency. Whitham calls Γ a *pseudo phase variable*. The variational equations (2.46), (2.47), (2.49) and (2.50) can be cast in optimal form via a Legendre transformation to the Hamiltonian representation as in Section A. We will defer this development to the detailed application of the theory to the CSE in the next chapter.

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Chapter 3

Whitham's averaged variational principle was formally introduced in the preceding chapter. In this chapter we shall apply the principle to the cubic Schrödinger equation. The variational approach can be applied systematically if we adopt the ordering scheme of Chapter II and restrict our study to waves that differ from uniform waves on spatial and temporal scales of $O(\varepsilon^{-1})$ relative to a local uniform wave. With this ordering we can use the variational principle in its optimal form. In the optimal form all averages are computed using explicit knowledge of the uniform wave. Thus our first task, accomplished in Section A, shall be to obtain analytic expressions for uniform waves. The results of Section A give the uniform waves as functions of four parameters. In Section B we use the variational principle to derive the evolution equations for the four uniform wave parameters on the slow temporal and spatial scales. The detailed reduction of these working equations and a discussion of their properties in various limiting cases is deferred to the following chapters.

A. Uniform waves.

1. Reduction to quadrature in the uniform phase variable θ .

By definition, uniform wave solutions of the CSE depend on space and time through a single phase variable $\theta=kx-\omega t$. We obtain this special class of solutions by rewriting the Lagrangian for the CSE in terms of the independent phase variable θ and integrating the corresponding Euler-Lagrange equations. The Lagrangian for the CSE is deduced from the requirement that the variational derivative $\delta\mathcal{L}(E, E^*, E_x, E_x^*, E_t, E_t^*)/\delta E^*$ yield the CSE. We have

$$\frac{\delta\mathcal{L}}{\delta E^*} = \mathcal{L}_{E^*} - (\mathcal{L}_{E_x^*})_x - (\mathcal{L}_{E_t^*})_t = 0 \quad 3.1$$

and require that the RHS (the Euler-Lagrange equation) yield the CSE. The Lagrangian is therefore given by

$$\mathcal{L} = i \frac{E_t E^* - E_t^* E}{2} - p |E_x|^2 + q \frac{|E|^4}{2} \quad 3.2$$

It is convenient to replace the complex phasor E with two real functions A and φ representing the amplitude and phase of E :

$$E = A e^{i\varphi} \quad 3.3$$

In this representation the Lagrangian is

$$\mathcal{L} = -(\varphi_t + p\varphi_x^2)A^2 - pA_x^2 + q\frac{A^4}{2} \quad 3.4$$

Note that $\mathcal{L}$ depends on φ only through the derivatives φ_t and φ_x . Thus, following the discussion of Chapter II, for full generality the uniform waves are parameterized as follows

$$\varphi = \Gamma + \psi(\theta) \quad 3.5a$$

$$A = A(\theta) \quad 3.5b$$

with $\Gamma = \beta x - \gamma t$ and $\theta = kx - \omega t$. Furthermore, we require that the waves be periodic with $A(\theta) = A(\theta + 2\pi)$ and $\psi(\theta) = \psi(\theta + 2\pi)$. We denote the new Lagrangian for the restricted problem of uniform waves with a single independent variable θ by L . Using (3.5a,b) we have

$$L = [\gamma + \omega\psi_\theta - p(\beta + k\psi_\theta)^2]A^2 - pk^2A_\theta^2 + q\frac{A^4}{2} \quad 3.6$$

L is now an ordinary differential expression analogous to the Lagrangian of a conservative mechanical system with coordinates A and ψ and 'time' variable θ . We exploit this analogy and integrate the equations of motion for A and ψ . First we define the momenta conjugate to A and ψ :

$$\Pi = L_{A_\theta} = -2pk^2A_\theta \quad 3.7$$

$$\Omega = L_{\psi_\theta} = [\omega - 2pk(\beta + k\psi_\theta)]A^2 \quad 3.8$$

The Euler-Lagrange equations yield the equation of motion for A and ψ :

$$\frac{\delta L}{\delta A} = L_A - (L_{A_\theta})_\theta = 0 \quad 3.9$$

giving

$$[\gamma + \omega\psi_\theta - p(\beta + k\psi_\theta)^2]A + 2qA^3 + pk^2A_{\theta\theta} = 0 \quad 3.10$$

and

$$\frac{\delta L}{\delta \psi} = L_\psi - (L_{\psi_\theta})_\theta = 0 \quad 3.11$$

yielding

$$[\omega - 2pk(\beta + k\psi_\theta)]A^2 = \Omega, \text{ constant} \quad 3.12$$

Clearly, ψ is an ignorable coordinate so that the conjugate momentum Ω is a constant. Thus we can solve (3.12) for ψ_θ in terms of A.

$$\psi_\theta = \frac{1}{2k} \left(\frac{u}{p} - 2\beta \right) - \frac{\Omega}{2pk^2A^2}, \quad u = \frac{\omega}{k} \quad 3.13$$

and eliminate ψ_θ in equation (3.10) obtaining

$$(\gamma - \beta u + \frac{u^2}{4p})A - \frac{\Omega^2}{4pk^2A^2} + pk^2A_{\theta\theta} + A^3 = 0 \quad 3.14$$

which integrates to

$$(\gamma - \beta u + \frac{u^2}{4p})A^2 + \frac{\Omega^2}{4pk^2A^2} + pk^2A_{\theta}^2 + q\frac{A^4}{2} = D \quad 3.15$$

It is convenient to define an intensity variable and rewrite (3.15) as follows

$$I_{\theta}^2 = \frac{2q}{pk^2} \left[-I^3 + \frac{2}{q}(\beta u - \gamma - \frac{u^2}{4p})I^2 + \frac{2D}{q}I - \frac{\Omega^2}{2qpk^2} \right] \quad 3.16$$

Integration of (3.16) and (3.13) will yield explicit expressions for the general waveforms.

2. Graphical solution for uniform waves.

We can gain qualitative and specific information about the solutions of (3.16) by examining them from a graphical viewpoint. First we note that the RHS of (3.16) is a cubic polynomial in I and can be written

$$I_{\theta}^2 = \frac{2q}{pk^2} (e_1 - I)(e_2 - I)(e_3 - I) = P(I) \quad 3.17$$

with e_1, e_2, e_3 . Equating the RHS of (3.17) and (3.16) yields the following relations

$$e_1 + e_2 + e_3 = -\frac{2}{q}\left(\gamma + \frac{u^2}{4p} - \beta u\right) \quad 3.18a$$

$$e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{2D}{q} \quad 3.18b$$

$$e_1 e_2 e_3 = -\frac{\Omega^2}{2pqk^2} \quad 3.18c$$

Relation (3.18c) introduces a degeneracy when solved for Ω that is conveniently removed by introducing the symbol $\sigma = \pm 1$. Our convention is to write Ω as follows:

$$\Omega = -\sigma k(-2pqe_1 e_2 e_3)^{1/2} \quad 3.19$$

The graphical solution of (3.17) can be obtained by plotting I_θ^2 as the ordinate and I as the abscissa. Physical solutions must satisfy the inequalities $I_\theta^2 \geq 0$ and $I \geq 0$. Also note that I must have minimum and maximum values where $I_\theta = 0$. Thus a solution to (3.17) exists over a positive arc of $P(I)$ between two positive zeros of $P(I)$. Now $P(I)$ is a cubic; if it has two real zeros, it also has a third. We have given these roots the ordering $e_1 \geq e_2 \geq e_3$. If we examine (3.18c) we distinguish two cases:

$$pq > 0: \quad e_1 \geq e_2 \geq e_3 \quad 3.20a$$

$$pq < 0: \quad e_1 > e_2 > e_3 > 0 \quad 3.20b$$

The corresponding graphical solutions are shown in Figure 1a,b. Note that these two cases have different behavior in the limits $e_3 \rightarrow e_2$ for $pq > 0$ and $e_2 \rightarrow e_1$ for the case $pq < 0$. These limiting cases are represented by the dashed curves in Figure 1a,b. In the case $pq > 0$, e_2 and e_3 must satisfy $e_3 < 0 < e_2$ so that two parameters are removed as $e_2 \rightarrow e_1 \rightarrow 0$. In the case $pq < 0$, the limit $e_2 \rightarrow e_1$ removes only one parameter. Both of these limits correspond to isolated pulses or solitary waves and are the limiting wave forms of periodic waves where the period is infinite. We establish this fact analytically in the next sections.

3. Analytic solution for the case $pq > 0$.

The solution of (3.17) for the case $pq > 0$ can be written implicitly as an integral. We write

$$\int_{x_0}^x \frac{dl}{[(e_1 - l)(e_2 - l)(e_3 - l)]^{1/2}} = \left(\frac{2q}{p}\right)^{1/2} (x - ut - x_0) \quad 3.21$$

The graphical solutions of the preceding section are of immediate value because they reveal the correct limits to use in the integral (3.21). For a physical solution $l \in [e_2, e_1]$, with $e_3 < 0 < e_2 < e_1$. Thus we have, setting $x_0 = 0$:

Figure 1. Graphical solution for the intensity I for the case (a) $pq > 0$ and (b) $pq < 0$. The dashed lines are solitary wave constructions.

$$\int_1^{e_1} \frac{dl}{[(e_1-l)(e_2-l)(e_3-l)]^{1/2}} = \left(\frac{2q}{p}\right)^{1/2}(x - ut) \quad 3.22$$

The integral (3.22) is of the elliptic type and can be inverted through the use of Jacobian elliptic functions. To this end we transform (3.22) into canonical form. Let

$$y^2 = \frac{l - e_2}{e_1 - e_2} \quad 3.23$$

With this transformation y ranges over the interval $[0,1]$ as l ranges over the interval $[e_2, e_1]$ and (3.22) becomes

$$x - ut = \left(\frac{2p}{q}\right)^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} \int_y^1 \frac{dy}{[(1-y^2)(my^2+1-m)]^{1/2}} \quad 3.24$$

with parameter m given by

$$m = \frac{e_1 - e_2}{e_1 - e_3} \in [0,1] \quad 3.25$$

The integral on the RHS of equation (3.24) can be expressed in closed form in terms of the incomplete elliptic integral of the first kind ($K(\theta, m)$):

$$K(\cos^{-1}(y), m) = \int_0^{\cos^{-1}(y)} \frac{d\theta}{[1 - m \sin^2 \theta]^{1/2}} =$$

$$\int_0^{\sqrt{1-y^2}} \frac{ds}{[(1-s^2)(1-ms^2)]^{1/2}} = \int_y^1 \frac{dy}{[(1-y^2)(my^2+1-m)]^{1/2}} \quad 3.26$$

with

$$K(m) = K\left(\frac{\pi}{2}, m\right) \quad 3.27$$

where $K(m)$ is the complete elliptic integral of the first kind. Using (3.24), (3.26) and (3.27) we obtain an expression for the period l of the nonlinear wave

$$\frac{l}{2} = \left(\frac{2p}{q}\right)^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} K(m) \quad 3.28$$

The nonlinear wavenumber k is therefore given by

$$k = \frac{2\pi}{l} = \pi \left(\frac{q}{2p}\right)^{1/2} \frac{(e_1 - e_3)^{1/2}}{K(m)} \quad 3.29$$

The inversion of (3.22) or (3.24) for l is facilitated by the following identity for the Jacobian elliptic function $\operatorname{cn}(y|m)$:

$$\operatorname{cn}(K(\cos^{-1}(y), m)|m) = y \quad 3.30$$

which is often written alternatively as

$$w = \int_0^1 \frac{dy}{cn(w|m) [(1-y^2)(my^2+1-m)]^{1/2}} \quad 3.31$$

Using (3.30), (3.23) and (3.24) we solve for I obtaining

$$I = e_2 + (e_1 + e_2)cn^2[(e_1 - e_3)^{1/2}(\frac{q}{2p})^{1/2}(x - ut)|m] \quad 3.32$$

In the theory we regard the velocity u as a fundamental parameter so that the nonlinear frequency ω is given from (3.13) by $\omega = uk$. The parameter β can be related to e_1 , e_2 , e_3 and u using (3.13) and the periodicity property of ψ . As $\psi(\theta)$ is periodic in θ with period 2π , the integral of ψ_θ over this interval must vanish. Thus integrating (3.13) over 2π we obtain:

$$u = \frac{\omega}{k} = 2\beta p + \frac{\Omega}{e_1 \pi} \int_0^{l/2} \frac{dx}{l} \quad 3.33$$

which together with (3.32) gives

$$u = 2\beta p + \frac{\Omega}{e_1 \pi} \left(\frac{2p}{q}\right)^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} \int_0^{K(m)} \frac{dw}{\frac{e_2}{e_1} + (1 - \frac{e_2}{e_1})cn^2(w|m)} \quad 3.34$$

To further reduce (3.34) we use the following identity for the incomplete elliptic integral of the third kind ($\Pi(n; w, m)$):

$$\Pi(n;w,m) = \int_0^w \frac{dw}{1-n+ncn^2(w|m)} \quad 3.35$$

with

$$\Pi(n,m) = \Pi(n;K(m),m) \quad 3.36$$

where $\Pi(n,m)$ is the complete elliptic integral of the third kind. Using (3.35), (3.36) and (3.19), we rewrite (3.34) as follows:

$$2\beta p = u + \sigma \left(\frac{-2pqe_2e_3}{e_1} \right)^{1/2} \frac{\Pi(n,m)}{K(m)} \quad 3.37$$

$$n = \frac{e_1 - e_2}{e_1}, \quad 0 \leq n \leq 1 \quad 3.38$$

The parameter γ can be found in terms of e_1 , e_2 , e_3 and u using (3.18a).

To complete our solution for uniform waves, we express the phase φ in closed form. Using (3.5a) and (3.13) we have

$$\varphi = \frac{u}{2p}x + (\beta u - \frac{u^2}{2p} - \gamma)t - \frac{\Omega}{2pk} \int_0^{x-ut} \frac{dx}{l} \quad 3.39$$

Using (3.18a), (3.32) and (3.35) we further reduce (3.39) to

$$\varphi = \left[\frac{q}{2}(e_1 + e_2 + e_3) - \frac{u^2}{4p} \right] t + \frac{u}{2p}x + \sigma \frac{p}{|p|} \left[\frac{e_2e_3}{e_1(e_3 - e_1)} \right]^{1/2} \Pi(n;m,w) \quad 3.40$$

$$w = \left(\frac{q}{2p}\right)^{1/2} (e_1 - e_3)^{1/2} (x - ut) \quad 3.41$$

In Appendix II we include a list of the limiting expressions of the elliptic integrals used in the modulational theory. Using these expressions we deduce the following simple limits for the cnoidal waves:

$$e_1 - e_2 = \varepsilon \rightarrow 0: \quad I = e_1 + \varepsilon \cos^2 \left[\left(\frac{q}{2p}\right)^{1/2} (e_1 - e_3)^{1/2} (x - ut) \right] \quad 3.42a$$

$$\varphi = (qe_1 - q\frac{e_3}{2} - \frac{u^2}{4p})t + \frac{u}{2p}x + \frac{\sigma}{p} \left(\frac{-pqe_3}{2}\right)^{1/2} (x - ut) \quad 3.42b$$

and

$$e_2, e_3 \rightarrow 0: \quad I = e_1 \operatorname{sech}^2 \left[\left(\frac{e_1 q}{2p}\right)^{1/2} (x - ut) \right] \quad 3.43a$$

$$\varphi = \left(\frac{qe_1}{2} - \frac{u^2}{4p}\right)t + \frac{u}{2p}x \quad 3.43b$$

The limit $e_2, e_3 \rightarrow 0$ gives the solitary waveforms mentioned earlier.

4. Analytic solution for the case $pq < 0$.

Our analysis of the case $pq < 0$ parallels that for $pq > 0$. We again use the graphical solution to establish the correct limits for the integration of (3.17). We obtain

$$\int_1^{e_2} \frac{dl}{[(e_1-l)(e_2-l)(e_3-l)]^{1/2}} = \left| \frac{2q}{p} \right|^{1/2} (x - ut) \quad 3.44$$

with $1 \leq [e_3, e_2]$, $0 \leq e_3 \leq e_2 \leq e_1$. The integral in (3.44) is reduced to canonical form using the transformation

$$y^2 = \frac{e_2 - l}{e_2 - e_3} \quad 3.45$$

and parameter m

$$m = \frac{e_2 - e_3}{e_1 - e_3} \in [0, 1] \quad 3.46$$

yielding

$$x - ut = \left| \frac{2p}{q} \right|^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} \int_y^1 \frac{dy}{[(1-y^2)(m^2 y^2 + 1 - m)]^{1/2}} \quad 3.47$$

The identities (3.26) and (3.27) are used to express the nonlinear wavenumber k in terms of the complete elliptic integral of the first kind

$$k = \pi \left| \frac{q}{2p} \right|^{1/2} \frac{(e_1 - e_3)^{1/2}}{K(m)} \quad 3.48$$

while the indefinite integral (3.47) is inverted using the identity (3.30) giving

$$i = e_2 - (e_2 - e_3)cn^2[(e_1 - e_3)^{1/2} | \frac{q}{2p} |^{1/2} (x - ut) | m] \quad 3.49$$

An expression for β in terms of e_1 , e_2 , e_3 and u is derived from (3.13) exactly as for the case $pq > 0$ in terms of the incomplete elliptic integral of the third kind

$$2\beta = \dots + \left(\frac{-2pq e_1 e_2}{e_3} \right)^{1/2} \frac{\Pi(n, m)}{K(m)} \quad 3.50$$

where the parameter n is now given by

$$n = \frac{e_3 - e_2}{e_3}, \quad n \leq 0 \quad 3.51$$

The expression for the phase φ is deduced from (3.5a), (3.13), (3.19) (3.18a) and (3.35):

$$\varphi = \left[\frac{q}{2}(e_1 + e_2 + e_3) - \frac{u^2}{4p} \right] t + \frac{u}{2p} x + \sigma \frac{p}{|p|} \left(\frac{e_1 e_2}{e_3(e_1 - e_3)} \right)^{1/2} \Pi(n; w, m) \quad 3.52$$

$$w = (e_1 - e_3)^{1/2} | \frac{q}{2p} |^{1/2} (x - ut) \quad 3.53$$

The two simple limits of the cnoidal waves $pq < 0$ are

$$e_2 - e_3 \rightarrow 0: \quad I = e_2 - \varepsilon \cos^2 \left[\left| \frac{q}{2p} \right|^{1/2} (e_1 - e_2)^{1/2} (x - ut) \right] \quad 3.54a$$

$$\varphi = \left[\frac{qe_1}{2} + qe_2 - \frac{u^2}{4p} \right] t + \frac{u}{2p} x + \frac{\sigma}{p} \left| \frac{pqe_1}{2} \right|^{1/2} (x - ut) \quad 3.54b$$

$$e_1 - e_2 \rightarrow 0: \quad I = e_1 - (e_1 - e_3) \operatorname{sech}^2 \left[\left| \frac{q}{2p} \right|^{1/2} (e_1 - e_3)^{1/2} (x - ut) \right] \quad 3.55a$$

$$\begin{aligned} \varphi = & \left[\frac{qe_3}{2} + qe_1 - \frac{u^2}{4p} \right] t + \frac{u}{2p} x \\ & + \frac{\sigma}{p} \left[\left| \frac{pqe_3}{2} \right|^{1/2} (x - ut) + |p| \tanh^{-1} \left[\left(\frac{e_1 - e_3}{e_3} \right)^{1/2} \tanh(w) \right] \right] \end{aligned} \quad 3.55b$$

$$w = (e_1 - e_3)^{1/2} \left| \frac{q}{2p} \right|^{1/2} (x - ut)$$

5. Further discussion of the uniform cnoidal waves.

Our final expressions for the intensity $I=A^2$ and phase of the uniform waves ((3.32), (3.40), (3.49) and (3.52)) are given in terms of the four parameters e_1 , e_2 , e_3 and u . The complete expression for the phasor E is given in terms of A and φ by (3.3). The real part of the complex phasor E for typical cnoidal waves $pq>0$ and $pq<0$ is plotted in Figure 2a,b. In the modulational theory of the cnoidal waves, the parameters e_1 , e_2 , e_3 and u will become dependent variables that vary slowly in space and time.

Figure 2. Uniform cnoidal waves for the case (a) $pq > 0$ and (b) $pq < 0$. $\text{Re}(E) = A \cos \varphi$. The wave envelope is shaded and is bounded by the curves $\pm A$.

Although e_1 , e_2 , e_3 and u are the natural variables for the modulational theory, they are not the most physically appealing variables. An optimal set for labeling a graph of a cnoidal wave field might be β , k , e_1 and e_2 for the case $pq < 0$, or β , k , e_2 and e_3 for the case $pq > 0$. The parameter β has a physical interpretation derived from (3.5a) and the definition $\Gamma = \beta x - \gamma t$:

$$\beta = \langle \varphi_x \rangle, \quad \langle \cdot \rangle = \oint \frac{d\theta}{2\pi} \quad 3.56$$

Thus β is the average local wavenumber of the phasor E and is closely associated with our usual concepts of a local wavenumber for linear waves. However, because our current perspective treats the cnoidal waves in intensity as fundamental, we have chosen to call k the nonlinear wavenumber. The significance of these variables is illustrated in Figure 3a,b. In Figure 3a,b we have purposely exaggerated the scale of change of these parameters in order to give a visualization of the degrees of freedom in the modulational theory to be derived in the next sections.

B. Nonuniform waves; derivation of the modulational equations.

In Section A we derived the uniform cnoidal wave solutions of the CSE and found that they depend on four parameters e_1 , e_2 , e_3 and u . We shall now use the averaged variational principle to obtain the system of closed equations satisfied by e_1 , e_2 , e_3 and u when they are regarded as slowly varying in space and time.

Figure 3. Nonuniform cnoidal waves for the case (a) $pq > 0$ and (b) $pq < 0$. $\text{Re}(E) = A \cos \varphi$. The new curves are the slowly varying parameters of the cnoidal wave.

1. *Parameterization of nonuniform waves.*

Nonuniform waves are parameterized so that a close analogy is maintained to the uniform waves. The uniform phase function $\theta=kx-\omega t$ and the auxiliary function $\Gamma=\beta x-\gamma t$ are generalized to nonuniform functions Γ and θ satisfying

$$\omega = -\theta_t \quad , \quad k = \theta_x \quad 3.57$$

and

$$\gamma = -\Gamma_t \quad , \quad \beta = \Gamma_x \quad 3.58$$

Furthermore, the amplitude A and phase φ are generalized from the expressions (3.5a,b) for uniform waves to the following for nonuniform waves:

$$\varphi = \Gamma(x,t) + \psi(\theta,x,t) \quad 3.59$$

$$A = A(\theta,x,t) \quad 3.60$$

We require that the phase variable θ continue to mediate the oscillatory behavior of A and ψ so that $\psi(\theta,x,t)=\psi(\theta+2\pi,x,t)$ and $A(\theta,x,t)=A(\theta+2\pi,x,t)$. The additional explicit x,t arguments of A and ψ are needed to parameterize their nonperiodic behavior.

The next step in Whitham's procedure is to enlarge the space of independent variables from x and t to x , t and θ . This step is reversible by the process of synchronization described in Chapter II. In order that our extended representation be compatible with the original representation, we must simultaneously redefine the differential operators $\partial/\partial t$ and $\partial/\partial x$ to reflect the original dependence of θ on x and t . Thus we replace these operations according to the following rules:

$$\frac{\partial}{\partial t} \rightarrow -\omega \frac{\partial}{\partial \theta} + \frac{\partial}{\partial t} \quad 3.61$$

$$\frac{\partial}{\partial x} \rightarrow k \frac{\partial}{\partial \theta} + \frac{\partial}{\partial x} \quad 3.62$$

In the following, x , t and θ are treated as independent variables after formulas have been transcribed in accord with the rules (3.61) and (3.62).

We now establish an ordering scheme that defines a weakly nonuniform system. The natural scale length for the modulational theory is the local period of the nonuniform waves, l . The length l is assumed $O(\varepsilon^0)$ so that $\theta_x = k \sim 2\pi/l \sim O(\varepsilon^0)$. We require ω , γ and β to be of the same order; that is, we require that the partial differential operators $\partial/\partial t$ and $\partial/\partial x$ yield $O(\varepsilon^0)$ quantities when applied to θ and Γ . To complete our ordering scheme we assume that the operators $\partial/\partial t$ and $\partial/\partial x$ yield quantities $O(\varepsilon)$ when applied to the remaining variables A and ψ .

while the operator $\partial/\partial\theta$ is to yield quantities of $O(\varepsilon^0)$ when applied to these same variables. To indicate this ordering in a convenient manner, we rewrite the expressions (3.61) and (3.62) using the order parameter ε explicitly:

$$\frac{\partial}{\partial t} \rightarrow -\omega \frac{\partial}{\partial \theta} + \varepsilon \frac{\partial}{\partial t} \quad 3.63$$

$$\frac{\partial}{\partial x} \rightarrow k \frac{\partial}{\partial \theta} + \varepsilon \frac{\partial}{\partial x} \quad 3.64$$

Using (3.59) and (3.60) together with (3.63) and (3.64) we can write down the Lagrangian which is the generalization of (3.6):

$$L = [\gamma + \omega\psi_\theta - \varepsilon\psi_t - p(\beta + k\psi_\theta + \varepsilon\psi_x)^2]A^2 - p(kA_\theta^2 + \varepsilon A_x)^2 + \frac{A^4}{q_2^2} \quad 3.65$$

Weak nonuniformity is formally imposed by setting $\varepsilon=0$ in (3.65). The resulting Lagrangian is written L^0 and is superficially identical to the Lagrangian (3.7) for a uniform wave. However, L^0 is now a functional of θ and Γ as well as A and ψ . The Euler-Lagrange equations $\delta L^0/\delta A$ and $\delta L^0/\delta \psi$ result in precisely the same equations ((3.10) and (3.12)) as in Section A. As k , ω , β and γ are functions of the independent variables x and t , they are held constant during differentiation and integration with respect to the variable θ . Thus all the calculations of Section A for uniform waves carry over to weakly nonuniform waves. In particular, the exact analytic expressions for cnoidal waves ((3.32)

and (3.49)) now apply to nonuniform waves. Clearly the nonuniformity arises through the slowly varying parameters e_1 , e_2 , e_3 and u . Whitham's averaged variational principle provides additional conditions needed to synchronize the parameters e_1 , e_2 , e_3 and u . The Lagrangian L^0 is first averaged over a period of θ yielding the functional $\langle L^0 \rangle$. The synchronization conditions are the Euler-Lagrange equations corresponding to variation of $\langle L^0 \rangle$ with respect to θ and Γ ; that is, the requirement that $\delta \langle L^0 \rangle / \delta \theta$ and $\delta \langle L^0 \rangle / \delta \Gamma$ vanish. We defer the imposition of these conditions until a closed form for $\langle L^0 \rangle$ has been derived. In the next section we shall derive a simple expression for $\langle L^0 \rangle$ that makes optimal use of the integrals of the Euler-Lagrange equations $\delta L^0 / \delta A$ and $\delta L^0 / \delta \psi = 0$. This is Whitham's optimal procedure. In Section B.3 we shall show that Whitham's optimal procedure is equivalent to a canonical transformation in which the variables A and ψ are replaced by nonoscillatory variables. We shall show that $\langle L^0 \rangle$ is fully equivalent to L^0 under the canonical transformation.

2. The optimal form of $\langle L^0 \rangle$.

If we regard L^0 as a function of independent variable θ only, then L^0 is identical to L and we can use the results of Section A. As θ is not an explicit argument of L^0 , the Hamiltonian function H^0 is a constant (in our present context this means that H^0 is a function of the independent variables x and t only). We construct the Hamiltonian H^0 using the appropriate Legendre transformation and the definitions (3.10) and (3.12) of the conjugate momenta to A and ψ .

$$H^0 = \Pi A_\theta + \Omega \psi_\theta - L^0 = \frac{\Omega}{2kp} (u - 2\beta) - D \quad 3.67$$

where we have used the integral (3.15) and our expression (3.66) for L^0 .

Solving for L^0 we obtain

$$L^0 = \Pi A_\theta + \Omega \psi_\theta - \frac{\Omega}{2kp} (u - 2\beta p) + D \quad 3.68$$

This form for L^0 is particularly simple to average, we have

$$\langle L^0 \rangle = \oint \frac{\Pi dA}{2\pi} - \frac{\Omega}{2kp} (u - 2\beta p) + D \quad 3.69$$

The integral on the RHS of (3.69) is the action variable corresponding to the periodic coordinate A and conjugate momentum Π . We write

$$J = \oint \frac{\Pi dA}{2\pi} = -2kW \quad 3.70$$

with

$$\begin{aligned} W &= \frac{P}{4\pi} \oint \frac{dI}{I} \left[\left(\frac{q}{2p} \right) (-I^3 + \frac{2}{q} (\beta u - \gamma - \frac{u^2}{4p}) I^2 + \frac{2D}{q} I - \frac{\Omega^2}{2pqk^2}) \right]^{1/2} \\ &= \frac{P}{4\pi} \oint \frac{dI}{I} \left[(e_1 - I)(e_2 - I)(e_3 - I) \left(\frac{q}{2p} \right) \right]^{1/2} \end{aligned} \quad 3.71$$

so that $\langle L^0 \rangle$ is written

$$\langle L^0 \rangle = D + \frac{\Omega}{2kp} (2\beta p - u) - 2kW \quad 3.72$$

The function W is a particularly useful generating function from which other important averages in the modulational theory may be generated by differentiation. W and its derivatives are related to complete elliptic integrals in Appendix II.

Our Lagrangian L^0 has been transformed to Whitham's optimal form which makes use of the readily obtained integrals to express the averaged Lagrangian in closed form. As a result we have two new functions of x and t to synchronize, namely the integration 'constants' D and Ω . We synchronize D and Ω by requiring that the variational derivative of D and Ω vanish. We have

$$\frac{\delta \langle L^0 \rangle}{\delta D} = 1 - 2kW_D = 0 \quad 3.73$$

and

$$\frac{\delta \langle L^0 \rangle}{\delta \Omega} = \frac{1}{2kp} (2\beta p - u) - 2kW_\Omega = 0 \quad 3.74$$

The derivatives W_D and W_Ω are expressed in terms of complete elliptic integrals in Appendix II. It is easily verified that (3.73) is identical to our earlier derived expressions for the nonlinear wavenumber k , (3.29) and (3.48). Similarly, (3.74) is identical to the

earlier derived expressions (3.37) and (3.50). We shall refer to (3.74) as the nonlinear dispersion relation.

3. The canonical transformation $L^0 \rightarrow \langle L^0 \rangle$.

The analysis of this section is not intended as a replacement for Whitham's optimal averaging procedure. Rather, our intent is to clarify the significance of averaging by showing that it is equivalent to a canonical transformation. In the optimal procedure we used information contained in the Euler-Lagrange equations $\delta L^0 / \delta A = 0$ and $\delta L^0 / \delta \psi = 0$ to integrate A and ψ and to derive an averaging prescription for L^0 . In mechanics there exists a canonical method for eliminating oscillatory coordinates such as A and ψ from the Hamiltonian H^0 . A canonical transformation is constructed to action-angle variables corresponding to A and ψ . This transformation induces a new representation of the Lagrangian. In our case the resulting Lagrangian turns out to be precisely $\langle L^0 \rangle$. The procedure is an alternative explanation of the shift in relevance from L^0 to $\langle L^0 \rangle$. Loosely speaking, the high frequency information is extracted from L^0 and then used to renormalize L^0 to the appropriate functional $\langle L^0 \rangle$ of the remaining low frequency variables.

We desire a canonical transformation from old coordinates and momenta to new coordinates and momenta:

$$\begin{aligned}
 q_1 &= A & a'_1 &= ? \\
 p_1 &= \Pi & p'_1 &= J = \oint \Pi dA / 2\pi \\
 q_2 &= \psi & q'_2 &= ? \\
 p_2 &= \Omega & p'_2 &= \Omega
 \end{aligned}
 \tag{3.75}$$

The generating function F for the transformation is a function of old coordinates and new momenta and satisfies the following relations

$$F_A = \Pi, \quad F_\psi = \Omega \tag{3.76a}$$

$$F_\Omega = q'_2, \quad F_J = q'_1 \tag{3.76b}$$

The relations (3.76a) allow us to express F as an integral

$$F = \int_0^A \Pi dA + \Omega \psi \tag{3.77}$$

In order to compute q'_1 and q'_2 we must find F and therefore Π as a function of J and Ω . This can be done using the integral (3.15) in differential form. We first substitute Π for $-2pk^2 A_\theta$ in (3.15) and then form the differential

$$\frac{\Pi d\Pi}{2pk^2} = [D_\Omega|_J - \frac{\Omega}{2pk^2 A^2}]d\Omega + D_J|_\Omega dJ \tag{3.78}$$

The partial derivatives $\Pi_\Omega|_J$ and $\Pi_J|_\Omega$ are easily obtained from (3.78). Eliminating Π using $-2pk^2 A_\theta$, we have

$$\Pi_{\Omega}|_J = \frac{\frac{\Omega}{2pk^2 A^2} - D_{\Omega}|_J}{A_{\theta}} \quad 3.79a$$

$$\Pi_J|_{\Omega} = -\frac{D_J|_{\Omega}}{A_{\theta}} \quad 3.79b$$

Both of these expressions can be further simplified. First we rewrite (3.79a) using

$$J_D|_{\Omega} d\Omega + J_{\Omega}|_D dJ = dJ = 0 \quad 3.80$$

to obtain

$$D_{\Omega}|_J = -\frac{J_{\Omega}}{J_D} = -\frac{W_{\Omega}}{W_D} = \frac{1}{2kp} (u - 2\beta p) \quad 3.81$$

where we have used the relations (3.73) and (3.74). Using (3.81) and (3.13) we rewrite (3.79a) as

$$\Pi_{\Omega}|_J = -\frac{\psi_{\theta}}{A_{\theta}} \quad 3.82$$

The remaining expression (3.79b) is reduced using (3.73) and (3.70), i.e. $J_D|_{\Omega} = 1/D_J|_{\Omega} = -1$:

$$\Pi_J|_{\Omega} = \frac{1}{A_{\theta}} \quad 3.83$$

The new coordinates are now easily computed using (3.79a,b), (3.82) and (3.83):

$$q_1' = \int \Pi_J|_{\Omega} dA = \int d\theta = \theta + \theta_0 \quad 3.84a$$

$$q_2' = \int \Pi_{\Omega}|_J dA + \psi = - \int \psi_{\theta} d\theta + \psi = \psi_0 \quad 3.84b$$

where θ_0 and ψ_0 are constants. The complete canonical transformation is thus:

$$\begin{aligned} q_1 &= A & q_1' &= \theta + \theta_0 \\ p_1 &= \Pi & p_1' &= J = \oint \Pi dA / 2\pi \\ q_2 &= \psi & q_2' &= \psi_0 \\ p_2 &= \Omega & p_2' &= \Omega \end{aligned} \quad 3.85$$

The transformed Lagrangian is

$$\begin{aligned} L^{0'} &= q_1' p_1' + q_2' p_2' - H^{0'} = J - H^{0'} = J - H^0 \\ &= J + D - \frac{\Omega}{2kp} (u - 2\beta p) = \langle L^0 \rangle \end{aligned} \quad 3.86$$

by virtue of the numerical equality of the transformed Hamiltonian $H^{0'}$ and H^0 under the canonical transformation. Thus we have established the

equivalence of L^0 and $\langle L^0 \rangle$; the two are related by a canonical transformation to action-angle variables.

4. *The modulational equations for the variables $\omega, k, \gamma, \beta, \Omega$ and D .*

The first modulational equation is derived from the Euler-Lagrange equation corresponding to variation of $\langle L^0 \rangle$ with respect to Γ . We have

$$-\frac{\delta \langle L^0 \rangle}{\delta \Gamma} = \langle L_\gamma^0 \rangle_t - \langle L_\beta^0 \rangle_x = 0 \quad 3.87$$

and the consistency requirement

$$\beta_t + \gamma_x = 0 \quad 3.88$$

Using (3.72) and (3.71) we have

$$\langle L_\gamma^0 \rangle = -2kW_\gamma \quad 3.89$$

$$\langle L_\beta^0 \rangle = \frac{\Omega}{k} - 2kW_\beta = \frac{\Omega}{k} + 2\omega W_\gamma \quad 3.90$$

so that (3.87) can be written

$$\frac{\partial}{\partial t}(2kW_\gamma) + \frac{\partial}{\partial x}(2\omega W_\gamma + \frac{\Omega}{k}) = 0 \quad 3.91$$

The second modulational equation is derived similarly from the

Euler-Lagrange equation corresponding to the variation of $\langle L^0 \rangle$ with respect to θ . We have

$$-\frac{\delta \langle L^0 \rangle}{\delta \theta} = \langle L_\omega^0 \rangle_t - \langle L_k^0 \rangle_x = 0 \quad 3.92$$

and the consistency requirement

$$k_t + \omega_x = 0 \quad 3.93$$

It is convenient to replace (3.92) by an equivalent conservation law. The reason is that a cancellation of terms is facilitated. The appropriate conservation law is that for momentum and can be derived from Noether's theorem and the requirement that the action $\langle L^0 \rangle dx \wedge dt$ remain invariant under displacements of the origin of the coordinate x . We have

$$\frac{\partial}{\partial t} [k \langle L_\omega^0 \rangle + \beta \langle L_\gamma^0 \rangle] + \frac{\partial}{\partial x} [\langle L^0 \rangle - k \langle L_k^0 \rangle - \beta \langle L_\beta^0 \rangle] = 0 \quad 3.94$$

It is easy to verify that (3.94) is equivalent to (3.92). The proof is as follows. We first expand (3.94) and obtain

$$\begin{aligned} & \beta [\langle L_\gamma^0 \rangle_t - \langle L_\beta^0 \rangle_x] + \langle L_\gamma^0 \rangle [\beta_t + \gamma_x] \\ & + k [\langle L_\omega^0 \rangle_t - \langle L_k^0 \rangle_x] + \langle L_\omega^0 \rangle [k_t + \omega_x] \\ & - \langle L_\beta^0 \rangle \beta_x - \langle L_\gamma^0 \rangle \gamma_x - \langle L_k^0 \rangle k_x - \langle L_\omega^0 \rangle \omega_x + \langle L^0 \rangle_x = 0 \end{aligned} \quad 3.95$$

The last line cancels identically. The remaining bracketed terms are

our four modulational equations. If three of them hold, then (3.94) implies that the fourth holds as well. Thus we may use (3.94) in place of (3.92). The explicit expression for $\langle L_\omega^0 \rangle$ is

$$\langle L_\omega^0 \rangle = -\frac{\Omega}{2pk^2} - 2kW_\omega = -\frac{\Omega}{2pk^2} + (2\beta - \frac{u}{p})W_\gamma \quad 3.96$$

and for $\langle L_k^0 \rangle$

$$\langle L_k^0 \rangle = \frac{\Omega}{k^2}(\frac{u}{p} - \beta) - 2W - 2kW_k \quad 3.97$$

but

$$W_k = -\frac{\Omega}{k}W_\Omega + \frac{u}{2k}(2\beta - \frac{u}{p})W_\gamma \quad 3.98$$

so that

$$\langle L_k^0 \rangle = \frac{\Omega}{k^2}(\frac{u}{p} - \beta) - 2W + 2\Omega W_\Omega + u(\frac{u}{p} - 2\beta)W_\gamma \quad 3.99$$

Combining (3.96) and (3.89) we have

$$k\langle L_\omega^0 \rangle + \beta\langle L_\gamma^0 \rangle = -\frac{1}{p}(\frac{\Omega}{2k} + \omega W_\gamma) \quad 3.100$$

and using (3.99) and (3.90) we have

$$\langle L^0 \rangle - k \langle L_k^0 \rangle - \beta \langle L_\beta^0 \rangle = D - \frac{3\Omega_1}{2pk} + \frac{\Omega_1^2}{k} - 2k\omega W_\Omega - \frac{ku^2}{p} W_\gamma \quad 3.101$$

W_Ω is eliminated using (3.74) yielding

$$\langle L^0 \rangle - k \langle L_k^0 \rangle - \beta \langle L_\beta^0 \rangle = D - \frac{\Omega_1}{pk} - \frac{ku^2}{p} W_\gamma \quad 3.102$$

so that the final modulational equation is

$$\frac{\partial}{\partial t} \left(\frac{\Omega}{2k} + \omega W_\gamma \right) + \frac{\partial}{\partial x} \left(-pD + \frac{\Omega_1}{k} + ku^2 W_\gamma \right) \quad 3.103$$

5. The modulational equations for the variables e_1 , e_2 , e_3 and u .

Our closed system consists of the four evolution equations (3.88), (3.91), (3.93), (3.103) and the two constraints (3.74), (3.73). If we use the alternative variable

$$E = - \frac{\Omega}{2k} \quad 3.104$$

then the nonlinear dispersion relation (3.74) can be written

$$2\beta p = u - p \frac{W_E}{W_D} \quad 3.105$$

Using the variable E and the relations (3.105) and (3.73) in (3.88),

(3.91), (3.93) and (3.103) we obtain a system of four quasilinear equations in four unknowns.

$$\frac{\partial}{\partial t} \left(\frac{W_\gamma}{W_D} \right) + \frac{\partial}{\partial x} \left(u \frac{W_\gamma}{W_D} - 2E \right) = 0 \quad 3.106$$

$$\frac{\partial}{\partial t} \left(u - p \frac{W_E}{W_D} \right) + \frac{\partial}{\partial x} (2p\gamma) = 0 \quad 3.107$$

$$\frac{\partial}{\partial t} \left(E - \frac{u}{2} \frac{W_\gamma}{W_D} \right) + \frac{\partial}{\partial x} \left(pD + 2uE - \frac{u^2}{2} \frac{W_\gamma}{W_D} \right) = 0 \quad 3.108$$

$$\frac{\partial}{\partial t} \left(\frac{1}{W_D} \right) + \frac{\partial}{\partial x} \left(\frac{u}{W_D} \right) = 0 \quad 3.109$$

We now transform this system from the variables γ , D , E and u to the variables e_1 , e_2 , e_3 and u ; where e_1 , e_2 , e_3 are the roots of the polynomial (3.17) here rewritten using E :

$$(1-e_1)(1-e_2)(1-e_3) = I^3 + \frac{2}{q}(\gamma + \frac{u^2}{4p} - \beta u)I^2 - \frac{2D}{q}I + \frac{2E^2}{pq} = 0 \quad 3.110$$

so that

$$2\gamma = \frac{u^2}{4p} - u \frac{W_E}{W_D} - q(e_1 + e_2 + e_3) \quad 3.111a$$

$$2D = -q(e_1e_2 + e_2e_3 + e_3e_1) \quad 3.111b$$

$$2E = \sigma(-2pqe_1e_2e_3)^{1/2} \quad 3.111c$$

Recall that for $pq > 0$, $e_1 \geq e_2 \geq 0 \geq e_3$; and for $pq < 0$, $e_1 \geq e_2 \geq e_3 \geq 0$ so that $-pqe_1e_2e_3$ is always greater than zero, so that E is real. Also, using (3.105) we have eliminated β in (3.111a). With these expressions we can further reduce our modulational equations. Our final system is:

$$\frac{D}{Dt} W_y = \sigma W_D \frac{\partial}{\partial x} (-2pqe_1e_2e_3)^{1/2} \quad 3.112$$

$$\frac{Du}{Dt} - p \frac{D}{Dt} \frac{W_F}{W_D} - p \frac{W_F}{W_D} \frac{\partial u}{\partial x} - pq \frac{\partial}{\partial x} (e_1 + e_2 + e_3) = 0 \quad 3.113$$

$$\begin{aligned} \frac{D}{Dt} (\sigma(-2pqe_1e_2e_3)^{1/2} + 2\sigma(-2pqe_1e_2e_3)^{1/2} \frac{\partial u}{\partial x} \\ - pq \frac{\partial}{\partial x} (e_1e_2 + e_2e_3 + e_3e_1) - \frac{W_F}{W_D} \frac{Du}{Dt}) = 0 \end{aligned} \quad 3.114$$

$$\frac{D}{Dt} W_D = W_D \frac{\partial u}{\partial x} \quad 3.115$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \quad 3.116$$

Explicit expressions for W_γ , W_D , and W_E in terms of complete elliptic integrals and as functions of e_1 , e_2 and e_3 are given in Appendix II. Our next task will be to solve this system. We will do so in several limits and then for the general case over the full range of the governing parameters.

References

- 1 H. Goldstein, Classical Mechanics, (Addison-Wesley, Reading MA, 1950), pp. 288-299.
- 2 Courant and Hilbert, Methods of Mathematical Physics, (Wiley, New York, 1962), pp. 262-268.

Chapter 4

In Chapter 3 we derived the uniform cnoidal waves of the cubic Schrödinger equation. These waves were completely specified, apart from phase constants, by the values of four parameters. Whitham's averaged variational principle was then used to derive a system of equations for the parameters of the wave when they vary slowly in space and time. In this way we obtained a system of evolution equations for weakly modulated cnoidal waves. We refer to this system as the *modulational equations*.

The stability of perturbed cnoidal waves is the single most important fact to determine from the modulational theory. Of additional interest are the characteristic velocities of propagation and physical nature of traveling long wavelength perturbations of a uniform cnoidal wave in the case where the motion is stable. We shall begin to extract this information from the theory in this chapter.

We start in Section A with a review of the mathematical properties of quasilinear systems of partial differential equations that are

applicable to the modulational equations. In Sections B and C we apply the tools and concepts introduced in Section A to the limiting case where the cnoidal wave has become an infinitesimal sinusoidal ripple on an otherwise uniform intensity carrier. The general case will be examined in the next chapter.

A. *Determined quasilinear systems and hyperbolicity'.*

The system of modulational equations (3.112) through (3.115) can be classified as hyperbolic or elliptic in accord with standard methods for partial differential equations. We will outline these procedures in this section and apply them in the remainder of this chapter. The first step is to reduce our modulational equations to standard form by carrying through all differentiations with respect to the dependent variables. The result of this operation is a system that can be written in matrix form:

$$A\dot{\Psi}_t + B\Psi_x = 0 \quad 4.1$$

A and B are four by four matrices and $\Psi=(e_1, e_2, e_3, u)$ is a column vector composed of the dependent variables. The system (4.1) is quasilinear; i.e. linear in the derivatives and nonlinear in the dependent variables. In general (4.1) contains four independent relations between the four dependent variables (in which case the system is also determined). We begin our analysis of equation (4.1) by showing that under certain conditions (4.1) uniquely determines a set of curves in

the x,t plane that are directly associated with intuitive notions of wave motion. These curves, provided they exist, are the *characteristics* of the partial differential equation and are determined by a system of ordinary differential equations defined by a simple eigenvalue problem. We anticipate this result by representing the characteristic curves parametrically by the following ordinary differential equations:

$$\frac{dx}{d\xi} = \xi(\Psi) \quad 4.2a$$

$$\frac{dt}{d\xi} = \tau(\Psi) \quad 4.2b$$

The functions ξ and τ are defined by the following eigenvalue problem for left eigenvectors L and m :

$$LA = \tau m, \quad LB = \xi m \quad 4.3$$

so that

$$L(\xi A - \tau B) = 0 \quad 4.4$$

The left eigenvalue problem (4.4) is equivalent to (4.3) and determines eigenvectors L^k and eigenfunctions ξ^k and τ^k with a unique ratio $v^k = \xi^k / \tau^k$ for $k=1,4$. Upon re-examination of our parametric equations (4.2) we see that it is necessary to specify four curves with

individual parameters $\tau^k=1,4$. The functions ξ^k and τ^k are the solutions of the secular equation corresponding to (4.4)

$$\text{Det}(\xi A - \tau B) = 0 \quad 4.5$$

The significance of these curves is clearest when Ψ represents a simple wave, that is, if we assume that Ψ depends on x and t through a single phase variable $\theta(x,t)$. In this case (4.1) becomes

$$(A\theta_t + B\theta_x)\Psi_\theta = 0 \quad 4.6$$

Equation (4.6) defines a right eigenvalue problem with right eigenvector R ,

$$(A\theta_t + B\theta_x)R = 0, \quad \Psi_\theta = R \quad 4.7$$

The corresponding eigenvalues θ_t^k and θ_x^k for $k=1,4$ are thus solutions of the following secular equation:

$$\text{Det}(A\theta_t + B\theta_x) = 0 \quad 4.8$$

which is similar to (4.5). If we compare (4.5) and (4.8) we see that θ_t^k, θ_x^k and ξ^k, τ^k must be related. They can differ in general by a common multiplicative factor but their ratio must be the same. We have

$$\frac{dx}{dt} = \frac{\xi^k}{\tau^k} = -\frac{\theta_t^k}{\theta_x^k} = v^k \quad 4.9$$

or alternatively

$$\xi^k \theta_x^k + \tau^k \theta_t^k = \frac{d\theta^k}{d\varepsilon^k} = 0 \quad 4.10$$

and conclude that the simple wave phase function θ^k is constant on the characteristic curves of the differential equation (4.1).

For a simple wave it is easy to solve (4.1). Solution of the eigenvalue problem (4.7) gives R^k . To obtain Ψ we solve the following system of ordinary differential equations:

$$\frac{d\Psi}{d\theta^k} = R^k[\Psi(\theta^k)] \quad 4.11$$

Once (4.11) is integrated, we know Ψ as a function of θ^k . But θ^k is a constant on characteristic curves given by

$$\frac{dx}{dt} = v^k(\theta^k) \quad 4.12$$

Thus knowledge of Ψ at one point on the curve (4.12) gives Ψ at all points on the curve. Also, the curves θ^k are straight lines with constant slopes given by v^k . Thus for simple waves we can advance Ψ away from a boundary curve $t=0$ on the characteristic θ^k and into the neighborhood of the boundary curve provided the characteristic in question crosses the boundary curve, a condition that is assured if $\tau^k \neq 0$. *This direct local causal relationship (actually the identity relation for a simple wave) of Ψ to neighboring points on a characteristic curve corresponds to our intuitive notion of wave*

propagation and information flow. Armed with this insight we examine the general case described by (4.1) and look for local causal relationships that will allow us to construct Ψ from knowledge of Ψ on a neighboring boundary curve. We operate on (4.1) with left eigenvectors L^k

$$L^k(A\Psi_t + B\Psi_x) = m^k \cdot (\tau^k \Psi_t + \xi^k \Psi_x) = m^k \cdot D^k \Psi^k = 0 \quad 4.13$$

where we use the following notation for the directional derivative along a characteristic curve

$$D^k = \xi^k \frac{\partial}{\partial x} + \tau^k \frac{\partial}{\partial t} \quad 4.14$$

If the left eigenvectors L^k are linearly independent, and if the characteristic velocities v^k are real, then (4.13) represents four independent relations that specify Ψ uniquely in a neighborhood of a boundary curve in terms of the value of Ψ on neighboring points of the boundary curve, provided that all of the characteristic curves cross the boundary and pass into the neighborhood. Proofs of these assertions and further discussion are given in standard treatises².

A hyperbolic system is defined to conform to the notions introduced above. System (4.1) is hyperbolic if there exist four linearly independent left eigenvectors L^k and if the eigenfunctions ξ^k , τ^k are real with $(\xi^k)^2 + (\tau^k)^2 \neq 0$.

A considerable simplification occurs for the general case if, in addition to satisfying the conditions for hyperbolicity, there exist scalar functions λ^k and A^k such that

$$m^k \cdot D^k \Psi^k = \lambda^k D^k A^k = 0 \quad 4.15$$

Under these circumstances the functions A^k are constant on their respective characteristic curves. The A^k can be used as new dependent variables and the original system (4.1) is fully equivalent to a coupled set of ordinary differential equations, namely

$$\frac{dA^k}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = v^k[A^1, A^2, A^3, A^4] \quad 4.16$$

The functions A^k are called *Riemann invariants*.

Note that the assumption of a simple wave solution results in the same eigenvalue problem for the characteristic velocities as does the general prescription for determining hyperbolicity. Thus in the sequel we will test for hyperbolicity by making the simple wave ansatz and checking for real eigenvelocities. The existence of distinct real velocities is a sufficient condition for linear independence of the associated eigenvectors L^k .

We note that all of the above analysis centered upon the local behavior of the system (4.1). There are problems that arise when considering the global evolution of Ψ away from specific initial data on a boundary curve. The problem is most easily demonstrated for a

simple wave (for which characteristic curves are straight lines). For simplicity we assume that we have initial data smoothly distributed on a boundary curve $t=0$. For regions of the boundary curve where $\partial v^k / \partial x$ is less than zero the characteristic curves must necessarily cross at the loci of points in the region of the x, t plane $t > 0$. As a point of crossing is approached, the monotonic phase function θ^k must necessarily steepen so that $\partial \theta^k / \partial x \rightarrow \infty$. The phase function θ^k can be chosen as $\theta^k = x - v^k t$ so that differences in phase on two characteristic curves at a particular time is

$$\Delta \theta^k = \Delta x - \Delta v^k t \quad 4.17$$

In the limit $\Delta \theta^k \rightarrow 0$

$$1 = \frac{\partial x}{\partial \theta^k} - \frac{\partial v^k}{\partial \theta^k} t \quad 4.18$$

At a point of intersection $\partial x / \partial \theta^k = 0$ and we obtain

$$t_b = - \frac{\partial \theta^k}{\partial v^k} = - \frac{\partial x}{\partial v^k} \Big|_{t=0} \quad 4.19$$

The time $t_b > 0$ is defined for each characteristic curve where $\partial x / \partial v^k < 0$ on the boundary curve and is the moment in the future when the simple wave develops infinite slope in the neighborhood of the characteristic. This phenomena is commonly referred to as wave breaking. Thus the minimum value of t_b over the entire boundary curve $t=0$ gives a time limit beyond which our simple wave is not defined. We can expect

similar behavior for a general wave. For our modulational theory the occurrence of ∞ derivatives in the dependent variables violates our fundamental assumption that the scale length for the derivatives of the dependent variables e_1, e_2, e_3 and u be much greater than the period $2\pi/k$ of the cnoidal wave.

If some of the eigenvelocities are complex, then they must be so in conjugate pairs, and the system (4.1) is elliptic. This corresponds to instability for the associated linearized problem. That is, if the system (4.1) is linearized by assigning constant values to the entries in the matrices A and B , and the resulting system is Fourier analyzed (via the replacements $\partial/\partial t \rightarrow i\omega$, $\partial/\partial x \rightarrow ik$), the identical eigenvalue problem is obtained as for λ , i.e. $\lambda = \omega/k$, only with the parameters e_1 , e_2 , e_3 and u constants evaluated at a particular point of space and time. Thus the frequency corresponding to an infinitesimal modulation of wavelength k is given by $\omega = \lambda k$. Complex values for λ will clearly give exponentially growing or decaying modes.

B. Infinitesimal waves on a uniform nonlinear carrier; the limit $e_2 \neq e_3$ for the (hyperbolic) case $pq < 0$.

1. The modulational equations in the limit $e_2 \neq e_3$.

Local cnoidal waves for the case $pq < 0$ are given by (3.48). In the limit $\delta = e_2 - e_3 \rightarrow 0$ the local wavenumber $k \rightarrow (e_1 - e_2)^{1/2} |q/2p|^{1/2}$ and the cnoidal wave degenerates to

$$I = e_2 - \delta \cos^2(kx - \omega t) \quad 4.20$$

where we have used (3.53a). The parameters e_2 , k and ω vary slowly in space and time in accord with the modulational equations (3.112) through (3.115). We can evaluate the modulational equations in the limit $\delta \rightarrow 0$ by using the limiting forms of the functions W_D , W_E and W_γ occurring in (3.112) through (3.115). In Appendix II these functions are expressed in terms of complete elliptic integrals of the first, second and third kind; denoted respectively K , E and Π . The required expressions are

$$W_D = \frac{1}{\pi} \left| \frac{p}{2q} \right|^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} K(m) \quad 4.21a$$

$$W_\gamma = \frac{1}{\pi} \left| \frac{p}{2q} \right|^{1/2} (e_1 - e_3)^{1/2} \left[E(m) - \frac{e_1}{e_1 - e_3} K(m) \right] \quad 4.21b$$

$$W_E = \frac{\sigma |q|}{\pi q} \left(\frac{e_1 e_2}{e_3} \right)^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} \Pi(n, m) \quad 4.21c$$

$$m = \frac{e_2 - e_3}{e_1 - e_3} \quad n = \frac{e_3 - e_2}{e_3} \quad 4.21d$$

Using these expressions the modulational equations (3.112) through (3.115) become:

$$\frac{D}{Dt}[(e_1 - e_2)^{1/2} E - \frac{e_1}{(e_1 - e_3)^{1/2}} K] = \frac{\sigma K}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} (-2pq e_1 e_2 e_3)^{1/2} \quad 4.22$$

$$\begin{aligned} \frac{Du}{Dt} + \sigma |pq|^{1/2} \frac{D}{Dt} \left(\frac{2e_1 e_2}{e_3} \right)^{1/2} \frac{\Pi}{K} + \sigma |pq|^{1/2} \left(\frac{2e_1 e_2}{e_3} \right)^{1/2} \frac{\Pi}{K} \frac{\partial u}{\partial x} \\ - pq \frac{\partial}{\partial x} (e_1 + e_2 + e_3) = 0 \end{aligned} \quad 4.23$$

$$\begin{aligned} \frac{D}{Dt} \sigma^{1/2} \frac{\partial u}{\partial x} + 2\sigma^{1/2} \frac{\partial u}{\partial x} - pq \frac{\partial}{\partial x} (e_1 e_2 + e_2 e_3 + e_3 e_1) \\ + [e_1 - (e_1 - e_3) \frac{E}{K}] \frac{Du}{Dt} = 0 \end{aligned} \quad 4.24$$

$$\frac{D}{Dt} \frac{K}{(e_1 - e_3)^{1/2}} = \frac{K}{(e_1 - e_3)^{1/2}} \frac{\partial u}{\partial x} \quad 4.25$$

The numerical factor pq can be removed from (4.22) through (4.25) by the scaling transformation $|pq|e_i \rightarrow e_i$ for $i = 1, 3$. No generality is lost using this transformation as it can be reversed at any time by the inverse transformation. The effect is equivalent to setting $pq = -1$ in (4.22) through (4.25). We assume that this has been done in the remainder of this chapter. In the limit $e_2 \rightarrow e_3$ $E=K=\pi/2$ and our modulational equations reduce to

$$\frac{D}{Dt} \left[(e_1 - e_3)^{1/2} - \frac{e_1}{e_1 - e_3} \right] = \frac{\sigma}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} e_3 (2e_1)^{1/2} \quad 4.26a$$

$$\frac{De_1}{Dt} + \sigma \frac{D}{Dt} (2e_1)^{1/2} + \sigma (2e_1)^{1/2} \frac{\partial u}{\partial x} + 2 \frac{\partial e_3}{\partial x} + \frac{\partial e_1}{\partial x} = 0 \quad 4.26b$$

$$\sigma \frac{D}{Dt} e_3 (2e_1)^{1/2} + 2\sigma e_3 (2e_1)^{1/2} \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} (e_3^2 + 2e_1 e_3) + e_3 \frac{Du}{Dt} = 0 \quad 4.26c$$

$$\frac{D}{Dt} \frac{1}{(e_1 - e_3)^{1/2}} = \frac{1}{(e_1 - e_3)^{1/2}} \frac{\partial u}{\partial x} \quad 4.26d$$

In the limit we have formally removed one parameter through the identification $e_2 = e_3$ so that the equations (4.26a-d) are overdetermined unless they are linearly dependent and equivalent to a system of reduced order. This proves to be the case. First we define a new variable (recall $\sigma = \pm 1$)

$$v = \sigma (2e_1)^{1/2} \quad 4.27$$

Equations (4.26a), (4.26b), and (4.26d) can then be combined to yield

$$\frac{De_3}{Dt} + \sigma v \frac{\partial e_3}{\partial x} + \sigma e_3 \frac{\partial v}{\partial x} + e_3 \frac{\partial u}{\partial x} = 0 \quad 4.28a$$

$$\frac{Dv}{Dt} + \sigma \frac{\partial e_3}{\partial x} + \sigma \frac{e_3}{v} \frac{\partial v}{\partial x} + (v - \frac{e_3}{v}) \frac{\partial u}{\partial x} = 0 \quad 4.28b$$

$$\frac{Du}{Dt} + \frac{\partial e_3}{\partial x} + (v - \frac{e_3}{v}) \frac{\partial v}{\partial x} + \sigma \frac{e_3}{v} \frac{\partial u}{\partial x} = 0 \quad 4.28c$$

It is easy to demonstrate that equations (4.28a,b,c) imply (4.26c) so that the reduced system (4.28a,b,c) represents the modulational equations in the limit $e_2 \rightarrow e_3$.

2. Analysis of the limiting modulational equations as a hyperbolic system; Riemann invariants and simple waves.

At this point we express our reduced modulational equations in matrix form conforming to the standard procedure for a quasilinear system introduced in Section A. We write

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} e_3 \\ v \\ u \end{bmatrix} + \begin{bmatrix} u+v & e_3 & e_3 \\ 1 & u+\frac{e_3}{v} & v-\frac{e_3}{v} \\ 1 & v-\frac{e_3}{v} & u+\frac{e_3}{v} \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} e_3 \\ v \\ u \end{bmatrix} = 0 \quad 4.29$$

and define $\Psi = (e_3, v, u)$. The hyperbolicity of this system is determined using the simple wave assumption that Ψ depends on a single phase variable θ with eigenvelocity $\lambda = dx/dt = -\theta_1/\theta_x$. The following eigenvalue problem for eigenvalue λ and eigenvector $R = \Psi_\theta$ is obtained:

$$\begin{bmatrix} u+v-\lambda & e_3 & e_3 \\ 1 & u+\frac{e_3}{v} & -\lambda & v-\frac{e_3}{v} \\ 1 & v-\frac{e_3}{v} & u+\frac{e_3}{v} & -\lambda \end{bmatrix} \begin{bmatrix} \frac{d}{d\theta} \\ v \\ u \end{bmatrix} = 0 \quad 4.30$$

The secular determinant associated with (4.30) yields the following characteristic polynomial for eigenvelocity λ

$$(\lambda - u + v - \frac{2e_3}{v})(\lambda - u - v - (2e_3)^{1/2})(\lambda - u - v + (2e_3)^{1/2}) = 0 \quad 4.31$$

so that the eigenvelocities are

$$\lambda^0 = u - v + \frac{2e_3}{v} \quad 4.32a$$

$$\lambda^\pm = u + v \pm (2e_3)^{1/2} \quad 4.32b$$

As the characteristic velocities are real and distinct, we conclude that there are three linearly independent left eigenvectors and that the system is hyperbolic. To compute the left eigenvector L^0 corresponding to λ^0 , we substitute Ψ^0 in matrix (4.30) and solve the left eigenvector equation

$$L^0 \begin{bmatrix} 2(v - \frac{2e_3}{v}) & e_3 & e_3 \\ 1 & v - \frac{2e_3}{v} & v - \frac{2e_3}{v} \\ 1 & v - \frac{2e_3}{v} & v - \frac{2e_3}{v} \end{bmatrix} = 0 \quad 4.33$$

and obtain $L^0 = (0, 1, -1)$. Similarly for $\lambda^\pm$ we have

$$L^\pm \begin{bmatrix} -(2e_3)^{1/2} & -(2e_3)^{1/2} & 0 \\ 1 & 1 & v - \frac{2e_3}{v} \pm \frac{(2e_3)^{1/2}}{2} \\ 1 & 1 & -(v - \frac{2e_3}{v} \pm \frac{(2e_3)^{1/2}}{2}) \end{bmatrix} = 0 \quad 4.34$$

yielding $L^\pm = (1, \pm(2e_3)^{1/2}/2, \pm(2e_3)^{1/2}/2)$.

As pointed out in Section A, the left eigenvectors L^0 , $L^\pm$ have a general significance as integrating operators for the original quasilinear system (4.29). Consider first the action of L^0 when applied to (4.29):

$$(0, 1, -1) \frac{\partial}{\partial t} \begin{bmatrix} e_3 \\ v \\ u \end{bmatrix} + (0, u - v + \frac{2e_3}{v}, v - u - \frac{2e_3}{v}) \frac{\partial}{\partial x} \begin{bmatrix} e_3 \\ v \\ u \end{bmatrix} = 0 \quad 4.35a$$

$$\text{or } \frac{d}{dt}(v - u) = 0 \quad \text{on} \quad \frac{dx}{dt} = u - v + \frac{2e_3}{v} \quad 4.35b$$

Thus $\Lambda^0 = v - u$ is the Riemann invariant corresponding to characteristic

velocity λ^0 and left eigenvector L^0 . Consider next the action of $L^\pm$ on the system (4.29):

$$\begin{aligned} & \left(1, \pm \frac{(2e_3)^{1/2}}{2}, \pm \frac{(2e_3)^{1/2}}{2}\right) \frac{\partial}{\partial t} \begin{bmatrix} e_3 \\ v \\ u \end{bmatrix} + \\ & (u \pm v \pm (2e_3)^{1/2}) \left[1, \pm \frac{(2e_3)^{1/2}}{2}, \pm \frac{(2e_3)^{1/2}}{2}\right] \frac{\partial}{\partial x} \begin{bmatrix} e_3 \\ v \\ u \end{bmatrix} = 0 \end{aligned} \quad 4.36a$$

$$\text{or } \frac{d}{dt} \left[\frac{u+v}{2} \pm (2e_3)^{1/2} \right] = 0 \quad \text{on} \quad \frac{dx}{dt} = u - v \pm (2e_3)^{1/2} \quad 4.36b$$

Thus $A^\pm = (u \pm v) / 2 \pm (e_3)^{1/2}$ are the Riemann invariants corresponding to characteristic velocities $\lambda^\pm$. In summary, our quasilinear system (4.29) is equivalent to the following system of coupled ordinary differential equations

$$A^0 = v - u \quad \text{on} \quad \frac{dx}{dt} = \lambda^0 \quad 4.37a$$

$$A^\pm = \frac{u \pm v}{2} \pm (2e_3)^{1/2} \quad \text{on} \quad \frac{dx}{dt} = \lambda^\pm \quad 4.37b$$

with

$$\lambda^0 = u - v + \frac{2e_3}{v} = \frac{A^+ - A^-}{2(A^+ + A^- + A^0)} - A^0 \quad 4.38a$$

$$\lambda^\pm = u + v \pm (2e_3)^{1/2} = A^+ + A^- \pm \frac{A^+ - A^-}{2} \quad 4.38b$$

Additional insight into the dynamics associated with the natural velocities and invariants of the system can be obtained by solving for the analytic form of the simple waves. To this end we return to the right eigenvalue problem (4.30) and solve for $R^0, R^\pm$. For R^0 we substitute λ^0 for λ in (4.30) and obtain

$$\begin{bmatrix} 2(v - \frac{e_3}{v}) & e_3 & e_3 \\ 1 & v - \frac{e_3}{v} & v - \frac{e_3}{v} \\ 1 & v - \frac{e_3}{v} & v - \frac{e_3}{v} \end{bmatrix} \begin{bmatrix} R_1^0 \\ R_2^0 \\ R_3^0 \end{bmatrix} = 0 \quad 4.39$$

implying $R^0 = (0, R_2(\Psi), -R_2(\Psi))$. For simple waves we solve

$$\frac{d}{d\theta} \begin{bmatrix} e_3 \\ v \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ R_2 \\ -R_2 \end{bmatrix} \quad 4.40$$

which implies that $d/d\theta(v+u)=0$ and $de_3/d\theta=0$ or equivalently $d(A^+ + A^-)/d\theta=0$ and $d(A^+ - A^-)/d\theta=0$. Thus the simple wave Ψ^0 is equivalent to the constancy of the Riemann invariants A^+ and A^- throughout the

entire x, t plane. If we use the Riemann invariants A^+, A^- as constants of integration, the simple wave corresponding to λ^0 can be written in terms of one unknown function. If we choose $u(\theta)$ as the arbitrary function then ψ^0 can be written

$$\psi^0 = \begin{bmatrix} \frac{(A^+ - A^-)^2}{8} \\ A^+ - A^- - u(\theta) \\ u(\theta) \end{bmatrix} = \begin{bmatrix} e_3 \\ v \\ u \end{bmatrix} \quad 4.41$$

For the simple waves corresponding to $\lambda^\pm$ we obtain the following right eigenvalue problem

$$\begin{bmatrix} -(2e_3)^{1/2} & \pm e_3 & \pm e_3 \\ 1 & -\left(v - \frac{e_3}{v}\right) \pm (2e_3)^{1/2} & v - \frac{e_3}{v} \\ 1 & v - \frac{e_3}{v} & -\left(v - \frac{e_3}{v}\right) \pm (2e_3)^{1/2} \end{bmatrix} \begin{bmatrix} R_1^\pm \\ R_2^\pm \\ R_3^\pm \end{bmatrix} = 0 \quad 4.42$$

which reduces to

$$\begin{bmatrix} 0 & 0 & 0 \\ \pm 2 & -(2e_3)^{1/2} & -(2e_3)^{1/2} \\ 0 & v - \frac{e_3}{v} \pm \frac{(2e_3)^{1/2}}{2} & -\left(v - \frac{e_3}{v} \pm \frac{(2e_3)^{1/2}}{2}\right) \end{bmatrix} \begin{bmatrix} R_1^\pm \\ R_2^\pm \\ R_3^\pm \end{bmatrix} = 0 \quad 4.43$$

and implies $R^\pm = (\pm(2e_3)^{1/2} R_2^\pm, R_2^\pm, R_2^\pm)$. For simple waves we solve

$$\frac{d}{d\theta} \begin{bmatrix} e_3 \\ v \\ u \end{bmatrix} = \begin{bmatrix} \pm(2e_3)^{1/2} R_2^\pm \\ R_2^\pm \\ R_2^\pm \end{bmatrix} \quad 4.44$$

which is equivalent to $d(v-u)/d\theta=0$ and $d[(v+u)/2\mp(2e_3)^{1/2}]=0$. These two conditions are equivalent to the constancy of the two Riemann invariants $A^0=v-u$ and $\bar{A}^\mp=(v+u)/2\mp(2e_3)^{1/2}$ throughout the x,t plane. As before, we can use these Riemann invariants as constants of integration to express our simple wave in terms of a single unknown function. For convenience we choose $e_3(\theta)$ as our arbitrary function and write

$$\Psi^\pm = \begin{bmatrix} e_3(\theta) \\ \pm(2e_3)^{1/2} + \bar{A}^\mp - \frac{A^0}{2} \\ \pm(2e_3)^{1/2} + \bar{A}^\mp + \frac{A^0}{2} \end{bmatrix} = \begin{bmatrix} e_3 \\ v \\ u \end{bmatrix} \quad 4.45$$

Recall that for a simple wave the characteristic velocities λ^k are constant on θ^k and $\theta^k=x-\lambda^k t$. Thus the simple waves above can be propagated forward in time if the functions $u(\theta)$ or $e_3(\theta)$ are specified on an initial boundary curve (i.e. the $\theta^k|_{t=0} = x$ axis). As discussed in Section A, these solutions will become singular as the waves steepen and break. The breaking time is given for each characteristic by the formula (4.19).

3. Reversion to physical variables and discussion.

The results of the last section are obscured by the natural variables of the analysis and we must revert to more physical variables to gain additional insight. The first step is to recover the general case by undoing the scaling transformation introduced in Section B. In particular, we redefine the dependent variables e_1 and e_3 via $e_i \rightarrow |pq|e_i$ and transcribe $v \rightarrow \sigma(2|pq|e_1)^{1/2}$. Our expressions for the characteristics and Riemann invariants are then as follows

$$\begin{aligned} A^0 &= u - \sigma(2|pq|e_1)^{1/2} \quad \text{constant on} \\ \frac{dx}{dt} &= u - \sigma(2|pq|e_1)^{1/2} + \frac{2\sigma|pq|e_3}{(2|pq|e_1)^{1/2}} = \lambda^0 \end{aligned} \quad 4.46$$

$$\begin{aligned} A^\pm &= \frac{u + \sigma(2|pq|e_1)^{1/2}}{2} \pm (2|pq|e_3)^{1/2} \quad \text{constant on} \\ \frac{dx}{dt} &= u + \sigma(2|pq|e_1)^{1/2} \pm (2|pq|e_3)^{1/2} = \lambda^\pm \end{aligned} \quad 4.47$$

Physical variables are recovered from the definitions of Chapter 3 and the limiting forms in Appendix II. In particular, we have for cnoidal waves $pq < 0$:

$$I = e_3 \quad 4.48$$

as $e_2 \rightarrow e_3 \rightarrow 0$. Also, from our nonlinear dispersion relationships (3.65) and (3.66) and Appendix II we obtain

$$2k = \frac{1}{w_D} \rightarrow 2 \left| \frac{2q}{p} \right|^{1/2} (e_1 - e_3)^{1/2} \quad 4.49$$

$$\omega = 2\beta pk + pk \frac{w_E}{w_D} \rightarrow 2\beta pk - \sigma(2|pq|e_1)^{1/2}k \quad 4.50$$

as $e_2 - e_3 \rightarrow 0$. From (4.49) and (4.50) we conclude that

$$2|pq|e_1 = p^2k^2 + 2|pq|I \quad 4.51$$

$$\omega = 2\beta pk - \sigma(2|pq|I + p^2k^2)^{1/2}k \quad 4.52$$

In the present limit ω and k have there usual significance as Fourier variables. The dispersion relation (4.52) should be compared to the heuristic result (1.14b) for the case $pq < 0$ in the equivalent limit. Using (4.48), (4.51) and (4.52) we write the Riemann invariants and characteristic velocities as follows:

$$A^0 = 2\beta p - 2\sigma(2|pq|I + p^2k^2)^{1/2} \quad 4.53$$

$$\lambda^0 = 2\beta p - \sigma(2|pq|I + p^2k^2)^{1/2} - \frac{\sigma p^2k^2}{(2|pq|I + p^2k^2)^{1/2}} \quad 4.54$$

$$A^{\pm} = \beta p \pm (2|pq|I)^{1/2} \quad 4.55$$

$$\lambda^{\pm} = 2\beta p \pm (2|pq|l)^{1/2} \quad 4.56$$

The physical significance of the velocities (4.54) and (4.56) can be understood using the nonlinear dispersion relation (4.52). In the limit $e_2 \rightarrow e_3$ the cnoidal wave degenerates into an infinitesimal sinusoidal modulation with local wavenumber k and frequency ω . For this wave the usual concept of a group velocity is well defined by $d\omega/dk$. Certainly one characteristic velocity of the modulations of the wave should correspond to this group velocity. Accordingly, we compute $d\omega/dk$ using (4.52):

$$\frac{d\omega}{dk} = 2\beta p - \sigma(2|pq|l + p^2k^2)^{1/2} - \frac{\sigma p^2k^2}{(2|pq|l + p^2k^2)^{1/2}} \quad 4.57$$

Clearly (4.57) corresponds to λ^0 . For the remaining characteristic velocities, we examine the dispersion relation (4.52) in the long wavelength limit $k \rightarrow 0$:

$$\frac{\omega}{k} = 2\beta p - \sigma(2|pq|l)^{1/2} \quad 4.58$$

Recall that our modulational theory was restricted to modulations with characteristic scales much larger than the local wave period $2\pi/k$. One such class of modulations are waves of the same type as the particular wave with wavenumber k that we are studying but with relative wavenumbers much smaller than k , that is with $k \rightarrow 0$. Such a wave would

have a phase velocity given by (4.58). The label σ is fixed for the carrier and is therefore fixed for the group velocity of the carrier. However, the waves in the present discussion are independent of the carrier and we relax the meaning of the label σ and let it take on both values ± 1 . With this remark (4.58) is identified with the remaining two characteristic velocities $\lambda^{\pm}$.

The simple wave (4.41) has a particularly simple form in terms of physical variables. Noting that $A^+ + A^- = 2\beta^0$ and $A^+ - A^- = (2|pq|I^0)^{1/2}$ we have

$$\Psi^0 = \begin{bmatrix} |pq|I^0 \\ 2\beta^0 - u \\ u \end{bmatrix} = \begin{bmatrix} |pq|I^0 \\ \sigma(p^2k^2 + 2|pq|I^0)^{1/2} \\ u \end{bmatrix} \quad 4.59$$

Clearly this simple wave corresponds to a linearization about a uniform carrier wave of intensity I^0 and wavenumber $2\beta^0$. The quantities varying from characteristic to characteristic are the phase velocity u or equivalently the wavenumber k of linear waves, and these quantities are constant on a characteristic corresponding to a frame moving at the local group velocity. The second component of Ψ^0 just restates the linear dispersion relation that connects k and u . We note that *wave breaking or the crossing of characteristic curves in this case is well defined and corresponds to the overtaking of linear packets with lower group velocities by those with higher group velocities*. In this particular case the scaling constraints of the modulational theory,

which restricts the validity of the modulational evolution equations to times much less than the breaking time, can be relaxed. Double valuedness in the linear wave parameters is meaningful in the sense described above. Two or more linear packets with distinct wavenumbers and group velocities can coexist at the same point of space and time.

This is in marked contrast to the simple waves (4.45). Using $A^+ + A^0/2 = B^+$, $A^+ - A^0/2 = C^+$ (4.45) can be written

$$\Psi^\pm = \begin{bmatrix} |pq||l| \\ \pm(2|pq||l|)^{1/2} + B^\pm \\ \pm(2|pq||l|)^{1/2} + C^\pm \end{bmatrix} = \begin{bmatrix} |pq||l| \\ \sigma(p^2k^2 + 2|pq||l|)^{1/2} \\ u \end{bmatrix} \quad 4.60$$

For this simple wave the intensity is modulated. In a frame moving with the local characteristic velocity (the phase velocity of a $k=0$ intensity modulation) l is a constant. Also, a superimposed infinitesimal wave with parameter k varying from characteristic to characteristic as a function of l as given, has stationary values for k in the same local frame. This local frame velocity is not at the group velocity corresponding to k and l . The relationship between k and l contained in the second component of $\Psi^\pm$ can be obtained by insisting that the phase velocity u differ from the characteristic velocity $\lambda^\pm$ by a constant. Infinitesimal packets moving at their group velocities will accelerate and decelerate as they move over the modulations of the carrier and the local value of k will change in accord with the local dispersion relation. Thus for an intensity modulated carrier the group

velocity frame is not a frame of constant k . However, for the particular simple wave above, there is a frame where both I and the infinitesimal wave parameters k and u are stationary.

In this case wave breaking or crossing of characteristics is accompanied by an unphysical double valuedness in wave intensity. Fortunately, the scaling restrictions of the modulational theory on gradients in I invalidate our derivation as the breaking time is approached. Intuitively, we expect higher order dispersive effects to come into play to prevent breaking by balancing steepening and producing cnoidal or solitary waves. This expectation is solidly rooted: intensity modulations with an infinitesimal sinusoidal wave superimposed must behave as though the infinitesimal wave were not present and therefore in accord with the dynamics of the CSE itself, which we know supports cnoidal and solitary waves.

It is instructive to proceed further in our discussion of these two simple wave types by adopting a perturbation theory for the CSE especially suited to the limit $e_2 \rightarrow e_3$ and to the dynamics of simple waves. In this theory the contrasting behavior of the above simple waves at the breaking time, i.e. the overtaking of linear packets in the former and nonlinear steepening yielding nonphysical double valuedness in I in the later, are unified in a weakly nonlinear theory based on the Korteweg de Vries equation.

4. Derivation of the Korteweg de Vries equation as an asymptotic approximation to the dynamics of the simple waves $\psi^\pm, \psi^0$.

For a dynamical system governed by hyperbolic equations it can be shown that regions of the x, t plane adjoining a region of constant values of the dependent variables must be dynamically of the form of simple waves³. In this this sense simple waves describe the far fields of localized disturbances of a medium. Simple waves are unidirectional disturbances carried on a particular class of characteristics. We shall now derive a perturbation theory tailored to balance dispersive and nonlinear dynamics as higher order effects to lower order dynamics driving the disturbance at the characteristic velocity. We shall follow the analysis of Taniuti *et al*^{4,5}. The reductive perturbation theory of Taniuti⁶ applies to the far field dynamics of hyperbolic systems when the underlying dynamics has separated a localized disturbance into it's constituent simple waves.

For simplicity we use specific values for the constitutive parameters p and q in the CSE. Using $p=1$ and $q=-1$ and the phasor representation (3.3) the CSE splits into the following two equations

$$(A^2)_t + (2\varphi_x A^2)_x = 0 \quad 4.61$$

$$(\varphi_t + \varphi_x^2)A - A_{xx} + A^3 = 0 \quad 4.62$$

We adopt the following notation

$$A^2 = I = n \quad 4.63$$

$$2\varphi_x = v \quad 4.64$$

so that (4.61) and (4.62) are written

$$n_t + (nv)_x = 0 \quad 4.65$$

$$v_t + vv_x = -2n_x - \frac{2n_x n_{xx}}{n^2} + \frac{n_x^3}{2n^3} + \frac{n_{xxx}}{n} \quad 4.66$$

The notation in (4.63) and (4.64) has been chosen to accentuate an analogy in (4.65) and (4.66) with a compressible fluid with density n and flow velocity v and an unusual pressure term involving derivatives of the density.

We choose our zero order state of the perturbation theory to correspond as closely as possible to the simple wave types Ψ^0 , Ψ^+ and Ψ^- . Accordingly the zero order dynamics are chosen with the Riemann invariants A^+ and A^- constant everywhere. These conditions correspond to a simple wave of type Ψ^0 with intensity I^0 and average wavenumber β^0 constant everywhere. Also, we will assume that $k \sim \partial/\partial x$ is of higher order (eventually $O(\varepsilon^{1/2})$) and note that for $k \sim 0$

$$A^0 = 2\beta - 2\sigma(2I)^{1/2} \quad , \quad \lambda^0 = 2\beta - \sigma(2I)^{1/2} \quad 4.67$$

$$A^\pm = \beta \pm (2I)^{1/2} \quad , \quad \lambda^\pm = 2\beta \pm (2I)^{1/2} \quad 4.68$$

so that the characteristic velocities λ^0, λ^+ (for $\sigma=-1$) or λ^0, λ^- (for $\sigma=+1$) coincide. The coincidence of characteristics at longwavelengths suggests that the perturbation theory should be capable of picking up dynamics of a mixed character involving aspects of Ψ^0, Ψ^+ or Ψ^0, Ψ^- . We adopt the following ordering for n and v based on the above discussion:

$$n = I^0 + \varepsilon n^1 + \varepsilon^2 n^2 + \dots \quad 4.69$$

$$v = 2\beta^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \dots \quad 4.70$$

Next, we linearize (4.65) and (4.66) based on (4.69) and (4.70) obtaining:

$$n_t^1 + 2\beta^0 n_x^1 + I^0 v_x^1 = 0 \quad 4.71$$

$$v_t^1 + 2\beta^0 v_x^1 - 2n_x^1 - \frac{n_{xx}^1}{I^0} = 0 \quad 4.72$$

Which yields, upon Fourier analysis, the following dispersion relation

$$u = \frac{\omega}{k} = 2\beta^0 \pm (2I^0 + k^2)^{1/2} \quad 4.73$$

which is in agreement with (4.52). Equation (4.73) tells us that long wavelength disturbances propagate at velocities shifted by $\pm(2I^0)^{1/2}$ relative to a reference velocity $2\beta^0$. Using this information we adopt the following stretched coordinate system which has been tailored to examine unidirectional dynamics of disturbances near a natural velocity $2\beta^0 + (2I^0)^{1/2}$ (corresponding to $k=0$, $\sigma=-1$ in (4.54) or (4.56)):

$$\xi = \varepsilon^{1/2} [x - (2\beta^0 + \sqrt{2I^0})t] \quad 4.74$$

$$\tau = \varepsilon^{3/2} t \quad 4.75$$

from which we obtain

$$\frac{\partial}{\partial t} = \varepsilon^{3/2} \frac{\partial}{\partial \tau} - \varepsilon^{1/2} (2\beta^0 + \sqrt{2I^0}) \frac{\partial}{\partial \xi} \quad 4.76$$

$$\frac{\partial}{\partial x} = \varepsilon^{1/2} \frac{\partial}{\partial \xi} \quad 4.77$$

Using (4.76) and (4.77) together with (4.69) and (4.70) we obtain the following expression for (4.65) to $O(\varepsilon^{5/2})$:

$$\begin{aligned} \varepsilon^{5/2} n_\tau^1 - \varepsilon^{3/2} (2\beta^0 + \sqrt{2} \bar{I}^0) n_\xi^1 - \varepsilon^{5/2} (2\beta^0 + \sqrt{2} \bar{I}^0) n_\xi^2 + \varepsilon^{3/2} 2\beta^0 n_\xi^1 \\ + \varepsilon^{5/2} 2\beta^0 n_\xi^2 + \varepsilon^{5/2} v^1 n_\xi^1 + \varepsilon^{5/2} n^1 v_\xi^1 + \varepsilon^{3/2} I^0 v_\xi^1 + \varepsilon^{5/2} I^0 v_\xi^2 = 0 \end{aligned} \quad 4.78$$

and for (4.66) to $O(\varepsilon^{5/2})$:

$$\begin{aligned} \varepsilon^{5/2} v_\tau^1 - \varepsilon^{3/2} (2\beta^0 + \sqrt{2} \bar{I}^0) v_\xi^1 - \varepsilon^{5/2} (2\beta^0 + \sqrt{2} \bar{I}^0) v_\xi^2 + \varepsilon^{3/2} 2\beta^0 v_\xi^1 \\ + \varepsilon^{5/2} 2\beta^0 v_\xi^2 + \varepsilon^{5/2} v^1 v_\xi^1 + \varepsilon^{3/2} 2n_\xi^1 + \varepsilon^{5/2} 2n_\xi^2 - \varepsilon^{5/2} \frac{n^1}{I^0} = 0 \end{aligned} \quad 4.79$$

Both (4.78) and (4.79) give to $O(\varepsilon^{3/2})$:

$$v_\xi^1 = \left(\frac{2}{I^0} \right)^{1/2} n_\xi^1 \quad 4.80$$

from which we conclude

$$v^1 = \left(\frac{2}{I^0} \right)^{1/2} n^1 \quad 4.81$$

Using (4.81) to eliminate v^1 in (4.78) and (4.79) we obtain the following equations from terms of $O(\varepsilon^{5/2})$:

$$\sqrt{2} \bar{I}^0 n_\tau^1 - 2 I^0 n_\xi^2 + 4 n^1 n_\xi^1 + \sqrt{2} \bar{I}^0 I^0 v_\xi^2 = 0 \quad 4.82$$

$$\sqrt{2} \bar{I}^0 n_\tau^1 - I^0 \sqrt{2} \bar{I}^0 v_\xi^2 + 2 n^1 n_\xi^1 + 2 I^0 n_\xi^2 - n_\xi^1 n_\xi^1 = 0 \quad 4.83$$

Adding (4.82) and (4.83) and dividing the result by two we obtain

$$\sqrt{2}I^0 n_\tau^1 + 3n_\xi^1 n_\xi^1 - \frac{1}{2}n_\xi^1 \xi \xi = 0 \quad 4.84$$

which is the Korteweg-deVries equation⁷. Thus we have established that the stretched coordinate system and ordering scheme that we have adopted yield a nonlinear dynamical equation with nonlinear ($n^1 n_\xi^1$) and dispersive ($n_\xi^1 \xi \xi$) terms of the same magnitude. An initial state evolving in accord with the CSE in the scaling regime outlined above should be adequately modeled by the KdV equation for times much larger than the breaking time because (4.84) is known to have well behaved solutions as $t \rightarrow \infty$ in the form of multiple solitons and spreading self-similar packets⁸. In particular the KdV equation (4.84) has soliton solutions of the form:

$$n^1 = -\delta \operatorname{sech}^2 \left[\left(\frac{\delta}{2} \right)^{1/2} \left(\xi + \frac{\delta}{\sqrt{2}I^0} \tau \right) \right] \quad 4.85$$

So that

$$I = I^0 - \delta \operatorname{sech}^2 \left\{ \left(\frac{\delta}{2} \right)^{1/2} \left[x - \left(2\beta^0 + \sqrt{2}I^0 - \frac{\delta}{\sqrt{2}I^0} \right) t \right] \right\} \quad 4.86$$

This is in agreement with our fundamental solution (3.54a) if we identify $\delta = (e_2 - e_3)$ and $I^0 = e_2$ in the limit $e_2 \rightarrow e_1$. For consistency we must have agreement for u also. We check u in the limit $e_2 \rightarrow e_1$ for $\sigma = 1$ using (3.105) and Appendix II:

$$u = 2\phi^0 + (2I^0 - 2\delta)^{1/2} \quad 4.87$$

Our scaling is such that δ is itself a small quantity, so we take the limit $\delta \rightarrow 0$ obtaining

$$u = 2\phi^0 + \sqrt{2I^0} - \frac{\delta}{\sqrt{2I^0}} + O(\delta^2) \quad 4.88$$

which is in agreement with (4.86).

The soliton solutions of the KdV equation are known to preserve their identities during collisions. Thus we have the following consistent picture of the far field dynamics of the CSE corresponding to the modulational theory limit $e_2 \rightarrow e_3$ and the above ordering which unifies the contrasting nature of our simple waves at and past the breaking time: The unphysical double valuedness of intensity associated with the breaking of the simple waves $\Psi^\pm$ is indeed prevented by dispersive forces yielding solitons while the superposition properties of the linear simple waves Ψ^0 during wave breaking are preserved in the dynamics of identity preserving soliton collisions.

C. Infinitesimal waves on a uniform nonlinear carrier; the limit $e_2 \rightarrow e_1$ for the (elliptic) case $pq > 0$.

1. The modulational equations for $pq > 0$ in the limit $e_2 \rightarrow e_1$.

Much of the analysis of this section parallels that of Section B for the case $pq < 0$. The local cnoidal waves for $pq > 0$ are given by

(3.30). In the limit $\delta=(e_1-e_2)\rightarrow 0$ the wavenumber $k\rightarrow(e_1-e_3)^{1/2}(q/2p)^{1/2}$ and the local cnoidal wave degenerates to

$$I = e_1 + \delta \cos^2(kx - \omega t) \quad 4.89$$

To compute the evolution equations for the parameters e_1 , k and ω , we take the limit $e_2 \rightarrow e_1$ of the modulational equations (3.109) through (3.115) using the appropriate limiting forms of the functions W_D , W_E and W_γ from Appendix II. The expressions for W_D , W_E and W_γ in terms of complete elliptic integrals differ from the case $pq < 0$. They are

$$W_D = \frac{1}{\pi} \left(\frac{p}{2q} \right)^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} K(m) \quad 4.90a$$

$$W_\gamma = -\frac{1}{\pi} \left(\frac{p}{2q} \right)^{1/2} (e_1 - e_3)^{1/2} \left[E(m) + \frac{e_3}{e_1 - e_3} K(m) \right] \quad 4.90b$$

$$W_E = -\frac{\sigma |q|}{\pi q} \left(\frac{-e_2 e_3}{e_1} \right)^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} \Pi(n, m) \quad 4.90c$$

$$n = \frac{e_1 - e_2}{e_1}, \quad m = \frac{e_1 - e_2}{e_1 - e_3} \quad 4.90d$$

where K , E and Π are the complete elliptic integrals of the first, second and third kind. Using (4.90a,b,c) in (3.109) through (3.115) for the case $pq > 0$ we obtain the modulational system

$$\frac{D}{Dt}[(e_1 - e_3)E + \frac{e_3}{e_1 - e_3}K] = \frac{-\sigma K}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} (-2pqe_1 e_2 e_3)^{1/2} \quad 4.91$$

$$\begin{aligned} \frac{Du}{Dt} + \sigma(pq)^{1/2} \frac{D}{Dt} \left(\frac{-2e_2 e_3}{e_1} \right)^{1/2} \frac{\Pi}{K} + \sigma(pq)^{1/2} \left(\frac{-2e_2 e_3}{e_1} \right)^{1/2} \frac{\Pi}{K} \frac{\partial u}{\partial x} \\ - pq \frac{\partial}{\partial x} (e_1 + e_2 + e_3) = 0 \end{aligned} \quad 4.92$$

$$\begin{aligned} \frac{D}{Dt} \sigma(-2pqe_1 e_2 e_3)^{1/2} + 2\sigma(-2pqe_1 e_2 e_3)^{1/2} \frac{\partial u}{\partial x} - pq \frac{\partial}{\partial x} (e_1 e_2 + e_2 e_3 + e_3 e_1) \\ + [(e_1 - e_3) \frac{E}{K} + e_3] \frac{Du}{Dt} = 0 \end{aligned} \quad 4.93$$

$$\frac{D}{Dt} \frac{K}{(e_1 - e_3)^{1/2}} = \frac{K}{(e_1 - e_3)^{1/2}} \frac{\partial u}{\partial x} \quad 4.94$$

The numerical factor pq is removed in (4.91) through (4.94) by the scaling transformation $pqe_i \rightarrow e_i$ for $i=1,3$. As for the case $pq < 0$, no generality is lost using this transformation as it is reversible. In the limit $e_2 \rightarrow e_1$, $K=E=\pi/2$ and the modulational system (4.91) through (4.94) reduces to

$$\frac{D}{Dt} [(e_1 - e_3)^{1/2} + \frac{1}{(e_1 - e_3)^{1/2}}] = \frac{-\sigma}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} (-2e_3)^{1/2} e_1 \quad 4.95a$$

(3.30). In the limit $\delta=(e_1-e_2) \rightarrow 0$ the wavenumber $k \rightarrow (e_1-e_3)^{1/2}(q/2p)^{1/2}$ and the local cnoidal wave degenerates to

$$I = c_1 + \delta \cos^2(kx - \omega t) \quad 4.89$$

To compute the evolution equations for the parameters e_1 , k and ω , we take the limit $e_2 \rightarrow e_1$ of the modulational equations (3.109) through (3.115) using the appropriate limiting forms of the functions W_D , W_E and W_γ from Appendix II. The expressions for W_D , W_E and W_γ in terms of complete elliptic integrals differ from the case $pq < 0$. They are

$$W_D = \frac{1}{\pi} \left(\frac{p}{2q} \right)^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} K(m) \quad 4.90a$$

$$W_\gamma = - \frac{1}{\pi} \left(\frac{p}{2q} \right)^{1/2} (e_1 - e_3)^{1/2} [E(m) + \frac{e_3}{e_1 - e_3} K(m)] \quad 4.90b$$

$$W_E = - \frac{\sigma |q|}{\pi q} \left(\frac{-e_2 e_3}{e_1} \right)^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} \Pi(n, m) \quad 4.90c$$

$$n = \frac{e_1 - e_3}{e_1}, \quad m = \frac{e_1 - e_2}{e_1 - e_3} \quad 4.90d$$

where K , E and Π are the complete elliptic integrals of the first, second and third kind. Using (4.90a,b,c) in (3.109) through (3.115) for the case $pq > 0$ we obtain the modulational system

$$\frac{D}{Dt}[(e_1 - e_3)E + \frac{e_3}{e_1 - e_3}K] = \frac{-\sigma K}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} (-2pqe_1 e_2 e_3)^{1/2} \quad 4.91$$

$$\begin{aligned} \frac{Du}{Dt} + \sigma(pq)^{1/2} \frac{D}{Dt} \left(\frac{-2e_2 e_3}{e_1} \right)^{1/2} \frac{\Pi}{K} + \sigma(pq)^{1/2} \left(\frac{-2e_2 e_3}{e_1} \right)^{1/2} \frac{\Pi}{K} \frac{\partial u}{\partial x} \\ - pq \frac{\partial}{\partial x} (e_1 + e_2 + e_3) = 0 \end{aligned} \quad 4.92$$

$$\begin{aligned} \frac{D}{Dt} \sigma(-2pqe_1 e_2 e_3)^{1/2} + 2\sigma(-2pqe_1 e_2 e_3)^{1/2} \frac{\partial u}{\partial x} - pq \frac{\partial}{\partial x} (e_1 e_2 + e_2 e_3 + e_3 e_1) \\ + [(e_1 - e_3) \frac{E}{K} + e_3] \frac{Du}{Dt} = 0 \end{aligned} \quad 4.93$$

$$\frac{D}{Dt} \frac{K}{(e_1 - e_3)^{1/2}} = \frac{K}{(e_1 - e_3)^{1/2}} \frac{\partial u}{\partial x} \quad 4.94$$

The numerical factor pq is removed in (4.91) through (4.94) by the scaling transformation $pq e_i \rightarrow e_i$ for $i=1,3$. As for the case $pq < 0$, no generality is lost using this transformation as it is reversible. In the limit $e_2 \rightarrow e_1$, $K \rightarrow E = \pi/2$ and the modulational system (4.91) through (4.94) reduces to

$$\frac{D}{Dt} [(e_1 - e_3)^{1/2} + \frac{1}{(e_1 - e_3)^{1/2}}] = \frac{-\sigma}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} (-2e_3)^{1/2} e_1 \quad 4.95a$$

$$\frac{Du}{Dt} + \sigma \frac{D}{Dt} (-2e_3)^{1/2} + \sigma (-2e_3)^{1/2} \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} (2e_1 + e_3) = 0 \quad 4.95b$$

$$\frac{D}{Dt} \sigma (-2e_3)^{1/2} e_1 + 2\sigma (-2e_3)^{1/2} e_1 \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} (e_1^2 + 2e_1 e_3) + e_1 \frac{\partial u}{\partial x} = 0 \quad 4.95c$$

$$\frac{D}{Dt} \frac{1}{(e_1 - e_3)^{1/2}} = \frac{1}{(e_1 - e_3)^{1/2}} \frac{\partial u}{\partial x} \quad 4.95d$$

As before, one parameter has been removed by the identification $e_2 = e_1$ so that equations (4.95) are overdetermined unless they are linearly dependent and equivalent to a system of reduced order. Before we establish linear dependence we define a new variable (recall $\sigma = \pm 1$)

$$v = \sigma (-2e_3)^{1/2} \quad 4.96$$

Equations (4.95a), (4.95b) and (4.95d) can be combined to yield

$$\frac{D}{Dt} e_1 + e_1 \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} e_1 v = 0 \quad 4.97a$$

$$\frac{Dv}{Dt} + v \frac{\partial u}{\partial x} + \frac{e_1}{v} \frac{\partial u}{\partial x} - \frac{e_1}{v} \frac{\partial v}{\partial x} - \frac{\partial e_1}{\partial x} = 0 \quad 4.97b$$

$$\frac{Du}{Dt} + \frac{e_1}{v} \frac{\partial v}{\partial x} - \frac{e_1}{v} \frac{\partial u}{\partial x} - \frac{\partial e_1}{\partial x} + v \frac{\partial v}{\partial x} = 0 \quad 4.97c$$

In an analogous manner to the case $pq < 0$, (4.97a,b,c) together imply (4.95c) so that system (4.97a,b,c) represents the modulational equations in the limit $e_2 \rightarrow e_3$.

2. Analysis of the limiting modulational equations for the case $pq > 0$ as an elliptic system; modulational instability.

The reduced system (4.97) is expressed in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} e_1 \\ v \\ u \end{bmatrix} + \begin{bmatrix} u+v & e_1 & e_1 \\ 1 & u-\frac{e_1}{v} & v+\frac{e_1}{v} \\ 1 & v+\frac{e_1}{v} & u-\frac{e_1}{v} \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} e_1 \\ v \\ u \end{bmatrix} = 0 \quad 4.98$$

where $\Psi = (e_1, v, u)$. The simple wave ansatz assumes Ψ is a function of a single phase variable θ with eigenvelocity $\lambda = -\theta_t / \theta_x$. The following eigenvalue problem for $R = \Psi_\theta$ and eigenvalue λ is obtained:

$$\begin{bmatrix} u-v-\lambda & e_1 & e_1 \\ -1 & u-\frac{e_1}{v} & -\lambda & v-\frac{e_1}{v} \\ -1 & v+\frac{e_1}{v} & u-\frac{e_1}{v} & -\lambda \end{bmatrix} \frac{d}{d\theta} \begin{bmatrix} e_1 \\ v \\ u \end{bmatrix} = 0 \quad 4.99$$

The secular determinant associated with (4.99) yields the characteristic polynomial

$$(\lambda - u + v - \frac{2e_1}{v})(\lambda - u - v + (-2e_1)^{1/2})(\lambda - u - v - (-2e_1)^{1/2}) = 0 \quad 4.100$$

so that the eigenvelocities are

$$\lambda^0 = u - v + \frac{2e_1}{v} \quad 4.101a$$

$$\lambda^{\pm} = u + v \pm (-2e_1)^{1/2} \quad 4.101b$$

The eigenvelocities (4.101b) are complex and so our system (4.98) is elliptic. This implies the linear instability of the cnoidal waves for $pq > 0$. The elliptic nature of our system does not allow the construction of an equivalent system of ordinary differential equations; i.e. the method of integration along characteristics fails. However, we do have a prediction of instability and values for growth rates. Additional insight is obtained by reverting to physical variables.

3. Reversion to physical variables and discussion.

Physical variables can be recovered by reversing the simplifying transformation of the last section and then identifying the variables e_1 and e_3 with the appropriate limiting forms from Chapter 3. The

scaling transformation is reversed via $e_1 \rightarrow pqe_1$ and e_3 is recovered via $v \rightarrow \sigma(-2pqe_3)^{1/2}$. The intensity I is simply related to e_1 using (4.90)

$$I = e_1 \quad 4.102$$

and the nonlinear dispersion relation (3.66) and (3.67) along with the appropriate limits in Appendix II yield

$$2k = \frac{1}{W_D} \rightarrow 2\left(\frac{2p}{q}\right)^{1/2}(e_1 - e_3)^{1/2} \quad 4.103$$

$$\omega = 2\beta k + pk \frac{W_E}{W_D} \rightarrow 2\beta k - \sigma(-2pqe_3)^{1/2} \quad 4.104$$

(recall that $e_3 \geq 0$ for $pq > 0$). Using (4.102), (4.103) and (4.104) we can write

$$p^2 k^2 = 2pq(I - e_3) \quad 4.105$$

$$\omega = 2\beta - \sigma(p^2 k^2 - 2pqI)^{1/2} \quad 4.106$$

which in turn allows us to express the characteristic velocities in physical variables

$$\lambda^0 = 2\beta - \sigma(p^2 k^2 - 2pqI)^{1/2} - \frac{\sigma p^2 k^2}{(p^2 k^2 - 2pqI)^{1/2}} \quad 4.107$$

$$\lambda^{\pm} = 2\beta \pm (-2pq|)^{1/2} \quad 4.108$$

The example of the case $pq < 0$ suggests that λ^0 is the group velocity corresponding to the nonlinear dispersion relation (4.106). To verify this again for the case $pq > 0$ we compute

$$\frac{d\omega}{dk} = 2\beta - \sigma(p^2k^2 - 2pq|)^{1/2} - \frac{\sigma p^2k^2}{(p^2k^2 - 2pq|)^{1/2}} \quad 4.109$$

which is precisely λ^0 . Both the phase velocity and group velocity of the mode are well defined real quantities. As we are clearly in a near-linear regime, another class of long wavelength modulations of the particular wave under study can arise from superposition of waves of the same type satisfying (4.106) only with $k \sim 0$. The symbol σ is no longer constrained to be + or - as a label for our particular wave, so we substitute both values $\sigma = \pm 1$. The dispersion relation (4.106) becomes

$$\frac{\omega}{k} = 2\beta \pm (-2pq|)^{1/2} \quad 4.110$$

which is clearly equal to $\lambda^{\pm}$. Thus the modulational instability of infinitesimal waves satisfying $p^2k^2 > 2pq|$ for the case $pq > 0$ is due to the intrinsic instability of the nonlinear carrier wave to long wavelength perturbations satisfying $p^2k^2 < 2pq|$.

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Chapter 5

In this chapter we further reduce the modulational equations to compact matrix form. The stability of the modulational equations is then determined over the entire range of parameters of the local cnoidal waves by numerical solution of an eigenvalue problem. We conclude that the case $pq < 0$ is always hyperbolic and therefore stable while the case $pq > 0$ is always elliptic and therefore unstable.

We also examine the solitary wave chain limit and extract Riemann invariants for this limit for the case $pq < 0$. Also, our results are compared in several nontrivial limiting cases with those obtained by different methods in earlier studies. We obtain detailed agreement.

A. The general modulational system $pq < 0$.

1. Reduction of the modulational equations for the case $pq < 0$ to matrix form.

The starting point for our final reduction of the modulational equations for the general case $pq < 0$ are equations (4.22) through (4.25).

These equations are first simplified by the reversible scaling transformation $|pq|e_i \rightarrow e_i$ yielding

$$\frac{D}{Dt} [(e_1 - e_3)^{1/2} E - \frac{e_1}{(e_1 - e_3)^{1/2}} K] = \frac{\sigma K}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} (2e_1 e_2 e_3)^{1/2} \quad 5.1$$

$$\frac{Du}{Dt} + \sigma \frac{D}{Dt} \left(\frac{2e_1 e_2}{e_3} \right)^{1/2} \frac{\Pi}{K} + \sigma \left(\frac{2e_1 e_2}{e_3} \right)^{1/2} \frac{\Pi \partial u}{K \partial x} + \frac{\partial}{\partial x} (e_1 + e_2 + e_3) = 0 \quad 5.2$$

$$\begin{aligned} \frac{D}{Dt} (2e_1 e_2 e_3)^{1/2} + 2(2e_1 e_2 e_3)^{1/2} \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} (e_1 e_2 + e_2 e_3 + e_3 e_1) \\ + [e_1 - (e_1 - e_3)] \frac{E}{K} \frac{D}{Dt} u = 0 \end{aligned} \quad 5.3$$

$$\frac{D}{Dt} \frac{K}{(e_1 - e_3)^{1/2}} = \frac{K}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} u \quad 5.4$$

where $D/Dt = \partial/\partial t + u\partial/\partial x$, and then reduced to a matrix equation by operating on the dependent variables e_1 , e_2 , e_3 and u . The calculations are carried out in detail in Appendix III for the case $pq > 0$ using the expressions for the derivatives of complete elliptic integrals provided in Appendix II. The calculations for the case $pq < 0$ are very similar. After considerable algebra the matrix system (5.5) is obtained. A surprising and fortunate property of the matrix system (5.5) is the absence of the complete elliptic integral of the third kind and the occurrence of K and E only through the ratio $D=E/K$. The matrix system (5.5) and the analogous expression for the case $pq > 0$ are the most

important results of the modulational theory because of the wealth of information they contain.

2. The solitary wave chain; the limit $e_2 \rightarrow e_1$ for the case $pq < 0$.

The solitary wave chain limit $e_2 \rightarrow e_1$ is most easily obtained from the matrix system (5.5). For this reason we did not examine this limit in Chapter 4. Our procedure is to simplify the matrices in (5.5) by eliminating e_1 temporarily via the scaling transformations $|e_1| \rightarrow |pq|e_1$ followed by $|pq| \rightarrow 1/e_1$. We then define $\varepsilon = e_1 - e_2 = 1 - e_2$ and note that $\sigma(2e_1e_2e_3)^{1/2} \rightarrow \sigma(2e_3)^{1/2}(1-\varepsilon/2) + O(\varepsilon^2)$. As $\varepsilon \rightarrow 0$, $D \rightarrow 0$ logarithmically and so will be taken to be $O(\varepsilon^0)$. Using these expressions, e_2 is eliminated in favor of ε . We then perform the row manipulations (row 3) $\rightarrow$ (row 3) - (row 4) - e_3 (row 1) followed by the limit $\varepsilon \rightarrow 0$ and obtain the quasilinear system (5.6). If we define a new variable $v = \sigma(2e_3)^{1/2}$ and simplify (5.6) further we obtain:

$$\begin{bmatrix} 1-D & 1 & vD & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ vD/2 & 0 & 2-D & 0 \end{bmatrix} \frac{D}{Dt} \begin{bmatrix} e_1 \\ e_2 \\ u \end{bmatrix} + \begin{bmatrix} v & v & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & v & v \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} e_1 \\ e_2 \\ v \\ u \end{bmatrix} = 0 \quad 5.7$$

Row 2 and row 3 show that u is a double characteristic velocity and that there are two independent Riemann invariants:

$$\frac{d}{dt}(e_1 - e_2) = 0 \quad \text{on} \quad \frac{dx}{dt} = u \quad 5.8a$$

$$\begin{bmatrix} 1-D & 1 & D & 0 \\ (e_3-e_2)D & (e_1-e_3)D+e_2-e_1 & (e_1-e_2)(1-D) & 0 \\ e_2e_3 & e_1e_3 & e_1e_2 & [e_1-(e_1-e_3)D]\Sigma \\ e_3D & 0 & e_1(1-D) & \Sigma \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \frac{D}{Dt} e_3 \\ u \end{bmatrix}$$

+

$$\begin{bmatrix} \Sigma/e_1 & \Sigma/e_2 & \Sigma/e_3 & 2e_2 \\ 0 & 0 & 0 & 2(e_3-e_2)(e_1-e_2) \\ (e_2+e_3)\Sigma & (e_1+e_3)\Sigma & (e_1+e_2)\Sigma & 4e_1e_2e_3 \\ \Sigma & \Sigma & \Sigma & 2e_1e_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \frac{\partial}{\partial x} e_3 \\ u \end{bmatrix} = 0 \quad 5.5$$

$$\Sigma = \sigma(2e_1e_2e_3)^{1/2}, \quad D = E/K$$

$$\begin{bmatrix} 1-D & 1 & D & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -\sigma(2e_3)^{1/2} \\ e_3D & 0 & 1-D & \sigma(2e_3)^{1/2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \frac{D}{Dt} e_3 \\ u \end{bmatrix}$$

+

5.6

$$\begin{bmatrix} \sigma(2e_3)^{1/2} & \sigma(2e_3)^{1/2} & 2\sigma/(2e_3)^{1/2} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma(2e_3)^{1/2} & \sigma(2e_3)^{1/2} & \sigma(2e_3)^{1/2} & 2e_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \frac{\partial}{\partial x} e_3 \\ u \end{bmatrix} = 0$$

$$\frac{d}{dt}(v - u) = 0 \quad \text{on} \quad \frac{dx}{dt} = u \quad 5.8b$$

The result (5.8a) is consistent with the limit $e_1 - e_2 \sim \varepsilon \rightarrow 0$ if we assume that $e_1 - e_2 \sim \varepsilon$ everywhere at $t=0$, for then (5.8a) assures us that $e_1 - e_2$ will remain $O(\varepsilon)$. This fact allows us to reduce the matrix system (5.7) to a three by three system by identifying e_1 and e_2 in the column vector. Thus to lowest order we obtain:

$$\begin{bmatrix} 2-D & vD & 0 \\ 0 & 1 & -1 \\ vD/2 & 2-D & 0 \end{bmatrix} \frac{D}{Dt} \begin{bmatrix} e_2 \\ v \\ u \end{bmatrix} + \begin{bmatrix} 2v & 2 & 2 \\ 0 & 0 & 0 \\ 2 & v & v \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} e_2 \\ v \\ u \end{bmatrix} = 0 \quad 5.9$$

The characteristic velocities of (5.9) are obtained by making the usual simple wave assumption $\Psi = \Psi(\theta)$ where $\Psi = (e_2, v, u)$ with eigenvelocity λ defined by

$$\frac{dx}{dt} = \lambda = -\frac{\theta_t}{\theta_x} \quad 5.10$$

We thus convert (5.9) to an eigenvalue problem with right eigenvector $R = \Psi_x$ and left eigenvector L .

$$L \begin{bmatrix} (u-\lambda) \end{bmatrix} \begin{bmatrix} 2-D & vD & 0 \\ 0 & 1 & -1 \\ vD/2 & 2-D & 0 \end{bmatrix} + \begin{bmatrix} 2v & 2 & 2 \\ 0 & 0 & 0 \\ 2 & v & v \end{bmatrix} \frac{\partial}{\partial x} \Psi = 0 \quad 5.11$$

It can be easily verified that the eigenvelocities of (5.11) are

$$\lambda^0 = u, \quad \lambda^\pm = \frac{2v \pm 2\sqrt{2}}{2-D \pm vD/\sqrt{2}} \quad 5.12$$

and that the corresponding left eigenvectors are

$$L^0 = (0, 1, 0) \quad 5.13a$$

$$L^\pm = (1, \pm(D-2)/\sqrt{2-Dv}, \pm\sqrt{2}) \quad 5.13b$$

We now check the eigenvectors $L^0, L^\pm$ and simultaneously obtain the Riemann invariants $A^0, A^\pm$ by operating on (5.9) with (5.13a) and (5.13b). Operating on (5.9) with L^0 we obtain

$$(0, 1, -1) \frac{\partial}{\partial t} \begin{bmatrix} e_2 \\ v \\ u \end{bmatrix} + (0, u, -u) \frac{\partial}{\partial x} \begin{bmatrix} e_2 \\ v \\ u \end{bmatrix} = 0 \quad 5.14$$

reproducing the result (5.8b), i.e.

$$\frac{d}{dt}(v - u) \quad \text{on} \quad \frac{dx}{dt} = u \quad 5.15$$

and operating on (5.9) with $L^\pm$ we obtain:

$$(2-D\pm vD/\sqrt{2})(1,\pm 1/\sqrt{2},\pm 1/\sqrt{2})\frac{D}{Dt}\begin{bmatrix} e_2 \\ v \\ u \end{bmatrix}$$

+

5.16

$$(2v\pm 2\sqrt{2})(1,\pm 1/\sqrt{2},\pm 1/\sqrt{2})\frac{\partial}{\partial x}\begin{bmatrix} e_2 \\ v \\ u \end{bmatrix} = 0$$

or

$$\frac{d}{dt}(e_2\pm v/\sqrt{2}\pm u/\sqrt{2}) = 0 \quad \text{on} \quad \frac{dx}{dt} = u + \frac{2v \pm 2\sqrt{2}}{2-D \pm vD/\sqrt{2}} \quad 5.17$$

The general results are recovered by reversing the scaling transformation particular to the eigenvelocities, i.e.

$$\lambda \rightarrow \lambda/\sqrt{e_1} \quad , \quad e_2 \rightarrow e_2/e_1 \quad , \quad e_3 \rightarrow e_3/e_1 \quad 5.18$$

followed by $e_1 \rightarrow |pq|e_1$ in the expressions for the eigenvelocities and Riemann invariants. The final system of Riemann invariants and characteristics for the solitary wave chain are:

$$\begin{aligned} \frac{d}{dt}(\sqrt{|pq|}e_3 - u) &= 0 \\ \frac{d}{dt}(e_1 - e_2) &= 0 \end{aligned} \quad \text{on} \quad \frac{dx}{dt} = u \quad 5.19$$

$$\frac{d}{dt}(|pq|e_2 \pm \sqrt{|pq|e_3} \pm u/\sqrt{2}) = 0$$

on

5.20

$$\frac{dx}{dt} = u + 2\sqrt{2} \left[\frac{\sqrt{e_3/e_2} \pm 1}{2 - D \pm D\sqrt{e_3/e_2}} \right]$$

For later comparison we also write the eigenvelocities $\lambda^\pm$ to lowest order in D recalling that D is a (logarithmically) small quantity:

$$\lambda^\pm = \sqrt{2|pq|e_3} \pm \sqrt{2|pq|e_2} [1 + D(1 - e_3/e_2)/2] + O(D^2, \varepsilon) \quad 5.21$$

3. *Numerical solution of the eigenvalue problem; determination of the global hyperbolic character of the case $pq < 0$.*

The quasilinear system (5.5) is amenable to the analysis so far applied only to the limiting cases. System (5.5) can be written more compactly as

$$A \frac{D}{Dt} \Psi + B \frac{\partial}{\partial x} \Psi = 0 \quad 5.22$$

where A and B are the matrices in (5.5) and Ψ is the column vector composed of the dependent variables. The simple wave ansatz $\Psi = \Psi(\theta)$ with $\lambda = -\theta_t / \theta_x$ yields the generalized eigenvalue problem

$$[(u-\lambda)A + B] R = 0 \quad 5.23$$

with eigenvector $R=\psi_x$ and eigenvelocity λ . The eigenvelocity λ is determined by (5.23) to be one of four functions of e_1, e_2, e_3 and u ; however, the infinite domain of these parameters is an unnecessarily large range to survey numerically. Symmetries of the system allow one to categorize the eigenvelocities and eigenvectors as members of a bounded two parameter family. The first and most obvious symmetry we exploit is the invariance of the eigenvectors and simple shifting of the eigenvelocities λ under a Galilean transformation. The most convenient frame of reference for our analysis is clearly a frame where the phase velocity $u=0$. Thus, as a result of Galilean invariance, the parameter range is reduced to the three fold continuum $0 \leq e_3 \leq e_2 \leq e_1$. This domain can be further reduced by exploiting the symmetry inherent in the scaling transformation $|pq|e_i \rightarrow e_i$.

Recall that (5.5) has been simplified by the transformation $|pq|e_i \rightarrow e_i$. This means that a particular solution of (5.5) actually represents an entire family of solutions with different values of pq . Solutions differ qualitatively only for different values of the ratios of the e_i 's. Thus we choose a convenient reference system for which $|pq|=1/e_1$. In summary, the simplifying transformation consists of the replacements $e_1 \rightarrow 1, e_2/e_1 \rightarrow e_2, e_3/e_1 \rightarrow e_3$ and $(\lambda-u)/\sqrt{e_1} \rightarrow \lambda$. Our unbounded domain collapses to the triangular region $0 \leq e_3 \leq e_2 \leq 1$. Using the above symmetry considerations, the generalized eigenvalue problem reduces to (5.24).

$$\begin{aligned}
 & \begin{bmatrix} \Sigma & \Sigma/e_2 & \Sigma/e_3 & 2e_2 \\ 0 & 0 & 0 & 2(e_3-e_2)(1-e_2) \\ \Sigma(e_2+e_3) & \Sigma(1+e_3) & \Sigma(1+e_2) & 4e_2e_3 \\ \Sigma & \Sigma & \Sigma & 2e_3 \end{bmatrix} \Psi_x \\
 & = \\
 & \lambda \begin{bmatrix} 1-D & 1 & D & 0 \\ (e_3-e_2)D & (1-e_3)D+e_2-1 & (1-e_2)(1-D) & 0 \\ e_2e_3 & e_3 & e_2 & [1-(1-e_3)D]\Sigma \\ e_3D & 0 & e_1(1-D) & \Sigma \end{bmatrix} \Psi_x
 \end{aligned}
 \tag{5.24}$$

The system (5.24) was solved as is using the EISPACK generalized eigenvalue problem routine over the full range of the rescaled parameters. In Figure 5.1a the equivalent parameter domain $0 \leq e_3/e_2 \leq 1$, $0 \leq e_2 \leq 1$ is represented as a square in the $e_2, e_3/e_2$ plane. The contours are level curves of constant $|pk|$ which indicate the general trend to long wavelength waves ($k \rightarrow 0$ as $e_2 \rightarrow 1$). The points A, B, C, D on the boundary and the interior point E are labels for the waveforms in Figure 5.1b. The illustrations in Figure 5.1b indicate the qualitative form of the nonlinear carrier wave in the various regions of parameter space. Thus the limit $e_2 \rightarrow 0$ corresponds to modulations of a linear carrier; the limit $e_3/e_2 \rightarrow 1$ to linear waves on a finite carrier; the limit $e_2 \rightarrow 1$ to modulations of solitary wave chain on a finite carrier; the limit $e_3/e_2 \rightarrow 0$ to modulations of a nonlinear carrier wave with vanishing

intensity minima (and uniform phase); and a general interior point to modulations of a general nonlinear carrier.

The characteristic velocities corresponding to a particular value of e_2 and e_3 are plotted as a vertical coordinate above the $e_2, e_3/e_2$ plane. Figures 5.2a,b,c are vertical planes defined by the value of e_3/e_2 and show the curves of intersection with the characteristic surfaces of the eigenvelocities. Note that the characteristic velocities are real throughout the entire $e_2, e_3/e_2$ plane. They are also distinct except for limiting cases of measure zero. We conclude that *the modulational equations for the case $pq < 0$ are everywhere hyperbolic and therefore stable*. The numerical values for the eigenvelocities on the boundaries agree with the limiting cases of Chapter 4.

Note also the fine structure of the top and bottom eigenvelocity curves in Figure 5.2a,b,c as $e_2 \rightarrow 1$. There is a very rapid convergence to the asymptotic values for these curves at $e_2 = 1$; observe the nearly vertical segment at the terminus of the curves. This feature is real and is due to the logarithmic dependence of these eigenvelocities upon the parameter e_2 near the boundary. Recall that $D \rightarrow 0$ logarithmically with e_2 and that (5.21) shows that the eigenvelocities $\lambda^\pm$ are asymptotically linear in D .

4. The analytic expression for form of the characteristic polynomial of the eigenvelocity λ .

Figure 5.1. (a) Reduced parameter domain and (b) associated qualitative waves forms for the case $pq < 0$. The lines in (a) are level curves of $|pk|$.

Figure 5.2. Planes of intersection with the real eigenvelocity surfaces for the values (a) $e_2/e_3=0$, (b) $e_2/e_3=.5$ and (c) $e_2/e_3=1.0$.

In the preceeding section the eigenvalue problem (5.24) was solved numerically. The characteristic polynomial for the eigenvelocities can be obtained by computing the secular determinant algebraically and is particularly useful for extracting limiting forms as a check with the results of Chapter 4 and the numerical results of the last section. The algebra was carried out using the MIT MACSYMA symbolic interpreter available on the ARPA network. The usual tedious and error prone calculations were reduced using MACSYMA. The following forth order polynomial in λ is the result of these calculations:

$$\begin{aligned}
 & [(D^2 e_3 - e_2)^2 - 2(D-1)^2(e_2 + e_3 D^2) + (D-1)^4](2e_2 e_3)^{1/2} \lambda^4 \\
 & + \\
 & -16(D-1)De_2 e_3 (De_3 - e_2 + 1 - D) \lambda^3 \\
 & + \\
 & -4 \left[\begin{aligned} & D^2 e_2 e_3^2 + D^2 e_3^2 + D^2 e_2^2 e_3 - 4De_2^2 e_3 + e_2^2 e_3 \\ & - 6D^2 e_2 e_3 + 6De_2 e_3 - 2e_2 e_3 + D^2 e_3 - 2De_3 + e_3 \\ & + e_2^3 + D^2 e_2^2 + 2De_2^2 - 2e_2^2 + D^2 e_2 - 2De_2 + e_2 \end{aligned} \right] (2e_2 e_3)^{1/2} \lambda^2 \\
 & + \\
 & 4(e_2 - 1)^2 (e_3 - e_2)^2 (2e_2 e_3)^{1/2} = 0
 \end{aligned} \tag{5.25}$$

There are four natural limits of (5.25) to consider that serve as check on the numerical results of the last section. These limits were introduced in Figures 5.1a,b where they were labeled by the letters A, B, C, and D. The following expressions are the corresponding limits of (5.25):

A) $e_2 \rightarrow 0$; e_2/e_3 constant.

$$(\lambda^2 - 2)^2 = 0 \quad 5.26$$

B) $e_3/e_2 \rightarrow 1$; e_2 constant.

$$(\lambda - \sqrt{2 + \sqrt{2e_2}})(\lambda - \sqrt{2 - \sqrt{2e_2}})(\lambda + \sqrt{2(1 - e_2)})^2 = 0 \quad 5.27$$

C) $e_2 \rightarrow 1$; e_3/e_2 constant.

$$\lambda^2 [\lambda - \sqrt{2e_3} \pm \sqrt{2(1 - (1 - e_3)D/2)}] = 0 \quad 5.28$$

D) $e_3/e_2 \rightarrow 0$, e_2 constant.

$$\begin{aligned} & - [e_2 - (D-1)^2] \lambda^4 + 4e_2 [e_2^2 + D^2 e_2 + 2D(2 - 2e_2 + D^2 - 2D + 1)] \lambda^2 \\ & - 4(e_2 - 1)^2 e_2^2 = 0 \end{aligned} \quad 5.29$$

The four cases A) through D) agree quantitatively with the results of the last section. Note in particular the linear dependence on D in case C) as $e_2 \rightarrow 1$ in agreement with (5.21).

The most general parameterization of (5.25) and the limits (5.26) through (5.29) are recovered via $\lambda \rightarrow (\lambda - u)/\sqrt{e_1}$, $e_2 \rightarrow e_2/e_1$, $e_3 \rightarrow e_3/e_1$ followed by $e_1 \rightarrow |pq|e_1$.

5. Comparison with earlier results.

G. Rowlands¹ studied the stability of the cnoidal waves of the CSE for both cases $pq < 0$ and $pq > 0$. He considered long wavelength

perturbations about uniform cnoidal wave solutions of the CSE. The particular uniform carrier waves examined by Rowlands correspond, for the case $pq < 0$, to our limit $e_3/e_2 \rightarrow 0$. These are the wave forms illustrated in frame D Figure 5.1b. This is a nontrivial limit, as can be appreciated by examining the characteristic polynomial (5.29) for the eigenvelocities λ . Nevertheless, Rowlands' results are in exact agreement with ours for this limiting case. This is best illustrated by proving the equivalence of Rowlands' dispersion relationship for the velocities of long wavelength perturbations with the characteristic polynomial determining the eigenvelocities of our long wavelength modulations.

Rowlands' dispersion relationship for perturbations is written (using his notation):

$$\left(\frac{\lambda}{l}\right)^4 + D\left(\frac{\lambda}{l}\right)^2 + A = 0 \quad 5.30$$

Where λ_1 is the frequency and l is the wavelength of long wavelength perturbations. We identify our characteristic velocity λ with Rowlands' phase velocity λ_1/l and show that (5.30) reduces to (5.29) when the coefficients D and A are converted to our variables. Rowlands gives the following expressions for the coefficients A and D in (5.30):

$$A = a_1 b_1 \quad , \quad D = a_1 + b_1 + a_2 b_2 \quad 5.31a$$

$$a_1 = \frac{4|b|k^2}{1+k^2} \left(\frac{-A_1}{A_2 H} \right) \quad , \quad b_1 = \frac{4|b|k^2}{1+k^2} \left(\frac{-A_2}{A_1 H} \right) \quad 5.31b$$

$$a_2 b_2 = \frac{4|b|k^2}{1+k^2} - \frac{(A_2 - A_1 A_3)^2 k^2}{A_1 A_2 H^2 (1-k^2)} \quad 5.31c$$

where, in our variables,

$$A_1 = 1-D \quad , \quad A_2 = 1 - \frac{(1+e_2/e_1)D}{1-e_2/e_1} \quad 5.32a$$

$$A_3 = 1 - \frac{2D}{1-e_2/e_1} \quad , \quad H = 1 - \frac{e_1 + e_2}{2} \quad 5.32b$$

$$2|b| = e_1 + e_2 \quad , \quad k^2 = e_2/e_1 \quad , \quad D = D(k^2) \quad 5.32c$$

We insert the above expressions into (5.30) and then multiply (5.30) by $-H^2(e_2-e_1)^2$. To compare the result with (5.27), we evaluate (5.30) for the value $e_1=1$; the resulting equations are identical. It is a satisfying confirmation of both Whitham's and Rowlands' independent methods that identical results for a nontrivial limit are obtained.

B. The general modulational system $pq>0$.

1. Reduction of the modulational equations to matrix form for the case $pq>0$.

Our study of the general case $pq > 0$ commences with the four separate modulational equations (4.92) through (4.95). If these equations are simplified via the reversible scaling transformation $pqe_1 \rightarrow e_1$, we obtain

$$\frac{D}{Dt} [(e_1 - e_3)^{1/2} E + \frac{e_3}{(e_1 - e_3)^{1/2}} K] = \frac{-\sigma K}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} (-2e_1 e_2 e_3)^{1/2} \quad 5.33$$

$$\frac{Du}{Dt} + \sigma \frac{D}{Dt} \left(\frac{-2e_2 e_3}{e_1} \right)^{1/2} \frac{\Pi}{K} + \sigma \left(\frac{-2e_2 e_3}{e_1} \right)^{1/2} \frac{\Pi}{K} \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} (e_1 + e_2 + e_3) = 0 \quad 5.34$$

$$\begin{aligned} \frac{D}{Dt} \sigma (-2e_1 e_2 e_3)^{1/2} + 2\sigma (-2e_1 e_2 e_3)^{1/2} \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} (e_1 e_2 + e_2 e_3 + e_3 e_1) \\ + [e_3 + (e_1 - e_3) \frac{E}{K}] \frac{D}{Dt} u = 0 \end{aligned} \quad 5.35$$

$$\frac{D}{Dt} \frac{K}{(e_1 - e_3)^{1/2}} = \frac{K}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} u \quad 5.36$$

where $D/Dt = \partial/\partial t + u\partial/\partial x$. As for the case $pq < 0$, (5.33) through (5.36) can be written in matrix form if the differentiations, are carried out as operations of the dependent variables e_1, e_2, e_3 and u . In order to carry out the differentiations we again need the expressions for derivatives of complete elliptic integrals from Appendix II. The detailed reduction of (5.33) through (5.36) is carried out in Appendix III. Matrix system (5.37) is the result of that analysis.

$$\begin{bmatrix} D & 1 & 1-D & 0 \\ (e_2-e_3)(1-D) & (e_1-e_3)D+e_3-e_2 & (e_2-e_1)D & 0 \\ e_2e_3 & e_1e_3 & e_1e_2 & -[e_3+(e_1-e_3)D]\Sigma \\ e_3(1-D) & 0 & e_1D & -\Sigma \end{bmatrix} \frac{D}{Dt} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ u \end{bmatrix} + \begin{bmatrix} \Sigma/e_1 & \Sigma/e_2 & \Sigma/e_3 & 2e_2 \\ 0 & 0 & 0 & 2(e_1-e_3)(e_2-e_3) \\ (e_2+e_3)\Sigma & (e_1+e_3)\Sigma & (e_1+e_2)\Sigma & 4e_1e_2e_3 \\ \Sigma & \Sigma & \Sigma & 2e_1e_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ u \end{bmatrix} = 0 \quad 5.37$$

$$\Sigma = \sigma(-2e_1e_2e_3)^{1/2} \quad , \quad D = E/K$$

2. The solitary wave chain limit $e_2, e_3 \rightarrow 0$ for the case $pq > 0$.

The solitary wave chain is obtained when the roots e_1, e_3 coalesce. Because $e_3 < 0$ and $e_2 > 0$, the solitary wave limit is obtained only when e_2 and e_3 both approach zero. Perhaps the easiest way to extract this limit of the matrix system (5.37) is to introduce the ordering parameter ε into (5.37) via $e_2, e_3 \rightarrow \varepsilon e_2, \varepsilon e_3$. The ordering parameter ε is removed from the column vectors (5.37) by multiplying columns two and three by an additional factor ε . The matrices in (5.37) are then examined in the limit $\varepsilon \rightarrow 0$ after common factors of ε are removed from rows two, three and four. The resulting system can be converted to an eigenvalue problem via the simple wave ansatz with eigenvelocity $\lambda = -\theta_1/\theta_x$ and is given explicitly in (6.38).

$$L \left[(u-\lambda) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{D} \begin{bmatrix} 0 & 0 & 0 & 0 \\ v(D-1) & 0 & 0 & 2e_2 \\ v(D-1) & 0 & 0 & 2e_3 \\ -1 & 0 & 0 & 0 \end{bmatrix} \right] R = 0 \quad 5.38$$

$$v = \sigma(-2e_2e_3)^{1/2} \quad , \quad R = \Psi_x$$

The secular determinant corresponding to eigenvalue problem (5.38) yields

$$(u - \lambda)^4 = 0 \quad 5.39$$

and reveals that u is a four fold degenerate eigenvelocity. However, (5.38) cannot be classified as hyperbolic, as only two linearly independent left eigenvectors exist. The two left eigenvectors L are

$$L = \left\{ \begin{array}{l} (1, 0, 0, 0) \\ (0, 1, -e_2/e_3, (1-e_2/e_3)v(D-1)/D) \end{array} \right\} \quad 5.40$$

In the next sections we shall see that (5.38) is the limit of an elliptic system with imaginary components for the eigenvelocities of $O(\varepsilon^{1/2})$ for $|e_3| < e_2$ and $O(\varepsilon)$ for $|e_3| > e_2$.

All attempts to find an ordering for which (5.37) reduces to a hyperbolic system in the lowest order have failed. Thus we conclude that the solitary wave chain is not hyperbolic and only marginally stable. This may seem to be in contradiction to the known properties of the multisoliton solutions of the CSE which show a remarkable degree of

stability. However, the solitary wave chain limit of the modulational theory is a degenerate case of general soliton chains because the ordering of the modulational theory demands that adjacent solitary waves be nearly identical in all parameters. The persistence of an elliptic character may indicate a tendency of adjacent nearly identical solitary waves to coalesce. We shall not pursue this problem further but move on to an examination of the general case $pq > 0$.

3. Determination of the global elliptic character of the case $pq > 0$; numerical solution of the general eigenvalue problem.

The general matrix system (5.37) can be converted to a local eigenvalue problem suitable for the study of the local character of the waves. In order to study the local character of the dynamics over the full range of parameters we again make use of the symmetries of the system. We first use Galilean invariance and shift our local frame to one with $u=0$. As with the case $pq < 0$, we also make use of the scaling properties of (5.37) which allows us to renormalize the local magnitudes of the roots e_1 , e_2 and e_3 to the local value of e_1 . Thus we restore the general case via $e_i \rightarrow pqe_i$ in the matrices of (5.37) and then set $pq=1/e_1$ - equivalent to the transformation $e_1 \rightarrow 1$, $e_2/e_1 \rightarrow e_2$, $e_3/e_1 \rightarrow e_3$ and $(\lambda-u)/\sqrt{e_1} \rightarrow \lambda$. The simple wave ansatz yields the following generalized eigenvalue problem with eigenvelocity $\lambda = -\theta_t/\theta_x$:

$$\begin{aligned}
 & \begin{bmatrix} \Sigma & \Sigma/e_2 & \Sigma/e_3 & 2e_2 \\ 0 & 0 & 0 & 2(1-e_2)(e_2-e_3) \\ \Sigma(e_2+e_3) & \Sigma(1+e_3) & \Sigma(1+e_2) & 4e_2e_3 \\ \Sigma & \Sigma & \Sigma & 2e_3 \end{bmatrix} \Psi_x \\
 & = \\
 & \lambda \begin{bmatrix} D & 1 & 1-D & 0 \\ (e_2-e_3)(1-D) & (1-e_3)De_3-e_2 & (e_2-1)D & 0 \\ e_2e_3 & e_3 & e_2 & -[(1-e_3)D+e_3]\Sigma \\ e_3(1-D) & 0 & D & -\Sigma \end{bmatrix} \Psi_x
 \end{aligned}
 \tag{5.41}$$

with $\Psi = (e_1, e_2, e_3, u)$. The generalized eigenvalue problem (5.41) was solved numerically over the reduced parameter domain $0 \leq e_2 \leq 1, 0 \leq 3/(e_3-1) \leq 1$. We use the quantity $e_2/(e_3-1)$ because e_3 ranges from negative infinity to zero. The parameter domain is shown in Figure 5.3a along with contour lines that are the level curves of $|pk|$. The contours clearly show that the solitary wave limit is isolated to the single point $e_3=e_2=0$. The points labeled A,B,C,D and E correspond to the qualitative wave forms in Figure 5.3b. As for the case $pq < 0$, the eigenvelocities are shown as loci of intersection with vertical planes above the $e_2, e_3/(e_3-1)$ plane for fixed values of e_2 . Figures 5.4a through 5.4f show the real parts of the eigenvelocities as thick lines and the imaginary parts as thin lines. Note that the solitary wave limit is the only point where the imaginary part of the eigenvelocities

vanish. We conclude that the case $pq > 0$ is everywhere elliptic and therefore unstable.

4. The analytic expression for the characteristic polynomial of the eigenvelocities for the case $pq > 0$.

As for the case $pq < 0$, the eigenvalue problem (5.40) can be solved analytically for the eigenvelocities. Again, the MIT MACSYMA symbolic interpreter proved to be very helpful in deducing the following result for λ :

$$\begin{aligned}
 & \{ [e_3(D-1)^2 - D^2]^2 - 2e_2e_3(D-1)^2 + (e_2 - D^2)^2 - D^4 \} (-2e_2e_3)^{1/2} \lambda^4 \\
 & + \\
 & 16(1-D)De_2e_3(De_3 - e_3 + e_2 - D)\lambda^3 \\
 & + \\
 & 4 \left[\begin{aligned} & D^2e_2e_3^2 - 2De_2e_3^2 + e_2e_3^2 + D^2e_3^2 - 2De_3^2 + e_3^2 \\ & + D^2e_2^2e_3 + 2De_2^2e_3 - 6D^2e_2e_3 + 6De_2e_3 - 2e_2e_3 \\ & + D^2e_3 + e_3^3 - D^2e_2^2 - 4De_2^2 + e_2^2 + D^2e_2 \end{aligned} \right] (-2e_2e_3)^{1/2} \lambda^2 \\
 & + \\
 & 4(e_2-1)^2(e_3-e_2)^2(-2e_2e_3)^{1/2} = 0
 \end{aligned} \tag{5.42}$$

The physically most interesting limits of (5.42) correspond to the cases A, C and D of Figure 5.3a. They are:

A) $e_2 \rightarrow 1$; e_3 constant:

Figure 5.3. (a) Reduced parameter domain and (b) associated qualitative wave forms for the case $pq > 0$. The lines in (a) are level curves of $|pk|$.

Figure 5.4. Planes of intersection with the real eigenvelocity surfaces (thick lines) and imaginary eigenvelocity surfaces (thin lines) for the values $e_2 = 0, .01, .1, .5, .9$ and 1 . ((a) through (f)).

$$(\lambda + \sqrt{2}e_3 - \frac{2}{\sqrt{2}e_3})^2(\lambda - \sqrt{2}e_3 + \sqrt{-2})(\lambda - \sqrt{2}e_3 - \sqrt{-2}) \quad 5.43$$

C) $e_2=0$; e_3 constant:

$$\lambda^2 = \frac{|2e_3|[D^2+e_3(D-1)^2]\pm 4|e_3|D(1-D)\sqrt{-|e_3|}}{[D^2-e_3(D-1)^2]^2} \quad 5.44$$

D) $e_2, e_3 \rightarrow 0(\varepsilon)$:

$$\lambda^2 = \frac{2(|e_3|-e_2)\pm 4\sqrt{-e_2}|e_3| - |e_3|^3/D^2 + e_2^3/D^2}{D^2} + O(\varepsilon^{3/2}) \quad 5.45$$

The limiting case B) of Figure 5.3a corresponds to $e_3 \rightarrow \infty$ and has been excluded as highly singular and unphysical. Not only does $k \rightarrow \infty$ but (3.43) shows that the phase of the cnoidal waves is also highly singular in this limit. The general parameterization of (5.42) through (5.45) is recovered via $\lambda \rightarrow \{\lambda - u\}/\sqrt{e_1}$, $e_2 \rightarrow e_2/e_1$, $e_3 \rightarrow e_3/e_1$ followed by $e_1 \rightarrow pqe_1$.

5. Comparison with other results.

G. Rowlands¹ also studied the case $pq > 0$. For comparison, we shall examine the imaginary part of the eigenvelocities in the solitary wave limit. Rowlands' results for solitary waves are limiting cases of cnoidal waves for $pq > 0$ with uniform phase. These waves correspond to our special cases $e_2=0$, $e_3 \neq 0$ and $e_3=0$, $e_2 \neq 0$. He reports the imaginary

part of the eigenvelocities for the case $e_2=0$, $e_3 \rightarrow 0$ and in private communication the results for $e_3=0$, $e_2 \rightarrow 0$. The results are

$$\begin{aligned} e_2=0, e_3 \rightarrow 0: \quad \text{Im}\left(\frac{\lambda_1}{l}\right) &\sim (\sqrt{|e_3|} \ln \sqrt{|e_3|})^{3/2} \\ e_3=0, e_2 \rightarrow 0: \quad \text{Im}\left(\frac{\lambda_1}{l}\right) &\sim \sqrt{e_2} \ln \sqrt{e_2} \end{aligned} \tag{5.46}$$

where our λ is Rowlands' λ_1/l . These results agree with the corresponding limits of (5.45).

References

- 1 G. Rowlands, *On the stability of the non-linear Schrödinger equation*, J. Inst. Maths. Applics. 13, 367(1974).

Chapter 6

The modulational theory of Chapters 2 through 5 applies strictly to wave motion in one spatial dimension. As such, the theory accurately describes the dynamics of weakly modulated waves (1) in wave guides where transverse coordinates are integrated out of the dynamics or (2) that are initiated as one dimensional in lowest order and that maintain a hyperbolic character so that weak transverse modulations of $O(\varepsilon)$ remain $O(\varepsilon)$ as time progresses. As an example that violates this later requirement, consider the soliton solutions of the one dimensional CSE for $pq > 0$ initiated in a fully two dimensional space. Although the solitons are highly stable in one dimension, they are known to be modulationally unstable to transverse perturbations^{1,2}. Furthermore, this instability does not saturate and evolves towards a set of isolated singularities and decaying oscillations. (For a numerical example of this instability, see Section B of Chapter 7.)

These conclusions are not obtained from the IST theory because it does not apply to fully two dimensional dynamics. Rather, it can be shown^{3,4} using a very general inequality, that broad classes of initial

conditions must evolve towards singularities. The onset of singularities means that the ordering of terms leading to the CSE have been violated and that higher order dissipative, inertial or other saturation effects are important. These effects are highly specific to the physical problem that is under study. (References 5 and 6 give typical examples of the more complex models needed to treat these effects in the physics of high frequency oscillations of plasmas.)

In this chapter we shall be concerned with the dynamics of the CSE for the case $pq > 0$ generalized to two dimensions by modification of the dispersive term via the simple replacement $\partial^2/\partial x^2 \rightarrow \partial^2/\partial x^2 + \partial^2/\partial y^2$. We study the case $pq > 0$ because it predominates in applications in nonlinear optics and plasma physics. Frequently, especially in applications to nonlinear optics, the CSE is written with the role of slow modulations in time replaced by slow modulations along the primary axis of beam propagation^{7,8}. This is an example of the simple relation of boundary value problems and initial value problems common in wave propagation. The required modification of the two dimensional CSE reduces to the replacement $\partial/\partial t \rightarrow \partial/\partial z$ where z is the axial beam coordinate. Clearly, all of the results of this chapter will carry over to this case by the simple replacement $t \rightarrow z$ and an obvious change in interpretation.

As the dynamics of the CSE in two dimensions are dominated by non-saturable modulational instabilities leading to singularities, we shall be concerned with deriving and categorizing the different types of self-similar solutions of the CSE. This approach is in line with a general philosophy of solutions of partial differential equations⁹:

self-similar solutions are related to asymptotic developments of general motions when the dynamics are such that natural scales of a disturbance have undergone a significant change. That is, it is frequently the case that intrinsic dynamics associated with a differential equation and its symmetry properties predominate over the details of initial conditions and cause self-similar solutions to emerge in lowest order in an asymptotic expansion. (As a graphic example, we note the universal character of shock waves and smoke rings.) As these notions are somewhat heuristic, the role of self-similar solutions is often clarified by observations from numerical experiments. Thus we shall complement the results of this chapter with several numerical experiments in Chapter 7. We shall also introduce a systematic procedure for obtaining conservation laws for a physical system possessing a Lagrangian and use it to obtain nine integral invariants for the CSE.

A. Mathematical Preliminaries

In this section we introduce a specific mathematization of symmetry notions that we shall use to obtain particular solutions and conservation laws of the CSE. The methods that we use are presented in detail, together with a variety of applications, in Reference 10. References to the original work of Sophus Lie and to modern developments, especially by Russian workers, can be found in the bibliography of the aforementioned reference.

By a symmetry of a partial differential equation we shall mean a definite transformation of the independent and dependent variables that is continuously dependent on a real parameter ε and that leaves the form of the partial differential equation invariant. The transformation is a mapping $\mathbf{T}_\varepsilon$ parameterized by ε from original variables (say x, t, u) to transformed variables (say $\bar{x}, \bar{t}, \bar{u}$) specified by a set of functions and containing the identity transformation for the value $\varepsilon = 0$. It is convenient to indicate the transformation in the following way:

$$\begin{aligned}\mathbf{T}_\varepsilon: \quad \bar{x} &= X(x, t, u; \varepsilon) = x + \varepsilon \xi(x, t, u) \\ \bar{t} &= T(x, t, u; \varepsilon) = t + \varepsilon \tau(x, t, u) + O(\varepsilon^2) \\ \bar{u} &= U(x, t, u; \varepsilon) = u + \varepsilon \eta(x, t, u)\end{aligned}\tag{6.1}$$

As the transformation $\mathbf{T}_\varepsilon$ is continuously dependent on the parameter ε , the functions that constitute the transformation have power series expansions in ε . The coefficients of the linear terms in the expansion of the functions completely specify $\mathbf{T}_\varepsilon$. This can be easily demonstrated by examining the mapping $\mathbf{T}_\varepsilon$ of a general function $f(x, t, u)$ defined simply as a replacement of the (unbarred) arguments of f by their (barred) transformed counterparts, the functional form of f remaining unchanged. Consider the Taylor series expansion of f

$$\begin{aligned}f(\bar{x}, \bar{t}, \bar{u}) &= \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left[\frac{d}{d\varepsilon} \right]^n f(\bar{x}, \bar{t}, \bar{u}) \Big|_{\varepsilon=0} \\ &= \sum_{n=0}^{\infty} \frac{\Delta_\varepsilon^n}{n!} f(x, t, u)\end{aligned}\tag{6.2}$$

where

$$\begin{aligned}\Delta_\varepsilon &= \varepsilon \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} = \varepsilon \left[\frac{d\bar{x}}{d\varepsilon} \frac{\partial}{\partial \bar{x}} + \frac{d\bar{t}}{d\varepsilon} \frac{\partial}{\partial \bar{t}} + \frac{d\bar{u}}{d\varepsilon} \frac{\partial}{\partial \bar{u}} \right] \Big|_{\varepsilon=0} \\ &= \varepsilon \left[\xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} \right]\end{aligned}\tag{6.3}$$

so that we can write the Taylor series representation symbolically

$$f(\bar{x}, \bar{t}, \bar{u}) = e^{\Delta_\varepsilon} f(x, t, u) \tag{6.4}$$

Clearly the operator Δ_ε contains all of the information needed to construct the mapping $\mathbf{T}_\varepsilon$. The important functions ξ , τ and η are called the *generators* of the transformation. In the sequel we shall drop the ε subscript on the Δ operator. The generator for a variable is simply the result of the Δ operation on that variable, for example

$$\begin{aligned}\Delta x &= \varepsilon \xi \\ \Delta t &= \varepsilon \tau \\ \Delta u &= \varepsilon \eta\end{aligned}\tag{6.5}$$

In the sequel we shall study the effects of the mapping $\mathbf{T}_\varepsilon$ on a partial differential equation (the CSE) denoted

$$H(x, t, u, u_t, u_x, u_{tt}, u_{xt}, \dots) = 0 \tag{6.6}$$

with a particular solution of (6.6) written

$$u = \theta(x, t) \quad 6.7$$

The argument list in the partial differential form H includes partial derivatives of the dependent variable u . The derivative of a dependent variable, like any other function, must have an associated generator. The mapping of the partial differential equation H is facilitated if we augment $\mathbf{T}_\epsilon$ with the generators of the various partial derivatives appearing in H . We write

$$\begin{aligned} \mathbf{T}_\epsilon: \quad \bar{x} &= x + \epsilon \xi(x, t, u)|_{u=\theta(x, t)} \\ \bar{t} &= t + \epsilon \tau(x, t, u)|_{u=\theta(x, t)} \\ \bar{u} &= u + \epsilon \eta(x, t, u)|_{u=\theta(x, t)} + O(\epsilon^2) \\ \bar{u}_x &= u_x + \epsilon \Pi \\ &\vdots \end{aligned} \quad 6.8$$

with $\epsilon \Pi = \Delta u_x$. The argument list of Π is left unspecified for the moment as we shall see shortly that Π can be expressed in terms of the derivatives of the fundamental generators ξ , τ and η and derivatives of the dependent variable. Note also that the transformation $\mathbf{T}_\epsilon$ is defined only when the solution function $\theta(x, t)$ is substituted for the dependent variable symbol u . The fact that the solution function $\theta(x, t)$ is involved in the definition of $\mathbf{T}_\epsilon$ will eventually allow us to incorporate the constraint equation $H = 0$ in $\mathbf{T}_\epsilon$ liberating θ to the role of an arbitrary test function. This freedom will allow us to determine the functional form of the generators when $\mathbf{T}_\epsilon$ leaves the form of the partial differential equation $H = 0$ invariant.

A compact algorithm has been found by this author for the computation of the generators of derivatives of dependent variables like Π . We shall need a new operator¹¹ δ defined as the linear part of the mapping $\mathbf{T}_\varepsilon$ with the independent variables x, t held fixed. For example

$$\delta u = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{ \bar{u}(x, t) - u(x, t) \} \quad 6.9a$$

whereas

$$\Delta u = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{ \bar{u}(\bar{x}, \bar{t}) - u(x, t) \} \quad 6.9b$$

Note in particular that $\delta x = \delta t = 0$. It is trivial to show that

$$\delta = \Delta - \Delta x \cdot \nabla \quad 6.10$$

where

$$\Delta x = \Delta(x, t) = \varepsilon(\xi, \tau) = \varepsilon \xi \quad 6.11$$

and

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) \quad 6.12$$

The δ operator is useful because it commutes with partial differentiation with respect to the independent variables, i.e. $\delta u_x = (\delta u)_x$. We also introduce an additional notation defining partial

differentiation denoted as a subscript. The following conventions are adopted:

$$\xi_x = \xi_\alpha[\alpha, t, u]|_{\alpha=x, u=\theta(x, t)} \quad 6.13$$

$$(\xi)_x = \frac{\partial}{\partial x} \xi = \xi_x + \xi_u \theta_x \quad 6.14$$

We are now prepared to compute an expression for the generator Π .

We have

$$\begin{aligned} \Delta u_x &= \delta u_x + \Delta x \cdot \nabla u_x \\ &= (\delta u)_x + \Delta x \cdot \nabla u_x \end{aligned} \quad 6.15$$

but

$$(\Delta u)_x = (\delta u)_x + (\Delta x)_x \cdot u + \Delta x \cdot \nabla u_x \quad 6.16$$

so that if we subtract (6.16) from (6.15) we obtain

$$\Delta u_x = (\Delta u)_x - (\Delta x)_x \cdot \nabla u \quad 6.17$$

or

$$\Pi = (\eta)_x - (\xi)_x \cdot \nabla u \quad 6.18$$

Written out fully, (6.18) is

$$\Pi = \eta_x + \eta_u \theta_x - \xi_x \theta_x - \xi_u \theta_x^2 - \tau_x \theta_t - \tau_u \theta_x \theta_t \quad 6.19$$

The generators for higher order derivatives can be computed in a fully analogous manner. The generalization of the arguments of this section to the case three independent and two dependent variables is straightforward. We shall now derive the totality of transformations $\mathbf{T}_t$ that leave the CSE invariant. These transformations form a group $\mathbf{G}$ under composition of mappings. However, we shall not make use of any formal group theoretical properties of the transformations.

B. Derivation of the group $\mathbf{G}$ of continuous symmetry transformations of the CSE in two dimensions.

1. The Lie algorithm .

A symmetry transformation $\mathbf{T}_t \in \mathbf{G}$ of the CSE in two dimensions is specified by its generators:

$$\begin{aligned} \mathbf{T}_t: \quad \bar{x} &= x + \varepsilon \xi(x, y, t, E, E^*) \\ \bar{y} &= y + \varepsilon \sigma(x, y, t, E, E^*) \\ \bar{t} &= t + \varepsilon \tau(x, y, t, E, E^*) \\ E &= E + \varepsilon \eta(x, y, t, E, E^*) + O(\varepsilon^2) \\ E^* &= E^* + \varepsilon \eta^* \\ E_{\bar{t}} &= E_t + \varepsilon \Pi \\ E_{\bar{x}\bar{x}} &= E_{xx} + \varepsilon \Omega \\ E_{\bar{y}\bar{y}} &= E_{yy} + \varepsilon \Psi \end{aligned} \quad 6.20$$

Recall that the dependent variables E, E^* in the argument list of the generators must be evaluated at $E=\theta(x,y,t), E^*=\theta(x,y,t)^*$ before applying $\mathbf{T}_\epsilon$ to the CSE $H=0$ or to its solution function $\theta(x,y,t)$. $\mathbf{T}_\epsilon$ by definition leaves the CSE invariant, that is

$$H = H(E, E^*, E_t, E_{xx}, E_{yy})|_{E=\theta(\bar{x}, \bar{y}, \bar{t})} = 0 \quad 6.21$$

iff

$$H(E, E^*, E_t, E_{xx}, E_{yy})|_{E=\theta(x,y,t)} = 0 \quad 6.22$$

where

$$H = iE_t + E_{xx} + E_{yy} + |E|^2E \quad 6.23$$

and $E = \theta(x,y,t)$ is the solution of (6.23). If we represent (6.21) as a formal power series in ϵ it is clear that (6.21) holds provided $\Delta H = 0$. Writing out ΔH in terms of the generators of $\mathbf{T}_\epsilon$ we obtain:

$$\begin{aligned} & [H_E \eta + H_{E^*} \eta^* + H_{E_t} \Pi + H_{E_{xx}} \Omega + H_{E_{yy}} \Psi]_{E=\theta(x,y,t)} \\ & = 2\theta \theta^* \eta + \theta^2 \eta^* + i\Pi + \Omega + \Psi = 0 \end{aligned} \quad 6.24$$

The conditions (6.22) and (6.24) together are sufficient to determine the generators ξ, σ, τ and η and therefore $\mathbf{T}_\epsilon$. The generators Ω and Ψ are written out in Appendix IV and involve derivatives of the generators ξ, σ, τ and η as well as derivatives of the solution function

θ . The constraint (6.22) is incorporated into (6.24) by eliminating the combination $\theta_{xx} + \theta_{yy}$ via

$$\theta_{xx} + \theta_{yy} = -i\theta_t - |\theta|^2\theta \quad 6.25$$

whenever the combination $\theta_{xx} + \theta_{yy}$ occurs as a factor. Also, when θ_{yy} occurs as an isolated factor, it is eliminated in favor of θ_{xx} via

$$\theta_{yy} = -i\theta_t - \theta_{xx} - |\theta|^2\theta \quad 6.26$$

Once these substitutions have been accomplished, (6.24) can be regarded as a constraint on the functional form of the generators ξ , σ , τ and η for an arbitrary test function θ . Thus (6.24) holds in general only if the coefficients of distinct powers of θ , derivatives of θ and products of derivatives of θ vanish. The resulting equations are called the determining equations for the generators of the group **G**.

2. Solution of the determining equations for the generators of symmetry transformations of the CSE in two dimensions.

The full set of determining equations for the generators are listed in Appendix IV. This set is overdetermined and can be reduced to the following equivalent system of equations:

$$\begin{aligned}\tau &= \tau(t) \\ \xi &= \xi(x,y,t) \\ \sigma &= \sigma(x,y,t) \\ \eta &= f(x,y,t)E\end{aligned}\tag{6.27a}$$

$$f + f^* + 2\sigma_y = 0\tag{6.27b}$$

$$if_t + f_{xx} + f_{yy} = 0\tag{6.27c}$$

$$i\xi_t + \xi_{xx} + \xi_{yy} - 2f_x = 0\tag{6.27d}$$

$$i\sigma_t + \sigma_{xx} + \sigma_{yy} - 2f_y = 0\tag{6.27e}$$

$$\tau_t - 2\sigma_y = 0\tag{6.27f}$$

$$\xi_x - \sigma_y = 0\tag{6.27g}$$

$$\sigma_x + \xi_y = 0\tag{6.27h}$$

Care must be exercised when solving the system (6.27a) through (6.27h) to ensure that the most general solutions consistent with the

equations are obtained. To proceed with the solution of the reduced determining equations — we examine (6.27a,g,h,i) and conclude

$$\begin{aligned}\xi &= A(t)x + B(t)y + C(t) \\ \sigma &= A(t)y - B(t)x + D(t)\end{aligned}\tag{6.28}$$

so that $\xi_{xx}=\xi_{yy}=\sigma_{xx}=\sigma_{yy}=0$. Using (6.28) together with (6.27d,e) we find that

$$f = i \left[A_t \frac{x^2}{4} + D_t \frac{x}{2} + E_t \frac{y}{2} + A_t \frac{y^2}{4} + H(t) \right] \tag{6.29}$$

and that

$$B = a_0 \tag{6.30}$$

where a_0 is a complex or real number. Using (6.29) in (6.27c) we obtain

$$A_{tt} \frac{x^2}{4} + C_{tt} \frac{x}{2} + D_{tt} \frac{y}{2} + A_{tt} \frac{y^2}{4} + E_t - iA_t = 0 \tag{6.31}$$

which implies that

$$A_{tt} = C_{tt} = D_{tt} = 0 \tag{6.32a}$$

and

$$E_t = iA_t \tag{6.32b}$$

or

$$A = a_1 + a_2 t \quad 6.33a$$

$$C = a_3 + a_4 t \quad 6.33b$$

$$D = a_5 + a_6 t \quad 6.33c$$

$$E = i a_2 t + \beta \quad 6.33d$$

and so

$$f = i \left[a_2 \frac{x^2}{4} + a_4 \frac{x}{2} + a_6 \frac{y}{2} + a_2 \frac{y^2}{4} + \beta \right] - a_2 t \quad 6.34$$

But (6.27b) requires that

$$\operatorname{Re}(f) = -a_1 - a_2 t \quad 6.35$$

which restricts a_1, a_2, a_4, a_6 , and a_7 to be real numbers and gives

$$\beta = a_7 + i a_1 \quad 6.36$$

Using (6.27f) we have

$$\tau = a_8 + 2a_1 t + a_2 t^2 \quad 6.37$$

Thus the symmetry transformations T_t of the CSE in two dimensions contains nine arbitrary constants $a_0, a_1, \dots, a_8$ with a_1, a_2, a_4, a_6 and a_7 real and a_0, a_3, a_5 and a_8 complex numbers. The final expressions for the generators are:

$$\begin{aligned}\xi &= (a_1 + a_2 t)x + a_0 y + a_3 + a_4 t \\ \sigma &= (a_1 + a_2 t)y - a_0 x + a_5 + a_6 t \\ \tau &= 2a_1 t + a_2 t^2 + a_8 \\ \eta &= \{ i[a_2 \frac{x^2}{4} + a_4 \frac{x}{2} + a_6 \frac{y}{2} + a_2 \frac{y^2}{4} + a_7] - a_1 - a_2 t \} E\end{aligned}\tag{6.38}$$

These generators specify a nine parameter Lie group of symmetry transformations. In the sequel we will consider all of the parameters to be real constants and shall not explore the consequences of complex values for a_0, a_3, a_5 and a_8 .

C. Similarity solutions of the CSE in two dimensions.

Now that the symmetry group of the CSE has been obtained in terms of explicit functional forms for the generators, it is a straight forward matter to obtain particular solutions $E=\theta(x,y,t)$ intimately associated with the symmetries.

If T_t is a symmetry of the CSE then if

$$E = \theta(x, y, t)\tag{6.39}$$

is a solution of the CSE we must have

$$\bar{E} = \theta(\bar{x}, \bar{y}, \bar{t}) \quad 6.40$$

as a solution of the transformed equation. Expanding (6.40) as a power series in ε we obtain

$$E + \varepsilon \eta = \theta + \varepsilon \theta_x \xi + \varepsilon \theta_y \sigma + \varepsilon \theta_t \tau + O(\varepsilon^2) \quad 6.41$$

Using (6.39) we have the condition

$$\eta = \theta_x \xi + \theta_y \sigma + \theta_t \tau \quad 6.42$$

which can be written in a compact way using the operator Δ :

$$\Delta [E - \theta(x, y, t)] = 0 \quad 6.43$$

Equation (6.42) is a quasilinear partial differential equation, as the generators are known functions of x, y, t and θ . The standard method for integrating (6.42) is to convert it to a system of four ordinary differential equations — the method of *integration along characteristic curves*. We encountered this ~~same~~ technique for the more complex quasilinear systems of Chapter 4.

It is convenient to consider the solution of (6.42) as a three dimensional surface in the four dimensional space (θ, x, y, t) . Such a surface can be parameterized by any convenient coordinate system (s, r, w) provided that the Jacobian determinant $J(s, r, w/x, y, t)$ does not vanish. That is we assume

$$\theta[s,r,w] = \theta(x(s,r,w), y(s,r,w), t(s,r,w)) \quad 6.44$$

The curves $s_0=s(x,y,t)$, $r_0=r(x,y,t)$ and $w_0=w(x,y,t)$ lie in the (x,y,t) subspace and can be chosen to individually share the symmetry of the CSE, i.e.

$$\Delta s = \Delta r = \Delta w = 0 \quad 6.45$$

Functions so defined are *similarity variables*. Proceeding with the solution of (6.42), we consider the derivative of (6.44) with respect to the variable s :

$$\frac{d\theta}{ds} = \theta_x \frac{dx}{ds} + \theta_y \frac{dy}{ds} + \theta_t \frac{dt}{ds} \quad 6.46$$

Comparing (6.46) with (6.42) we identify the following

$$\begin{aligned} \frac{d\theta}{ds} &= \eta(x(s,r,w), y(s,r,w), t(s,r,w), \theta(s,r,w)) & a) \\ \frac{dx}{ds} &= \xi(x(s,r,w), y(s,r,w), t(s,r,w)) & b) \\ \frac{dy}{ds} &= \sigma(x(s,r,w), y(s,r,w), t(s,r,w)) & c) \\ \frac{dt}{ds} &= \tau(x(s,r,w), y(s,r,w), t(s,r,w)) & d) \end{aligned} \quad 6.47$$

Note that (6.38) shows that ξ , σ and τ do not depend on θ and so (6.47b,c,d) can be integrated separately to yield $x(s,r,w)$, $y(s,r,w)$ and

$t(s,r,w)$. Subsequently (6.47a) can be integrated. The integration of (6.47a) is facilitated by the fact that η can be written

$$\eta = f(x,y,t)\theta = f[s,r,w]\theta \quad 6.48$$

so that

$$\frac{d \ln \theta}{ds} = f[s,r,w] \quad 6.49$$

yielding

$$E = \theta = g[r,w] \int ds f[s,r,w] \quad 6.50$$

with f given by (6.34) and g an arbitrary function of (r,w) . The function θ specified in (6.50) has the symmetry of the CSE. It must also satisfy the CSE itself. Thus the next step in the procedure is to substitute (6.50) into the CSE. The result of this substitution is a new differential equation for the unknown function $g[r,w]$. In the case of the CSE, the partial differential equation for $g[r,w]$ can be further simplified by separation in polar coordinates $\rho^2 = r^2 + w^2$, $\varphi = \arctan(r/w)$. This, of course, restricts our study to self-similar waves with axial symmetry. To obtain the final expressions for the similarity solution E , we eliminate s,r,w in favor of x,y,t . The details of the procedure are in Appendix V. The results are summarized in the four cases listed on the following two pages.

Similarity Solutions of the CSE in Two Dimensions

Case I: $A = a_2 a_8 - a_1^2 \neq 0$, $a_2 \neq 0$.

$$\rho^2 = \frac{x^2 + y^2}{1 + \frac{(a_2 t + a_1)^2}{A}} \quad ; \quad m = 0, \pm 1, \pm 2, \dots$$

$$E = \frac{h(\rho) \exp i \left[m\theta + \frac{a_2 \rho^2}{4A} (a_2 t + a_1) + \frac{(a_7 - a_0 m)}{A^{1/2}} \tan^{-1} \left(\frac{a_2 t + a_1}{A^{1/2}} \right) \right]}{\left(1 + \frac{(a_2 t + a_1)^2}{A} \right)^{1/2}} \quad 6.51$$

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} h - \frac{m^2}{\rho^2} h - \frac{a_2 (a_7 - a_0 m)}{A} h - \frac{a_2^2 \rho^2}{4A} h + h^3 = 0$$

Case II: $A = a_2 a_8 - a_1^2 = 0$, $a_2 \neq 0$.

$$\rho^2 = \frac{x^2 + y^2}{(a_1 + a_2 t)^2} \quad ; \quad m = 0, \pm 1, \pm 2, \dots$$

$$E = \frac{h(\rho) \exp i \left[m\theta + \frac{a_2 \rho^2}{4} (a_1 + a_2 t) + \frac{a_0 m - a_7}{a_1 + a_2 t} \right]}{a_1 + a_2 t} \quad 6.52$$

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} h - \frac{m^2}{\rho^2} h - a_2 (a_7 - a_0 m) h + h^3 = 0$$

Case III: $a_2 = 0$, $a_1 \neq 0$.

$$\rho^2 = \frac{x^2 + y^2}{2a_1 t + a_8} \quad ; \quad m = 0, \pm 1, \pm 2, \dots$$

$$E = \frac{h(\rho) \exp i \left[m\theta + \frac{a_1}{4} \rho^2 + \frac{a_7 - a_8}{2a_1} \ln(2a_1 t + a_8) \right]}{(2a_1 t + a_8)^{1/2}} \quad 6.53$$

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} h - \frac{m^2}{\rho^2} h - (a_7 - a_8 m) h + \frac{a_1}{4} \rho^2 h + h^3 = 0$$

Case IV: $a_2 = 0$, $a_1 = 0$.

$$\rho^2 = x^2 + y^2 \quad ; \quad m = 0, \pm 1, \pm 2, \dots$$

$$E = h(\rho) \exp i \left[m\theta + \frac{a_7 - a_8 m}{a_8} t \right] \quad 6.54$$

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} h - \frac{m^2}{\rho^2} h - \frac{a_7 - a_8 m}{a_8} h + h^3 = 0$$

D. Conservation laws for the CSE in two dimensions.

The algorithm of Section B has been adapted by this author to derive conservation laws for the CSE. Our starting point is the fundamental theorem due to E. Noether¹² establishing the connection between a symmetry of the action corresponding to a physical system that

possesses a Lagrangian and a conservation law. It is convenient to consider an action element defined as the product of the Lagrangian and a volume element of the space of the independent variables x, y, t :

$$L(E, E^*, \nabla E, \nabla E^*) dx dy dt \quad 6.55$$

with

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t} \right) \quad 6.56$$

Now suppose that T_ξ is a symmetry of the action integral or equivalently an action element. Then we must have

$$\Delta L dx dy dt = 0 \quad 6.57$$

It is straight forward to show that

$$\Delta x dy dt = dx dy dt \nabla \cdot \Delta x \quad 6.58$$

where

$$\Delta x = \Delta(x, y, t) = \varepsilon(\xi, \sigma, \tau) = \varepsilon \xi \quad 6.59$$

so that (6.57) becomes

$$\Delta L + L \nabla \cdot \Delta x = 0 \quad 6.60$$

Expanding (6.60) we write

$$L_E \Delta E + L_E^* \Delta E^* + L_{\nabla E} \cdot \Delta \nabla E + L_{\nabla E}^* \cdot \Delta \nabla E^* + L_V \cdot \Delta x = 0 \quad 6.61$$

At this point we use $\Delta = \delta + \Delta x \cdot \nabla$ to replace the Δ operator in all but the last term in (6.61). Recall that the δ operation commutes with partial differentiation so that $\nabla \delta = \delta \nabla$ and allows us to write

$$L_{\nabla E} \cdot \delta \nabla E = \nabla \cdot L_{\nabla E} \delta E - \delta E \nabla \cdot L_{\nabla E} \quad 6.62$$

and a similar expression for the conjugate relation. Using (6.62) and its conjugate in (6.61) we obtain the final important expression:

$$\delta E \frac{\delta L}{\delta E} + \delta E^* \frac{\delta L}{\delta E^*} + \nabla \cdot (L \Delta x + L_{\nabla E} \delta E + L_{\nabla E}^* \delta E^*) = 0 \quad 6.63$$

where the symbol $\delta L / \delta E$ is used for the Euler-Lagrange variational derivatives

$$\frac{\delta L}{\delta E} = L_E - \nabla \cdot L_{\nabla E} \quad 6.64a$$

and

$$\frac{\delta L}{\delta E^*} = L_E^* - \nabla \cdot L_{\nabla E}^* \quad 6.64b$$

For the CSE the Lagrangian is defined so that (6.64a) and (6.64b) yield the CSE and its conjugate equation. For a general physical system

the Lagrangian, if it exists, is defined similarly so that the Euler-Lagrange equations yield the dynamical equations of motion and therefore vanish. Thus Noether's theorem in the form (6.63) reveals that a symmetry of the action integral necessarily implies

$$\nabla \cdot [L \Delta x + L_{\nabla E} \delta E + L_{\nabla E^*} \delta E^*] = 0 \quad 6.65$$

which is in the form of a general conservation law.

It is only natural to enquire at this point what the symmetries of the action are. Once a symmetry $\mathbf{T}_\epsilon$ is known we can write a conservation law by inserting the generators of $\mathbf{T}_\epsilon$ into (6.65). The algorithm of Section B provides a prescription for systematically deducing a class of symmetry transformations. We return to equation (6.61) and regard it now in a new light — as the determining equation for the generators of a symmetry transformation. It is convenient to first list the generators

$$\begin{aligned} \Delta x &= \epsilon \xi \\ \Delta y &= \epsilon \sigma & \Delta E_t &= \epsilon \Pi & \Delta E_t^* &= \epsilon \Pi^* \\ \Delta t &= \epsilon \tau & \Delta E_x &= \epsilon \Phi & \Delta E_x^* &= \epsilon \Phi^* \\ \Delta E &= \epsilon \eta & \Delta E_y &= \epsilon \Theta & \Delta E_y^* &= \epsilon \Theta^* \\ \Delta E^* &= \epsilon \eta^* \end{aligned} \quad 6.66$$

and to define the compact notation

$$\Phi = (\phi, \Theta, \Pi) \quad \xi = (\xi, \sigma, \tau) \quad 6.67$$

so that (6.61) is written

$$L_E \eta + L_{E^*} \eta^* + L_{\nabla E} \cdot \Phi + L_{\nabla E^*} \cdot \Phi^* + L \nabla \cdot \xi = 0 \quad 6.68$$

Our simple prescription for Φ in terms of ξ , E and E^* of Section A can be written in dyadic form as follows:

$$\Phi = \nabla(\eta) - \nabla(\xi) : \nabla E \quad 6.69$$

Following the algorithm of Section B, we expand (6.68) and insert the result into (6.68) using the Lagrangian

$$L = i \frac{E_t E^*}{2} - i \frac{E_t^* E}{2} - \nabla E^2 + \frac{|E|^4}{2} \quad 6.70$$

for the CSE in two dimensions. The resulting expression can be regarded as a constraint on the functional form of the generators ξ . The fact that the Euler-Lagrange equations $\delta L / \delta E$, $\delta L / \delta E^*$ must vanish for a physical system is a separate problem from determining the symmetry group of the action. Thus we do not require that they vanish. Thus we can treat $E(x, y, t)$, and $E(x, y, t)^*$ as unconstrained test functions. As a consequence of these considerations and the general validity of (6.68), the coefficients of distinct products, derivatives and products of derivatives of the test functions E, E^* must vanish. The resulting system of equations are again the determining equations for the group of symmetry transformations of the action.

The calculation for the generators of the symmetries of the action is summarized in Appendix IV. The result is the same group as for the CSE obtained in Section B. As the group contains nine parameters, we can write down nine conservation laws by inserting the generators obtained by setting all parameters but one in succession equal to zero. Although the construction of the complete conservation laws in differential form is now a simple matter of substitution, we shall list only the integral invariants associated with the laws. Integral invariants can be derived from (6.65) by examining (6.65) in the expanded form

$$\begin{aligned} \frac{\partial}{\partial t}(L\Delta t + L_{E_t}\delta E + L_{E_t^*}\delta E^*) + \frac{\partial}{\partial x}(L\Delta x + L_{E_x}\delta E + L_{E_x^*}\delta E^*) \\ + \frac{\partial}{\partial y}(L\Delta y + L_{E_y}\delta E + L_{E_y^*}\delta E^*) = 0 \end{aligned} \quad 6.71$$

It is customary to define fluxes Q_i , R_i and a density P_i and identify them with the quantities in parenthesis so that (6.71) can be written symbolically

$$\frac{\partial}{\partial t}P_i + \frac{\partial}{\partial x}Q_i + \frac{\partial}{\partial y}R_i = 0 \quad 6.72$$

where $i = 1, 9$ labels the individual conservation laws. If (6.72) is integrated over the entire domain of the independent variables x and y then the flux terms vanish under two conditions: i) The x, y domain is infinite and the fluxes vanish as $x, y \rightarrow \infty$, ii) The x, y domain is periodic

so that fluxes at opposite sides of the fundamental rectangle cancel one another. If in addition the dependent variables are smooth functions of x, y we have

$$\frac{\partial}{\partial t} \iint dx dy P_i = \frac{\partial}{\partial t} I_i = 0 \quad 6.73$$

Thus the quantities I_i are integral invariants of the motion in time. In the following list we enumerate the nine integral invariants of the CSE in two dimensions. Each invariant is identified by the parameter a_i occurring in the generators (6.38) of the complete Lie group of the CSE. (To construct the generator of the symmetry yielding I_i , we set all of the a_i 's other than a_i equal to zero.) We also include a mnemonic that suggests the intuitive symmetry:

a_0 , rotation (angular momentum):

$$I_0 = \iint dx dy \left[i \frac{E^*}{2} (xE_y - yE_x) + \text{c.c.} \right] \quad 6.74a$$

a_1 , stretching (rescaling):

$$I_1 = \iint dx dy \left\{ \left[i \frac{E}{2} (xE_x^* + yE_y^*) + \text{c.c.} \right] - 2t (|E_x|^2 + |E_y|^2 - \frac{|E|^4}{2}) \right\} \quad 6.74b$$

a_2 , projective:

$$I_2 = \iint dx dy \{ [-it \frac{E}{2} (xE_x^* + yE_y^*) + c.c.] + \frac{|E|^2}{4} (x^2 + y^2) + t^2 (|E_x|^2 + |E_y|^2 - \frac{|E|^4}{2}) \}$$

6.74c

a_3 , displacements in x (linear momentum):

$$I_3 = \iint dx dy i (EE_x^* - E^*E_x)$$

6.74d

a_4 , galilean translation in x:

$$I_4 = \iint dx dy [-it (EE_x^* - E^*E_x) + |E|^2 x]$$

6.74e

a_5 , displacements in y (linear momentum):

$$I_5 = \iint dx dy i (EE_y^* - E^*E_y)$$

6.74f

a_6 , galilean translation in y:

$$I_6 = E^*E_y [-it (EE_y^* + E^*E_y) + |E|^2 y]$$

6.74g

a_7 , phase shifts:

$$I_7 = \iint dx dy |E|^2$$

6.74h

a_8 , displacements in time (energy):

$$I_8 = \iint dx dy (|E_x|^2 + |E_y|^2 - \frac{|E|^4}{2})$$

6.74i

E. An application of the integral invariants of the CSE in two dimensions.

The integral invariants listed above can be combined to yield useful information about the average dynamics of localized packets in two dimensions. We first define an intensity weighted averaging operator:

$$\langle f \rangle = \frac{\iint dx dy |E|^2 f}{\iint dx dy} \quad 6.75$$

The following relations follow from (6.74a) through (6.74i) and the definition (6.75):

$$\begin{aligned} \frac{\langle x \rangle}{4} &= \frac{I_4 + I_3 t}{I_7} \\ \frac{\langle y \rangle}{4} &= \frac{I_6 + I_5 t}{I_7} \end{aligned} \quad 6.76$$

$$\begin{aligned} \frac{\langle (x - \langle x \rangle)^2 \rangle}{4} + \frac{\langle (y - \langle y \rangle)^2 \rangle}{4} = \\ \frac{I_2 + I_1 t + I_8 t^2}{I_7} - 4 \left(\frac{I_4 + I_3 t}{I_7} \right)^2 - 4 \left(\frac{I_6 + I_5 t}{I_7} \right)^2 \end{aligned} \quad 6.77$$

If our frame of reference is such that $\langle x \rangle = \langle y \rangle = 0$ then (6.77) is written

$$\frac{\langle x^2 + y^2 \rangle}{4} = \frac{I_2 + I_1 t + I_8 t^2}{I_7} \quad 6.78$$

The relation (6.78) can also be obtained by the procedures used to derive Ehrenfest relations¹³ in Quantum Mechanics and is equivalent to that derived in References 3 and 4. The present derivation, however, clarifies the simple relation of the constants to particular integral invariants connected with specific symmetries of the differential equation.

The self-similar solutions and conservation laws derived in this chapter were obtained by a single unified approach. In Chapter 7 we shall use this information together with the data from select numerical experiments to deduce the general character of modulational instability in two dimensions.

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Chapter 7

The purpose of computing is insight, not numbers

- Hamming¹

We shall follow Hamming's credo in this chapter and explore the synergistic relationship of the symmetry considerations of Chapter 6 and the simple numerical experiments of this chapter. Our philosophy is that of an experimentalist looking for general characterization of the phenomena of modulational instability in two dimensions.

In Section A we shall solve the eigenvalue problem implicit in the similarity solutions (6.51) through (6.54). The solution of the eigenvalue problem will yield a family of eigenmodes that depend continuously on a parameter λ and that can be labeled by two positive integers n and m . For the complete self-similar solutions, λ is a particular function of time. Thus the dynamics of the CSE continuously transform perfect members of a n,m family of similarity solutions into other members of the same family.

In Section B we shall solve the two dimensional CSE numerically using a code written by the author. The initial data for two of the numerical experiments are eigenmodes of type IV self-similar waves. We shall directly observe the modulational instability of these modes and follow the course of the dynamics through the nonlinear stage. The intensity maxima of the unstable envelope asymptotically approach singularities of an apparently universal type. (The modulational instability of the self-similar eigenmodes is the reason we inserted the qualifier perfect in the last paragraph.) We shall also present a reasonable upper bound for the number of singularities that can emerge from the modulational instability of a localized wave that will also serve as a ball-park estimate for the number that actually emerge.

In Section C we shall follow in detail the collapse of a Gaussian packet towards a singularity through a change of scale of one order of magnitude. In order to do this we use a one dimensional code written in cylindrical coordinates. As the Gaussian collapses, we monitor the intensity-weighted variance and the maximum amplitude at the origin to determine the scaling with time and also overplot an appropriately rescaled profile of a $n,m=0$ eigenmode. We shall verify that, over the time of observation, the Gaussian approaches a shape close to that of a $n,m=0$ eigenmode and adopts a temporal scaling consistent with a type III self-similar mode.

A. The self-similar eigenmodes.

It is convenient to combine the two ordinary differential equations that occur in the definitions of the self-similar solutions (6.51) through (6.54) obtained in Chapter 6 into one equation.

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dh}{d\rho} \right) - \frac{m^2}{\rho^2} h - \alpha h - \beta \rho^2 h + h^3 = 0 \quad 7.1$$

As written, equation (7.1) corresponds to cases I and III. Cases II and IV are included for the value $\beta=0$. We look for localized solutions that are nonsingular at $\rho=0$ and that are normalizable. The norm of a solution h is defined to be the value of the integral $N = \int dx dy h^2 = \int 2\pi \rho d\rho h(\rho)^2$. We require that the norm of h yield a finite result. These requirements can be posed as the boundary conditions $\rho h(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$ and $0 \leq h(\rho)$ as $\rho \rightarrow 0$. If we assume that a power series expansion for $h(\rho)$ exists at the origin $\rho=0$, then $h(\rho)$ takes the following form as $\rho \rightarrow 0$

$$\begin{aligned} h(\rho) &\sim a \rho^m, \quad m=1,2,3,\dots \\ h(\rho) &\sim b - b \frac{(b^2 - \alpha)}{4} \rho^2, \quad m=0 \end{aligned} \quad 7.2$$

where a and b are positive constants. We choose $a=b=1$ and use (7.2) to obtain the first few points in a finite difference algorithm for the numerical solution of (7.2). We search for particular values of α and β for which $h(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$. With β a parameter, this defines an eigenvalue problem with eigenvalues $\alpha_{n,m}$. The range of possible α and β for the eigenvalue problem can be determined by examining the asymptotic properties of the linearized version of (7.1). Our first step in the

derivation of the asymptotic solution to the linearized equation is to transform (7.1) to canonical form² via the transformation $u=\rho^2$. We obtain

$$\frac{d^2h}{du^2} + \frac{1}{u} \frac{dh}{du} - \left[\frac{\beta}{4} + \frac{\alpha}{4u} + \frac{m^2}{4u^2} \right] h = 0 \quad 7.3$$

The coefficients of (7.3) are now bounded as $u \rightarrow \infty$. The asymptotic solution of (7.3) now falls into one of three categories:

$$\beta \neq 0, \alpha \neq 0: \quad h \sim e^{\sqrt{\beta}u/2} u^{-1/2-\alpha/4\sqrt{\beta}} P + e^{-\sqrt{\beta}u/2} u^{-1/2+\alpha/4\sqrt{\beta}} Q \quad 7.4a$$

$$\beta = 0, \alpha \neq 0: \quad h \sim e^{\sqrt{\alpha}u} u^{-1/4} P + e^{-\sqrt{\alpha}u} u^{-1/4} Q \quad 7.4b$$

$$\beta = 0, \alpha = 0: \quad h = p_0 u^{m/2} + q_0 u^{-m/2} \quad 7.4c$$

where P and Q are asymptotic series of the form

$$\begin{aligned} \beta \neq 0, \alpha \neq 0: \quad P, Q &= \sum_{n=0}^{\infty} p_n, q_n u^{-n} \\ \beta = 0, \alpha \neq 0: \quad P, Q &= \sum_{n=0}^{\infty} p_n, q_n u^{-n/2} \end{aligned} \quad 7.5$$

The coefficients p_n and q_n for $n > 0$ are determined recursively in terms of the arbitrary constants p_0 and q_0 . Thus the asymptotic form of h is a superposition of two linearly independent solutions. For the particular case $\alpha = 0$, (7.3) reduces to Bessel's equation, and the

solution is determined for all u in closed form in terms of Bessel's functions of imaginary argument, I_γ and K_γ ; in particular

$$\beta \neq 0, \alpha = 0: \quad h = p_0 I_{m/2}(\frac{\sqrt{\beta} u}{2}) + q_0 K_{m/2}(\frac{\sqrt{\beta} u}{2}) \quad 7.6$$

Upon examination of the asymptotic solutions (7.4a,b,c), we conclude that eigenfunctions can exist only for $\beta > 0$ or $\beta = 0$ and $\alpha > 0$. The solution of the nonlinear problem is thus reduced to finding particular values for α and β within these ranges that asymptotically connect to the solution of the linearized problem for which $P=0$.

For $\beta=0$ and the boundary conditions (7.2), there exist a countable set of eigenvalues $\alpha_{n,m}$ for $n=0,1,2,\dots$ for which $\rho h(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$. For $\beta > 0$, the eigenvalues $\alpha_{n,m}$ monotonically decrease with β until $\alpha_{n,m}=0$ which occurs for $\beta=\beta_{n,m}$. Also, the scaling transformation $h \rightarrow \lambda h$, $\rho \rightarrow \rho/\lambda$, $c \rightarrow \lambda^2 c$ and $\beta \rightarrow \lambda^4 \beta$ leaves (7.1) invariant. This implies that if $h=\theta(\rho)$ is a particular solution of (7.1) then so is $h=\theta(\rho/\lambda)/\lambda$ for the same function θ . In particular the eigenfunctions for a given n , m and β are members of a λ family and we may label them $h_{n,m}^\beta(\lambda; \rho)$. It is convenient to define $h_{n,m}^\beta(\rho) = h_{n,m}^\beta(1; \rho)$ and to note that $h_{n,m}^\beta(\rho/\lambda)/\lambda = h_{n,m}^\beta(\lambda; \rho)$. The value of the norm $N_{n,m}^\beta$ is unique for a λ family:

$$N_{n,m}^\beta = \int 2\pi \rho d\rho \frac{h_{n,m}^\beta(\rho/\lambda)^2}{\lambda^2} = \int 2\pi r dr h_{n,m}^\beta(r)^2 \quad 7.7$$

The norm $N_{n,m}^\beta$ monotonically increases with β and takes on its lowest value $N_{n,m}^0$ for $\beta=0$. The index n is equal to the number of finite radial zeros of $h_{n,m}$. In Figures 7.1a,b we plot $h_{n,m}^0$ for $n=0,1; m=0,1,2,3$. In Table 7.1 we give the value of $N_{n,m}^0$ for these same eigenfunctions. For the complete self-similar solutions the value of the norm $N_{n,m}^\beta$ is identical to the integral invariant I_7 derived in Chapter 6 and so is conserved in time. Thus for any localized initial condition with a finite value of the invariant I_7 , the value of I_7 must exceed $N_{n,m}^0$ in order for the system to evolve towards a self-similar solution of topological type n,m^3 .

$$N_{n,m}^0$$

m	0	1	2	3
n = 0	11.71	48.30	89.75	132.4
n = 1	77.18	150.2	228.2	309.0

Table 7.1

The $n,m=0$ family of type IV self-similar waves was first obtained by Chiao⁴ et. al. who postulated the existence of self-trapped non-spreading optical beams in nonlinear media. These authors also noted the scaling properties of (7.1) (for $m,\beta=0$) and the invariant value of the norm - which they wrote in the form of a critical power for

self-trapping of a cylindrical beam. H. Haus⁵ noted that there are higher order solutions (our $m=0, n=0, 1, 2, \dots$ sequence) exhibiting concentric rings of intensity. The solutions discussed in the preceeding paragraph are simple generalizations of these modes to cases with angular and radial phase structure.

B. Some numerical experiments involving self-similar eigenmodes.

The numerical experiments of this section were performed on a CDC 7600 using a code written by the author that solves the CSE on a 64×64 spatial grid with periodic boundary conditions. High accuracy was maintained by using fast Fourier transform algorithms to advance the linear terms of the CSE. The accuracy of the algorithm was monitored by computing the values of two invariants of the CSE, in particular $I_7 = \iint dx dy |E|^2$ and $I_8 = \iint dx dy (|E_x|^2 + |E_y|^2 - |E|^4/2)$. The invariants remain constant to accuracies better than 1% for the experiments recorded in this section.

1. Collision of solitary waves.

The initial conditions for this experiment were the following two waveform.

$$E_1 = a \operatorname{sech} \left[\frac{a}{\sqrt{2}} \left(x - \frac{3L}{4} \right) \right] e^{-iux} \quad 7.8$$

Figure 7.1. Plots of the eigenfunction $h_{n,m}^0(r)$ for (a) $n=0$, $m=0, 1, 2, 3$ and (b) $n=1$, $m=0, 1, 2, 3$.

$$E_2 = bh_{0,0}^0 \{ b[(x - \frac{L}{4})^2 + (y - \frac{L}{2})^2]^{1/2} \} e^{iux} \quad 7.9$$

with $a=0.15$, $b=0.3$, $L=100$, $u=1.0$. The total field $E_T=E_1+E_2$ gives the two separated waves shown in the first frame of Figure 7.2. The frames of Figure 7.2 show the intensity of the field at various times. The initial velocities of the two waves are such that they return to their original positions at $t=100$ (recall that the boundary conditions are periodic). Thus the waveforms undergo collisions centered at the times $t=25$ and $t=75$.

E_1 is a one dimensional solitary wave and would be a true soliton if the dynamics were one dimensional. In two dimensions these solitary waves are known to be unstable to transverse perturbations (see References 1 and 2 Chapter 6). E_2 is a member of the type IV self-similar family of eigenmodes. In this experiment we perturb the two wave forms by a mutual collision. The intensity $|E|^2$ is normalized in each frame to the current maximum intensity on the grid.

Two features of the interaction are immediately apparent from the figure. First note the strong interference effects as the waves overlap. This feature would also be present for linear waves. The nonlinear phase shifts that occur during the interaction manifest themselves as the waves emerge from the collision and separate. The solitary wave has been strongly perturbed and is well into the nonlinear stages of instability by the time $t=50$. Recall that $|E|^2$ is normalized by the maximum intensity on the grid. By $t=100$ (after one additional

collision event) the central core of the evolving singularity of the solitary wave dominates all other features. Although the $h_{0,0}^0$ self-similar mode has also been perturbed and is also slowly collapsing, it is doing so on a much slower time scale and is relatively unchanged at $t=100$.

2. *Modulational instability of an $n=0, m=1$ eigenmode.*

For the second numerical experiment we choose as our initial condition a type IV $n=0, m=1$ self-similar eigenmode. The initial condition is:

$$E = ah_{0,1}^0 \{ a[(x - \frac{L}{2})^2 + (y - \frac{L}{2})^2]^{1/2} \} e^{i \arctan(x/y)} \quad 7.10$$

with $a=0.3$ and $L=40.0$. This eigenmode is pictured in the three frames of Figure 7.3. The intensity profile is annular and the phase structure rotates in a clockwise direction.

If the discrete grid were infinitely fine, the numerical algorithm perfectly accurate, and our initial condition a perfect eigenmode, then the dynamics of the CSE would advance the mode forward in time in accord with the full self-similar solution of type IV Chapter 6, and the intensity profile would remain unchanged indefinitely. However, the small errors implicit when interpolating $h_{0,0}^0$ onto the 64×64 grid are equivalent to a perturbed initial condition suitable for observing the instability of the mode. The annular envelope of the mode develops a modulation that propagates counter-clockwise (retrograde to the

Figure 7.2. The collision of a one dimensional solitary wave and a $n,m = 0$ type IV self-similar eigenmode. $|E|^2$ is shown at $t = 0, 10, 20, 40, 50$ and 100 .

Figure 7.3. The modulational instability of a $n=0, m=1$ type IV self-similar eigemode. $|E|^2$ and $\text{Re}(E)$ are plotted at $t = 0, 60$ and 70 . Only the positive contours of $\text{Re}(E)$ are shown.

Figure 7.4. The modulational instability of a $n=1, m=1$ type IV self-similar eigenmode. $|E|^2$ and $\text{Re}(E)$ are plotted at $t = 0, 80$ and 90 . Only the positive contours of $\text{Re}(E)$ are shown.

phase). By $t=50$ the instability has evolved into a rotating triple of localized intensity maxima that are rapidly evolving towards singularities. These singularities clearly resemble the type observed in the first numerical experiment.

It is natural to conjecture that the singularities are self-similar modes of a $n,m=0$ family of type I, II or III. The threshold value of the norm for this family is $N_{0,0}^0$ given in Table 7.1. The value of l_7 for this experiment is $N_{0,1}^0$. Dividing $N_{0,1}^0$ by $N_{0,0}^0$ we obtain the number 4.13. Thus the upper bound that we expect for the number of singularities that can emerge is 4. It is reasonable that a portion of the evolving waveform is not captured by the singularities, thus the fact that only three singularities do emerge is also reasonable.

3. *Modulational instability of an $n=1, m=1$ eigenmode.*

The third numerical experiment concerns an $n=1, m=1$ eigenmode corresponding to a type IV self-similar mode. The initial condition is therefore

$$E = ah_{1,1}^0 \{ a[(x - \frac{L}{2})^2 + (y - \frac{L}{2})^2]^{1/2} \} e^{i \arctan(x/y)} \quad 7.11$$

with $a=0.3$ and $L=50$. The evolution of this mode is depicted in the frames of Figure 7.4 at $t=0, 80, 90$. The inner annulus evolves in a manner identical with the case $n=0, m=1$. The modulational instability of the outer annulus is also just apparent in the last frame at $t=90$. Four distinct maxima can be discerned on the ridge of the outer annulus

aligned with the faces of the grid. These outer modulations are rotating counter-clockwise at a slower rate than those of the inner annulus. The fact that they have precipitated aligned with the faces of the grid suggest that they are more strongly correlated with the perturbed initial condition. It is possible that a more accurate treatment with the grid edges further removed from the mode would a different result with a larger number of maxima from the outset. Again, we can estimate the largest number of singularities that can emerge from the mode by dividing $N_{1,1}^0$ by $N_{0,0}^0$. The result is 12.8 so that the upper bound on the maximum number of singularities that can emerge is 12. At $t=90$, only 7 tentative and 3 obvious singularities have emerged. If the numerical calculation could be continued it is possible that the four maxima in the outer annulus would split to yield 8 singularities bringing the total number to 11.

C. Details of collapse of a Gaussian packet.

The three numerical experiments of the last section demonstrate that the self-similar solutions of type IV are modulationally unstable and that the intensity maxima of the modulations evolve into singularities. As a working hypothesis, we assume that these singularities are of a universal type. In this section we will attempt to identify this universal singularity with one of the singular self-similar solutions of type I, II or III given respectively by the expressions (6.51), (6.52) and (6.53). We further assume that the character of the universal singularity can be obtained by following the dynamics of a Gaussian wave that is initialized in such a manner that it

is guaranteed to collapse by virtue of the sign of the energy-like invariant I_0 appearing in the expression (6.78).

The Gaussian packet shares several common features with the observed singularities. They are single-humped functions with no angular phase structure in a local frame centered at the intensity maxima. Thus our candidates for the singularity among our self-similar waves are $n, m=0$ modes associated with the eigenvalue problem (7.1) and the related families of cases I, II, and III of Chapter 6.

The singular waves of type I must have a negative value for the constant A in (6.51), and this in turn implies that the associated eigenvalue problem (7.1) has a value $\beta < 0$ for the constant β . This would seem to rule out this case, as a negative value for β does not define a proper eigenvalue problem, for in this case the asymptotic expression (7.4a) reveals that h oscillates and decays like ρ^{-1} and therefore has an infinite norm. A similar argument holds for type III waves given by (6.53), for which β is always negative. Thus we are left with the type II wave for which $\beta=0$. This wave has a definite scaling with time that can be compared with our numerical experiment with the Gaussian. The results of the computation do not agree. In order to find satisfactory agreement with numerical experiment, we shall have to modify the arguments just given and admit self-similar waves that are not true describe eigenmodes with finite values of the norm $N_{0,0}^2$. We shall find best agreement when we assume that the singularity is actually a member of a continuum of modes with a small negative value for β that are very close to the eigenmode $h_{0,0}^0$ within the finite core region.

In order to follow the dynamics of a Gaussian accurately through a change of scale of one order of magnitude, we need a numerical grid with a high resolution. This is accomplished by solving the CSE in cylindrical coordinates and restricting the class of solutions to those with no angular dependence. This effectively reduces the two dimensional problem to one with only a single radial coordinate. We solve the CSE as a standard parabolic equation using an implicit tridiagonal inversion algorithm⁶ written in the single cylindrical coordinate r . The Gaussian initial condition is

$$E = ae^{-r^2/b^2} \quad 7.12$$

with $a = .3$ and $b = 10.0$ on a grid with $r_{\max} = 40.0$. The initial condition is plotted in Figure 7.5a and at $t=42$ in Figure 7.5b. The dotted line superimposed in Figures 7.5a,b is the magnitude of a $n,m=0$ eigenmode ($h_{0,0}^0$) rescaled to match the magnitude of the Gaussian at $r = 0$. Note that the scale of Figure 7.5b has been magnified by a factor of 10, so that the wave has collapsed through a change of scale of one order of magnitude. Figure 7.6 is actually two plots. The dashed curve shows the value of $|E|^{-1}$ versus time on the left hand scale indicating a singularity at $t=42^+$. The solid curve shows the value of the variance $\langle r^2 \rangle$ versus time and is referenced to the right hand scale. The dotted line is the computed value for $\langle r^2 \rangle$ from the initial conditions and expression (6.78). Note that the predicted and computed values for $\langle r^2 \rangle$ are in agreement.

Figure 7.5. $|E(r)|$ for a collapsing gaussian at time (a) $t = 0$ and (b) $t = 42$. The dashed line is $|E|$ for a $h_{0,0}^0$ eigenfunction (scaled to match the collapsing gaussian at $r=0$).

Figure 7.6. Time plots of $|E(r=0)|^{-1}$ (left scale) and $\langle r^2 \rangle$ (right scale) for a collapsing gaussian. The dotted line gives $\langle r^2 \rangle$ as it is predicted from the initial conditions.

It is apparent from Figure 7.6 that the core of the Gaussian is collapsing to a singularity (at $t \sim 42^+$) faster than is expected from (6.78) (which shows a singularity at $t \sim 70^+$). Evidently the core collapses rapidly leaving a residual tail of sufficient magnitude that the variance expression (6.78) is satisfied. This means that the core singularity must not decay exponentially as a $n, m=0$ must; rather, it must decay rather slowly and smoothly connect to the residual field that is left behind. However, Figure 7.5b shows that the precise form of the core is still very close to that of a $n, m=0$ mode.

Next we note the rapid acceleration of the collapse that is manifest in the rapid turn-over of the curve $|E|^{-1}$ in Figure 7.6. If we assume that $|E|$ obeys a power law, then we can extract the value for the time at which the singularity occurs (τ) and the exponent (γ) of the power law. These calculations are summarized in Table 7.2.

Using data from $t=30$ to $t=42$ we obtain $\gamma = .4828 \pm .002$ and $\tau = 42.202 \pm .002$. The value of the integral invariant I_7 was maintained to high accuracy during the calculation ($I_7 = 14.137$ at $t=0$, 14.137 at $t=42$). The invariant I_7 was conserved well until the very end of the calculation ($I_8 = -.0353$ at $t=0$, $-.0359$ at $t=40$ and $-.0479$ at $t=42$). Thus the values in Table 7.2 are questionable for $t > 41$.

Is this data consistent with our self-similar solutions? Clearly a type II wave is ruled out because the scaling of $|E|^{-1}$ in this case should be linear with time. Consider next a type III wave with a small but finite negative value for β (we write $\beta \sim 0^-$). A wave of this type

t	$\ln E $	$\frac{dt}{d\ln E }$	$-\frac{d}{dt}\left[\frac{dt}{d\ln E }\right]$	γ	τ
30	-0.437959				
31	-0.396640	23.1524			
32	-0.351575	21.1186	2.0472	0.4884	42.196
33	-0.301937	19.1186	2.0678	0.4836	42.201
34	-0.246632	16.9831	2.0755	0.4818	42.199
35	-0.184173	14.9071	2.0736	0.4822	42.197
36	0.112468	12.8359	2.0683	0.4835	42.197
37	0.028366	10.7706	2.0631	0.4847	42.200
38	0.073223	8.7098	2.0585	0.4858	42.205
39	0.20128	6.65366	2.0622	0.4849	42.212
40	0.37381	4.58533	2.1276	0.4700	42.214
41	0.63743	2.39839			
42	1.2077				

Table 7.2

If we assume $|E| = A(\tau - t)^{-\gamma}$ as $t \rightarrow \tau$ then:

$$\tau = t + \gamma \frac{dt}{d\ln|E|}, \quad \gamma = \frac{-1}{\frac{d}{dt}\left[\frac{dt}{d\ln|E|}\right]}$$

The constants γ and τ are computed from data between $t=30$ and $t=42$. The results are, $\langle\gamma\rangle = .4828 \pm 0.002$, $\langle\tau\rangle = 42.202 \pm 0.002$.

can be constructed to agree closely with a $n,m=0$ eigenmode for $\beta=0$ within any finite region - e.g. the collapsing core region. The fact the $\beta \sim 0^-$ pseudo eigenmode must eventually decay slower than exponential is consistent with the physical reality of the residual field that is left behind as the core collapses and to which the core must smoothly connect. Furthermore, the scaling of $|E|^{-1}$ with time for a type III wave gives a value $\gamma=0.5$ which is in fair agreement with the observed value $\gamma=0.48$. Thus it is a reasonable assumption that the singularities observed are limiting members ($\beta \sim 0^-$) of a type III self-similar family. They are quasi-eigenmodes because $\alpha \sim \alpha_{0,0}$ so that the core region is very close in form to $h_{0,0}^0$.

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Appendix I

In Chapter I we used simple physical arguments to derive the CSE as a model equation for the envelope of an electromagnetic wave propagating at right angles to a uniform magnetic field in a cold plasma. In this appendix we shall use a perturbation theory approach to solve the same problem.

We use a right handed coordinate system with $\hat{z}=\hat{x}\times\hat{y}$ and with a uniform magnetic field produced by external sources aligned with the positive z axis. All dependent quantities will be represented by a power series in the dimensionless ordering parameter $\varepsilon\ll 1$. The driving field is an $O(\varepsilon)$ electromagnetic wave with the electric field vector aligned parallel to a zero order uniform magnetic field $B_0=B_0\hat{z}$. The electromagnetic wave is assumed to be a function of x only and to be traveling in the positive $\hat{x}$ direction. All gradients are in the $\hat{x}$ direction and are assumed to be $O(\varepsilon)$ (i.e. we consider only long wavelength waves). All superscripts on dependent variables of $O(\varepsilon)$ or higher denote the order in ε unless the variable is enclosed in parenthesis () or bars | |, in which case the superscript denotes a

power. Variables that are $O(\varepsilon^0)$ will be denoted with a subscript and superscripts will denote powers.

We begin by combining Maxwell's equations yielding

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} = -\frac{4\pi}{c} \frac{\partial}{\partial t} \mathbf{J} \quad 1.1$$

where the current is given by

$$\mathbf{J} = en_i \mathbf{v}_i - en_e \mathbf{v}_e \quad 1.2$$

The kinetic energy density of the ions and electrons due to thermal motion is assumed to be negligible compared with the magnetic field energy density. In addition, the effects of collisions are assumed to be negligible. Under these assumptions we may use the cold fluid equations for the electrons and ions. The densities therefore obey the equation of continuity

$$\frac{\partial}{\partial t} n_\alpha + \nabla \cdot n_\alpha \mathbf{v}_\alpha = 0 \quad 1.3$$

and the dynamics of the charged particles are governed by

$$\frac{\partial}{\partial t} \mathbf{v}_\alpha + \mathbf{v}_\alpha \cdot \nabla \mathbf{v}_\alpha = \frac{q_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{\mathbf{v}_\alpha \times \mathbf{B}}{c} \right) \quad 1.4$$

It is convenient to use a notation that allows easy identification of vectors parallel and perpendicular to the external magnetic field $\mathbf{B}_0$.

Accordingly, we split the currents, velocities and electric fields into two components as follows:

$$\mathbf{J}, \mathbf{E}, \mathbf{v} \rightarrow \mathbf{J}, \mathbf{E}, \mathbf{v} \parallel \hat{\mathbf{z}} \quad , \quad \mathbf{K}, \mathbf{A}, \mathbf{a} \perp \hat{\mathbf{z}} \quad 1.5$$

so that (1.1) is represented by two coupled equations

$$-\nabla^2 \mathbf{E} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} = -\frac{4\pi}{c^2} \frac{\partial}{\partial t} \mathbf{J} \quad 1.6a$$

$$-\nabla^2 \mathbf{A} + \nabla \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} = -\frac{4\pi}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{K} \quad 1.6b$$

The driving field for both equations (6a,b) is the $O(\varepsilon)$ electric field $\mathbf{E}^1 = E^1 \hat{\mathbf{z}}$, as all higher fields will be determined in terms of $\mathbf{E}^1$. For $\mathbf{E}^1$ we choose a representation wherein the slow temporal and spatial dependence of $\mathbf{E}^1$ is incorporated into a complex envelope function, or *phasor*, while the rapid oscillations are accounted for by a multiplicative factor $e^{i\omega_0 t}$:

$$2\mathbf{E}^1 \rightarrow \mathbf{E}^1(x, t) e^{i\omega_0 t} + \text{c.c.} \quad 1.7$$

The time dependence of the phasor is assumed $O(\varepsilon^2)$ while that of the exponential is $O(\varepsilon)$. The product rule for differentiation motivates the following convenient operator replacement for partial differentiations with respect to time:

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \tau} + \frac{\partial}{\partial t} \quad 1.8$$

where $\partial/\partial\tau$ operates only on the slowly varying phasor and is $O(\varepsilon^2)$ while $\partial/\partial t$ operates on the or high frequency part of a variable and is $O(\varepsilon^0)$.

We proceed first to derive the high frequency field equation for E^1 by expanding (1.6a) keeping terms of $O(\varepsilon^3)$:

$$-\nabla^2 E^1 - \frac{\omega_0^2}{c^2} E^1 + \frac{2i\omega_0}{c^2} \frac{\partial}{\partial \tau} E^1 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E^3 = -\frac{4\pi\partial}{c^2} J \quad 1.9$$

with

$$J = en_0(v_i^1 - v_e^1) + en_0(v_i^1 - v_e^3) + en^2(v_i^1 - v_e^1) \quad 1.10$$

We note the following:

i). $B^1 = 0$ as $\partial B^2/\partial t = -c\nabla \times E^1$ and so B^2 is the lowest order high frequency field driven by E^1 .

ii). $v_\alpha^2, E_\alpha^2 = 0$ because they are uncoupled from the lower order fields, i.e.

$$\frac{\partial}{\partial t} v_\alpha^2 = \frac{q_\alpha}{m_\alpha} E^2 \quad 1.11a$$

$$\frac{\partial}{\partial t} a_\alpha^2 = \frac{q_\alpha}{m_\alpha} (A^2 + \frac{a_\alpha^2}{c} \times B_0) \quad 1.11b$$

iii). $n_\alpha^1 = 0$ as $v_\alpha^1 \perp \nabla$.

$$\text{iv). } n^2 = n_\alpha^2 \text{ as } \nabla \cdot \mathbf{E}^1 = 4\pi e(n_1^2 - n_e^2) = 0.$$

Using the $O(\varepsilon)$ equation of motion for $\mathbf{v}_\alpha^1$, we have

$$\mathbf{v}_\alpha^1 = -\frac{i q_\alpha}{\omega_0 m_\alpha} \mathbf{E}^1 \quad \text{I.12}$$

and together with (I.10) we can rewrite (I.9) as

$$\begin{aligned} -\nabla^2 \mathbf{E}^1 - \frac{\omega_0^2}{c^2} \mathbf{E}^1 + \frac{2i\omega_0}{c} \frac{\partial}{\partial \tau} \mathbf{E}^1 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E}^3 = \\ - \frac{\omega_{pe}^2 + \omega_{pi}^2}{c^2} \mathbf{E}^1 + i \frac{\omega_{pe}^2 + \omega_{pi}^2}{\omega_0 c^2} \frac{\partial}{\partial \tau} \mathbf{E}^1 \\ + \\ - \frac{\omega_{pi}^2 + \omega_{pe}^2}{c^2} \frac{n^2}{n_0} \mathbf{E}^1 - \frac{4\pi en_0}{c^2} \left(\frac{\partial}{\partial t} \mathbf{v}_\alpha^3 - \frac{\partial}{\partial t} \mathbf{v}_e^3 \right) \end{aligned} \quad \text{I.13}$$

The $O(\varepsilon)$ terms cancel in (I.13) if we identify $\omega_0^2 = \omega_{pe}^2 + \omega_{pi}^2$. We obtain

$$-\nabla^2 \mathbf{E}^1 + \frac{i\omega_0}{c^2} \frac{\partial}{\partial \tau} \mathbf{E}^1 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E}^3 = -\frac{\omega_0^2}{c^2} \frac{n^2}{n_0} \mathbf{E}^1 - \frac{4\pi en_0}{c^2} \left(\frac{\partial}{\partial t} \mathbf{v}_i^3 - \frac{\partial}{\partial t} \mathbf{v}_e^3 \right) \quad \text{I.14}$$

To continue the analysis we must examine the $O(\varepsilon^3)$ equations of motion:

$$\frac{\partial}{\partial \tau} \mathbf{v}_\alpha^1 + \frac{\partial}{\partial t} (\mathbf{v}_\alpha^3 + \mathbf{a}_\alpha^3) = \frac{q_\alpha}{m_\alpha} [(\mathbf{E}^3 + \mathbf{A}^3) + \frac{\mathbf{a}_\alpha^3}{c} \times \mathbf{B}_0] + \frac{q_\alpha}{m_\alpha} \frac{\mathbf{v}_\alpha^1}{c} \times \mathbf{B}^2 \quad \text{I.15}$$

but

$$\frac{1}{c} \frac{\partial}{\partial t} B^2 = \nabla \times E^1 = \nabla \times \left(\frac{m_\alpha}{q_\alpha} \frac{\partial}{\partial t} v_\alpha^1 \right) \quad 1.16$$

so that

$$B^2 = -c \nabla \times \left(\frac{m_\alpha}{q_\alpha} v_\alpha^1 \right) \quad 1.17$$

and (1.15) splits into two equations:

$$\frac{\partial}{\partial t} v_\alpha^3 + \frac{\partial}{\partial \tau} v_\alpha^1 = \frac{q_\alpha}{m_\alpha} E^3 \quad 1.18a$$

$$\frac{\partial}{\partial t} a_\alpha^3 = \frac{q_\alpha}{m_\alpha} \left[A^3 + \frac{a_\alpha^3}{c} \times B_0 \right] - \nabla \left(\frac{v_\alpha^1 \cdot v_\alpha^1}{2} \right) \quad 1.18b$$

The driving term in (1.18a) is $\partial v_\alpha^1 / \partial \tau$ so that E^3 and v_α^3 are phasors synchronized with E^1 . If we eliminate $\partial v_\alpha^3 / \partial t$ in (1.14) using (1.18a) and (1.12) we obtain:

$$-\nabla^2 E^1 + \frac{2i\omega_0}{c^2} \frac{\partial}{\partial \tau} E^1 + \frac{\omega_0^2}{c^2} \frac{n^2}{n_0} E^1 = 0 \quad 1.19$$

This is our final expression for the high frequency field E^1 . As $\partial / \partial \tau$ does not operate on the exponential high frequency factor, (1.19) can be regarded as an equation for the phasor alone.

We now consider (I.18b), examining first the nature of the driving term:

$$\nabla \left(\frac{\mathbf{v}_a^1 \cdot \mathbf{v}_a^1}{2} \right) = \frac{q_a^2}{4m_a^2 \omega_0^2} \nabla |\mathbf{E}^1|^2 + \text{terms} \sim e^{2i\omega_0 t} \quad \text{I.20}$$

We are now concerned only with the low frequency motion. Thus we use only the low frequency part of the driver and regard A^3 and a_α^3 as low frequency fields. Equation (I.15) then becomes

$$0 = \frac{q_a}{m_a} \left[A^3 + \frac{a_\alpha^3}{c} \times B_0 \right] - \frac{q_a^2}{4m_a^2 \omega_0^2} \nabla |\mathbf{E}^1|^2 \quad \text{I.21}$$

The physically significant quantity that we need from (I.21) is the net current $e n_0 (a_i^3 - a_e^3)$. The three terms in (I.21) are all $O(\epsilon^3)$ and we could solve for a_i^3 and a_e^3 directly. However, there is a hidden free parameter in (I.21) that can raise the order of the next lower order low frequency inertial term on the LHS of (I.15). To extract the most general equation of motion we rewrite (I.21) as follows:

$$\frac{\partial}{\partial \tau} a_\alpha^3 = \frac{q_a}{m_a} \left[A^3 + \frac{a_\alpha^3}{c} \times B_0 \right] - \frac{q_a^2}{4m_a^2 \omega_0^2} \nabla |\mathbf{E}^1|^2 \quad \text{I.22}$$

Iterating (I.22) once we obtain the following expression

$$\begin{aligned} \frac{\partial^2}{\partial \tau^2} a_\alpha^2 + \Omega_\alpha^2 a_\alpha^2 &= \frac{q_\alpha}{m_\alpha} \frac{\partial}{\partial \tau} A^3 - \frac{\Omega_\alpha^2 c^2}{4\omega_0^2} \frac{\partial}{\partial \tau} \nabla \left(\frac{E^1}{B_0} \right)^2 \\ &+ \frac{\Omega_\alpha^2 c}{B_0} \hat{z} \times A^3 + \frac{\Omega_\alpha^2 c^2}{\omega_0^2} \frac{\Omega_\alpha}{4} \nabla \left(\frac{E^1}{B_0} \right)^2 \end{aligned} \quad 1.23a$$

$$\Omega_\alpha = \frac{q_\alpha B_0}{m_\alpha c} \hat{z} \quad 1.23b$$

The two terms on the LHS of (1.23a) can be unambiguously ordered relative to one another, and we take $\partial/\partial \tau \ll |\Omega_i| < |\Omega_e|$ obtaining

$$\Omega_\alpha^2 a_\alpha^3 = \frac{q_\alpha}{m_\alpha} \frac{\partial}{\partial \tau} A^3 - \frac{\Omega_\alpha^2 c^2}{4\omega_0^2} \frac{\partial}{\partial \tau} \nabla \left(\frac{E^1}{B_0} \right)^2 + \frac{\Omega_\alpha^2 c}{B_0} \hat{z} \times A^3 + \frac{\Omega_\alpha^2 c^2}{\omega_0^2} \frac{\Omega_\alpha}{4} \nabla \left(\frac{E^1}{B_0} \right)^2 \quad 1.24$$

Using (1.24) we construct K^3 , the total perpendicular current:

$$K^3 = 4\pi n_0 (a_i^3 - a_e^3) = \frac{4\pi n_0 (m_i + m_e)}{B_0} c^2 \frac{\partial}{\partial \tau} A^3 + \frac{B_0 c}{4} \hat{z} \times \nabla \left(\frac{E^1}{B_0} \right)^2 \quad 1.25$$

The second and third terms on the RHS of (1.24) have cancelled. We recognize the *Alphén* or *magnetosonic* velocity $a_0^2 = 4\pi n_0 (m_i + m_e) / B_0^2$ as a new parameter in the problem and write (1.25) in the following form

$$K^3 = B_0 \left(\frac{c^2}{a_0^2} \frac{\partial}{\partial \tau} A^3 + \frac{c}{4} \hat{z} \times \nabla \left(\frac{E^1}{B_0} \right)^2 \right) \quad 1.26$$

Clearly, if a_0^2 is $O(\varepsilon^2)$ then the inertial term on the RHS of (1.26) contributes to the $O(\varepsilon^3)$ current K^3 . For generality we include this

term and insert (1.26) into the source term in the perpendicular wave equation (1.6b) obtaining:

$$\frac{\partial^2}{\partial x^2} \left(\frac{A_y^3}{B_0} \right) - \left(\frac{1}{a_0^2} + \frac{1}{c^2} \right) \frac{\partial^2}{\partial \tau^2} \left(\frac{A_y^3}{B_0} \right) = \frac{1}{4c} \frac{\partial^2}{\partial x \partial \tau} \left| \frac{E^1}{B_0} \right|^2$$

$$A_x^3 = 0 \quad 1.27$$

Equation (1.27) can be written as an equation for B^2 using

$$\nabla \times (A_y^3 \hat{y}) = - \frac{1}{c} \frac{\partial}{\partial \tau} B^2 \hat{z} = \frac{\partial}{\partial x} A_y^3 \hat{z} \quad 1.28$$

where the low frequency magnetic field $B^2 \hat{z}$ is to be distinguished from the second order high frequency magnetic field $B^2 \hat{y}$. We obtain:

$$\left(1 + \frac{a_0^2}{c^2} \right) \frac{\partial^2}{\partial \tau^2} \left(\frac{B^2}{B_0} \right) - a_0^2 \frac{\partial^2}{\partial x^2} \left(\frac{B^2}{B_0} \right) = \frac{a_0^2}{4} \frac{\partial^2}{\partial x^2} \left| \frac{E^1}{B_0} \right|^2 \quad 1.29$$

Under the condition $\partial/\partial \tau \ll \Omega_i$, the frozen flux relation

$$\frac{B^2}{B_0} = \frac{n^2}{n_0} \quad 1.30$$

is valid so that (1.29) can be written:

$$\left(1 + \frac{a_0^2}{c^2} \right) \frac{\partial^2}{\partial \tau^2} \left(\frac{n^2}{n_0} \right) - a_0^2 \frac{\partial^2}{\partial x^2} \left(\frac{n^2}{n_0} \right) = \frac{a_0^2}{4} \frac{\partial^2}{\partial x^2} \left| \frac{E^1}{B_0} \right|^2 \quad 1.31$$

Equation (1.31) has a direct physical interpretation. If the driving term on the RHS is set equal to zero, then (1.31) is the equation for

magnetic sound waves. Our final system of coupled wave equations are (1.31) together with (1.18) and represent coupled electromagnetic - magnetosonic waves. If a_0^2 is $O(\varepsilon)$ or higher, then (1.31) is simplified to

$$\frac{n^2}{n_0} = \frac{1}{4} \left| \frac{E^1}{B_0} \right|^2 \quad 1.32$$

and (1.18) reduces to the CSE:

$$i \frac{\partial}{\partial \tau} E^1 - \frac{c^2 \partial^2}{2 \omega_0 \partial x^2} E^1 - \frac{\omega_0}{8} \left| \frac{E^1}{B_0} \right|^2 E^1 = 0 \quad 1.33$$

Appendix II

A. The derivatives of the generating function W in terms of complete elliptic integrals.

1. The case $pq > 0$.

$$\begin{aligned} W &= \frac{P}{2\pi} \left(\frac{q}{2p}\right)^{1/2} \int_{e_2}^{e_1} \frac{dI}{I} \left[-I^3 + \frac{2}{q} \left(\beta - \gamma - \frac{u^2}{4p} \right) I^2 + \frac{2D}{q} I - \frac{2E^2}{pq} \right]^{1/2} \\ &= \frac{P}{2\pi} \left(\frac{q}{2p}\right)^{1/2} \int_{e_2}^{e_1} \frac{dI}{I} [(e_1 - I)(e_2 - I)(e_3 - I)]^{1/2} \end{aligned} \quad \text{II.1}$$

The derivatives of W with respect to D, γ and $E = -\Omega/2k$ are all reduced to canonical form via the following transformation:

$$\cos^2 \theta = \frac{I - e_2}{e_1 - e_2} \quad \text{II.2}$$

For W_D we have

$$W_D = \frac{1}{2\pi} \left(\frac{P}{2q} \right)^{1/2} \int_{e_2}^{e_1} \frac{dI}{[(e_1-I)(e_2-I)(e_3-I)]^{1/2}} \quad \text{II.3}$$

which, using the transformation (1.2), reduces to

$$W_D = \frac{1}{\pi} \left(\frac{P}{2q} \right)^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} K(m) \quad \text{II.4}$$

where

$$m = \frac{e_1 - e_2}{e_1 - e_3}, \quad 0 \leq m \leq 1 \quad \text{II.5}$$

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{(1 - m \sin^2 \theta)^{1/2}} \quad \text{II.6}$$

and $K(m)$ is the complete elliptic integral of the first kind. For W_γ we obtain

$$W_\gamma = -\frac{1}{2\pi} \left(\frac{P}{2q} \right)^{1/2} \int_{e_2}^{e_1} \frac{IdI}{[(e_1-I)(e_2-I)(e_3-I)]^{1/2}} \quad \text{II.7}$$

which, using (1.2) reduces to

$$W_\gamma = -\frac{1}{\pi} \left(\frac{P}{2q} \right)^{1/2} (e_1 - e_3)^{1/2} [E(m) + \frac{e_3}{e_1 - e_3} K(m)] \quad \text{II.8}$$

where

$$E(m) = \int_0^{\pi/2} d\theta (1 - m \sin^2 \theta)^{1/2} \quad \text{II.9}$$

is the complete elliptic integral of the second kind. For W_E we have

$$W_E = -\frac{E}{\pi q} \left(\frac{q}{2p}\right)^{1/2} \int_{e_2}^{e_1} \frac{dI}{I} \frac{1}{[(e_1 - I)(e_2 - I)(e_3 - I)]^{1/2}} \quad \text{II.10}$$

and using (1.2) and $2E = \sigma(-2pq e_1 e_2 e_3)^{1/2}$ we obtain

$$W_E = -\frac{\sigma|q|}{\pi q} \left[\frac{e_2 e_3}{e_1(e_3 - e_1)} \right]^{1/2} \Pi(n, m) \quad \text{II.11}$$

where

$$\Pi(n, m) = \int_0^{\pi/2} \frac{d\theta}{[1 - n \sin^2 \theta][1 - m \sin^2 \theta]^{1/2}} \quad \text{II.12}$$

is the complete elliptic integral of the third kind with

$$n = \frac{e_1 - e_2}{e_1}, \quad 0 \leq n \leq 1 \quad \text{II.13}$$

2. The case $pq < 0$.

$$\begin{aligned} W &= \frac{p}{2\pi} \left| \frac{q}{2p} \right|^{1/2} \int_{e_3}^{e_2} \frac{dI}{I} \left[I^3 + \frac{2}{q} \left(\gamma + \frac{u^2}{4p} - \beta u \right) I^2 - \frac{2D}{q} I + \frac{2E^2}{pq} \right]^{1/2} \\ &= \frac{p}{2\pi} \left| \frac{q}{2p} \right|^{1/2} \int_{e_3}^{e_2} \frac{dI}{I} [(e_1 - I)(e_2 - I)(e_3 - I)]^{1/2} \end{aligned} \quad \text{II.14}$$

For the case $pq < 0$, the derivatives of W are reduced by the following

transformation

$$\cos^2 \theta = \frac{e_2 - I}{e_2 - e_3} \quad \text{II.15}$$

Thus for W_D we have

$$W_D = \frac{1}{2\pi} \left| \frac{p}{2q} \right|^{1/2} \int_{e_3}^{e_2} \frac{dl}{[(e_1 - I)(e_2 - I)(e_3 - I)]^{1/2}} \quad \text{II.16}$$

which reduces to

$$W_D = \frac{1}{\pi} \left| \frac{p}{2q} \right|^{1/2} \frac{1}{(e_1 - e_3)^{1/2}} K(m) \quad \text{II.17}$$

$$m = \frac{e_2 - e_3}{e_1 - e_3}, \quad 0 \leq m \leq 1 \quad \text{II.18}$$

For W_γ we obtain

$$W_\gamma = - \frac{1}{2\pi} \left| \frac{p}{2q} \right|^{1/2} \int_{e_3}^{e_2} \frac{Idl}{[(e_1 - I)(e_2 - I)(e_3 - I)]^{1/2}} \quad \text{II.19}$$

which reduces to

$$W_\gamma = - \frac{1}{\pi} \left| \frac{p}{2q} \right|^{1/2} (e_1 - e_3)^{1/2} \left[\frac{e_1}{e_1 - e_3} K(m) - E(m) \right] \quad \text{II.20}$$

Finally, for W_E we have

$$W_E = \frac{E}{\pi q} \left| \frac{q}{2p} \right|^{1/2} \int_{e_3}^{e_2} \frac{dl}{l} \frac{1}{[(e_1-l)(e_2-l)(e_3-l)]^{1/2}} \quad \text{II.21}$$

which reduces to

$$W_E = \frac{\sigma |q|}{\pi q} \left[\frac{e_1 e_2}{e_3 (e_1 - e_3)} \right]^{1/2} \Pi(n, m) \quad \text{II.22}$$

where

$$n = \frac{e_3 - e_2}{e_3} \quad , \quad -\infty \leq n \leq 0 \quad \text{II.23}$$

B. The derivatives of the complete elliptic integrals.

The derivatives of the complete elliptic integrals with respect to the parameters m and n are:

$$\frac{dE(m)}{dm} = \frac{1}{2m} [E(m) - K(m)] \quad \text{II.24}$$

$$\frac{dK(m)}{dm} = \frac{1}{2m(1-m)} [E(m) - (1-m)K(m)] \quad \text{II.25}$$

$$\frac{d\Pi(n, m)}{dm} = \frac{1}{2(n-m)} [\Pi(n, m) - \frac{E(m)}{1-m}] \quad \text{II.26}$$

$$\frac{d\Pi(n,m)}{dn} = \frac{1}{2n(m-n)(1-n)} [(n-m)K(m) - nE(m) + (m-n^2)\Pi(n,m)] \quad \text{II.27}$$

C. Limiting expressions for the complete elliptic integrals.

The limiting expressions for K , E and Π as $m, n \rightarrow 0$ are:

$$K(m) \rightarrow \frac{\pi}{2} \left[1 + \frac{m}{4} + O(m^2) \right] \quad \text{II.28}$$

$$E(m) \rightarrow \frac{\pi}{2} \left[1 - \frac{m}{4} + O(m^2) \right] \quad \text{II.29}$$

$$\Pi(m,n) \rightarrow \frac{\pi}{2} \left[1 + \frac{1}{2} \left(n + \frac{m}{2} \right) + O(m^2, n^2, mn) \right] \quad \text{II.30}$$

The limiting expressions for K , E and Π as $m \rightarrow 1$ are:

$$K(m) \rightarrow \frac{1}{2} \ln \left(\frac{16}{1-m} \right) + O(1-m) \quad \text{II.31}$$

$$E(m) \rightarrow 1 + \frac{1}{4} \left[\ln \left(\frac{16}{1-m} \right) - 1 \right] (1-m) + O((1-m)^2) \quad \text{II.32}$$

$$\Pi(n,m) \rightarrow \frac{1}{1-n} \left[\frac{1}{2} \ln \frac{16}{1-m} - \sqrt{n} \tanh^{-1}(\sqrt{n}) \right] + O(1-m) , \quad n > 0 \quad \text{II.33a}$$

$$\Pi(n,m) \rightarrow \frac{1}{1-n} \left[\frac{1}{2} \ln \frac{16}{1-m} + \sqrt{|n|} \tanh^{-1}(\sqrt{|n|}) \right] + O(1-m) , \quad n < 0 \quad \text{II.33b}$$

In Chapter III we also make use of the limiting expression for the incomplete elliptic integral of the third kind as $m \rightarrow 1$:

$$\Pi(n;w,m) \rightarrow \frac{1}{1-n} \left[w - \sqrt{n} \tanh^{-1}(\sqrt{n} \tanh(w)) \right] + O(1-m) , \quad n > 0 \quad \text{II.34a}$$

$$\Pi(n;w,m) \rightarrow \frac{1}{1-n} \left[w + \sqrt{|n|} \tanh^{-1}(\sqrt{|n|} \tanh(w)) \right] + O(1-m) , \quad n < 0 \quad \text{II.34b}$$

Note that (II.34a,b) reduce to (II.33a,b) for $W=K(m) \rightarrow \ln(16/(1-m))/2$.

References

P. F. Byrd and M. D. Friedman, Handbook of Elliptic Integrals for Engineers and Scientists, (Springer-Verlag, New York, 1971).

Appendix III

A. Reduction of the modulational equations for the general case $pq > 0$ to matrix form.

We wish to reduce the modulational equations (5.33) through (5.36) to matrix form. To accomplish this we carry through all differentiations on the dependent variables e_1, e_2, e_3 and u so that each equation reduces to a quasilinear differential operator on the column vector (e_1, e_2, e_3, u) . The four equations can then be combined into a single matrix equation.

It is convenient to write down our four modulational equations (5.33) through (5.36). We have

$$\frac{D}{Dt} [(e_1 - e_3)^{1/2} E + \frac{e_3}{(e_1 - e_3)^{1/2}} K] = \frac{-\sigma K}{(e_1 - e_3)^{1/2}} \frac{\partial}{\partial x} (-2e_1 e_2 e_3)^{1/2} \quad \text{III.1}$$

$$\frac{Du}{Dt} + \sigma \frac{D}{Dt} \left(\frac{-2e_2 e_3}{e_1} \right)^{1/2} \frac{\Pi}{K} + \sigma \left(\frac{-2e_2 e_3}{e_1} \right)^{1/2} \frac{\Pi}{K} \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} (e_1 + e_2 + e_3) = 0 \quad \text{III.2}$$

$$\begin{aligned} \frac{D}{Dt} \sigma (-2e_1 e_2 e_3)^{1/2} + 2\sigma (-2e_1 e_2 e_3)^{1/2} \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} (e_1 e_2 + e_2 e_3 + e_3 e_1) \\ + [e_3 + (e_1 - e_3) \frac{E}{K}] \frac{Du}{Dt} = 0 \end{aligned} \quad \text{III.3}$$

$$\frac{D}{Dt} \frac{K}{(e_1 - e_3)^{1/2}} = \frac{K}{(e_1 - e_3)^{1/2}} \frac{\partial u}{\partial x} \quad \text{III.4}$$

1. The reduction of equation (III.1).

To reduce equation (III.1) we use the expressions (II.24) and (II.25) for the derivatives dE/dm and dK/dm . Recall that

$$m = \frac{e_1 - e_2}{e_1 - e_3}, \quad n = \frac{e_1 - e_2}{e_1} \quad \text{III.5}$$

so that

$$dm = \frac{1}{e_1 - e_3} [(1-m)de_1 - de_2 + mde_3], \quad dn = \frac{e_2}{e_1^2} de_1 - \frac{1}{e_2} de_2 \quad \text{III.6}$$

Also, recall that $e_3 \ll e_2 \ll e_1$ so that

$$\frac{\partial}{\partial x} (-2e_1 e_2 e_3)^{1/2} = \left(\frac{-e_2 e_3}{2e_1} \right)^{1/2} \frac{\partial e_1}{\partial x} + \left(\frac{-e_1 e_3}{2e_2} \right)^{1/2} \frac{\partial e_2}{\partial x} - \left(\frac{-e_1 e_2}{2e_3} \right)^{1/2} \frac{\partial e_3}{\partial x} \quad \text{III.7}$$

If the above relations are used in (III.1) and if the coefficients of the partial derivatives are collected and simplified, the result is

$$\begin{aligned} & (e_1 E - e_2 K) \frac{De_1}{Dt} + \sigma(e_1 - e_2) K \left(\frac{-2e_2 e_3}{e_1} \right)^{1/2} \frac{\partial e_1}{\partial x} \\ & + [e_1 K - e_2 \left(\frac{e_1 - e_3}{e_2 - e_3} \right) E] \frac{De_2}{Dt} + \sigma(e_1 - e_2) K \left(\frac{-2e_1 e_3}{e_2} \right)^{1/2} \frac{\partial e_2}{\partial x} \\ & + (e_1 - e_2) \left[K + \left(\frac{e_3}{e_2 - e_3} \right) E \right] \frac{De_3}{Dt} + \sigma(e_1 - e_2) K \left(\frac{-2e_1 e_2}{e_3} \right)^{1/2} \frac{\partial e_3}{\partial x} = 0 \end{aligned} \quad \text{III.8}$$

2. The reduction of equation (III.2).

To reduce equation (III.2), we first differentiate K and Π with respect to the parameters m and n and then multiply by $(-2e_1 e_2 e_3)^{1/2} K$. The result is

$$\begin{aligned} & (-2e_1 e_2 e_3)^{1/2} e_1 K \frac{Du}{Dt} + \sigma \Pi [e_2 e_3 \frac{De_1}{Dt} - e_1 e_3 \frac{De_2}{Dt} - e_1 e_2 \frac{De_3}{Dt}] \\ & + 2\sigma e_1 e_2 e_3 \left[\frac{\Pi}{K} \frac{dK}{dn} - \frac{d\Pi}{dn} \frac{Dm}{Dt} - \frac{d\Pi}{dn} \frac{Dn}{Dt} \right] \\ & - e_1 K (-2e_1 e_2 e_3)^{1/2} \frac{\partial}{\partial x} (e_1 + e_2 + e_3) - 2\sigma e_1 e_2 e_3 \Pi \frac{\partial u}{\partial x} = 0 \end{aligned} \quad \text{III.9}$$

We next use (II.25) through (II.27) to eliminate the derivatives of K and Π and simplify the resulting equation to the following form

$$\begin{aligned}
& (-2e_1e_2e_3)^{1/2}e_1K \frac{Du}{Dt} + \sigma \frac{e_2e_3}{n}(E-K)\left(\frac{\Pi}{K} - 1\right)\frac{De_1}{Dt} \\
& \sigma \frac{e_1e_3}{n}\left(\frac{E}{1-m}K\right)(1 - (1-n)\frac{\Pi}{K})\frac{De_2}{Dt} + \sigma \frac{e_1e_2}{n}\frac{E}{1-m}\left[(m-n)\frac{\Pi}{K} - m\right]\frac{De_3}{Dt} \\
& - e_1K(-2e_1e_2e_3)^{1/2}\frac{\partial}{\partial x}(e_1 + e_2 + e_3) - 2\sigma e_1e_2e_3\Pi \frac{\partial u}{\partial x} = 0
\end{aligned} \tag{III.10}$$

We now collect all of the terms that multiply Π into the following expression

$$\begin{aligned}
& e_2e_3\frac{\partial \Pi}{\partial K}\left[(E-K)\frac{De_1}{Dt} + \left[K - \left(\frac{e_1-e_3}{e_2-e_3}\right)E\right]\frac{De_2}{Dt} + \left(\frac{e_1-e_2}{e_2-e_3}\right)E\frac{De_3}{Dt}\right. \\
& \left. - 2(e_1-e_2)K\frac{\partial u}{\partial x}\right]
\end{aligned} \tag{III.11}$$

The term in brackets in (III.11) is precisely the reduced form of (III.4) (that is, it is identical to (III.14)) and therefore vanishes. Thus we may set Π equal to zero in (III.10). This step eliminates the complete elliptic integral of the third kind from the modulational equations. If the remaining terms of (III.10) are multiplied by σn ($\sigma^2=1$), we obtain

$$\begin{aligned}
& e_2 e_3 (K - E) \frac{De_1}{Dt} - \sigma(e_1 - e_2)(-2e_1 e_2 e_3)^{1/2} K \frac{\partial e_1}{\partial x} \\
& + e_1 e_3 \left[\left(\frac{e_1 - e_3}{e_2 - e_3} \right) E - K \right] \frac{De_2}{Dt} - \sigma(e_1 - e_2)(-2e_1 e_2 e_3)^{1/2} K \frac{\partial e_2}{\partial x} \\
& - e_1 e_2 \left(\frac{e_1 - e_2}{e_2 - e_3} \right) E \frac{De_3}{Dt} - \sigma(e_1 - e_2)(-2e_1 e_2 e_3)^{1/2} K \frac{\partial e_3}{\partial x} \\
& + \sigma(e_1 - e_2)(-2e_1 e_2 e_3)^{1/2} K \frac{Du}{Dt} = 0
\end{aligned}
\tag{III.12}$$

which is the reduced form of (III.2).

3. The reduction of the equation (III.3).

The reduction of (III.3) is trivial provided we expand the first term correctly (see (II.1)). Expanding the first and third terms and multiplying by $-\sigma(-2e_1 e_2 e_3)^{1/2}$ and recalling $\sigma^2=1$, we obtain

$$\begin{aligned}
& e_2 e_3 \frac{De_1}{Dt} + \sigma(-2e_1 e_2 e_3)^{1/2} (e_2 + e_3) \frac{\partial e_1}{\partial x} \\
& + e_1 e_3 \frac{De_2}{Dt} + \sigma(-2e_1 e_2 e_3)^{1/2} (e_1 + e_3) \frac{\partial e_2}{\partial x} \\
& + e_1 e_2 \frac{De_3}{Dt} + \sigma(-2e_1 e_2 e_3)^{1/2} (e_1 + e_2) \frac{\partial e_3}{\partial x} \\
& - \sigma(-2e_1 e_2 e_3)^{1/2} \left[e_3 + (e_1 - e_3) \frac{E}{K} \right] \frac{Du}{Dt} + 4e_1 e_2 e_3 \frac{\partial u}{\partial x} = 0
\end{aligned}
\tag{III.13}$$

4. The reduction of equation (III.4).

Equation (III.4) is reduced using (II.25) for the derivative dK/dm

and (III.8). The resulting equation is simplified when multiplied by $2(e_1 - e_2)(e_1 - e_3)^{1/2}$ to

$$(E - K) \frac{De_1}{Dt} + [K - (\frac{e_1 - e_3}{e_2 - e_3})E] \frac{De_2}{Dt} + (\frac{e_1 - e_2}{e_2 - e_3})E \frac{De_3}{Dt} - 2(e_1 - e_2)K \frac{\partial u}{\partial x} = 0 \quad \text{III.14}$$

5. Further simplification of the system (III.1) - (III.4).

Equations (III.8), (III.12), (III.13) and (III.14) can be combined into the following matrix equation:

$$\begin{bmatrix} e_1 D - e_2 & e_1 - e_2 (\frac{e_1 - e_3}{e_2 - e_3}) D & (e_1 - e_2) (1 + \frac{e_3}{e_2 - e_3}) D & 0 \\ D - 1 & 1 - (\frac{e_1 - e_3}{e_2 - e_3}) D & \frac{e_1 - e_2}{e_2 - e_3} D & 0 \\ e_2 e_3 & e_1 e_3 & e_1 e_2 & -\sigma [e_3 + (e_1 - e_3) D] \\ e_2 e_3 (1 - D) & e_1 e_3 [(\frac{e_1 - e_3}{e_2 - e_3}) D - 1] & -e_1 e_2 (\frac{e_3}{e_2 - e_3}) D & (e_1 - e_2) \Sigma \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ u \end{bmatrix} \frac{D}{Dt} +$$

$$\begin{bmatrix} \Sigma/e_1 & \Sigma/e_2 & \Sigma/e_3 & 0 \\ 0 & 0 & 0 & -2(e_1 + e_2) \\ (e_2 + e_3) \Sigma & (e_1 + e_3) \Sigma & (e_1 + e_2) \Sigma & 4e_1 e_2 e_3 \\ -(e_1 - e_2) \Sigma & -(e_1 - e_2) \Sigma & -(e_1 - e_2) \sigma & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ u \end{bmatrix} \frac{\partial}{\partial x} = 0 \quad \text{III.15}$$

$$\Sigma = \sigma (-2e_1 e_2 e_3)^{1/2}, \quad D = \frac{E(m)}{K(m)}$$

After several simple row operations, (III.15) is further simplified to the following matrix equation:

$$\begin{aligned}
& \begin{bmatrix} D & 1 & 1-D & 0 \\ (e_2-e_3)(1-D) & (e_1-e_3)D+e_3-e_2 & (e_2-e_1)D & 0 \\ e_2e_3 & e_1e_3 & e_1e_2 & -[e_3+(e_1-e_3)D]\Sigma \\ e_3(1-D) & 0 & e_1D & -\Sigma \end{bmatrix} \begin{bmatrix} D \\ \frac{\partial}{\partial t} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ u \end{bmatrix} \\
& + \\
& \begin{bmatrix} \Sigma/e_1 & \Sigma/e_2 & \Sigma/e_3 & 2e_2 \\ 0 & 0 & 0 & 2(e_1-e_3)(e_2-e_3) \\ (e_2+e_3)\Sigma & (e_1+e_3)\Sigma & (e_1+e_2)\Sigma & 4e_1e_2e_3 \\ \Sigma & \Sigma & \Sigma & 2e_1e_3 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ u \end{bmatrix} = 0
\end{aligned}$$

III.16

B. Reduction of the modulational equations for the general case $pq < 0$ to matrix form.

The modulational equations (5.1) through (5.4) are reduced to matrix form in a manner fully analogous to that for the case $pq > 0$ in the last section.

Appendix IV

In Chapter 6 we introduced a general algorithm for obtaining the symmetry transformations of the CSE in two dimensions. The algorithm requires the explicit expressions for the generators Π , Ω and Ψ of infinitesimal transformations of the derivatives of the dependent variable E . In Section A we give the formulas for Π and Ω (the expression for Ψ follows by analogy from that for Ω). In Section B we list the determining equations for the generators of the group of the CSE. Section C contains several details in the procedure used to obtain the symmetries of the action integral of the CSE.

A. *Explicit expressions for the generators Π and Ω .*

The expressions for the generators Π and Ω are easily constructed using the δ and Δ operators introduced in Chapter 6. It is straightforward to show:

$$\varepsilon \Pi = \Delta E_t = (\Delta E)_t - (\Delta x)_t \cdot \nabla E \quad \text{IV.1}$$

$$\varepsilon\Omega = \Delta E_{xx} = (\Delta E)_{xx} - (\Delta x)_{xx} \cdot \nabla E - 2(\Delta x)_x \cdot \nabla E_x \quad \text{IV.2}$$

so that

$$\begin{aligned} \Pi = & \eta_t + \eta_E \theta_t + \eta_E^* \theta_t^* - (\xi_t + \xi_E \theta_t + \xi_E^* \theta_t^*) \theta_x \\ & - (\sigma_t + \sigma_E \theta_t + \sigma_E^* \theta_t^*) \theta_y - (\tau_t + \tau_E \theta_t + \tau_E^* \theta_t^*) \theta_t \end{aligned} \quad \text{IV.3}$$

and

$$\begin{aligned} \Omega = & \eta_{xx} + \eta_{Ex} \theta_x + \eta_E \theta_{xx} + \eta_E^* \theta_x^* + \eta_E^* \theta_{xx}^* \\ & - (\xi_{xx} + \xi_{Ex} \theta_x + \xi_E \theta_{xx} + \xi_E^* \theta_x^* + \xi_E^* \theta_{xx}^*) \theta_x \\ & - (\sigma_{xx} + \sigma_{Ex} \theta_x + \sigma_E \theta_{xx} + \sigma_E^* \theta_x^* + \sigma_E^* \theta_{xx}^*) \theta_y \\ & - (\tau_{xx} + \tau_{Ex} \theta_x + \tau_E \theta_{xx} + \tau_E^* \theta_x^* + \tau_E^* \theta_{xx}^*) \theta_t \\ & - 2(\xi_x + \xi_E \theta_x + \xi_E^* \theta_x^*) \theta_{xx} \\ & - 2(\sigma_x + \sigma_E \theta_x + \sigma_E^* \theta_x^*) \theta_{xy} \\ & - 2(\tau_x + \tau_E \theta_x + \tau_E^* \theta_x^*) \theta_{xt} \end{aligned} \quad \text{IV.4}$$

The expression for Ψ ($\Psi = \Delta E_{yy}$) can be obtained from (IV.4) by the replacement $x \rightarrow y$.

B. The determining equations of the group of the CSE.

The determining equations for the generators of the symmetry transformations of the two dimensional CSE are obtained by setting the coefficients of the distinct products of derivatives of θ and θ^* in (6.24) equal to zero. This can be done only after the constraint (6.23) has been incorporated into (6.24) by elimination of factors $\theta_{xx} + \theta_{yy}$

and θ_{yy} via (6.25) and (6.26). The following (overdetermined) system is obtained:

$$\begin{aligned}
 \theta^0: \quad & i\eta_t + \eta_{xx} + \eta_{yy} - \eta_E \theta^2 \theta^* - \eta_E^* \theta^{*2} \theta + 2\sigma_y \theta^2 \theta^* + 2\theta \theta^* \eta + \\
 & \theta^2 \eta^* = 0 \\
 \theta_t: \quad & i\theta_E - i\tau_t - \tau_{xx} - \tau_{yy} - i\eta_E + 2i\sigma_y = 0 \\
 \theta_t^*: \quad & i\eta_E^* + i\eta_E^* = 0 \\
 \theta_x: \quad & -i\xi_t + 2\eta_{xE} - \xi_{xx} - \xi_{yy} = 0 \\
 \theta_x^*: \quad & 2\eta_{xE}^* = 0 \\
 \theta_y: \quad & -i\sigma_t + 2\eta_{yE} - \sigma_{xx} - \sigma_{yy} = 0 \\
 \theta_y^*: \quad & 2\eta_{yE}^* = 0 \\
 \theta_t^* \theta_t: \quad & -2i\tau_E^* = 0 \\
 \theta_t \theta_x: \quad & -2\tau_{xE} = 0 \\
 \theta_x \theta_t^*: \quad & -2i\xi_E^* = 0 \\
 \theta_t \theta_y: \quad & -i\sigma_E - 2\tau_{yE} + 3i\sigma_E = 0 \\
 \theta_t^* \theta_y: \quad & -2i\sigma_E^* = 0 \\
 \theta_{xx}: \quad & -2\xi_x + 2\sigma_y = 0 \\
 \theta_{xy}: \quad & -2\sigma_x - 2\xi_y = 0 \\
 \theta_{xt}: \quad & -2\tau_x = 0 \\
 \theta_{yt}: \quad & -2\tau_y = 0 \\
 \theta_y^2: \quad & \eta_{EE} - 2\sigma_{yE} = 0 \\
 \theta_x^2: \quad & \eta_{EE} - 2\xi_{xE} = 0 \\
 \theta_y \theta_y^*: \quad & 2\eta_{EE}^* - 2\sigma_{yE}^* = 0 \\
 \theta_x \theta_x^*: \quad & 2\eta_{EE}^* - 2\xi_{xE}^* = 0 \\
 \theta_y^{*2}: \quad & \eta_E^* \theta^* = 0
 \end{aligned}$$

IV.5

$$\begin{aligned}
 \theta_x^{*2}: \quad \eta_E^* E^* &= 0 \\
 \theta_x \theta_y: \quad -2\sigma_{xE} - 2\xi_{yE} &= 0 \\
 \theta_x^* \theta_y: \quad -2\sigma_{xE^*} &= 0 \\
 \theta_x^* \theta_t: \quad -2\tau_{xE^*} &= 0 \\
 \theta_y^* \theta_t: \quad -2\tau_{yE^*} + 2i\sigma_{E^*} &= 0 \\
 \theta_y^* \theta_x: \quad -2\xi_{yE^*} &= 0 \\
 \theta_x \theta_{xx}: \quad -2\xi_E &= 0 \\
 \theta_y \theta_{xx}: \quad 2\sigma_E &= 0 \\
 \theta_x \theta_{xy}: \quad -2\sigma_E &= 0 \\
 \theta_x \theta_{xy}: \quad -2\sigma_E &= 0 \\
 \theta_x \theta_{xt}: \quad -2\tau_E &= 0 \\
 \theta_x^* \theta_{xx}: \quad -\xi_{E^*} &= 0 \\
 \theta_x^* \theta_{xy}: \quad -\sigma_{E^*} &= 0
 \end{aligned}$$

The several remaining equations add nothing new to this already highly redundant set. The system (IV.5) can be reduced to the set (6.27a) through (6.27h). Care must be exercised to obtain the most general solutions for the generators consistent with the above.

C. The determining equations for the group of the action integral of the CSE.

The determining equations for the generators of the action integral of the CSE in two dimensions are contained in (6.68). Using the Lagrangian (6.70), we can write (6.68) in expanded form:

$$\begin{aligned}
 & (-i\frac{E_t^*}{2} + \theta^{*2}\theta)\eta + (i\frac{E_t}{2} + \theta^2\theta^*)\eta^* \\
 & - \theta_x^*\phi - \theta_x\phi^* - \theta_y^*\theta - \theta_y\theta^* + i\frac{\theta^*}{2}\Pi - i\frac{\theta}{2}\Pi^* \\
 & + [\frac{i}{2}(\theta_t\theta^* - \theta\theta_t^*) + \frac{1}{2}\theta^2\theta^{*2} - \theta_x^*\theta_x](\xi_x + \xi_E\theta_x + \xi_E^*\theta_x^* + \\
 & \sigma_y + \sigma_E\theta_y + \sigma_E^*\theta_x^* + \tau_t + \tau_E\theta_t + \tau_E^*\theta_t^*) = 0
 \end{aligned}
 \tag{IV.6}$$

The expressions for the generators ϕ , θ and Π (and their conjugates) are straightforward to obtain. The generator Π is given by (IV.3) and ϕ and θ can be obtained from Π by permuting subscripts. That is, to obtain ϕ , interchange t and x . Similarly, to obtain θ , interchange t and y .

After the expression for the generators have been inserted into (IV.6) we set all coefficients of distinct powers of derivatives of θ and θ^* equal to zero. As mentioned in Chapter 6, we do not need to assume that the Euler-Lagrange equations are satisfied. If they are (of course, the whole calculation would be pointless if they were not) then the symmetries have additional meaning as the generators of conservation laws.

It is clear that the determining equations of the group of the action integral of the CSE are easier to obtain than those for the differential equation itself. This is because the Lagrangian involves lower order derivatives of the dependent variables than the CSE. It is clear from a comparison of (IV.3) and (IV.4) that the generators of higher order derivatives can be quite complicated.

As the remaining analysis is exactly like that of Section B, we state the results only. The generators (and therefore the group) of the action is identical to that of the CSE itself. This can be verified by directly substituting (6.38) into (IV.6).

Appendix V

In this appendix we show how the four types of self-similar solutions of the CSE enumerated in Chapter 6 are obtained. As the algebra is similar for the four cases, we shall only work out the details for type 1 self-similar waves. The results for type 1 self-similar waves are summarized in (6.51).

Spatial displacements and Galilean transformations are particularly simple symmetry operations and can be incorporated into any solution that we obtain by a change of variables. We shall set the elements corresponding to these symmetries in the generator (6.38) equal to zero. This will simplify our calculations. Setting a_3 , a_4 , a_5 and a_6 equal to zero, the generator (6.38) becomes:

$$\eta = \{i[a_2(\frac{x^2 + y^2}{4}) + a_7] - a_1 - a_2t\}E \quad a)$$

$$\xi = (a_1 + a_2t)x + a_0y \quad b)$$

$$\sigma = (a_1 + a_2t)y - a_0x \quad c)$$

$$\tau = a_2t^2 + 2a_1t + a_8 \quad d)$$

V.1

As discussed in Chapter 6, similarity solutions are obtained by integrating the characteristics spanning a solution surface defined by the generators (V.1). The equations for the characteristic curves spanning the solution surface are:

$$\begin{aligned}
 \frac{dE(s,r,w)}{ds} &= \eta[x(s,r,w), y(s,r,w), t(s,r,w)] & a) \\
 \frac{dx(s,r,w)}{ds} &= \xi[x(s,r,w), y(s,r,w), t(s,r,w)] & b) \\
 \frac{dy(s,r,w)}{ds} &= \sigma[x(s,r,w), y(s,r,w), t(s,r,w)] & c) \\
 \frac{dt(s,r,w)}{ds} &= \tau[x(s,r,w), y(s,r,w), t(s,r,w)] & d)
 \end{aligned}
 \tag{V.2}$$

where s, r, w are arbitrary coordinates in the solution surface with non-vanishing Jacobian ($J(s, r, w/x, y, t) \neq 0$). Comparing (V.2d) and (V.1d), we see that the temporal characteristic can be immediately integrated. We define the discriminant $A = a_2 a_8 - a_1^2$ and note the following four inclusive cases:

$$A \neq 0, \quad a_2 \neq 0: \quad s = \frac{1}{\sqrt{A}} \tan^{-1} \left(\frac{a_2 t + a_1}{\sqrt{A}} \right) \tag{V.3a}$$

$$A = 0, \quad a_2 \neq 0: \quad s = \frac{-1}{a_2 t + a_1} \tag{V.3b}$$

$$a_2 = 0, a_1 \neq 0: \quad s = \frac{1}{2a_1} \ln(2a_1 t + a_0) \quad \text{V.3c}$$

$$a_2 = 0, a_1 = 0: \quad s = \frac{t}{a_0} \quad \text{V.3d}$$

These cases correspond to I, II, III and IV ((6.51) through (6.54)) of Chapter 6. Let us continue with Case I ($A \neq 0, a_2 \neq 0$) and assume for simplicity that $A > 0$. Solving (V.3a) for t and eliminating t from (V.1b,c) we can immediately integrate (V.2b,c). The result is

$$x = \cos(\sqrt{A}s)^{-1} [\sin(a_0 s) A(r, w) + \cos(a_0 s) B(r, w)] \quad \text{V.4a}$$

$$y = \cos(\sqrt{A}s)^{-1} [\sin(a_0 s) B(r, w) - \cos(a_0 s) A(r, w)] \quad \text{V.4b}$$

where A and B are independent functions of r and w but otherwise arbitrary. We choose $A = r$ and $B = w$ and note that

$$r^2 + w^2 = \cos^2(\sqrt{A}s) (x^2 + y^2) \quad \text{V.5}$$

We next integrate (V.2a), which can be written

$$\frac{dE(s,r,w)}{ds} = \{i[\frac{a_2(r^2 + w^2)}{4\cos^2(\sqrt{A}s)} + a_7] - \sqrt{A}\tan(\sqrt{A}s)\}E \quad V.6$$

yielding

$$E = U(r,w)\cos(\sqrt{A}s)\exp i[\frac{a_2(r^2 + w^2)\tan(\sqrt{A}s)}{4\sqrt{A}} + a_7s] \quad V.7$$

The function $U(r,w)$ is not arbitrary. In order to determine $U(r,w)$ we require that E satisfy the CSE. This requires that we convert the CSE to the new independent variables s, r, w and then substitute (V.7). The result is:

$$i\frac{a_2a_0}{A}(wU_r - rU_w) + U_{rr} + U_{ww} - \frac{a_2^2(r^2 + w^2)}{4A}U + |U|^2U = 0 \quad V.8$$

The solution of (V.8) will give the most general class of similarity solutions of type I. However, we are particularly interested in waves with axial symmetry for which a cylindrical coordinate system is most appropriate. Thus we define new coordinates ρ and φ :

$$r = \rho \cos \varphi \quad w = \rho \sin \varphi \quad V.9a$$

$$\varphi = \arctan(w/r) \quad \rho = \sqrt{r^2 + w^2} \quad V.9b$$

Using these new variables, equation (V.8) becomes:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} U + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} U - i \frac{a_2 a_0}{A} \frac{\partial U}{\partial \varphi} - \frac{a_2^2}{4A} \rho^2 U - \frac{a_2 a_1}{A} U + |U|^2 U = 0 \quad \text{V.10}$$

Axially symmetric waves are extracted from (V.10) by separation of variables. We assume that

$$U = h(\rho) e^{im\varphi} \quad ; \quad m = 0, 1, 2, 3, \dots \quad \text{V.11}$$

so that (V.10) reduces to

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} h - \frac{m^2}{\rho^2} h - \frac{a_2^2}{A} (a_1 - m a_0) h - \frac{a_2^2}{A} \rho^2 h + h^3 = 0 \quad \text{V.12}$$

The solution of the ordinary differential equation (V.12) completes the derivation of the similarity solutions of type I. The explicit form of the self-similar wave in terms of the original variables x, y and t can be found by examining the various transformations that we have used. From (V.3a), (V.5) and (V.9) we find

$$\rho^2 = \frac{x^2 + y^2}{1 + \frac{(a_2 t + a_1)^2}{A}} \quad \text{V.13}$$

Using (V.3a), (V.4) and (V.9) we have

$$\begin{aligned}\varphi &= \arctan \left[\frac{\cos(a_0 s)x + \sin(a_0 s)y}{\sin(a_0 s)x - \cos(a_0 s)y} \right] \\ &= \theta + \frac{\pi}{2} - \frac{a_0}{\sqrt{A}} \tan^{-1} \left(\frac{a_2 t + a_1}{\sqrt{A}} \right)\end{aligned}\tag{V.14}$$

where $\theta = \arctan(y/x)$. Finally, using (V.3a), (V.5), (V.7), (V.11) and (V.14) we have (ignoring the phase constant $\pi/2$):

$$E = \frac{h(\rho) \exp i \left[m\theta + \frac{a_2 \rho^2}{4A} (a_2 t + a_1) + \frac{a_2 - a_0 m}{\sqrt{A}} \tan^{-1} \left(\frac{a_2 t + a_1}{\sqrt{A}} \right) \right]}{\left(1 + \frac{(a_2 t + a_1)^2}{A} \right)^{1/2}}\tag{V.15}$$

Equations (V.12), (V.13) and (V.15) specify the type I self-similar waves. The type II, III and IV self-similar waves are derived in the same manner.