Tokamak Magnetic Islands in the Presence of Nonaxisymmetric Perturbations

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Abstract

The effects of a small, externally imposed, nonaxisymmetric magnetic field perturbation on magnetic islands are studied analytically, assuming zero β , tokamak ordering, and narrow islands. For the tearing stable case, the conditions under which the self-consistent plasma response is self-healing or amplifying are elucidated. For the tearing unstable case, the quasilinear theory of tearing modes is extended to a description of locked modes.

I. Introduction

A small nonaxisymmetric magnetic field can have profound effects on a tokamak equilibrium if it resonates with a low order rational surface. For a stable equilibrium, a magnetic island is produced whose width scales as the square root of the magnitude of the nonaxisymmetric field. In that case, one is interested in knowing whether self-consistent plasma currents amplify or oppose the effect. For a tearing unstable plasma, a magnetic island naturally appears even in the absence of an external perturbation, and one is then interested in knowing how a small external perturbation affects the island width. The answers to these questions are of practical interest in tokamak design, where one needs to place tolerances on allowable nonaxisymmetric field errors. On a more fundamental level, a small nonaxisymmetric field provides a probe that significantly perturbs the plasma only in a narrow region about the rational surface, allowing us to test our understanding of the physics of magnetic islands. We will also see that the presence of an infinitesimal non-axisymmetric perturbation provides a link between equilibrium and stability theory, suggesting an equilibrium approach to the computation of saturated tearing modes. In this paper, we study the effects of a small nonaxisymmetric perturbation analytically, assuming cylindrical geometry and narrow islands. A forthcoming paper will present three-dimensional numerical calculations.¹

For the tearing stable case, a crude model of the plasma response^{2,3} has convinced many in the fusion community that tokamak plasmas are generally self healing. We will see in Sec. V that this is not the case. Resonant perturbations can be strongly amplified or strongly diminished by the self-consistent plasma response, depending on the current profile. (In fairness to the authors of references 2 and 3, we note that their model was a reasonable first cut at a description of their experiments, and that the authors themselves made no such general claims concerning the implications of their calculations.)

The tearing unstable case is relevant to the calculation of locked modes.⁴ We will see that saturated island widths can be sensitive to the presence of small perturbations. Saturated islands have previously been calculated for narrow islands in a cylinder in the absence of error fields.⁵⁻⁸ This is the so-called "quasilinear" theory of tearing modes. (The word "quasilinear" here refers to the approximation that only one nonaxisymmetric Fourier com-

ponent needs to be retained in the calculation.) Our work on the tearing unstable case is an extension of this previous work to include an external perturbation.

An additional aspect of our analysis is the perspective it gives us on the calculation of saturated tearing modes. In the absence of an external perturbation, the onset of a tearing mode corresponds to a bifurcation of the axisymmetric equilibrium. Below the tearing threshold, there is a stable axisymmetric equilibrium solution. Above the threshold, there is an unstable axisymmetric equilibrium solution, and there is a nonaxisymmetric equilibrium solution corresponding to the saturated tearing mode, with the island width going to zero as the threshold is approached from above. We will see in Sec. IV that the presence of an infinitesimal nonaxisymmetric perturbation breaks the bifurcation. Below the tearing threshold, the (unique) equilibrium in the presence of the perturbation is nonaxisymmetric. This equilibrium goes smoothly into the saturated tearing mode equilibrium solution as we go through the threshold. This suggests that a natural way to calculate tearing mode effects is to track the equilibrium solution as a function of the current profile parameters in the presence of an infinitesimal nonaxisymmetric perturbation. The saturated tearing modes can be calculated directly, without following their time evolution. We will have more to say about this in Sec. VII.

In Sec. VII, we will also discuss the connection of our results with error field compensation experiments on DIII-D.⁹

II. Equilibrium with a Narrow Island in the Presence of a Resonant Perturbation

When an island width becomes greater than the width of the resistive layer, the island enters the Rutherford regime, where inertial effects may be neglected. (In practice, the width of the resistive layer is quite small, typically of the order of a millimeter.) The subsequent time development of the island occurs on the resistive time scale, with the field evolving through a sequence of equilibria as the current changes resistively. In this section we look at a snapshot at a fixed time, writing down the equation for the instantaneous equilibrium without distinguishing saturated from evolving islands. We show how the effects of an external resonant perturbation can be extracted into

a separate term in this equation, with the dependence on the perturbation amplitude made explicit.

Our analysis assumes cylindrical geometry, with the tokamak ordering $B_z \gg B_\theta$, $a/R \ll 1$. Equilibrium quantities are functions only of r . Perturbed quantities are functions of r and $m\theta - n\phi$. The helical symmetry allows the introduction of a helical flux function,

$$\psi(r, m\theta - n\phi) = \psi_0(r) - \psi_1(r) \cos(m\theta - n\phi), \quad (1)$$

with

$$\mathbf{B} = \nabla\psi \times \hat{z} + \frac{nr}{mR} B_z \hat{\theta} + B_z \hat{z}. \quad (2)$$

The flux surfaces described by Eq. (1) have an island of width

$$w = 4|\psi_1(r_s)/\psi_0''(r_s)|^{1/2} \quad (3)$$

at $r_s) = n/m$, where

$$s = RB_\theta/rB_z,$$

and r_s denotes the location of the rational surface.

We neglect the effects of pressure. The equilibrium equation,

$$\mathbf{j} \times \mathbf{B} = 0,$$

together with the tokamak ordering gives

$$\mathbf{j} = j(\psi)\hat{z}.$$

The equilibrium is determined by the solution of Ampere's law:

$$\nabla^2\psi = j(\psi) - \frac{2n}{mR} B_z, \quad (4)$$

where B_z is a constant.

In specifying the profile of j , it is necessary in practice to recognize the subtlety that $\psi(r)$ is not monotonic ($\psi' = 0$ on the rational surfaces). The values of j on different flux surfaces are independent constants of integration of the equilibrium equation, even if the value of ψ on the two surfaces is the same. It follows that j must actually be specified as a function of ψ and an additional discrete index. This complication will not be important to us in

this paper, so we will continue to denote the functional dependence of j as $j(\psi)$, with the additional dependence on a discrete index implicit.

We follow the conventional treatment of narrow islands in dividing up the domain in which we solve Eq. (4) into two regions, an exterior region where we can expand j perturbatively,

$$j(\psi) = j_0(\psi_0) - \frac{R}{rB_z} \frac{dj_0/dr}{\epsilon - n/m} \psi_1(r) \cos(m\theta - n\phi), \quad (5)$$

and a narrow inner region near the rational surface where this approximation breaks down. It is the resonant denominator, of course, that causes the perturbation expansion to break down in the neighborhood of the rational surface where $\epsilon = n/m$. Equation (5) has been found to be a good approximation right up to the edge of the island separatrix, so our regions are as shown in Fig. 1. The regions are bounded by

$$r_o = r_s + w - s \quad (6)$$

and

$$r_i = r_s - w - s, \quad (7)$$

where

$$s = .5[\psi'_1(r_o) + \psi'_1(r_i)]/\psi''_o. \quad (8)$$

The values of r_o and r_i can be determined in practice by iterating back and forth between equations (6,7) and the equation for s , Eq. (8).⁵

In the exterior region, equations (4) and (5) give the standard tearing mode equation for ψ_1 ,

$$\frac{d}{dr} r \frac{d\psi_1}{dr} - \frac{m^2}{r} \psi_1 = \frac{j'_0 R}{B_z(\epsilon - n/m)} \psi_1. \quad (9)$$

We will include an external perturbation, which will enter through the boundary condition

$$\psi_1(r=a) = \epsilon, \quad (10)$$

where $\epsilon \cos(n\phi - m\theta)$ is the given external perturbation. There is also a regularity condition at the origin which requires that

$$\psi_1 \propto r^m \text{ for } r \rightarrow 0. \quad (11)$$

We take advantage of the narrowness of the interior region by integrating the equation in that region with respect to r and turning it into a jump condition. Writing

$$j(r, n\phi - m\theta) = j_0(r) + j_1(r) \cos(n\phi - m\theta)$$

and integrating Eq. (4) across the island ($r = r_i$ to $r = r_o$) gives a jump condition

$$r_o \psi'_1(r_o) - r_i \psi'_1(r_i) = \mathcal{I}_1 \psi_1(r_s), \quad (12)$$

where w is the island width and

$$\mathcal{I}_1 = \left[\int_{r_i}^{r_o} r j_1(r) dr \right] / \psi_1(r_s) + m^2 w / r_s. \quad (13)$$

In obtaining the m^2/r_s term on the right hand side of Eq. (13), we have neglected the variation of $\psi_1(r)$ across the island. From Eq. (9) we see that ψ'_1/ψ_1 is of order $\ln w$, so the discarded term is of order $w^4 \ln w$. We will need the terms up to order w^3 in Eq. (12). Using the fact that j_1 is a continuous function of ψ , it is easily verified that j_1 is of order w , so that $\mathcal{I}_1$ is of order 1.

We rewrite the left hand side of Eq. (12), as $r_s \Delta'_\epsilon(w) \psi_1(r_s)$, where

$$r_s \Delta'_\epsilon(w) \equiv [r_o \psi'_1(r_o) - r_i \psi'_1(r_i)] / \psi_1(r_s) \quad (14)$$

is a generalization of the usual¹⁰ Δ' to include finite ϵ and w . The quantity $\Delta'_\epsilon(w)$ contains the effects of ϵ through the boundary condition Eq. (10). To make this dependence explicit, and to make contact with the usual $\Delta'(w)$ for $\epsilon = 0$, we introduce solutions u_i , u_1 , and u_2 satisfying Eq. (9) with boundary conditions,

$$u_1(r = a) = 0, \quad du_1/dr|_{r=a} = 1, \quad (15)$$

$$u_2(r = a) = 1, \quad du_2/dr|_{r=a} = 0, \quad (16)$$

$$u_i \longrightarrow r^m \text{ as } r \longrightarrow 0. \quad (17)$$

After some algebra we find

$$\Delta'_\epsilon(w) = \Delta'(w) - \frac{\epsilon}{\psi_1(r_s)} \sigma'(w), \quad (18)$$

where

$$\sigma'(w) = \left\{ 1 + \left[\frac{u_1'(r_o)}{u_1(r_o)} + \frac{u_i'(r_i)}{u_i(r_i)} + \frac{2}{r_s} - \frac{1}{4} \frac{j_0'(r_s)R}{B_z \psi'(r_s)} \right] \frac{w}{2} \right\} u_2(r_o) \left[\frac{u_1'(r_o)}{u_1(r_o)} - \frac{u_2'(r_o)}{u_2(r_o)} \right], \quad (19)$$

and the definition of $\Delta'(w)$ is the same as that of $\Delta'_\epsilon(w)$ (Eq. (14)) except that the boundary condition at the edge is $\psi_1 = 0$. Some of the details of the derivation are given in the appendix.

For the benefit of the reader who is unfamiliar with the literature on the quasilinear stabilization of tearing modes, we emphasize that we have three different Δ' quantities. The first is the familiar Δ' that determines linear tearing stability. It is standard in the work on quasilinear stabilization to introduce a nonlinear generalization of this Δ' , $\Delta'(w)$, which appears in expressions for the nonlinear evolution and saturation of tearing modes. Finally, we have introduced a new quantity, $\Delta'_\epsilon(w)$, which includes the effects of an imposed external magnetic field. Our definition of $\Delta'(w)$ agrees with that of Zakharov⁸ but differs slightly from that of White et al.,⁵ in that they do not include the factors of r . The definition we use gives a slightly more compact representation for the equations, and therefore simplifies some of the analysis. In the limit that $w \rightarrow 0$, both definitions reduce to the usual Δ' of the linearized analysis.¹⁰ We have the relations

$$\lim_{w \rightarrow 0} \Delta'_\epsilon(w) = \Delta'(w),$$

and

$$\lim_{w \rightarrow 0} \Delta'(w) = \Delta'.$$

We have written our equilibrium equations in terms of both $\psi_1(r_s)$ and w . We would like to use Eq. (3) to eliminate one in terms of the other. When $\epsilon = 0$, the sign of $\psi_1(r_s)$ is irrelevant, and it is possible to write everything in terms of w . That is what is done in the nonlinear tearing literature. When ϵ is nonzero, the sign of $\psi_1(r_s)$ relative to ϵ is important and must be retained in the equations. We therefore work in terms of $\psi_1(r_s)$. To simplify our expressions, we will define

$$\psi_{1s} \equiv \psi_1(r_s).$$

Equations (12), (14), and (18) give

$$\psi_{1s} [r_s \Delta'(|\psi_{1s}|) - \mathcal{I}_1(|\psi_{1s}|)] = \epsilon r_s \sigma'(|\psi_{1s}|). \quad (20)$$

Equation (20) is our final equilibrium equation. It can be regarded as an eigenvalue equation for $\psi_1(r_s)$, and therefore for the island width and phase, for a given current profile and a given ϵ . In general, Eq. (9) is to be integrated numerically to determine the functional form of $\Delta'(|\psi_{1s}|)$ and $\sigma'(|\psi_{1s}|)$. Similarly, the value of $\mathcal{I}_1$ is determined by the current profile in and near the island. Its calculation for various model $j(\psi)$ profiles is one of the main subjects of several papers on nonlinear tearing modes.^{5,8}

III. Time Evolution of the Island

In the previous section we solved for the equilibrium with a magnetic island in the presence of an external perturbation. In this section we determine how that equilibrium evolves on a resistive time scale. The effects of an external perturbation appear through the boundary condition on the equilibrium. In principle, we could simply substitute our expression for Δ'_c into the standard quasilinear treatments of tearing modes [references (5-8)] wherever Δ' appears. As discussed in more detail in this section, these standard treatments of quasilinear effects are not adequate for our purposes. One widely used treatment⁶ substitutes $\Delta'(w)$ for Δ' in Rutherford's theory¹¹ of tearing mode evolution. This approach runs into difficulty near the plasma edge, where $\Delta'(w)$ has a nonvanishing dependence on w even in the limit $j \rightarrow 0$. Zakharov has recently pointed out⁸ that the classic, original paper on quasilinear saturation by White et al⁵ neglects the quasilinear modification to the equilibrium current profile. Zakharov's paper includes this effect, but treats only the equilibrium problem. In this section we derive an evolution equation for the island which remains valid in the limit where the current density near the island gets small.

The evolution of the magnetic islands is determined by Faraday's law. Applying Eq. (1) and Ohm's law, and averaging over the flux surfaces to eliminate $\mathbf{v} \cdot \nabla \psi$, we get^{5,7}

$$\left\langle \frac{\partial \psi}{\partial t} \right\rangle = -\eta j(\psi) + E, \quad (21)$$

where E is a constant determined by the electric field at the boundary, and

where the flux surface average is defined by

$$\langle f \rangle \equiv \int f \mathcal{J} d\theta d\phi / \int \mathcal{J} d\theta d\phi,$$

with the Jacobian $\mathcal{J} = (\nabla\psi \cdot \nabla\theta \times \nabla\phi)^{-1}$. We have assumed that the resistivity, η , is a function of the flux surface. This is appropriate because the time scale for temperature equilibration on a flux surface is determined by the electron thermal velocity, and is fast compared to the time scale on which the island evolves.

Equation (21) gives us a relation between j_1 and $\partial\psi/\partial t$ in the interior region. Substituting into the equilibrium equation, Eq. (20), via Eq. (13), we get an evolution equation that can be solved order by order for the time dependence of the island,

$$\begin{aligned} \frac{1}{\pi} \int_{r_i}^{r_o} r \left[\int_0^{2\pi} \cos(\zeta) \frac{1}{\eta} \left\langle \frac{\partial\psi}{\partial t} \right\rangle d\zeta \right] dr = \\ - \psi_{1s} r_s \Delta'_\epsilon(|\psi_{1s}|) - \frac{E}{\pi} \int_{r_i}^{r_o} r \left[\int_0^{2\pi} \cos(\zeta) \frac{1}{\eta} d\zeta \right] dr - m^2 \frac{w}{r_s} \psi_1, \end{aligned} \quad (22)$$

where

$$\zeta \equiv n\phi - m\theta.$$

(The quasilinear modification of ψ_0 must be incorporated in evaluating the average over a flux surface in all orders above the lowest.⁸⁾)

To lowest order

$$\partial\psi/\partial t \approx -\dot{\psi}_1(r_s) \cos(m\theta - n\phi),$$

and Eq. (22) gives a generalization of Rutherford's result:^{11,7}

$$\begin{aligned} \frac{\partial\psi_{1s}}{\partial t} &= \pi\eta_x \Delta'_\epsilon(w=0) \psi_{1s}/w \\ &= \pi\eta_x [\Delta'_\epsilon \psi_{1s} - \epsilon\sigma^7]/w, \end{aligned} \quad (23)$$

where η_x is the value of η at the X-point. This is consistent with the equation obtained by Boozer¹² for the evolution of a magnetic island in the presence of a given fixed resonant field perturbation at a rational surface. A similar

expression was also obtained by Hahm and Kulsrud for a slab geometry with a flat current profile ($j' = 0$) and a perturbed boundary.¹³

Equation (23) breaks down when ϵ/ψ_1 is sufficiently large. Inertial effects become important, even with the island width greater than the width of the resistive layer. In this paper we will be mainly interested in solving for the steady-state solutions and their stability, so this case will not be of concern.

Equation (23) is adequate to determine the steady-state response to an external perturbation for the tearing stable case. For that case

$$\psi_1(r_s) \approx \epsilon \sigma'(w=0)/\Delta'(w=0).$$

As we approach the threshold for tearing instability, $\Delta' \rightarrow 0$, the next order term becomes important. It is this term that causes saturation when the tearing mode is unstable.

From Eq. (21) it follows that we can write

$$\int_{r_i}^{r_o} r j_1 dr = a w \psi_1 + b w \psi_1, \quad (24)$$

where a and b are coefficients to be determined. In obtaining Eq. (23), we have evaluated a to lowest order:

$$a \approx \frac{r_s}{\pi \eta_x} (1 + a_1 w). \quad (25)$$

Substituting equations (24) and (25) into equations (13) and (20) gives

$$(1 + a_1 w) \dot{\psi}_1 = \eta_x \pi \psi_1 [\Delta'(w) - b w / r_s - m^2 w / r_s^2] - \eta_x \pi \epsilon \sigma'(w). \quad (26)$$

The coefficients a_1 and b are independent of ϵ . Numerical evaluation for $\epsilon = 0$ has shown that a_1 and b can generally be neglected relative to the w dependence of $\Delta'(w)$.⁶ However, this does not give a reasonable answer in the limit where the current density goes to zero in the vicinity of the island. When the rational surface lies outside the plasma, for example, the growth of the island cannot lead to any local redistribution of current, so that the finite w corrections to Eq. (23) should vanish. An evaluation of $\Delta'(w)$ for this case gives

$$\Delta'(w) = \Delta' + 2 w m^2 / r_s^2.$$

The resolution of this apparent paradox comes from the observation that $b = m^2 / r_s$ in this case.^{5,8} In the limit that $j_0 \rightarrow 0$, the finite w dependence of

$\Delta'(w)$ is exactly canceled by two other terms: the b term, and the $m^2 w/r_s^2$ term in the expression for I_1 . For the numerical evaluations of Ref. 6, the rational surface is in the interior of the current distribution, and these terms can be neglected. To obtain an expression which is also valid near the plasma edge, we retain these terms. Our final equation describing the evolution of the island is

$$\frac{\partial \psi_{1s}}{\partial t} = \frac{\pi \eta_z}{w} \left\{ \psi_{1s} \left[\Delta'(w) - 2 \frac{m^2}{r_s^2} w \right] - \epsilon \sigma'(w) \right\}. \quad (27)$$

Steady-state islands are described by the equation

$$[\Delta'(w) - 2wm^2/r_s^2]\psi_{1s} = \epsilon\sigma'(w). \quad (28)$$

IV. Steady-State Solutions and Their Stability

In this section we will discuss the steady-state solutions of Eq. (27), and their stability. We will begin by describing the general qualitative features of the solution, leaving detailed descriptions and proofs for later in the section. We proceed in this way to give the reader an overall picture of what the solution looks like before getting into the details of the algebra. Also, we want to emphasize the generic nature of the main properties of the solution. Many of the significant properties of the solution follow from general arguments of bifurcation theory,¹⁴ and do not depend on the details of the equations. To make this clear, and to develop some additional intuitive understanding, we will also discuss an analogous model of a ball in a potential well.

For reference, let us first consider the case $\epsilon = 0$ (no external perturbation). Figure 2 shows a schematic plot of ψ_{1s} as a function of a profile parameter, which we will call γ . We assume that Δ' increases monotonically with γ , and that $\gamma = 0$ corresponds to $\Delta' = 0$. Of course, the other coefficients in Eq. (28) will also be functions of γ . For $\gamma < 0$ there is a unique solution, corresponding to a cylindrically symmetric, stable equilibrium. For $\gamma > 0$, the cylindrically symmetric equilibrium is unstable. There are also solutions corresponding to saturated tearing modes. These are stable, non-symmetric equilibria with islands. As $\gamma \rightarrow 0$ from the right, the island width goes to zero. The point $\gamma = 0$ corresponds to a bifurcation point of the solution. The equations are independent of the sign of ψ_{1s} , so Fig. 2 has up-down

symmetry. There is an additional degree of freedom not shown in this figure: the phase of ψ_1 is completely arbitrary.

Analogous situations for a ball in a potential well are shown in figures 3 and 4. To model the arbitrariness in the phase of ψ_1 , we think of each of these figures as a two-dimensional slice through a three-dimensional picture, with the potential well symmetric with respect to rotation about its center. Figure 3 corresponds to $\gamma < 0$. There is a unique, stable equilibrium. Increasing γ corresponds to raising the potential at the center. Figure 4 corresponds to $\gamma > 0$. The "symmetric" equilibrium, in which the ball sits at the center of the potential well, is unstable. There are now also "nonsymmetric" equilibrium solutions, and they are stable.

When $\epsilon = 0$, the solution of Eq. (28) is as shown in Fig. 5. There is still a unique, stable solution for $\gamma < 0$, but the equilibrium is now a nonsymmetric one with an island. For large γ there are again three solutions. The external perturbation breaks the bifurcation at $\gamma = 0$, so that the solution now consists of two independent branches. The branch that we are interested in is the one that extends from $\gamma < 0$.

To extend the analogy of the ball in a potential well to nonzero ϵ , we impose an additional uniform force in the negative x direction. The force breaks the rotational symmetry of the potential well. Figures 6 and 7 show the corresponding situations, with the arrow indicating the direction of the additional uniform force. Figure 6 corresponds to $\gamma < 0$. The time-independent solution is unique, and it is a stable equilibrium which is no longer located at $x = 0$. Figure 7 corresponds to $\gamma > 0$. The balls in the figure are located at the three time-independent solutions.

The middle ball in Fig. 7 is in an unstable equilibrium. Any small motion along the x axis is amplified. The two balls on the end are stable with respect to perturbations along x . However, the ball at the far right is unstable with respect to motions out of the plane. The ball at the far left is in a stable equilibrium. The stability properties are the same as those for a magnetic island, and are summarized in Fig. 5.

Now let's see how these properties come out of equations (28) and (27). For w small, we write

$$\Delta'(w) \approx \Delta' + 2wm^2/r_s^2 - \alpha w, \quad (29)$$

and

$$\sigma'(w) \approx \sigma'_0 + \sigma'_1 w. \quad (30)$$

We will assume that σ'_0 is nonzero, so that the $\sigma'_1 w$ correction can be neglected for w small. The αw term cannot be neglected because $\Delta' = 0$ for $\gamma = 0$. (When Δ' is small, $\epsilon\sigma'_0$ balances $\alpha w\psi_{1s}$, so that $\epsilon\sigma'_1 w$ is of order w^4 .) Rewriting the equation in terms of w and carrying the sign of $\psi_{1s}/\psi''_0(r_s)$ over to the right hand side of the equation, we get

$$\Delta' w^2 - \alpha w^3 = \pm \bar{\epsilon}, \quad (31)$$

where

$$\bar{\epsilon} = 16\sigma'_0\epsilon/\psi''_0(r_s). \quad (32)$$

The sign on the right hand side of Eq. (31) corresponds to that of $\psi_{1s}/\psi''_0(r_s)$, which may be either positive or negative. We will take $\bar{\epsilon} > 0$, without loss of generality. (We are free to choose the sign of ϵ .)

We assume throughout this section that the finite w correction to Δ' is stabilizing ($\alpha > 0$). If that is not the case, there are no saturated tearing modes within our theory. Unstable tearing modes grow until higher order nonlinearities become important. For finite ϵ , the island width becomes large as Δ' approaches zero from below. We will discuss the case $\alpha = 0$ in Sec. V.

Figure 8 shows a plot of $\Delta' w^2 - \alpha w^3$ vs. w for $\Delta' < 0$, $\alpha > 0$. The shaded region corresponds to $w < 0$, which is of no interest. It is clear that there is a unique equilibrium solution for any value of ϵ . For Δ' large and negative, the island width scales like $\epsilon^{1/2}$, $w \approx (-\bar{\epsilon}/\Delta')^{1/2}$. As $\Delta' \rightarrow 0$ we get $w \rightarrow (\bar{\epsilon}/\alpha)^{1/3}$.

Figure 9 shows a plot of $\Delta' w^2 - \alpha w^3$ vs. w for $\Delta' > 0$, $\alpha > 0$, $w \geq 0$. There are three roots when $(\Delta')^3 > 27\alpha^2\bar{\epsilon}/4$. For small ϵ we have roots at $w \approx \Delta'/\alpha$ and $w \approx (\bar{\epsilon}/\Delta')^{1/2}$. Two of the roots coalesce at $w = \Delta'/3\alpha$ for $(\Delta')^3 = 27\alpha^2\bar{\epsilon}/4$. There is a single root at smaller values of Δ' .

To determine the stability of the steady-state solutions, let us rewrite Eq. (27) using equations (29) and (30):

$$2w^2 dw/dt = \pi\eta_x(\Delta' w^2 - \alpha w^3 \mp \bar{\epsilon}), \quad (33)$$

where again the $\mp$ sign corresponds to that of $\psi_{1s}/\psi''_0(r_s)$. The sign of the right hand side of Eq. (33) can be read off the graphs of figures 8 and 9.

Consider first $\Delta' < 0$ (Fig. 8). In steady-state, the island width satisfies $\Delta'w^2 - \alpha w^3 + \bar{\epsilon} = 0$. Examination of Fig. 8 shows that for larger island widths the right hand side of Eq. (33) is negative, while, for smaller island widths the right hand side is positive. This steady-state solution is stable to a perturbation of w . For $\Delta' > 0$ (Fig. 9) the small w steady-state solution is unstable, while the other steady-state solutions are stable to perturbation of w .

In addition to perturbations of w , we also need to consider perturbations which rotate the island slightly relative to the fixed external nonaxisymmetric boundary condition. In that case the external field is slightly out of phase with the internal plasma currents, and the plasma is no longer in equilibrium. A calculation of the resulting force^{15,16} shows that it is proportional to $\sin(m\theta)$, where θ is the angle between the rotated magnetic island and the island produced by the external field. The solution in which the external field increases the island width is stable, while that in which the external field causes the island width to decrease is unstable.

V. Self-Healing and Amplification: A Simplified Model

In the previous section we discussed some general properties of the steady-state magnetic island solutions in the presence of an external perturbation. In this section, we take a closer look at the solutions for $\Delta' < 0$, focusing on a question that is of particular interest: under what conditions is the plasma response self-healing or amplifying? For this purpose we make a further simplification in our model, assuming that the island lies in the vacuum region outside the plasma. No other assumptions are made concerning the current profile.

We determine the island width from equations (31,32). With the island in the vacuum region, u_1 and u_2 can be explicitly determined for $r \geq r_s$ by solving Eq. (9) with $j_0' \rightarrow 0$, giving

$$\sigma_0' = \frac{2m}{r_s} \frac{1}{(r_s/a)^m - (r_s/a)^{-m}}. \quad (34)$$

The α coefficient vanishes, yielding an explicit expression for the island width,

$$w^2 = -16 \frac{\epsilon}{\psi_0''} \left(\frac{r_s}{a} \right)^m \frac{\Delta'_v}{\Delta'}, \quad (35)$$

where

$$\Delta'_v = \frac{2m}{r_s} \frac{1}{(r_s/a)^{2m} - 1} \quad (36)$$

is Δ' for $j'_0 \rightarrow 0$ in Eq. (9).

For $\Delta' = \Delta'_v$, we get the usual error field in vacuum. We can get amplification or self-healing, depending on whether $|\Delta'|$ is greater than or less than $|\Delta'_v|$. This result is intuitively reasonable. When Δ' is sufficiently large and negative, the plasma is self-healing. When $\Delta' < 0$ approaches zero, we get amplification. There is a critical value of Δ' at which the island width is just that due to the vacuum error field. Figure 10 is a plot of $\Delta'_v a$ vs r_s/a for an $m = 2$ island.

Let us examine the calculations reported in references 2 and 3 in the light of these results. Those calculations employed a model that represents the plasma current by a single current filament. The plasma response is modeled by allowing the current filament to follow the magnetic axis. In calculations with this model, the islands at the $q = 2$ and $q = 3$ surfaces were found to be much smaller than those computed with an axisymmetric current filament.

We note, first, that the magnetic axis is unaffected by a pure $m = 2$ or $m = 3$ perturbation, or by any perturbation consisting only of poloidal mode numbers greater than one. Although we have found that a tokamak can either amplify or diminish the amplitude of such a perturbation, the filament model does not incorporate this self-consistent effect.

The experiment on the ST tokamak included a substantial $m = 1$ perturbation, giving a large deformation of the magnetic axis. A filamentary current has $q < 1$ at the edge of the current (in fact $q \rightarrow 0$ as the width of the filament goes to zero). It follows that $m = 1$ modes are kink unstable, and self-consistent effects should reinforce any $m = 1$ perturbations. The calculations reported in Ref. 2 consider only $m = 2$ and $m = 3$ islands. These island widths are substantially decreased by the helical deformation of the filament. This result may indicate that the plasma response decreases the coupling between $m = 1$ and higher m perturbations. Even this inference is unclear, given the expected sensitivity of this result to the q profile, and given the fact that $q \approx 0$ for the filament case.

The model of references 2 and 3 was a reasonable first cut at a description of their experiment. A more sophisticated analysis indicates that its

conclusions have to be interpreted with care. A conclusion that plasmas are generally self-healing is unwarranted. The authors themselves made no such general claims concerning the implications of their calculation.

VI. A Slab Model

In this section we explicitly calculate the effects of a nonsymmetric perturbation for an analytically soluble slab model. To go from cylindrical geometry to slab geometry, we make the substitutions $r \rightarrow x$, $m\theta - n\phi \rightarrow ky$. The flux function is of the form

$$\psi(x, y) = \psi_0(x) - \psi_1(x) \cos(ky), \quad (37)$$

with ψ_0 and ψ_1 even in x . The magnetic field is

$$\mathbf{B} = \nabla\psi \times \hat{z} + B_z \hat{z}. \quad (38)$$

In the tearing mode equation for the exterior region, Eq. (9), we substitute the expression for ∇^2 in a slab. To maintain the symmetry of ψ_1 , we impose our perturbed boundary condition in a symmetric fashion:

$$\psi_1(x = \pm a) = \epsilon/2. \quad (39)$$

In deriving our relation between Δ'_ϵ and Δ' , eqs. (18) and (19), we used boundary conditions appropriate to cylindrical geometry [Eq. (17)]. In slab geometry we dispense with u_i . We use Eqs. (15) and (16), and we follow the same steps as in the appendix to obtain Eq. (18) with

$$\sigma' = u_2(0) \left[\frac{u'_1(0)}{u_1(0)} - \frac{u'_2(0)}{u_2(0)} \right]. \quad (40)$$

To first order, σ' is independent of w .

We will use the current profile of Furth et al.,¹⁰

$$j_0 = -(b/l) \text{sech}^2(x/l). \quad (41)$$

The solution to the tearing mode equation in the exterior region is

$$\psi_1(x, y) = C e^{-kx} \left[1 + \frac{\tanh(x/l)}{kl} \right] + D e^{kx} \left[1 - \frac{\tanh(x/l)}{kl} \right], \quad (42)$$

where C and D are constants to be determined from the boundary conditions.

We simplify the calculation of $\Delta'(w)$ and σ' by assuming $ka \gg 1$. To first order in w , we get

$$\Delta'(w) \approx -\frac{2}{kl^2}(k^2l^2 - 1) + \frac{2w}{l^2}(k^2l^2 - 2). \quad (43)$$

In calculating $u'_1(0)/u_1(0) - u'_2(0)/u_2(0)$, the lowest order term in $\exp(-ka)$ cancels. The next order term gives

$$\sigma' \approx -2\frac{k_1 + 1}{l}e^{-ka}. \quad (44)$$

Figure 11 shows a plot of the resulting island width as a function of $1/(kl)$. For each of the curves, we have fixed the value of the vacuum island width, that is, the island width in the absence of the self-consistent plasma response,

$$w_v \equiv 4 \left(e^{-ka} \epsilon / \psi_0'' \right)^{1/2}.$$

The slab equilibrium is tearing stable for $kl > 1$, and tearing unstable for $kl < 1$. This is reflected in the $w_v = 0$ curve, which has $w = 0$ when the equilibrium is stable, and has the finite saturated island width for the unstable tearing mode. In the large kl limit, $w \rightarrow w_v$ for each of the curves. For the finite w_v cases, the perturbation is amplified for finite values of kl , with the amplification becoming substantial in the neighborhood of the tearing threshold. In the small kl limit, w asymptotically approaches the $w_v = 0$ saturated island width.

At the instability threshold, $w \approx .097 \approx 3.2w_v$ for $w_v = .03$, corresponding to an amplification of over an order of magnitude for $\psi(x = 0)$. This amplification will depend on the current profile. When j_0 is small at the rational surface, α is also small, and the amplification gets large. In the limit that $j_0(r_*) \rightarrow 0$, the instability goes over to an external kink, and the nonlinear saturation due to the local modification of the current profile vanishes.

For our slab model, the external perturbation is always amplified. We saw in the previous section that this is not always the case, that in general we can get either amplification or self-healing.

VII. Discussion

In the absence of an external perturbation, the equilibrium solutions in a cylinder bifurcate at the tearing threshold (Fig. 2). Below the threshold there is a stable cylindrically symmetric solution. Above the threshold, the cylindrically symmetric equilibrium is unstable to tearing modes. The saturated tearing mode corresponds to an equilibrium solution with islands, which branches off from the cylindrically symmetric equilibrium at the tearing threshold.

An external perturbation breaks the bifurcation at the tearing threshold (Fig. 5). There is again a unique solution at $\Delta' < 0$, now corresponding to a nonsymmetric solution with an island. This stable equilibrium solution goes continuously into the stable equilibrium solution for $\Delta' > 0$.

The breaking of the bifurcation suggests an alternative approach to the calculation of saturated tearing modes (Fig. 12). The conventional approach, indicated in Fig. 12a, starts with an unstable equilibrium solution, and follows the growth and saturation of the island. An alternative approach, indicated in Fig. 12b, is to add a small perturbation and track the equilibrium solution from the $\Delta' < 0$ region into the $\Delta' > 0$ region. This approach has the virtue of allowing the direct calculation of saturated island widths for a range of plasma parameters.

For $\Delta' < 0$, we have employed a simplified model to study the conditions under which the plasma response to an external perturbation is self-healing or amplifying. This model assumes that the island lies in the vacuum region outside the plasma, but makes no further assumptions concerning the current profile. We find that for sufficiently large and negative Δ' the plasma response is self-healing. In some region below $\Delta' = 0$ the plasma amplifies the perturbation. There is a critical value of Δ' (given by Eq. (36)) that marks the boundary between the two regions. For this value of Δ' the island width is just what it would be in the absence of the self-consistent plasma response.

As Δ' approaches zero from below, the amplification can become large. This is to be contrasted with the case where there is no external perturbation, when the island width is zero for $\Delta' \leq 0$. The presence of an error field in a tokamak broadens the range of parameters over which deleterious effects due to tearing are seen.

This conclusion is consistent with the results of error field compensation experiments on DIII-D.⁹ A coil that is capable of partially canceling the error fields has been placed on the tokamak. The threshold for the low density locked mode has been found to vary systematically with the amplitude of the current in the error compensation coil. The lowest densities are obtained with a finite current in the coil, corresponding to a partial cancellation of the field errors. Increasing or decreasing the coil current from this optimum value increases the density threshold.

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Appendix. Relation Between $\Delta'_\epsilon(w)$ and $\Delta'(w)$

The quantity $\Delta'_\epsilon(w)$ introduced in Sec. II is a generalization of the usual Δ'^{10} to include finite ϵ and w . It contains the effects of ϵ through the boundary condition Eq. (10). We can make the dependence explicit by expressing $\Delta'_\epsilon(w)$ in terms of the usual nonlinear $\Delta'(w)$ for $\epsilon = 0$.^{5,7,8} For this purpose we introduce solutions u_i , u_1 , and u_2 satisfying Eq. (9) with boundary conditions given by equations (15-17). We can then write ψ_1 in the form

$$\psi_1 = \begin{cases} a_1 u_1 + \epsilon u_2 & \text{for } r > r_o, \\ a_i u_i & \text{for } r < r_i. \end{cases} \quad (\text{A1})$$

We have

$$r_s \psi_1(r_s) \Delta'_\epsilon(w) = r_o \epsilon u'_2(r_o) + r_o a_1 u'_1(r_o) - r_i a_i u'_i(r_i). \quad (\text{A2})$$

We can similarly write

$$r_s \psi_1(r_s) \Delta'(w) = r_{o0} a_{10} u'_1(r_{o0}) - r_{i0} a_{i0} u'_i(r_{i0}).$$

In these expressions, w is regarded as an independent variable, so $\psi_1(r_s)$ has a fixed value determined by Eq. (3). We have

$$\begin{aligned} \psi_1(r_s) &\approx a_i u_i(r_i) + \frac{w}{2} \psi''_0 s \\ &\approx a_{i0} u_i(r_{i0}) + \frac{w}{2} \psi''_0 s_0, \end{aligned} \quad (\text{A3})$$

where

$$\psi''_0 s = \frac{1}{2} [a_i u'_i(r_i) + a_1 u'_1(r_o) + \epsilon u'_2(r_o)], \quad (\text{A4})$$

and

$$\psi''_0 s_0 = \frac{1}{2} [a_{i0} u'_i(r_{i0}) + a_{10} u'_1(r_{o0})]. \quad (\text{A5})$$

We can also express $\psi_1(r_s)$ in terms of u_1 and u_2 ,

$$\begin{aligned} \psi_1(r_s) &\approx a_1 u_1(r_o) + \epsilon u_2(r_o) - \frac{w}{2} \psi''_0 s \\ &\approx a_{10} u_1(r_{o0}) - \frac{w}{2} \psi''_0 s_0. \end{aligned} \quad (\text{A6})$$

Taylor expanding, and using Eq. (9), we get the relation

$$(r_0 + \delta r)u'(r_0 + \delta r) \approx r_0 u'(r_0) + \delta r \frac{j'_0 R}{B_z \epsilon' w} u. \quad (\text{A7})$$

This gives

$$\begin{aligned} r_s \psi_1(r_s) \Delta'_i(w) &\approx r_s \psi_1(r_s) \Delta'(w) + r_o \epsilon u'_2(r_o) \\ &+ r_o \delta a_1 u'_1(r_o) - r_i \delta a_i u'_i(r_i) - \psi_1(r_s) \frac{j'_0 R}{B_z \epsilon' w} (\delta r_i + \delta r_o), \end{aligned} \quad (\text{A8})$$

with the definitions

$$\delta a \equiv a - a_0,$$

for each of the coefficients, and

$$\delta r \equiv r - r_0.$$

Equations (6) and (7) give

$$\delta r_o = \delta r_i = \delta s. \quad (\text{A9})$$

Expanding Eq. (A3), and retaining terms up to order w^3 , we find

$$\epsilon a_i u_i(r_i) = -w \psi_0'' \delta s / 2. \quad (\text{A10})$$

Equation (A6) gives

$$\delta a_1 u_1(r_o) = -\epsilon u_2(r_o) + w \psi_0'' \delta s / 2. \quad (\text{A11})$$

Substituting equations (A10,A11) into equations (A4,A5), we find

$$\psi_0'' \delta s = \epsilon [u'_2(r_o) - u_2(r_o) u'_1(r_o) / u_1(r_o)]. \quad (\text{A12})$$

Finally, we substitute equations (A9-A12) into Eq. (A8) to get equations (18) and (19).

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Figures

FIG. 1. Division of the domain into regions for the solution of Ampere's law. The interior region is stippled. It is further divided into subregions I and II, separated by the separatrix.

FIG. 2. Schematic plot of $\psi_1(r_s)$ as a function of a profile parameter γ for $\epsilon = 0$. We assume that Δ' increases monotonically with γ .

FIG. 3. Analogous model of a ball in a potential well. The figure shown corresponds to $\gamma < 0$.

FIG. 4. Model of ball in a potential well corresponding to $\gamma > 0$.

FIG. 5. Solution for $\psi_1(r_s)$ with finite ϵ . The dashed line indicates the solution in the $\epsilon = 0$ limit.

FIG. 6. Ball in a potential well with an additional uniform force imposed. This case corresponds to $\gamma < 0$. The arrow indicates the direction of the additional force. The ball is in a stable equilibrium.

FIG. 7. Balls in a potential well with additional uniform force, $\gamma > 0$. Locations of the balls indicate the three time-independent solutions.

FIG. 8. Plot of the steady-state equation for $\Delta' < 0$.

FIG. 9. Plot of the steady-state equation for $\Delta' > 0$.

FIG. 10. Plot of $\Delta'_0 a$ vs r_s/a for an $m = 2$ island lying outside the plasma.

FIG. 11. Island width for a slab model with $j_0 \propto \text{sech}^2$. For each curve, the vacuum island width w_0 is held fixed.

FIG. 12. Schematic view of saturated island calculation by a) conventional time-dependent methods, and b) tracking the equilibrium solution.

FIG. 1

FIG. 2

91T1003

FIG. 4

FIG. 5

91T1006

91T1007

FIG. 7

FIG. 8

FIG. 9

FIG. 10

FIG. 11

a) conventional time-dependent calculation

b) equilibrium calculation

FIG. 12